Black Holes and Moving Mirrors

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Abstract

In this report I treat the subject of black hole radiation. Until the early seventies black holes were considered to be eternal objects in our universe, but the striking analogy with the laws of thermodynamics and the explicit calculation of the black hole radiation, due to quantum effects, by Hawking, showed a different scenario. Hawking recognized already the problems concerning quantum purity due to thermal radiation off the black hole and the occurence of divergent energies due to the gravitational blue shift.

Backreaction effects were neglected by Hawking, but could lead to corrections of Hawkings result and resolve the information problem. Furthermore, including backreaction ensures energy conservation. I included backreaction effects in the simple moving mirror model and found that the temperature of the mirror is reduced by a factor 1/2 relative to the standard result. However, the WKB approximation I used breaks down in the case of dilaton gravity and therefore I could not extend the result to real black holes.

Contents

Preface

This thesis is the conclusion of a year work under supervision of prof.dr Herman Verlinde on the Institute of Theoretical Physics of the University of Amsterdam. The thesis is the final project of my study of theoretical physics.

I tried to write this thesis on the level of graduate students. At the beginning of my research I encountered some problems and questions (some of them trivial) without finding explicit answers. I tried to answer these questions and hope that it might help other students who are trying to understand the difficult subject of evaporating black holes.

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Chapter 1

Introduction

A great deal of effort has been made the last few decades to construct a unified theory of the forces of nature. In 1961 Glashow proposed the gauge symmetry SU(2)×U(1) as the unification of the weak interactions and the theory of quantum electrodynamics (also known as QED, which has gauge symmetry U(1)). The gauge theory with symmetry SU(3), quantum chromodynamics (QCD), became in 1973 a strong candidate for the strong interactions. The standard electroweak theory and QCD together form the standard model. A possible candidate for a grand unified theory (GUT, i.e. a gauge theory with a single coupling constant) is a theory with gauge symmetry SU(5). The last few years there has also been a revival of string theory, which incorporates gravity, and is considered as another serious candidate for the Theory of Everything.

Until now it has not been possible to include the fourth force, namely gravity. Problems arise while constructing a quantum theory of gravity, because it appears that the coupling constant is not dimensionless, making perturbation expansion impossible. Another problem is that gravity couples to everything, because it influences the metric. A graviton is thus both a source of gravitation and subject to gravitation.

Therefore one considers the gravitational field as a classical background, obtaining a semiclassical theory. Hawking considered quantum processes in a classical background geometry and found that vacuum fluctuations near the black hole horizon results in thermal radiation off the hole [1]. Because of those quantum processes the black hole looses mass and can eventually disappear, while black holes were considered classically as eternal objects.

Hawking's result opened a new area in physics and is derived in different ways by several authors. One of the strongest arguments to support the Hawking effect is the deep connection between the laws of black hole physics and the laws of thermodynamics. This is also the major reason why Hawking's discovery is now widely believed to be real.

The paper is organized as follows. I will start more or less historically by discussing the subject of black hole thermodynamics, developed in the late sixties, anticipating Hawking by suggesting that black holes behave as black bodies with a temperature.

In chapter 3 I discuss briefly the problems which arise assuming black holes having a temperature and emitting thermal radiation. The different points of view in literature are given. Instead of treating subsequently the Hawking effect, I will discuss the Unruh effect in chapter 4. The Unruh effect is in some sense more general, because it states that particle production occurs wherever a bifurcate killing horizon appears. Even an accelerating observer in *Minkowski* spacetime detects thermal radiation. The breakdown of Poincare invariance as we have in curved spacetime is already manifest in non-inertial systems in flat spacetime. In chapter 5 the Hawking effect is explicitly calculated, giving a strong support to the theory of black hole thermodynamics of chapter 2. In Hawking's calculation pure states can evolve into mixed states hereby violating the basic principle of unitary evolution in quantum mechanics. In chapter 3 it became clear that backreaction effects are important and may lead to corrections of Hawking's result. Furthermore, due to the gravitational redshift the energy near the horizon diverges. In order to investigate back reaction effects and to implement energy conservation I treat in chapter 6 a model, which mimics many of the features of gravitational collapse. With this *moving mirror* model I showed that including backreaction effects and thus energy conservation, leeds to a correction of the temperature by a factor half relative to the standard result.

Chapter 2

Black Hole Thermodynamics

2.1 Black Hole

Before we start talking about black holes I will give a short review of what a black hole is. I will use units where $\hbar = c = 1$.

In 1916 Einstein found his classical field equations of general relativity,

$$G^{\mu\nu} = 8\pi T^{\mu\nu}. \tag{2.1}$$

If one imposes the metric to be static and spherically symmetric one finds an exact solution for the vacuum field equations $G^{\mu\nu} = 0$, namely the Schwarzschild solution, which has line element

$$ds^2 = (1 - \frac{2m}{r})dt^2 - \frac{1}{1 - \frac{2m}{r}}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{2.2}$$

This solution is physically interesting, because the objects in our universe are mostly spherically symmetric and the surrounding space can be considered empty. We see that the line element is singular at the coordinates $r = 0$ and $r = 2m$. The *coordinate* singularity at $r = 2m$ is however removable by performing a coordinate transformation. The other singularity, at $r = 0$, is an *intrinsic* singularity and is not removable by coordinate transformations. Although $r = 2m$ is no real singularity, it remains an

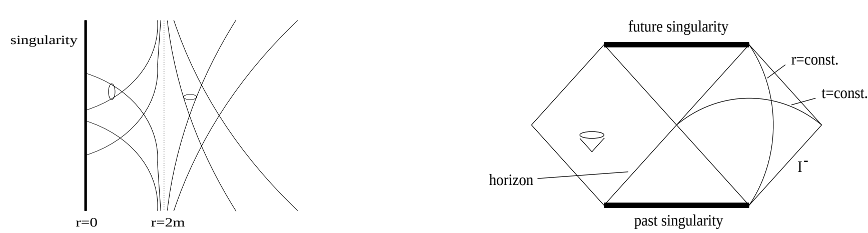

Figure 2.1: Two different figures of a black hole: a) Schwarzschild solution. b) Compactified Kruskal solution.

interesting point (or sphere), called the Schwarzschild radius. If one writes the solution in Eddington-Finkelstein coordinates, one sees the lightcones pointing towards the center of the hole and it becomes clear that nothing can come back after passing the Schwarzschild radius; $r = 2m$ is therefore called an event horizon. It is impossible to become aware of events that take place behind the horizon. Therefore this object is referred to as a "black hole".

By performing a suitable coordinate transformation one removes the coordinate singularity at $r = 2m$ obtaining the Eddington-Finkelstein coordinates. The Kruskal solution is the maximal analytic extension of the geometry (i.e. every geodesic ends in infinity or in a singularity) and can be compactified by mapping the infinite interval to a finite interval. In this way one obtains the penrose diagram of figure 2.1 b).

We can imagine different types of black holes. Beside the collapsing star, we can imagine huge black holes formed after a collapsing star cluster. On the other hand we can imagine so called primordial black holes with masses of $\sim 10^{12}$ kg. One often hears the question if elementary particles could be black holes. The answer is no, because the Compton wavelength of a particle is much bigger then its Schwarzschild radius. Those two length scales are equal (up to a factor $\sqrt{\pi}$) at the Planck mass:

$$\frac{2Gm}{c^2} = \frac{2\pi\hbar}{mc} \quad \Longrightarrow \quad m_{Pl} \approx 10^{-8} kg \tag{2.3}$$

By combining the fundamental constants $c, \hbar$ and G one finds units of length, time and mass, the so called *Planck units*:

$$m_{Pl} = \sqrt{\frac{c\hbar}{G}} \approx 2.18 \cdot 10^{-8}\, kg \tag{2.4}$$

$$l_{Pl} = \sqrt{\frac{G\hbar}{c^3}} \approx 1.62 \cdot 10^{-35}\, m$$

$$t_{Pl} = \sqrt{\frac{G\hbar}{c^5}} \approx 5.39 \cdot 10^{-44}\, s$$

$$E_{Pl} = m_{Pl} c^2 \approx 1.22 \cdot 10^{19}\, GeV$$

The Planck values mark the frontier where a full theory of quantum gravity is indispensable.

As we will see in the following chapters quantum effects alter the course of gravitational collapse. As seen by an asymptotic observer, collapsing matter evaporates before forming a real curvature singularity (see figure 3.1)! However, after stellar collapse the evaporation of a large heavy object proceeds very slowly, so we refer to this object as a black hole, keeping in mind that it is a rather classical notion.

2.2 Thermodynamics

In the late sixties and begin seventies a lot of work has been done on the theory of black holes. For example one has calculated stimulated emission in the ergo region of a rotating black hole (or even a neutron star), the so called *superradiance*. One of the most remarkable results obtained was the striking analogy between the laws of thermodynamics and the black hole laws (see for a recent review [2]).

It was found that classically energy extraction from black holes is possible as long as the black hole area does not decrease. This can be done for example by dropping an opposite charged particle in a charged black hole, by diminishing angular momentum of a rotating black hole or by letting two black holes collide, which form one bigger black hole. The analogy between entropy and the horizon area of the black hole was made.

$$\Delta A \geq 0 \quad \Longleftrightarrow \quad \Delta S \geq 0 \tag{2.5}$$

The Second Law of thermodynamics, stating that the total entropy never decreases, has now an analog in black hole physics with Hawking's area theorem.

It turns out that the analogy of the First Law ($dE = TdS$) in black hole theory becomes

$$dM = \frac{\kappa}{8\pi G} dA + \Omega dJ + \Phi dQ, \tag{2.6}$$

where A, J and Q are respectively the area, angular momentum and charge and Ω and Φ are the angular velocity and electric potential of the horizon. The surface gravity is denoted by κ ($= 1/4M$ for a non-rotating neutral black hole). This formula suggests that κ plays the role of temperature. By introducing quantummechanics Hawking found in 1975 that a black hole has indeed a temperature $kT_H = \frac{\kappa}{2\pi}$! This equivalent of the first law ensures energy conservation during black hole processes like superradiance or radiation from rotating and charged black holes whereby the hole looses angular momentum or charge.

The constancy of κ on the horizon is the analogy of the Zeroth Law, which states that the temperature is uniform everywhere in a system in thermal equilibrium.

The Third Law (the temperature can not be reduced to zero in a finite number of steps) has its analogy in the fact that zero surface gravity requires a black hole with infinite mass. There are however some difficulties with charged black holes. For an extremal black hole (a hole which is maximally charged, $e^2 = M^2$) the temperature would be reduced to zero (see (5.23)). The question is if the extremal limit can in principle be reached.

Concluding we can write down an explicit expression for the entropy by comparing mass and thermodynamic energy (which is not so strange knowing that the mass of a black hole is the same quantity as its energy), so we find for the entropy

$$dS = \frac{dE}{T} = 8\pi M k_B dM;$$

$$S_{bh} = 4\pi k_B M^2 = \frac{1}{4} k_B A. \tag{2.7}$$

This is the famous Bekenstein-Hawking entropy. When a black hole radiates, the mass of the black hole and thus the entropy decreases. To ensure the validity of the second law, which states that entropy always increases, we get the Generalized Second Law

$$S = S_{th} + S_{bh} \tag{2.8}$$

with S_{th} the thermodynamic entropy. The black hole entropy radiates away in the form of thermodynamic entropy. The laws of thermodynamics and the laws of black holes are so strongly related that the laws of black holes actually *are* the laws of thermodynamics.

In chapter 5 I will give the derivation of the Hawking effect and will show that in SI units the Hawking temperature is given by $T = \frac{\hbar c^3}{8\pi M k_B G}$, and thus the expression for the black hole entropy S_{bh} [1] becomes

$$S_{bh} = \frac{1}{4} k_B \frac{A}{G\hbar/c^3} \tag{2.9}$$

In this form it becomes visible that the area is expressed in units of the Planck length squared. Entropy is a measure of the containing information and therefore it is suggested that a Planck length squared contains a limited amount of information. This is in contradiction with the idea that a black hole can be made in unnumerable different ways and that therefore the entropy should be infinite. The problem is our lack of knowledge about Planck scale physics, but a possible candidate to resolve these problems is string theory. More details on the information problem are given in chapter 3. It is also visible that (2.9) is clearly a quantum mechanical result; in the classical limit $\hbar \to 0$ the entropy goes to infinity.

[1] The subscript "*bh*" conveniently stands for both "Bekenstein-Hawking" and "black hole".

Chapter 3

Information Problem

In the last chapter I mentioned the fact that black holes have a temperature $T_H = \frac{1}{8\pi M}$ and radiate thermal radiation just as a black body of temperature T_H. Imagine now forming a black hole out of pure quantum states. The black hole subsequently starts radiating thermal radiation which are *mixed* states. This is a serious violation of quantum mechanics, where pure states evolve unitarily to pure states, $U|\psi_0\rangle = e^{-i\omega t}|\psi_0\rangle = |\psi_t\rangle$. Furthermore, we see that probability is conserved when the evolution operator is unitary; $\langle\psi_t|\psi_t\rangle = \langle\psi_0|U^\dagger U|\psi_0\rangle = \langle\psi_0|\psi_0\rangle = 1$. One could also say that information is lost in the Hawking process.

You might wonder what the point is if you consider the following process where information is absorbed, thermalized and radiated away. Consider a cold piece of black cole which absorbs the signal of a coherent laser, starts burning and subsequently emits the laser signal in the form of thermal radiation. The difference between this process and the Hawking process, is that the former is a *macroscopic* description without specifying the microscopic process of the coal. In the black hole case the outside observer sees the information being stored somehow *microscopicly* on the stretched horizon before being radiated away. The stretched horizon is a physical membrane just outside (about a few Planck lengths) the event horizon.

To give some insight in how a pure state could apparently evolve in a mixed state I will give a general mixing mechanism, [3]. Consider a quantum theory with Hamiltonian $H(\alpha)$. A pure state $\psi(0)$ evolves for each α in a pure state $\psi(t)$ at time t:

$$\psi(t) = e^{-iH(\alpha)t}\psi(0). \tag{3.1}$$

But suppose that α is poorly known and that it has a probability distribution $P(\alpha)$. The expectation value of an operator O becomes a mixed state

$$\langle O\rangle = \sum_\alpha P(\alpha)\langle\psi(t)|O|\psi(t)\rangle. \tag{3.2}$$

A black hole can be made in approximately $e^{A/4}$ different ways, assuming that a black hole has entropy $A/4$. Recall that the entropy of a macrostate is equal to the logarithm of possible microstates, $S = \frac{A}{4} = ln(\#m.s.)$. The number of microstates can be identified with the different possibilities to make a black hole. The uncertainty in the Hamiltonian H (which 'forms' the black hole) possibly gives rise to a mixed state.

Discussions about the information problem resulted in three different directions which may lead to the solution of the problem.

3.1 Remnants

The first possibility is that the information is stored in the last 10^{-5} g, the Planck mass, [4]. Describing such a system would require knowledge of Planck scale physics, which we do not have [5].

What we *can* say is that if all the information is stored in the last 10^{-5} g, then the information could only be radiated away in the form of a very large number of socalled soft particles, for example photons with low energy. The time to radiate all the information in this way turns out to be very large

$$\tau_{10^{-5}g} \sim \left(\frac{M}{m_{pl}}\right)^4 t_{pl},$$

with M the original mass of the hole, which gives rise to the idea of a long-lived remnant. A black hole may even leave a *stable* remnant at the end of his live.

This solution might seem to be an easy way out, but at least the basic principles of quantum mechanics are not violated.

However, one other principle is now violated, namely crossing symmetry. A particle that couples to the electromagnetic field implies by crossing symmetry Schwinger pair production. The gravitational analog is Hawking radiation. The decay rate of a black hole is proportional to the number of species which can be produced. A small black hole should therefore decay with a high rate. Leaving a remnant would mean violation of crossing symmetry.

3.2 Nice Slices

Another possibility is that the black hole disappears and the information is simply lost [6]. An argument to support this idea is the socalled *nice slice argument.*

The argument is based on the fact that almost the entire evolution of a black hole involves low-energy particles, i.e. energies far below the Planck scale. Only in the final stage of the evaporation process one has to deal with high energies, but there would be too little particles left to carry all the information and is thus considered to be irrelevant. The history of a black hole may therefore be described by ordinary low-energy effective field theory, i.e. a field theory with a cutoff in order to avoid large energies.

To show that only low energy physics is involved we foliate the geometry with a family of Cauchy surfaces. We call a family of surfaces *nice slices* if the surfaces avoid regions of strong curvature and if both the infalling matter and the Hawking particles have low energies in the frame of the surface. Hence a nice slice does not contain the region inside the black hole. If one chooses the nice slices to agree with surfaces of constant time far from the hole, then one could parametrize the surfaces by t and an example of a full set of nice slices could be the following

$$uv = R^2, \qquad v < e^{t/2GM}u$$

$$e^{t/4GM}u + e^{-t/4GM}v = 2R, \qquad v > e^{t/2GM}u$$

At this point one may introduce the nice slice Hamiltonian which generates the time evolution. The Hamiltonian maps the state on one slice to a state on the neighboring slice,

$$i\partial_t|\Phi\rangle = H_{ns}|\Phi\rangle. \tag{3.3}$$

One has to realize that the nice slice argument involves information loss which implies a violation of unitarity (a breakdown of quantummechanics!).

The nice-slice theory is a local quantum field theory and therefore spacelike separated operators will commute. Consider a late time slice Σ and separate it in two parts, one part behind the horizon Σ_{in} and one outside the hole, Σ_{out}. According to (3.3) the states evolve linearly to another time slice, but according to the "no quantum Xerox principle" the initial information can not evolve *completely* to two *separated* sets of commuting degrees of freedom. The problem arises if one considers the infalling information being recorded inside the hole. Since an infalling observer sees nothing unusual happen while passing the horizon, little or no information can be found outside the horizon in the form of Hawking radiation or stored in the stretched horizon (a surface which is a few Planck lengths away from the event horizon). The information does not come out and is thus lost after the evaporation of the hole.

Furthermore, information transmission requires energy and therefore also energy conservation is violated. If evolution is not described by a unitary operator then the Hamiltonian is time dependent and energy is not conserved. But as we will see, none of the possible solutions are ideal.

3.3 Complementarity, S-Matrix Ansatz

The third possibility is that the information is stored in some way in the out-going Hawking radiation. The key-point is to take interactions into account, which were neglected by Hawking (see [7, 8, 9, 10]). Massless scalarfields can be regarded as a superposition of left- and rightmoving parts. Leftmoving waves after a certain time after gravitational collapse, will disappear behind the horizon. Those waves however could influence the outgoing waves and hence the thermal spectrum. Another way of seeing it is that incoming particles interact with the outgoing particles, because the infalling matter enlarges the Schwarzschild radius, or v_0, and changes the total black hole geometry. The variation in v_0 is very small, but the effect on the wave-function of an outgoing particle is enormous. Taking this small quantum contribution into account, one finds that the in- and outgoing fields no longer commute.

The idea is to *assume* that the evolution of the black hole Hilbert space is described by a unitary operator $U(t)$. One refers to this approach as the S-matrix Ansatz, where the S-matrix working on the in-state gives the out-state

$$|out\rangle = S|in\rangle \tag{3.4}$$

The S-matrix Ansatz only makes sense if one takes interactions into account; the in- and out-states are independent without interaction between them and if so there could be no matrix connecting them.

Because of the unitary evolution of black hole states, the Hilbert-space is only allowed to contain *either* all in-states[1] *or* all out-states, but not both sets simultaneously, since we expect the S-matrix to connect these Hilbert-spaces. The in-states evolve either into the black hole or out to null infinity J^+, dependent on the position of the observer. In fact, those two worlds are complementary, the information is both evolved into the hole as well as out to infinity! An infalling observer will see the infalling matter (inclusive information) propagate without interaction or perturbation into the hole. The observer at J^+, the asymptotic observer, sees Hawking radiation and the infalling matter appears to evaporate (inclusive information) completely before it falls in the hole. Actually, one might say that there is no singularity formed at all! It looks like a quantum copying machine, where the infalling matter is the complementary description, or the duplicate, of the outgoing matter. A justification for this view can be the fact that so called superobservers can not exist, i.e. it is not possible for ordinary people to do experiments simultaneously on both sides of the horizon. A similar paradox occurs in quantum mechanics where we can consider light also in two different ways, as a particle or as a wave.

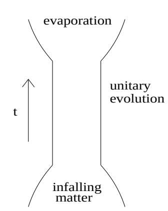

Figure 3.1: Infalling matter evolves unitarily to Hawking radiation without having formed a singularity.

General relativity seems to be violated by having created two different (complementary) worlds. There is not necessarily a relation between the *two* different Hamiltonians of the two worlds, i.e. there is not necessarily a transformation from one world to its complementary partner and reality is described by only *one* Hamiltonian. Furthermore, the fact that the in- and out-going fields no longer commute seems to violate locality and causality[2], because events behind the black hole horizon are spacelike separated

[1] Here we mean with in-states, those in-states which are evolved into the hole.

[2] The causality condition means that no signal can travel faster than the speed of light.

from events outside the horizon and therefore they can not influence each other. However, commutators in light front string theory remain large even when observers are separated by a macroscopic distance. String theory can possibly solve the paradox and is a strong candidate for a quantum theory of gravity.

Chapter 4

The Unruh Effect

As a result of efforts to understand the Hawking effect, Unruh discovered the effect that a uniformly accelerated observer detects thermal radiation of temperature $T_U = \frac{a}{2\pi}$, [11]. Although the Unruh effect was discovered *after* the Hawking effect I will treat first the Unruh effect because it is in some way underlying the Hawking effect. The point is that even our well known Minkowski vacuum has a thermal character.

Consider Minkowski spacetime, in 1+1 dimensions for convenience, with line element $ds^2 = dt^2 - dx^2$. We transform it conformally[1] to the so-called Rindler coordinates (η, ξ)

$$ds^2 = e^{2a\xi}(d\eta^2 - d\xi^2) \tag{4.1}$$

where the coordinate transformation is given by

$$t = \frac{1}{a}e^{a\xi}\sinh a\eta,$$

$$x = \frac{1}{a}e^{a\xi}\cosh a\eta. \tag{4.2}$$

In this coordinate system a uniform accelerating observer travels along a line with constant ξ (see figure 4.1), with proper acceleration $ae^{-a\xi}$. Furthermore we see that in this form the boost symmetry, generated by the killing vector $\frac{\partial}{\partial\eta}$, is manifest [2]. A non-infinitesimal Lorentz boost along the x-axis is a transformation from system S to system S' with relative velocity v in the x-direction between the systems. Note that the translation symmetries generated by $\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial x}$ are not manifest anymore.

We now have constructed the so-called *Rindler space*. This spacetime is said to be *globally hyperbolic* because it admits a *Cauchy surface*. A Cauchy surface is a sort of 'time slice'. More precisely, if every (past or future) inextendible causal curve (through an arbitrary point of spacetime) intersects a surface Σ, then Σ is said to be

[1] Conformal transformation: $\bar{g}_{ab} = \Omega^2 g_{ab}$. The lightcones remain unchanged in a conformal transformation.

[2] Recall that a boost along the x-axis is equivalent to an imaginary rotation in (x, it)-space; $\sinh\eta = i\sin i\eta$

a Cauchy surface. For examples the lines $\eta = const.$ are Cauchy surfaces in Rindler space. Because of the admittance of Cauchy surfaces, one has a well defined classical evolution on globally hyperbolic spacetimes and Rindler space is thus a spacetime in its own right. We can therefore apply the standard procedure to construct quantum fields. If both sets are complete one can expand one set in terms of the other set and conversely. For example, if we have two expansions ϕ and $\bar{\phi}$ with sets of mode solutions u_k and $\bar{u}_k$ one can write the relation between the two sets as follows:

$$\bar{u}_j = \sum_i (\alpha_{ji} u_i + \beta_{ji} u_i^*) \tag{4.3}$$

$$u_i = \sum_j (\alpha_{ji}^* \bar{u}_j - \beta_{ji} \bar{u}_j^*)$$

with the coefficients α and β

$$\alpha_{ij} = (\bar{u}_i, u_j), \qquad \beta_{ij} = -(\bar{u}_i, u_j^*) \tag{4.4}$$

These relations are known as *Bogolyubov transformations* with α_{ij} and β_{ij} the *Bogolyubov coefficients*. With these equations in hand, and knowing that ϕ and $\bar{\phi}$ describe the same quantumfield, one obtains relations between the creation and annihilation operators from the different expansions

$$a_i = \sum_j (\alpha_{ji} \bar{a}_j + \beta_{ji}^* \bar{a}_j^\dagger) \tag{4.5}$$

$$\bar{a}_j = \sum_i (\alpha_{ji}^* a_i - \beta_{ji}^* a_i^\dagger)$$

The coefficient β is important, because it gives you the number of u-mode particles present in the "vacuum" $|\bar{0}\rangle$:

$$\langle \bar{0}| a_i^\dagger a_i |\bar{0}\rangle = \sum_j |\beta_{ji}|^2 \langle \bar{0}| \bar{a}_j \bar{a}_j^\dagger |\bar{0}\rangle = \sum_j |\beta_{ji}|^2 \tag{4.6}$$

(Recall that the number operator $N_i = a_i^\dagger a_i$ counts the number of i-particles.)

Now I will construct the quantum fields explicitly. A massless quantum field in 1+1 dimensional Minkowski spacetime satisfies the Klein-Gordon equation

$$\Box \phi = (\partial_t^2 - \partial_x^2)\phi = 0 \tag{4.7}$$

with mode solutions

$$u_k(t,x) = \frac{1}{\sqrt{4\pi\omega}} e^{ikx - i\omega t} \quad , \omega = |k| > 0. \tag{4.8}$$

The modes are said to be positive frequency modes with respect to the Minkowski time t, because they are eigenfunctions of the killing vector $\frac{\partial}{\partial t}$. Note that if $\beta \neq 0$

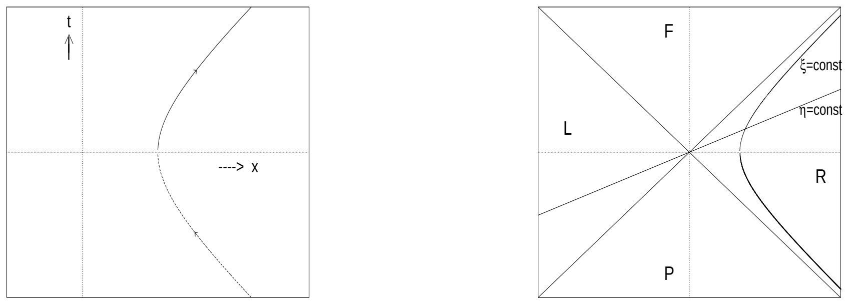

Figure 4.1: a) The trajectory of a uniformly accelerating observer. b) Rindler space with lines of constant ξ and η.

then $\bar{u}$ is a mixture of positive frequency modes u and negative frequency modes u^*, see (4.3). So if a pure positive frequency mode in one system corresponds to a mixture of positive and negative frequency modes in another system, then the vacua in both systems will differ.

The system is quantized by treating the field ϕ as an operator. ϕ may be expanded as follows:

$$\phi(t,x) = \sum_k [a_k u_k(t,x) + a_k^\dagger u_k^*(t,x)], \tag{4.9}$$

with u the mode solutions (4.8) and $a, a^\dagger$ the annihilation, respectively the creation operators. We define (ϕ_1, ϕ_2) as the usual scalar product and ϕ and a satisfy the canonical commutation relations

$$[\phi(t,x), \phi(t,x')] = 0 \tag{4.10}$$

$$[a_k, a_{k'}^\dagger] = \delta_{kk'}.$$

This procedure is also known as *second quantization* in order to distinguish the quantization in quantum field theory, with ϕ an operator, from the ("first") quantization in ordinary quantum mechanics, where one imposes commutation relations upon the observables and where ϕ is a wavefunction.

In exactly the same way we can construct quantum fields in Rindler coordinates. The equation satisfied by the (massless) Klein-Gordon field in Rindler coordinates is

$$e^{2a\xi}\Box\phi = (\partial_\eta^2 - \partial_\xi^2)\phi = 0, \tag{4.11}$$

with mode solutions

$$u_k = \frac{1}{\sqrt{4\pi\omega}} e^{ik\xi \pm i\omega\eta}. \tag{4.12}$$

Note that we obtained here two classes of mode solutions. The upper sign applies in region L, the lower sign in region R, because the killing vector is past-directed in region

L; with increasing η, the slope of the line $\eta = const.$ is increasing and one can see from figure 4.1 that $\frac{\partial}{\partial \eta}$ is then past-directed in region L. Note also that Rindler observers in region R are completely disconnected from region L and vice versa.
We can expand the field in this second set of modes:

$$\phi(x,t) = \sum_{k=-\infty}^{\infty} (b_k^L u_k^L + b_k^{L\dagger} u_k^{L*} + b_k^R u_k^R + b_k^{R\dagger} u_k^{R*}). \tag{4.13}$$

As both sets (4.9) and (4.13) are complete, one can expand the Rindler modes in terms of the Minkowski modes and conversely. Constructing the Bogolyubov transformations one gets the relations between the different sets of mode solutions. Equating (4.9) and (4.13) and using the Bogolyubov transformations, it is possible to deduce relations between the ladderoperators and thus one can express the Minkowski vacuum in terms of Rindler modes, as in (4.6).

Here I will use a slightly different method, which was used by Unruh, by *first* making use of analytic continuation across the horizons of the mode solutions (4.12) to the regions F and P (obtaining an imaginary a). *Then* we construct the Bogolyubov transformations between the new modes and the Rindler modes. The trajectory of a uniform accelerating observer approaches the lines $u = 0$ and $v = 0$ for which $\eta \to \pm\infty$. The Rindler modes are thus non-analytic there, but special linear combinations are. The new modes are analytically continued to the other side of the horizon. The appearance of horizons is crucial to get thermal radiation. The new modes have the same analycity properties as the Minkowski modes and therefore they share the same vacuum, provided that the analytic continuation is performed in the lower half complex plane, i.e. $\ln(-1) = -i\pi$, not $+i\pi$, fixing the sign of the exponent which gives rise to the thermal spectrum.
The two combinations are

$$u_k^R + e^{-\pi\omega/a} u_{-k}^{L*} \tag{4.14}$$

and

$$u_{-k}^{R*} + e^{\pi\omega/a} u_k^L.$$

One can see why these combinations are analytic and bounded in the lower half complex Minkowski plane, just as the Minkowski modes (4.8), by writing

$$e^{a\eta + a\xi} \approx v \quad \Longrightarrow \quad u_k^R + e^{-\pi\omega/a} u_{-k}^{L*} \approx \begin{cases} v^{-i\omega/a} & k < 0 \\ u^{i\omega/a} & k > 0 \end{cases} \tag{4.15}$$

with u, v the Minkowski null coordinates, $u = t - r$ and $v = t + r$.

One can expand the ϕ-field in the new modes as follows:

$$\phi = \sum_{k=-\infty}^{\infty} \frac{1}{\sqrt{2\sinh(\pi\omega/a)}} \times \tag{4.16}$$

$$\left[d_k^{(1)} (e^{\pi\omega/2a} u_k^R + e^{-\pi\omega/2a} u_{-k}^{R*}) \; + \; d_k^{(2)} (e^{-\pi\omega/2a} u_{-k}^{R*} + e^{\pi\omega/2a} u_k^L) \right] + h.c.$$

The Bogolyubov transformations between the new modes (4.14) and the Rindler modes (4.12) are easily obtained. By taking the inner products (ϕ, u_k^R), (ϕ, u_k^L) with ϕ from the new expansion and from the Rindler expansion, one finds the Bogolyubov transformation between the Rindler operators b and the operators d which act on the Minkowski vacuum:

$$b_k^{(1)} = \frac{1}{\sqrt{\sinh \pi\omega/a}}(e^{\pi\omega/2a} d_k^{(2)} + e^{-\pi\omega/2a} d_k^{(1)\dagger}) \tag{4.17}$$

and simular for $b_k^{(2)}$. To know what Rindler particles are present in the Minkowski vacuum, we calculate

$$\langle 0_M | b_k^{L,R\dagger} b_k^{L,R} | 0_M \rangle = \frac{e^{-\pi\omega/a}}{\sqrt{2\sinh(\pi\omega/a)}} = \frac{1}{e^{2\pi\omega/a} - 1} \tag{4.18}$$

which is precisely the Planck spectrum of a black body with temperature $T_U = \frac{a}{2\pi}$ (see appendix B).

A uniformly accelerated observer, or Rindler observer sees the red-shifted temperature

$$T_U = \frac{a e^{-a\xi}}{2\pi}. \tag{4.19}$$

The proper acceleration is $ae^{-a\xi}$, since the lines with constant η are the hyperbolae $x^2 - t^2 = \frac{1}{a^2 e^{-2a\xi}}$ and the temperature seen by the accelerating observer is proportional to the *proper* acceleration. Furthermore, the red-shifted temperature is $T = \frac{1}{\sqrt{g_{00}}} T$ and from (4.1) we see that $g_{00} = e^{2a\xi}$. In the same way one can show that an inertial observer sees himself immersed in a thermal bath of *Rindler* particles.

A way to understand the Unruh process is to imagine vacuum processes where pairs of virtual particles and anti-particles are created. One of the particles is detected by the accelerating observer, while its counterpart passes the horizon $u = 0$ and gets eternally disconnected from the world of the ever accelerating observer, wedge R.

In SI units the temperature is given by

$$T_U = \frac{\hbar a}{2\pi k_B c} = a \cdot 4.05 \cdot 10^{-21} \, \frac{K}{m/s^2} \tag{4.20}$$

which shows that it is very difficult, if not impossible to see the Unruh effect in the laboratory. However, recently the Unruh effect was mentioned in the context of sonoluminesence, a phenomenon which is still unexplained [12]. A related geometrical effect, which is detected experimentally, is the Casimir effect. By imposing certain boundary conditions one alters the topology and can give non-zero vacuum stress energy. Casimir calculated that two conducting surfaces at close distance ($\approx 10^{-6} m$) feel attractive forces in the vacuum, due to vacuum fluctuations. Between the plates we have the situation that not *all* the fluctuations are aloud, because the waves have to "fit" between the plates. Outside the plates there are more fluctuations which results in a pressure and a measurable force.

But there are more methods to derive the Unruh effect, [13]. In statistical mechanics we have the KMS-condition, named after Kubo, Martin and Schwinger, which says that the property of periodicity in imaginary time is associated with states of finite temperature in field theory. Formally one can write the KMS-condition as follows:

For each pair of local observables A, B, there is a function F(z) of the complex variable z, that is analytic in the strip $\mathrm{Im}\ z \in (0, \beta)$ *and continuous on its boundaries, such that* $F(t) = \rho(AB_t)$ *and* $F(t + i\beta) = \rho(A_t B)$ *for real t.*

Simply one writes:

$$\langle A_{-i\beta} B\rangle_\beta = \langle BA\rangle_\beta \tag{4.21}$$

defining $\langle A\rangle_\beta$ to be the expectation value $\mathrm{Tr}(\rho A)$, β the inverse temperature, and A_t to be the time translation by t of the operator A. I will prove that the KMS-condition is equivalent to that of local thermodynamical stability.

We have

$$A_t = e^{itH} A e^{-itH}$$

so, with $t = -i\beta$,

$$\langle A_{-i\beta} B\rangle_\beta = Z^{-1}\mathrm{Tr}(\rho e^{\beta H} A e^{-\beta H} B)$$

with ρ the density matrix. In order to satisfy the KMS condition we get the condition

$$\mathrm{Tr}(\rho e^{\beta H} A e^{-\beta H} B) = \mathrm{Tr}(\rho BA)$$

or, by using the cyclical property of the trace,

$$e^{\beta H}\rho B e^{-\beta H} = \rho B \quad \Longrightarrow \quad e^{\beta H}\rho B = B\rho e^{\beta H}.$$

By putting $B = I$, one sees that ρ commutes with $e^{\beta H}$. This means that $e^{\beta H}\rho$ commutes with arbitrary B and must be of the form cI with c a scalar. As $\mathrm{Tr}\rho = 1$, ρ must be the canonical density matrix $\rho_c = Z^{-1}e^{-\beta H}$. In other words, ρ_c is the only density matrix that satisfies the KMS-condition.

Because of $G(\eta, \eta') = f(\sinh(\eta' - \eta))$ and $\sinh(x - i\pi) = \sinh(-x)$, it can be shown that the two-point function $G(\eta, \eta') = \langle \phi(\eta), \phi(\eta')\rangle$ along a uniformly accelerated worldline in Minkowski vacuum satisfies the KMS condition!

$$G(\eta - \frac{2\pi i}{a}, \eta') = G(\eta', \eta) \tag{4.22}$$

The two-point correlation function is a Green function and is an important entity. The two-point function is defined as the vacuum expectation value of the time ordered product of the fields $\phi(x)$ and $\phi(y)$, $G(x, y) = \langle 0|T\phi(x)\phi(y)|0\rangle$ and give information about the system. With the two-point correlation function in hand one knows the evolution of the system and one can calculate the observables. Along a uniformly accelerated worldline the Minkowski vacuum is a thermal state.

$$|0_M\rangle = \prod_j \Big[N \sum_{n_j} e^{-\pi n_j \omega_j / a} |n_j, R\rangle \otimes |n_j, L\rangle \Big] \tag{4.23}$$

Another way to analyse the thermal properties of the Minkowski vacuum is to look at the intimate relation between the topological structure and the particle concept. We already noticed the importance of horizons. In fact, the key geometrical structure in the Minkowski spacetime analysis which gives rise to the thermal state property of the vacuum, is the *bifurcate killing horizon* generated by the Lorentz boost isometries. In Rindler, Kruskal and De Sitter spacetime two killing horizons appear which divide the spacetime into different sections or wedges. A pair of intersecting killing horizons is called the bifurcation horizon. The killing field $\partial/\partial\eta$ is normal to (null plane) horizons and therefore we get non-analytic behaviour across the horizon. Analytic continuation across the horizon gives rise to the factor $e^{-\pi\omega/k}$ which is responsible for the thermal spectrum.

The notion of vacuum has become subjective. We can not speak anymore about *the* vacuum. So when one speaks about "the" vacuum, what do they mean? The conventional vacuum, "the" vacuum, is the agreed vacuum for all inertial observers. Both the conventional vacuum and the set of inertial observers (in Minkowski space) are invariant under the poincaré group[3]. From (4.20) we see that the temperature indeed vanishes if the acceleration is equal to zero. Furthermore, "the" vacuum becomes only ill defined when huge accelerations or very strong gravitational fields are involved.

4.1 Particle Detector

In order to understand the Unruh effect better and to clarify the obscure notion of "particles", one has developed models of "particle detectors"[4]. The essential feature of a particle detector is the interaction with the field. After interaction the state of the detector has changed. One can show that a non-inertial detector will not detect the same particle density as an inertial one.

In literature one finds two models. One model is a particle detector which consists of an idealized point particle with internal energy levels. The interaction Lagrangian is described by $m(\tau)\phi[x(\tau)]$, with $m(\tau)$ the detectors monopole moment, and the detector couples via monopole interaction to the field ϕ. The other model (Unruh and Wald, [14]) is essentially a "particle in a box" detector, which is simplified by allowing the detector only two energy levels. I will treat the second model in more detail. We find

[3] A poincaré transformation is a proper Lorentz transformation followed by a translation. The generators of the poincaré group are the six generators of the lorentz group (3 from rotations and 3 from boosts) plus the four generators of the translation group. The poincaré group is thus the group of rotations (3), spacetime translations (4) and Lorentz boosts (3).

[4] Don't confuse here particle detectors with bubble chambers.

a remarkable paradox involving the change of the field energy during detection. In the following it will be shown that the detector "detects", i.e. the detector will be found in its excited state $|\uparrow\rangle$ when it absorbs a quantum of the field.

The total hamiltonian of the field-detector system is

$$H = H_F + H_D + H_I \tag{4.24}$$

where H_F is the free Klein-Gordon hamiltonian of the massless ϕ-field, $H = \int dx[\frac{1}{2}\pi^2 + \frac{1}{2}(\nabla\phi)^2]$. For the detector hamiltonian H_D we take

$$H_D = \Omega A^\dagger A. \tag{4.25}$$

The raising and lowering operators are defined by

$$A^\dagger|\downarrow\rangle = |\uparrow\rangle \tag{4.26}$$

$$A|\uparrow\rangle = |\downarrow\rangle$$

where the kets represent the state of the detector. The coupling of the detector to the ϕ-field is assumed to be given by the interaction Hamiltonian

$$H_I = \epsilon(t)\int \Phi(x)[\Psi(\vec{x})A + \Psi^*(\vec{x})A^\dagger]\sqrt{-g}d^3\vec{x} \tag{4.27}$$

where $\Psi(\vec{x})$ is a smooth function which vanishes outside the detector and $\epsilon(t)$ is the coupling constant with explicit time dependence in order to enable us to turn the detector on and off.
The system, i.e. a state $|s\rangle$ at time t_1 evolves to time t_2 via

$$|s, t_2\rangle = e^{-i\int_{t_1}^{t_2} H dt}|s, t_1\rangle. \tag{4.28}$$

Let us take the initial state at early times to be a state where the detector is in its lowest state while the field is containing n particles, thus $|n, \downarrow\rangle$. At late times the system will be found, in first order in ϵ, in the state

$$|s_\infty\rangle = |n; \downarrow\rangle - i\int e^{i\Omega t'}\epsilon(t')\Psi^*(\vec{x})\Phi_0(x')\sqrt{-g}d^3\vec{x}' dt'|n; \uparrow\rangle. \tag{4.29}$$

It can be shown with some tedious algebra that

$$\int f\Phi_0\sqrt{-g}d^4x \approx a \tag{4.30}$$

for any function f (for "example" $f = \epsilon e^{i\Omega t}\Psi^*$). Here a is the annihilation operator for the free field.

We can conclude that our simple model does function properly as a detector because the detector will be excited if and only if one quantum of the field is absorbed, due to the appearance of the annihilation operator a.

We now let the detector uniformly accelerate by taking the boost killing field as its killing time. Choosing the Minkowski vacuum $|0_M\rangle$ as initial state and letting the detector uniformly accelerate (which results in the fact that the annihilation operator in (4.30) becomes the *Rindler* annihilation operator in the right wedge R, a_R), we get

$$|s_\infty\rangle = |0_M, \downarrow\rangle - ia_R|0_M, \uparrow\rangle. \tag{4.31}$$

The Minkowski vacuum is equivalent to a thermal bath of Rindler particles (see (4.23)), and hence we see that the detector detects (absorbs) indeed particles, because of the annihilation operator a_R.
On the other hand we can, by using the Bogolyubov transformations, express the Rindler *operator* in terms of Minkowski *operators*, instead of expressing the Minkowski vacuum *state* in terms of Rindler *states*. One finds that the detection of a particle, i.e. the *absorption* of a (Rindler) quantum, corresponds to the *emission* of a Minkowski particle!

An inertial observer sees the emission of a particle, while an accelerating observer in the detector frame sees absorption. The apparent paradox whether the field wins or looses energy $\hbar\omega$, can be resolved as follows. According to the inertial observer the field energy increases by one quantum. The Rindler observer has a different interpretation, but agrees that the field energy increases. The initial state of the field is not an eigenstate of energy and thus the act of detection, the measurement, indicates that a larger number of particles then originally expected were present initially.

Chapter 5

The Hawking Effect

Closely related to the Unruh effect is the Hawking effect, which states that at sufficiently late times after gravitational collapse a black hole will emit thermal radiation at a temperature $T_H = \frac{\kappa}{2\pi}$, due to quantum effects. This effect strongly supports the relationship between thermodynamics and black hole physics, as described in chapter 2. Secondly, one would expect according to the principle of equivalence [1] that also in strong gravitational fields some thermal effects could occur.

In his calculation of the outgoing flux, Hawking worked with a fixed background geometry. He neglected backreaction effects of the quantum fields on the black hole. The reason for this approximation is the absence of a quantum theory of gravity. Before the theory of quantum electrodynamics was developed, one used successfully the same procedure by considering the electromagnetic field as a classical background field.

Without taking backreaction effects into account however, it seems unavoidable that pure states evolve into mixed states and thereby violating quantum mechanics. I discussed this problem in chapter 3.

Now I will give you the derivation of Hawking's result, following [15]. I will work in 1+1 dimensions with the advantage that it enables us to say something about the energy-stress-tensor. The result is the same as in four dimensions. Secondly we restrict our attention to massless scalarfields. Note that the solutions of the Klein-Gordon equation for massless scalarfields, $\partial_u \partial_v \phi = 0$, are superpositions of left- and rightmoving waves.

Consider a collapsing spherically symmetric ball of matter, forming a black hole. One can imagine nullrays $\phi_{in} = e^{-i\omega v}$ coming in from J^- (the rightmoving part is irrelevant), being reflected in the center of the ball and becoming outgoing waves $\phi_{out} = e^{-i\omega p(u)}$. However, nullrays which are emitted after $v = v_0$, will pass the horizon and fall into the singularity. (Hawking looked at the time reversal process where an outgoing wave $\phi_{out} = e^{-i\omega' u}$ reflects at the origin and becomes the ingoing wave $\phi_{in} = e^{-i\omega' f(v)}$.)

Here I have used a WKB approximation (see appendix A) by taking the classical

[1] "A frame linearly accelerated relative to an inertial frame in special relativity is locally identical to a frame at rest in a gravitational field."

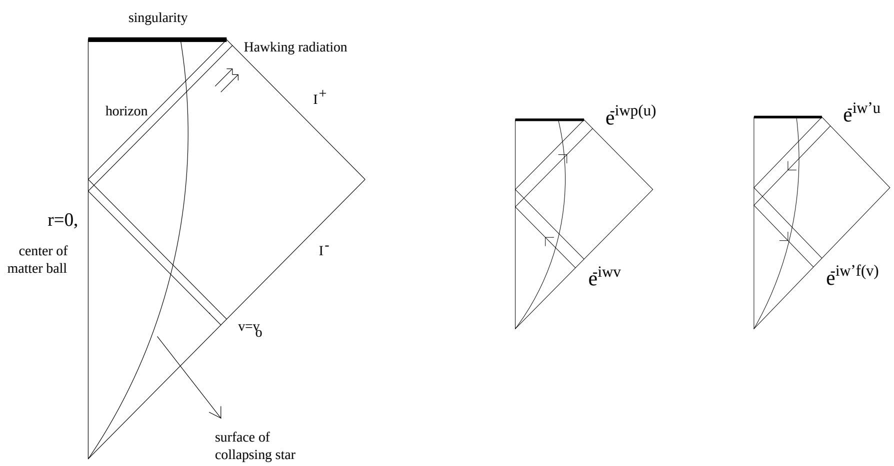

Figure 5.1: a) A collapsing star forming a black hole. b) Incoming test particles (photons for example) which reflect at $r = 0$. c) The time reversed process.

path $v \to p(u)$ of the reflecting wave, resulting in the waves described above. Hawking referred to this approach as the *geometrical optics* approximation, which means that one takes the waves at *one* frequency, without taking dispersion into account.

Before we can calculate the Bogolyubov coefficients, we will have to find the phase $p(u)$ of the out-going wave. The recipe is as follows.

- We begin by writing down the expressions of the line element for respectively the regions in- and outside the matter ball

$$ds^2 = A(U,V)dUdV \tag{5.1}$$

$$ds^2 = C(r)dudv \tag{5.2}$$

with $C = 1 - \frac{2M}{r}$ for a Schwarzschild black hole and

$$u = t - r^* + R_0^* \qquad v = t + r^* - R_0^* \tag{5.3}$$

$$U = \tau - r + R_0 \qquad V = \tau + r - R_0$$

where $dr^* \equiv \frac{dr}{C}$ since $Cdudv = Cdt^2 - \frac{1}{C}dr^2$. R_0 is the initial position of the surface of the collapsing ball and the relation between R_0 and R_0^* is the same as the relation between r and r^*.
The coordinate transformation is given by

$$U = \alpha(u) \qquad v = \beta(V).$$

At the center of the matter ball, at $r = 0$, one has $v = \beta(V) = \beta(U - 2R_0) = \beta[\alpha(u) - 2R_0] \equiv p(u)$ because the quantumfield has to obey the reflection condition, i.e. to vanish at the reflection point $r = 0$. In fact, the expression for the total massless scalarfield is a superposition of left- and rightmoving waves, $\phi = e^{-i\omega v} - e^{-i\omega p(u)}$. This shows that the quantumfield indeed vanishes at the reflection point. We neglect one term of the field, because it does not contribute to the reflection process.

- Secondly, the line elements have to match on the collapsing surface $r = R(\tau)$,

$$C(R)(dt^2 - dR^{*2}) = A(d\tau^2 - dR^2) \tag{5.4}$$

With this relation in hand we find

$$\frac{dU}{du} = \frac{d\tau - dr}{dt - dr^*} = \frac{d\tau(1-\dot{R})C}{Cdt - \dot{R}d\tau} = \frac{(1-\dot{R})C}{(AC(1-\dot{R}^2) + \dot{R}^2)^{\frac{1}{2}} - \dot{R}} \tag{5.5}$$

where $\dot{R} = \partial R/\partial\tau$.

- Finally note that we only consider the particle flux in the asymptotic region (i.e. near the horizon, $C \to 0$, or equivalent, at late times $u \to \infty$), hence we can approximate (5.5) by

$$\frac{dU}{du} \approx \frac{(\dot{R}-1)C(R)}{2\dot{R}}. \tag{5.6}$$

We find

$$\frac{1}{2}Cdu = dU\frac{\dot{R}}{\dot{R}-1} \tag{5.7}$$

In order to find an expression for $\dot{R}$ one can expand $R(\tau)$ near the horizon

$$R(\tau) = R_h + \dot{R}(\tau - \tau_h) + O((\tau-\tau_h)^2)$$

and thus:

$$\frac{1\partial C}{2\partial r}u = -ln|U + R_h - \tau_h| + const. \tag{5.8}$$

$$\Longrightarrow \alpha(u) = U \propto e^{-\kappa u} + const. \tag{5.9}$$

The phase $p(u)$ of our outgoing wave now finally becomes:

$$p(u) = Ce^{-\kappa u} + D \tag{5.10}$$

We used $v = \beta(V) \propto V$ because at late times an asymptotic observer sees the matter ball hardly change, due to time dilation.
The constant κ is the so called *surface gravity* of the horizon and is defined by

$$\kappa = \frac{1\partial C}{2\partial r}|_{r=R_h} = \frac{1}{4M}. \tag{5.11}$$

The surface gravity is a measure of the strength of the gravitational field at the horizon. More precisely, κ is the magnitude of the acceleration, with respect to killing time, of a stationary particle just outside the horizon. This is the same as the force per unit mass that must be applied at infinity in order to hold the particle on its path.

Formally the surface gravity is defined as the magnitude of the gradient of the norm of the horizon generating killing field χ^a (evaluated at the horizon):

$$\kappa^2 \equiv -(\nabla^a|\chi|)(\nabla_a|\chi|). \tag{5.12}$$

The constant C in (5.10) turns out to be $-\frac{1}{\kappa}$ and the constant D is the horizon v_0. As $u \to \infty$, v approaches indeed the horizon v_0, as we can see in figure 5.1. Furthermore, on physical grounds we *expected* already the phase factor of the outgoing wave, $p(u)$, to become exponentially small as the surface of the matter ball approaches the event horizon, due to the gravitational redshift.

If we consider an outgoing wave $\phi_{out} = e^{-i\omega' u}$, then the corresponding ingoing wave had to be $\phi_{in} = e^{-i\omega' p^{-1}(v)}$. We write $f(v)$ for the inverse function of $p(u)$

$$f(v) = -\frac{1}{\kappa}\ln(\kappa(v_0 - v)) - v_0. \tag{5.13}$$

We wish to calculate the Bogolyubov coefficients α and β which relate the different expansions of the quantum field. One of the expansions is in terms of outgoing modes and the other in terms of ingoing modes. But instead of the outgoing mode in the form $\phi_{out} = e^{-i\omega' u}$, we take its incoming form $\phi_{in} = e^{-i\omega' f(v)}$, which is of course the same (reflected) wave. The Bogolyubov coefficients are scalar products of wave modes and give therefore the overlap between the modes. We now wish to calculate the overlap between the incoming waves from the different expansions.

$$\begin{aligned}\beta_{\omega\omega'} &= -(\phi_{out}, \phi^*_{in}) = -(e^{-i\omega v}, e^{i\omega' f(v)}) \\ \alpha_{\omega\omega'} &= (\phi_{out}, \phi_{in}) = (e^{-i\omega v}, e^{-i\omega' f(v)})\end{aligned} \tag{5.14}$$

After an integration by parts and including the normalization factor, one obtains

$$\left.\begin{array}{c}\alpha_{\omega\omega'} \\ \beta_{\omega\omega'}\end{array}\right\} = \pm\frac{i}{2\pi}\sqrt{\frac{\omega}{\omega'}}\int_0^{v_0} e^{\pm i\omega' f(v) - i\omega v} dv = \tag{5.15}$$

$$= \pm\frac{i}{2\pi\kappa}\sqrt{\frac{\omega}{\omega'}}\int_0^{\kappa v_0} v'^{\mp i\omega'/\kappa} e^{iv_0(\omega'-\omega) + i\omega v'/\kappa} dv' =$$

$$= \mp\frac{1}{\sqrt{4\pi^2\omega\omega'}} e^{iv_0(\omega'-\omega)} \frac{i\kappa}{\omega}^{\mp i\omega'/\kappa} \int_0^{-i\omega' v_0} x^{\mp i\omega'/\kappa} e^{-x} dx.$$

With the definition of the Gamma function $\Gamma(n) = \int_0^\infty x^{n-1}e^{-x}dx$ and with the knowledge that lightrays after $v = v_0$ do not contribute to the integral, we find:

$$\left.\begin{array}{c}\alpha_{\omega\omega'} \\ \beta_{\omega\omega'}\end{array}\right\} = \mp\frac{1}{\sqrt{4\pi^2\omega\omega'}} e^{iv_0(\omega'-\omega)} e^{\pm\pi\omega'/2\kappa} (\frac{\omega}{\kappa})^{\pm i\omega'/\kappa} \frac{i\omega'}{\kappa}\Gamma(\mp\frac{i\omega'}{\kappa}). \tag{5.16}$$

As we saw in chapter 4, the out-going radiation spectrum is given by the Bogolyubov coefficient β,

$$\langle N_{\omega'} \rangle = F(\omega') = \sum_{\omega} |\beta_{\omega\omega'}|^2. \tag{5.17}$$

After a straightforward calculation we find for the flux coming out of a neutral, non-rotating black hole:

$$F(\omega') = \frac{1}{e^{2\pi\omega'/\kappa} - 1} \tag{5.18}$$

The same result is found if we use the (completeness) relation

$$\sum_{\omega}(|\alpha_{\omega\omega'}|^2 - |\beta_{\omega\omega'}|^2) = 1, \tag{5.19}$$

and the fact that $|\alpha_{\omega\omega'}|^2$ and $|\beta_{\omega\omega'}|^2$ are independent of ω:

$$F(\omega') = \sum_{\omega} |\beta_{\omega\omega'}|^2 = \frac{1}{\frac{|\alpha_{\omega\omega'}|^2}{|\beta_{\omega\omega'}|^2} - 1} = \frac{1}{e^{2\pi\omega'/\kappa} - 1}. \tag{5.20}$$

This is precisely the spectrum you would expect for (bosonic) thermal radiation at a temperature

$$T_H = \frac{\kappa}{2\pi k_B} = \frac{1}{8\pi k_B M} \approx 1.2 \cdot 10^{23} \frac{1}{M}\ K\,kg. \tag{5.21}$$

Of course we assumed that the black hole is surrounded by empty space, otherwise a black hole heavier then $0.007 M_{\oplus}$ (about half of the moon) would *absorb* the (3 K) background radiation. The notion of thermodynamic equilibrium becomes curious however, because a big, cold black hole would absorb the 3 K-radiation and instead of becoming hotter, as one would expect according to the Laws of thermodynamics, it becomes bigger and thus colder! This might seem as a violation of the Second Law, but it is not according to the Generalized Second Law. A black hole acts as a vacuum cleaner of entropy, storing it on its area.

Although one encounters some technical difficulties in the four dimensional case, the derivation is roughly the same and the result is identical. It is also possible to generalize the result to charged and rotating black holes. A rotating hole emits a flux

$$F(\omega') = \frac{1}{e^{2\pi(\omega' - m\Omega)/\kappa} - 1} \tag{5.22}$$

where m is the azimuthal quantum number ($m = -l, ..., +l$) and Ω the angular speed at the event horizon. The temperature of a charged black hole turns out to be

$$T = \frac{1}{8\pi k_B M}\Big(1 - \frac{16\pi^2 e^4}{A^2}\Big) \tag{5.23}$$

with A the area of the event horizon and e the charge of the hole.

We restricted our attention to massless scalarfields, which is actually quite unphysical (the only scalar particles we know of are Higgs bosons, which are still not found experimentally and have big masses). However, having the fundamental thermodynamic nature of quantum particle production in mind, the calculations should also be valid for higher spin fields. Because of the anticommuting nature of fermions we would get a +sign in (5.19) resulting in the appropriate Planck factor for Fermi statistics (see appendix B).

Due to the radiation, the black hole looses mass and gets hotter. For a hole with mass $M \approx 10^{14}$ kg the creation of electron-positron pairs becomes possible, at about $M \approx 10^{12}$ kg the Schwarzschild radius approaches the range of strong interactions and when the hole has reached the Planck mass $M \approx 10^{-8}$ kg nobody knows what the outcome is, a final burst of huge energy, or maybe a stable remnant, as described in section 3.1.

5.1 Stress Tensor

How real is this black hole radiation actually? A heuristic way to understand the Hawking process was given by Hawking personally. All the time and in the whole of space we have quantum processes in the vacuum, where pairs of virtual particles and anti-particles are created. These vacuum fluctuations are possible because the uncertainty relation of Heisenberg states that one can not measure the total energy with infinite accuracy at one moment, $\Delta E \Delta t \approx \hbar$, and therefore energy conservation can be violated in short time intervals. Hence these pairs annihilate immediately, typically in 10^{-23} s.

Now imagine that this happens just outside the event horizon. Due to the strong gravitational field these particles can be separated causing one particle to fall in the hole while the other particle manages to escape to infinity, constituting to the thermal radiation. Inside the black hole the virtual anti-particle becomes real *without loosing its negative energy*. This is possible because the time translation generating killing field becomes *spacelike* inside the horizon, allowing negative values. This is a heuristic description of the process and should not be taken too literally, because one could also imagine the *particle* falling into the hole and increasing the mass of the hole while the *antiparticle* escapes.

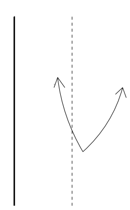

Figure 5.2: Pair creation of virtual particles and anti particles near the horizon.

Investigating the energy stress tensor $T_{\mu\nu}$ is a more rigorous and relevant method to get more insight in the physical existence of the Hawking radiation [17, 16]. We will find an expression for the energy-momentum tensor regularized by the point-splitting procedure. As before we will restrict our attention to the quantum theory of a massless scalar field in two dimensions.

The classical expression for the stress tensor is

$$T_{\mu\nu}(x) = \partial_\mu \phi \partial_\nu \phi - \frac{1}{2} g_{\mu\nu} \partial^\alpha \phi \partial_\alpha \phi \tag{5.24}$$

where ϕ are normal modes. The operator Φ can be expanded in those modes. For the *operator* $T_{\mu\nu}$ we get

$$T_{\mu\nu}(x) = \partial_\mu \Phi \partial_\nu \Phi - \frac{1}{2} g_{\mu\nu} \partial^\alpha \Phi \partial_\alpha \Phi. \tag{5.25}$$

Now consider a point x_ϵ at a proper distance $|\epsilon|$ from x. We denote the normalized tangent vector at x by

$$t^\alpha(\epsilon) \equiv \frac{dx_\epsilon^\alpha}{d\epsilon}.$$

Next we define the matrix $e_\mu^\alpha(\epsilon)$ as the matrix which maps the vectors at x to those at x_ϵ [2]. Evaluating one field at $x_{-\epsilon}$ and the other at x_ϵ we find for the expectation value of $T_{\mu\nu}$:

$$\langle T_{\mu\nu}(x;\epsilon)\rangle = \langle 0|\partial_\alpha \Phi(x_\epsilon) \partial_\beta \Phi(x_{-\epsilon})|0\rangle \Big(e_\mu^\alpha(\epsilon) e_\mu^\beta(-\epsilon) - \frac{1}{2} g_{\mu\nu} e_\sigma^\alpha(\epsilon) e^{\beta\sigma}(-\epsilon)\Big).$$

After an expansion in powers of ϵ and using the fact that every two dimensional metric is conformally flat (i.e. for any coordinate system we can write $ds^2 = C(u,v)dudv$) we get

$$\langle T_{\mu\nu}(x;\epsilon)\rangle = -\Big(\frac{1}{\epsilon^2 4\pi t_\alpha t^\alpha} + \frac{R}{24\pi}\Big)\Big(\frac{t_\mu t_\nu}{t_\alpha t^\alpha} - \frac{1}{2} g_{\mu\nu}\Big) + \theta_{\mu\nu} + O(\epsilon),$$

where R is the curvature scalar and the tensor $\theta_{\mu\nu}$ has the components (in the special null coordinates $(\bar{u},\bar{v})$ such that the normal modes are simple exponentials $e^{-i\omega\bar{v}}$, $e^{-i\omega\bar{u}}$ on J^- and J^+ respectively):

$$\theta_{\bar{u}\bar{u}} = -\frac{1}{12\pi}\sqrt{C}\,\partial_{\bar{u}}^2 (C)^{-\frac{1}{2}} \tag{5.26}$$

$$\theta_{\bar{v}\bar{v}} = -\frac{1}{12\pi}\sqrt{C}\,\partial_{\bar{v}}^2 (C)^{-\frac{1}{2}}$$

$$\theta_{\bar{u}\bar{v}} = \theta_{\bar{v}\bar{u}} = 0.$$

One sees immediately that the result diverges as $\epsilon \to 0$. Moreover, the result contains terms which depend on the (arbitrary!) direction of point separation. Therefore we will simply discard the terms containing ϵ or the tangent vector t_α:

$$\langle T_{\mu\nu}(x)\rangle = \theta_{\mu\nu} + \frac{R}{48\pi} g_{\mu\nu}. \tag{5.27}$$

[2] This means that $e_\mu^\alpha(\epsilon)$, for fixed μ in a given coordinate system, is the contravariant vector x_ϵ obtained by parallel transport - along the geodesic from x - of the basic vector in the μ direction.

As in (5.2) we consider the metric:

$$ds^2 = C(r)dudv$$

Because of the fact that any two dimensional coordinate system is conformally related to Minkowski spacetime which results in $R = 0$, and because of

$$\partial_u^2 = \frac{1}{4}(C^2\partial_r^2 + CC'\partial_r)$$

we find for the stress tensor:

$$\langle T_{uu}\rangle = \frac{1}{192\pi}(2CC'' - (C')^2) = \frac{1}{24\pi}\Big(\frac{3M^2}{2r^4} - \frac{M}{r^3}\Big). \tag{5.28}$$

So far there has been no collapse and still no black hole is formed. If there is a gravitational collapse, then the coordinates (u, v) are no longer appropriate, because the outgoing modes $e^{-i\omega u}$ become complicated exponentials $e^{-i\omega p(u)}$. In other coordinates $\bar{u} = p(u)$ and $\bar{v} = v$, the metric becomes

$$ds^2 = \bar{C}(\bar{u}, \bar{v})d\bar{u}d\bar{v}.$$

With $p(u) = e^{-u/4M} + v_0$ as in (5.10):

$$\bar{C}(\bar{u}, \bar{v}) = C(r)\frac{du}{d\bar{u}} = (1 - \frac{2M}{r})\frac{4M}{v_0 - \bar{u}} \tag{5.29}$$

we find for the energy tensor:

$$\langle T_{uu}\rangle = \frac{1}{24\pi}\Big(\frac{3M^2}{2r^4} - \frac{M}{r^3} + \frac{1}{32M^2}\Big). \tag{5.30}$$

For large r we see an outgoing flux

$$\langle T_{uu}\rangle = \frac{1}{768\pi M^2} = \frac{\pi}{12}\Big(\frac{1}{8\pi M}\Big)^2. \tag{5.31}$$

This is exactly the energy flux for a body with temperature $T = \frac{1}{8\pi M}$. Near the horizon however ($r \approx 2M$) we see a negative energy flux going *into* the horizon.

$$\langle T_{vv}\rangle = \frac{1}{24\pi}\Big(\frac{3M^2}{2r^4} - \frac{M}{r^3}\Big) \quad\longrightarrow\quad -\frac{13}{768\pi M^2} \tag{5.32}$$

Indeed one could in some way describe this process as the creation of particles and antiparticles near the horizon. Particles carrying the positive energy escape to infinity, while their counterparts with negative energy fall through the horizon into the black hole.

Chapter 6

Moving Mirrors

In 1976 Davies and Fulling [18] developed a simple *"moving mirror"* model, which provides a very useful analogy to the black hole situation (see also [19, 20, 21]). By choosing suitable trajectories, the moving mirror mimics many of the features of black hole radiance, with the difference that the quantum fields propagate in flat Minkowski space instead of complicated geometries with high curvatures. First we will treat the case with a fixed trajectory, which represents a fixed background geometry. Then we take the rest mass of the mirror *finite* and calculate classically the back reaction effect of the reflection of massless scalar fields off the boundary.

6.1 Fixed Trajectory

Consider a reflecting boundary in 1+1 dimensions following a trajectory at late times (for example $z(t) = \ln(\cosh t)$) that mimics the effect of the gravitational collapse geometry,

$$z(t) = -t - \frac{1}{\kappa}e^{-2\kappa t} + v_0, \tag{6.1}$$

In the black hole case the incoming waves travel through the collapsing matter and reflect at $r = 0$. The mirror trajectory represents thus the origin of the black hole geometry. The last term, v_0, is the horizon after which the lightrays travel undisturbed to the left, without being reflected. This represents the black hole event horizon. Furthermore we see that the mirror velocity approaches the speed of light exponentially fast

$$\mathrm{v}(t) \equiv \frac{dz(t)}{dt} = -1 + 2e^{-2\kappa t}. \tag{6.2}$$

Secondly we have a massless scalarfield ϕ, which reflects off the mirror. We impose the reflection boundary condition that the field has to vanish on the boundary,

$$\phi(t, z(t)) = 0 \tag{6.3}$$

(One could see that this is a reflection condition by imagining a running wave on a rope with a fixed end. The wave vanishes at the fixed end and reflects.)

A late outgoing wave along $u = \bar{u}$ hits the mirror approximately at time $t = \frac{1}{2}(\bar{u} + v_0)$. From (6.1) we deduce the following relation at the reflection point $(\bar{u}, \bar{v})$:

$$\bar{v} - v_0 = -\frac{1}{\kappa} e^{-\kappa(\bar{u}+v_0)} \quad \Longrightarrow \quad \bar{u} = -v_0 - \frac{1}{\kappa} \ln[-\kappa(\bar{v} - v_0)] \equiv f(\bar{v}) \tag{6.4}$$

This classical calculation gives us the classical trajectory $f(v) \to u$ of an monochromatic outgoing wavefunction (monochromatic means that the wave function is sharply peaked at one frequency), which reflects off the mirror. Taking the classical trajectory as the phase of the wavefunction corresponds to making use of the WKB approximation, as described in appendix A. Due to this classical approximation, we have the monochromatic out-wavefunction

$$\phi^{out} \sim e^{-i\omega' u} \tag{6.5}$$

and the corresponding complicated in-wave

$$\phi^{in} \sim e^{-i\omega' f(v)}.$$

Note that we have constructed the mirror trajectory in such way, that the function $f(v)$ is identical to the phase of the wavefunction in (5.13). The Bogolyubov coefficients are therefore the same as in (5.16)

$$\beta_{\omega\omega'} = \frac{1}{\sqrt{4\pi^2\omega\omega'}} e^{-\pi\omega'/2\kappa} e^{iv_0(\omega'-\omega)} (\frac{\omega}{\kappa})^{-i\omega'\kappa} \Gamma(1 + \frac{i\omega'}{\kappa}) \tag{6.6}$$

and we find for the out-going flux

$$F(\omega') = \sum_{\omega} |\beta_{\omega\omega'}|^2 = \frac{1}{e^{2\pi\omega'/\kappa} - 1} \tag{6.7}$$

which is the flux of the radiation of a black body with temperature $T = \frac{\kappa}{2\pi}$. However, κ is not the surface gravity but a measure of the acceleration.

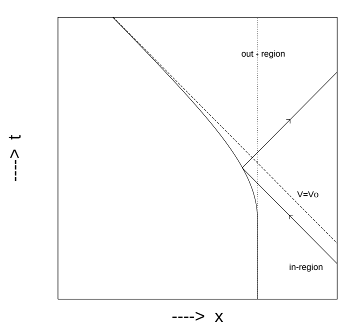

Figure 6.1: The mirror trajectory $z(t)$ with reflecting particle.

6.1.1 Saddlepoint Approximation

For later comparison I will calculate the Bogolyubov coefficients again, but now by using the method of saddlepoints. The idea is to approximate the integral (5.15) by taking only the biggest contribution of the integral into account. The biggest contribution of the integral is at the so-called saddlepoint and is found in our case where the derivative of the exponent vanishes. Note that it makes no difference if we take the outgoing forms of the waves $e^{-i\omega' u}$, $e^{-i\omega p(u)}$ or the ingoing forms $e^{-i\omega' f(v)}$, $e^{-i\omega v}$.

The saddlepoints for $\alpha_{\omega\omega'}$ and $\beta_{\omega\omega'}$ lie respectively at $u = \bar{u}_\pm$ with

$$\omega' = \pm\omega\partial_u p(u)|_{\bar{u}_\pm} \quad \Longrightarrow \quad \bar{u}_\pm = v_0 - \frac{1}{\kappa}\log(\pm\frac{\omega'}{\omega}) \tag{6.8}$$

The saddlepoint $u = \bar{u}_+$ that contributes to $\alpha_{\omega\omega'}$, corresponds to the physical reflection time, which is uniquely determined for given in and out-frequency via the Doppler relation

$$\omega' = \omega\frac{1+\mathrm{v}}{1-\mathrm{v}} \tag{6.9}$$

with v the velocity of the mirror given in (6.2). This means that one could also deduce (6.8) by using (6.9). The reflection point satisfies the equation $p(\bar{u}) = \bar{v}$, and we see:

$$\omega' = \omega\partial_u p(u) = \omega\partial_u v = \omega\frac{\dot{t}+\dot{r}}{\dot{t}-\dot{r}} = \omega\frac{1+\mathrm{v}}{1-\mathrm{v}}.$$

The saddlepoint $u = \bar{u}_-$ that contributes to $\beta_{\omega\omega'}$, on the other hand, has an imaginary part equal to π/κ, because of the term $\frac{1}{\kappa}\log(-1) = -i\pi/\kappa$. The saddlepoint $\bar{u}_-$ corresponds therefore to the virtual reflection process that gives rise to the particle creation phenomenon. These out-going reflection times correspond to ingoing times $\bar{v}_\pm$ with

$$\bar{v}_\pm = v_0 \mp \frac{\omega'}{\kappa\omega} \tag{6.10}$$

Hence for the Bogolyubov coefficients we find [1]

$$\alpha_{\omega\omega'} \approx e^{i\omega'\bar{u}_+ - i\omega\bar{v}_+} \approx e^{-\frac{i\omega'}{\kappa}\ln(\frac{\omega'}{\omega}) - iv_0(\omega-\omega') + \frac{i\omega'}{\kappa}}$$

$$\beta_{\omega\omega'} \approx e^{i\omega'\bar{u}_- + i\omega\bar{v}_-} \approx e^{-\frac{i\omega'}{\kappa}\ln(-\frac{\omega'}{\omega}) + iv_0(\omega+\omega') + \frac{i\omega'}{\kappa}}. \tag{6.11}$$

The out-going flux is again our well known Planck spectrum

$$F(\omega') = \frac{1}{|\frac{\alpha_{\omega\omega'}}{\beta_{\omega\omega'}}|^2 - 1} = \frac{1}{e^{2\pi\omega'/\kappa} - 1}. \tag{6.12}$$

[1] Instead of β we take here its conjugate. This means nothing more then looking at the outgoing wave where the *ingoing* wave had to have negative energy. In (5.15) we saw that the *outgoing* energy ω' changed sign. One could say that we look at the time reversed process. Looking at the time reversed process has no influence on the mirror trajectory, because we take it fixed. In the next section however, we will have to be consistent and take the mirror trajectory at late times determined in order to be able to look at the time reversed process.

6.1.2 Uniform Acceleration

If one takes the acceleration of the mirror *uniform* one finds that the Bogolyubov coefficient β is non-zero and that the mirror therefore emits particles. However, the energy stress tensor vanishes! Recall the relation for the expectation value of the stress tensor, (5.27),

$$\langle T_{\mu\nu} \rangle = \theta_{\mu\nu} \tag{6.13}$$

which becomes after a coordinate transformation

$$u = f(\bar{u}) \qquad v = \bar{v} \quad \Longleftrightarrow \quad \bar{u} = p(u) \qquad \bar{v} = v \tag{6.14}$$

to a coordinate system $(\bar{u}, \bar{v})$ in which the mirror remains at rest, $(\bar{x} = 0)$:

$$\langle T_{\bar{u}\bar{u}} \rangle = -\frac{1}{12\pi}(f'(\bar{u}))^{\frac{1}{2}} \partial_{\bar{u}}^2 (f'(\bar{u}))^{-\frac{1}{2}} \tag{6.15}$$

Transforming to the (u, v) system gives

$$\langle T_{uu} \rangle = -\frac{1}{12\pi}(f'(\bar{u}))^{-\frac{3}{2}} \partial_{u}^2 (f'(\bar{u}))^{-\frac{1}{2}} \tag{6.16}$$

and, with $z(t)$ the mirror trajectory, we find in the (x, t) system

$$\langle T_{tt} \rangle = \langle T_{xx} \rangle = -\frac{1}{12\pi} \frac{(1-\dot{z}^2)^{\frac{1}{2}}}{(1-\dot{z})^2} \frac{d}{d\tau_u} \frac{\ddot{z}}{(1-\dot{z}^2)^{\frac{3}{2}}} \tag{6.17}$$

For non-relativistic motion this is simply $-\frac{1}{12\pi}\frac{d}{dt}\ddot{z}$. We see that a uniform accelerating mirror will not radiate energy, although we are dealing with non-inertial motion of the mirror and particle creation off the mirror occurs! The problem is that in curved spacetime or involving non-inertial motions, the particle concept becomes bad defined and different from our conventional view. The statement "energy $\hbar\omega$ per quantum" does not apply in this case.

6.2 Backreaction on the Mirror

In this section we will repeat the calculation from section 6.1, but now by taking the backreaction of the radiation off the mirror into account. This section is largely based on [22]. Backreaction effects are important for two reasons. First we have the information problem as described in chapter 3. Including backreaction effects could lead to a solution of the problem (see section 3.3). Secondly there is the problem of divergent energies as $v \to v_0$. The radiation coming off a mirror or out of a black hole at late times, had to be infinitely blue-shifted incoming waves, due to the Dopplershift (or gravitational shift).

A mirror with infinite mass corresponds to a fixed trajectory, because an object with infinite mass does not feel the bounce from particles. I will therefore take the mirror mass to be *finite* and impose energy conservation upon reflection. However, the

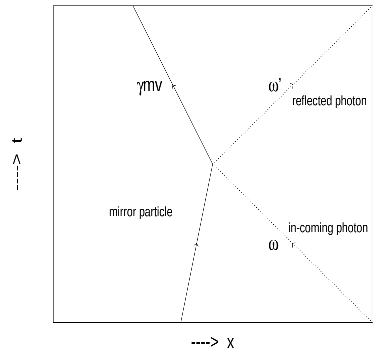

Figure 6.2: The reflection of a photon off a mirror particle with finite mass.

Bogolyubov coefficients are not exactly soluble anymore which forces us to use approximations. We will use here the saddlepoint approximation.

Consider an incoming photon with energy $\hbar\omega$ which collides elastically with a mirror particle with mass m. After the collision the mirror particle has energy $\gamma(\mathrm{v})mc^2$ and the photon reemerges with a energy $\hbar\omega'$. (From now we put again $\hbar = c = 1$). We want to express the energy ω' in terms of ω and the mirror velocity v just after the collision. With the condition for energy conservation in two dimensions, $p' + k' = p + k$, classically one finds:

$$k \cdot k' = p' \cdot (k' - k) \tag{6.18}$$

and with $k = (\omega, -\omega), k' = (\omega', \omega')$ and $p' = (m\gamma, -m\gamma|\mathrm{v}|)$:

$$\omega' = \omega \frac{1}{\delta^2 - \frac{2\omega}{m}\delta} \tag{6.19}$$

with δ the Doppler factor

$$\delta = \sqrt{\frac{1 - \mathrm{v}}{1 + \mathrm{v}}}. \tag{6.20}$$

Note that for $\mathrm{v} \to -1$ we get $\omega' \to 0$ and for $m \to \infty$ we find the quadratic Doppler-shift as expected (see (6.9)). We now wish to consider the same classical mirror trajectory as in the previous section. We imagine therefore that there is some external force that acts on the mirror particle, which in the absence of other forces (such as those due to possible (virtual) collisions with the photons) will keep it in the given trajectory (6.1). More specifically, in determining the relevant classical WKB-trajectory, we keep the mirror trajectory *after* the collision with the photon fixed, so that it always remains asymptotic to $v = v_0$. From (6.1) we deduce [2]

$$v = -\frac{1}{\kappa} e^{-2\kappa(t - v_0)} + v_0$$

[2] Note that we take now a shifted trajectory $t \to t - v_0$.

$$\kappa(v - v_0) = e^{-2\kappa(t-v_0)}$$

and by using (6.2) and (6.4) we find for late times

$$\delta \approx e^{\frac{\kappa}{2}(u-v_0)} \approx \frac{1}{\sqrt{\kappa(v_0 - v)}}. \tag{6.21}$$

Thus applying the general formula (6.19), we get the following relation between the in- and out-going energies in terms of the out-going time u

$$\omega' \approx \frac{\omega}{e^{\kappa(u-v_0)} - \frac{2\omega}{m} e^{\frac{\kappa}{2}(u-v_0)}} \tag{6.22}$$

and in terms of the incoming time we have

$$\omega \approx \frac{\omega'}{\kappa(v_0 - v) + \frac{2\omega'}{m}\sqrt{\kappa(v_0 - v)}} \tag{6.23}$$

We see now that the energy ω of the incoming photon can not become infinitely large, neither larger then the mirror energy after collision. Energy is indeed conserved. If we keep ω' fixed, then for late times ω approaches this maximal value

$$\omega \approx \frac{m}{2}\delta \approx \gamma m \tag{6.24}$$

since $\gamma \equiv \delta/(1 - \mathrm{v}) \approx \delta/2$ for late times.

We would again like to use a combination of a WKB approximation for determining the relation between the in- and out-going wave packets and a saddlepoint approximation for the integrals in the expressions of the (corrected) Bogolyubov coefficients $\tilde{\alpha}_{\omega\omega'}$ and $\tilde{\beta}_{\omega\omega'}$. In fact one could say that the two approximations are related, because the saddlepoint approximation gives us the classical path as does the WKB approximation. Following the same steps as in section 6.1, we can again express these coefficients in terms of the reflection points $(\tilde{u}_\pm, \tilde{v}_\pm)$ as

$$\tilde{\alpha}_{\omega\omega'} \approx e^{i\omega'\tilde{u}_+ - i\omega\tilde{v}_+}$$

$$\tilde{\beta}_{\omega\omega'} \approx e^{i\omega'\tilde{u}_- + i\omega\tilde{v}_-}. \tag{6.25}$$

That is, the coefficients are just equal to the "phase jump" between the incoming and outgoing wave at the reflection point. Note that the *incoming* frequency ω changes sign when we calculate the saddlepoints $\tilde{u}_-, \tilde{v}_-$ for teh Bogolyubov coefficient β, because we look at the time reversed process. Furthermore, I take +-solution, because in the limit $m \to \infty$ one would like to get the old result back and avoid negative values of the exponent.

Using (6.22), we find the new saddlepoints $\tilde{u}_\pm$ at

$$\tilde{u}_\pm = v_0 - \frac{1}{\kappa}\log(\frac{\omega'}{\omega}) + \frac{2}{\kappa}\log(\sqrt{\mathrm{x}^2 \pm 1} \pm \mathrm{x}) \tag{6.26}$$

and with (6.23) we obtain

$$\tilde{v}_{\pm} = v_0 - \frac{\omega'}{\kappa\omega}(\pm 1 + 2\mathrm{x}^2 - 2\mathrm{x}\sqrt{\mathrm{x}^2 \pm 1}) \tag{6.27}$$

with

$$\mathrm{x}^2 = \frac{\omega\omega'}{m^2}.$$

As before, the physical reflection point $(\tilde{u}_+, \tilde{v}_+)$ that contributes to the Bogolyubov coefficient $\tilde{\alpha}_{\omega\omega'}$ is always real, while the saddle-point $(\bar{u}_-, \bar{v}_-)$ that contributes to $\tilde{\beta}_{\omega\omega'}$ again has an imaginary part. This imaginary part, however, is no longer constant, but depends on the variable x. Note further that the virtual reflection point $(\bar{u}_-, \bar{v}_-)$ is related to the real one by changing the sign of the *in-going* frequency ω. Indeed, the out-going radiation is produced because negative energy in-coming modes can propagate via virtual trajectories into positive energy out-going modes.

It is easy to see that, if we keep the out-going frequency ω' constant (which is still physically reasonable), the incoming frequency ω will at late times still grow exponentially in the out-going time, but now with half the e-folding time. This implies in particular that at late times (that is for $\kappa(\bar{u}_+ - v_0) \gg 1$), we have the relation

$$\tilde{u}_+ \approx \frac{2}{\kappa}\log \mathrm{x}^2 + \mathrm{const.}$$

between the parameter x and the reflection time $\tilde{u}_+$. In the following we will make use of this relation to read off the time dependence of the radiation spectrum, given its expression in terms of the in and out-going frequencies.

A straightforward calculation now gives the following result for the new Bogolyubov coefficients:

$$\tilde{\alpha}_{\omega\omega'} \approx \alpha_{\omega\omega'}\ \exp\frac{2i\omega'}{\kappa}\Big[-\mathrm{x}(\sqrt{\mathrm{x}^2+1} - \mathrm{x}) + \log(\sqrt{\mathrm{x}^2+1} + \mathrm{x})\Big] \tag{6.28}$$

and

$$\tilde{\beta}_{\omega\omega'} \approx \beta_{\omega\omega'}\ \exp\frac{2i\omega'}{\kappa}\Big[-\frac{i\pi}{2} + \mathrm{x}(\sqrt{\mathrm{x}^2-1} - \mathrm{x}) + \log(\sqrt{\mathrm{x}^2-1} - \mathrm{x})\Big] \tag{6.29}$$

where $\alpha_{\omega\omega'}$ and $\beta_{\omega\omega'}$ denote the uncorrected coefficients given in section 6.1. Note that we indeed get the old result back in the limit $\mathrm{x} \to 0$, which corresponds to the infinite mass limit $m \to \infty$. For the ratio of the absolute values of the new coefficients we find [3]

$$\frac{|\tilde{\alpha}_{\omega\omega'}|^2}{|\tilde{\beta}_{\omega\omega'}|^2} \approx \exp\Big[\frac{4\omega'}{\kappa}R(\mathrm{x})\Big] \tag{6.30}$$

with

$$\begin{aligned} R(\mathrm{x}) &= \pi - \arccos \mathrm{x} + \mathrm{x}\sqrt{1-\mathrm{x}^2} && \text{for } \mathrm{x} < 1 \\ &= \pi && \text{for } \mathrm{x} > 1. \end{aligned} \tag{6.31}$$

[3] Note that the ω-dependency is transformed into the *time* dependency x, which justifies this procedure.

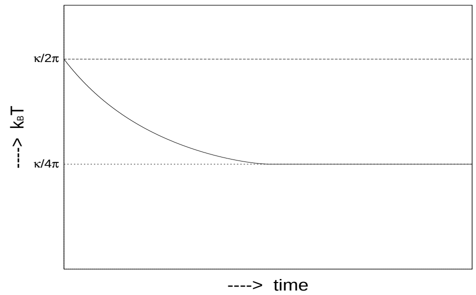

Figure 6.3: The temperature of the radiation as a function of the outgoing time $u = \frac{2}{\kappa}\log \mathrm{x}^2$.

where we used $\mathrm{Im}\log(-\mathrm{x}+i\sqrt{1-\mathrm{x}^2}) = \pi - \arctan(\mathrm{x}/\sqrt{1-\mathrm{x}^2}) = \pi - \arccos \mathrm{x}$. We now define the out-going spectrum $F(\omega', \mathrm{x})$ as a function of the time parameter x as

$$F(\omega', \mathrm{x}) = \frac{1}{\frac{|\alpha_{\omega\omega'}|^2}{|\beta_{\omega\omega'}|^2} - 1} = \frac{1}{e^{\omega'/k_B T(\mathrm{x})} - 1} \tag{6.32}$$

where we identified the time-dependent temperature as

$$k_B T(\mathrm{x}) = \frac{\kappa}{4R(\mathrm{x})}. \tag{6.33}$$

The temperature as a function of the outgoing time $\bar{u}$ is plotted in figure 6.3.

We read off that after a relatively short amount of time of the order of

$$\bar{u} \approx v_0 + \frac{2}{\kappa}\log(\frac{2m}{\kappa}) \tag{6.34}$$

the mirror radiates a constant flux of thermal radiation at a temperature $k_B T = \frac{\kappa}{4\pi}$, where I used (6.26) with $\mathrm{x} \approx 1$. Recall that the typical frequency of the out-going radiation is of the order of κ, see appendix B.

In conclusion, we see that introducing a reflecting boundary with *finite* mass and imposing energy conservation upon reflection, leads to back reaction effects with dramatic consequences: the temperature of the mirror is half of the original result. This result could have been anticipated from equation (6.23). For late times, i.e. in the limit $v \to v_0$, the second term in the denominator determines the corrected Doppler relation between the in- and out-frequencies. In other words, the magnitude of the ingoing frequency for given ω' grows linearly with the Doppler factor δ, instead of quadratically.

6.3 Dilaton Gravity

A close related model to the one of moving mirrors is *dilaton gravity*, see [23, 24, 25, 4]. This model is just complicated enough to contain black hole solutions as well as Hawking radiation. Again we restrict our attention to two dimensions.

The action which gives us the Einstein equations is

$$S = \int d^2x\sqrt{-g}[R - \frac{1}{2}(\Delta f)^2]$$

with g the metric, R the Ricci or curvature scalar and f the matter fields. The action for dilaton gravity is given by [4]

$$S = \frac{1}{2\pi}\int d^2x\sqrt{-g}(e^{-2\phi}[R + (2\nabla\phi)^2 + 4\lambda^2] - \frac{1}{2}(\nabla f)^2) \tag{6.35}$$

where ϕ is the dilaton and λ a cosmological constant, which sets the scale of the surface gravity, and e^{ϕ} plays the role of gravitational coupling[5]. A factor in front of R in the action scales the metric giving rise to dilatation of spacetime; ϕ is therefore called a dilaton. A crucial property is the fact that we can rescale the metric obtaining a *flat* spacetime

$$ds^2 = e^{2\rho}dx^+dx^-$$

$$\rho(x^+, x^-) = \phi(x^+, x^-)$$

with $x^{\pm} = x^0 \pm x^1$. Now one can solve the classical equations of motion for $e^{-2\phi}$

$$e^{-2\phi} = M - \lambda^2 x^+x^- - \int^{x^+} dy^+dz^+T_{++} - \int_{x^-} dy^-dz^-T_{--} \tag{6.36}$$

Putting the energy-momentum tensor equal to zero, we get the static black hole solution (see (2.1)). Without *any* matter present, i.e. $T_{\pm\pm} = 0, M = 0$, one gets the the two dimensional Minkowski vacuum, or *linear dilaton vacuum*,

$$ds^2 = \frac{dx^+dx^-}{x^+x^-}$$

which becomes with

$$x^{\pm} = \pm e^{\lambda(r\pm t)} \tag{6.37}$$

our familiar expression for the Minkowski line element

$$ds^2 = dt^2 - dx^2. \tag{6.38}$$

One obtains the Hawking radiation by considering infalling matter. Before the matters comes in, we have simple Minkowski space-time. The infalling matter forms a black hole in the model of dilaton gravity as can be seen from the metric after some coordinate transformations. If one now calculates the matter stress tensor of the right movers, one finds the thermal Hawking flux. In the left asymptotic region of the Minkowski plane, ($x^{\pm} \to 0 \leftrightarrow r \to -\infty$), the dilaton takes a large value, as can be seen from (6.36).

[4]We take this action, simply because it gives us a black hole solution. Furthermore, this action is closely related to an action for some sort of strings. The lowest order string effective action looks like $S = \int d^{d+1}x\sqrt{-g}[e^{2\phi}(R + (2\nabla\phi)^2 + ...) + V] + loops$.

[5]In Yang-Mills theory (the general gauge theories) for example, one writes the Lagrangian as $\mathcal{L} \sim -\frac{1}{g^2}...$, with g the coupling constant.

Including the reflection condition, i.e. energy conservation, and imposing a cutoff on ϕ, ϕ_{cr}, we find the boundary trajectory $x^{\pm}(\tau)$ where $e^{\phi(x^+,x^-)}$ takes its critical value $e^{\phi_{cr}}$. This is closely related to the moving mirror trajectory or the $r=0$ point in the black hole geometry.

Above we considered $T_{\pm\pm}=0$. If we now consider an incoming shock wave $T_{++}=p_+\delta(x^+-q^+)$ (as the photon in the previous section), one sees that the boundary trajectory acts as a mirror with *negative* mass.

In particular, one finds that all incoming particle wave of energy below some critical value ω_{crit} will reflect back to infinity, provided it does not disappear behind the horizon of an already existing black hole, see [21]. The trajectory is given by

$$(\lambda^2 x^- - p^+)x^+ = -\frac{m^2}{4\lambda^2} \tag{6.39}$$

$$x^-(\lambda^2 x^+ + p^-) = -\frac{m^2}{4\lambda^2}$$

with

$$p_- = \frac{\lambda^2 p_+ q^+}{m^2/4\lambda^2 q^+ - p^+}. \tag{6.40}$$

For incoming energies larger than $p_+q^+ = \frac{m^2}{4\lambda^2} \equiv \omega_{crit}$ we see that the trajectory after the collision becomes spacelike. For larger energies than the critical value ω_{crit}, the particle wave will never reflect back, but always lead to black hole formation. In the sub-critical regime one can calculate the relation between the in- and out-going frequencies by solving for the corresponding classical geometry. After shifting x^- by $x^- \to y^- + P^+$, defining $P^+ = e^{-\lambda v_0}$, and by using (6.37), we can rewrite the trajectory in the notation of the forgoing sections:

$$e^{-\lambda(\bar{u}-\bar{v})} = \frac{1}{\lambda}(\omega_{crit} e^{-\lambda(\bar{u}-v_0)} - \omega'). \tag{6.41}$$

The relation $x^+p_+ = y^-p_-$ becomes

$$\omega = \omega'(1-e^{-\lambda(v_0-\bar{u})}) \approx \omega' e^{\lambda(\bar{u}-v_0)}. \tag{6.42}$$

In the background geometry of a black hole of mass M, we send a signal with frequency ω' backwards in time from an outgoing time $\overline{u}$. It will bounce off the boundary, and produce an incoming signal at past infinity. The relation between the initial frequency and final frequency can thus be written for late times $\overline{u}$ as

$$\overline{u} = v_0 - \frac{1}{\lambda}\log(\frac{\omega'}{\omega}), \tag{6.43}$$

with λ the dilaton gravity "cosmological constant", which sets the scale of the surface gravity at the black hole horizon. The time v_0 is roughly the black hole formation time. The reflection off the boundary takes place at an ingoing time $\overline{v}$ given by

$$\overline{v} = v_0 + \frac{1}{\lambda}\log\Big(\frac{\omega_{crit} - \omega}{\lambda}\Big). \tag{6.44}$$

Somewhat surprisingly, we see in (6.43) that the backreaction of the geometry does not modify the old linearized relation between the in- and out-going frequencies. Hence, if we would use this relation to compute the Bogolyubov coefficients and the out-going spectrum, we find no interesting corrections to the out-going spectrum. We have for $\omega < \omega_{crit}$

$$\tilde{\alpha}_{\omega\omega'} \approx e^{i\omega'\overline{u}_+ - i\omega\overline{v}_+} \approx e^{iv_0(\omega'-\omega) - \frac{i\omega'}{\lambda}\log(\frac{\omega'}{\omega}) - \frac{i\omega}{\lambda}\log((\omega_{crit}-\omega)/\lambda)}.$$

$$\tilde{\beta}_{\omega\omega'} \approx e^{i\omega'\tilde{u}_- + i\omega\tilde{v}_-} \approx e^{-\frac{\pi\omega'}{\lambda} + iv_0(\omega'+\omega) - \frac{i\omega'}{\lambda}\log(\frac{\omega'}{\omega}) + \frac{i\omega}{\lambda}\log((\omega_{crit}+\omega)/\lambda)} \tag{6.45}$$

So that

$$\frac{|\alpha_{\omega\omega'}|^2}{|\beta_{\omega\omega'}|^2} = e^{\frac{2\pi\omega'}{\lambda}} \tag{6.46}$$

which seems to indicate that the out-going thermal spectrum receives no corrections whatsoever.

However, the result (6.43) for the classical reflection time is only valid in the sub-critical regime. As soon as

$$\omega > \omega_{crit} \tag{6.47}$$

we can no longer use the above formulas, because there no longer exists a classical trajectory that relates an outgoing wave of this frequency to a regular incoming wave. Indeed, as seen from (6.44), there is no longer a physical reflection time. Instead, the only classical solutions that contain out-going particles in the supercritical regime (6.47) are solutions that contain either white holes or naked singularities. In neither case, however, we know of good physical principles that provide concrete initial conditions for the out-going state. This super-critical regime (6.47) is reached very quickly, since in terms of the out-going frequency ω' the inequality reads

$$\omega' > \omega_{crit} e^{-\lambda(\overline{u}-v_0)}. \tag{6.48}$$

Thus we are forced to conclude that the WKB method breaks down when applied to two-dimensional dilaton gravity.

6.4 Discussion

In appendix A I discuss the WKB approximation. It is shown that the approximation is justified provided that the frequency varies slowly relative to its magnitude. In our case the frequency increases exponentially, and thus the variation is big. At the same time however, the frequency itself is very large, satisfying the condition.

Another justification for using this approximation is the fact that this same approximation was used by the physicists who invented the moving mirror model. I just included backreaction effects in the same calculation. However, I do not deny that other approximation schemes could give other interesting results.

The combination of the WKB approximation and the saddlepoint approximation should not be viewed as succesive approximations, but rather as related approximations.

As described in the appendix, the WKB approximation is a classical approximation by taking the classical path of the quantumfields. On the other hand the saddlepoints of the integral are the classical reflection points on the mirror.

Chapter 7

Conclusion

In this report I gave a short review of the facts which led to Hawkings discovery and the explicit calculation of the Hawking effect is presented. Through the years their has been done a lot of work on the subject and the effect is supported by many authors. At the same time one recognized the problems coming along with the acceptance of black hole evaporation. The violation of unitarity and the inevitable occurrence of divergent energies are problems which are still unresolved due to the absence of a theory of quantum gravity and the lack of knowledge of Planck scale physics.

I tried to include backreaction effects in the moving mirror model in order to implement energy conservation in the particle creation process and found that backreaction effects indeed affect the outcome.

In the calculations of the corrections of the original result, I was forced to use approximations. In itself the result is promising and could give an indication to other models. However, it remains a challenge to improve the approximations, in order to strengthen the result and gain more insight in the whole process. This area in physics, where general relativity and quantum mechanics meet each other, could give more insight in what properties a unified theory, or a quantum theory of gravity, should possess.

Appendix A

WKB Approximation

Consider the Schrödinger equation (in 1 dimension), where the potential energy does not have a simple form,

$$\frac{d^2\psi}{dx^2} + \frac{2\mu}{\hbar}(E - V(x))\psi = 0. \tag{A.1}$$

If $V = const.$ the equation (A.1) has simple solutions $e^{\pm\frac{i}{\hbar}kx}$. This suggests to try a solution of the form

$$\psi(x) = e^{\frac{i}{\hbar}u(x)}. \tag{A.2}$$

Substitution of (A.2) in (A.1) gives us an equation for the x-dependent phase, $u(x)$:

$$i\hbar\frac{d^2u}{dx^2} - \left(\frac{du}{dx}\right)^2 + \hbar^2k^2(x) = 0 \tag{A.3}$$

where

$$k(x) = \begin{cases} \sqrt{\frac{2\mu}{\hbar^2}(E - V(x))} & E > V(x) \\ -i\sqrt{\frac{2\mu}{\hbar^2}(V(x) - E)} & E < V(x). \end{cases}$$

At this point we make the approximation that the potential varies slowly. A first crude approximation is to omit the first term in (A.3). In first order, u_0, we find thus

$$u_0 = \pm\hbar\int^x k(x')dx' + C. \tag{A.4}$$

To improve the approximation in second order we write first

$$\frac{du}{dx} = \pm\hbar\sqrt{k^2(x) + \frac{i}{\hbar}u''(x)}$$

and make the approximation

$$u_1(x) \approx \pm\hbar\int^x \sqrt{k^2(x') + \frac{i}{\hbar}u_0''(x')}\,dx' + C_1 = \pm\hbar\int^x \sqrt{k^2(x') \pm i\hbar k'(x')}dx' + C_1 \tag{A.5}$$

The correction is baseless unless u_1 is close to $u_0(x)$, i.e. unless

$$|k'(x)| \ll |k^2(x)|. \tag{A.6}$$

This means that $k(x)$ may only vary slowly. With condition (A.6) we can approximate (A.5)

$$u_1(x) \approx \pm\hbar \int^x k(x') + \frac{i}{2}\frac{k'(x')}{k(x')}dx' + C_1 = \pm\hbar \int^x k(x')dx' + \frac{i\hbar}{2}\log k(x) + C_1 \tag{A.7}$$

which is known as the *WKB approximation*, named after Wentzel, Kramers and Brillouin. As we see in (A.2), $u(x)$ can be regarded as the action S. One expands S in powers of $\hbar$,

$$S = S_0 + \hbar S_1 + \frac{\hbar^2}{2}S_2 + ...$$

Note that $S_0 + \hbar S_1 = u_1(x)$.

The approximated wave function ψ becomes thus

$$\psi(x) \approx \begin{cases} e^{\pm i\int^x k(x)dx} & \text{in zeroth order in } \hbar \\ \frac{1}{\sqrt{k(x)}}e^{\pm i\int^x k(x)dx} & \text{in first order in } \hbar \end{cases} \tag{A.8}$$

The WKB approximation is also refered to as classical approximation, stationary state approximation or geometrical optics approximation. In the classical limit $\hbar \to 0$ we obtain the classical solution and the rays associated with ψ (orthogonal trajectories to the surfaces with constant phase) are the possible paths of the classical particle. If we compare those wavefunctions with the wavefunctions in chapters 5 and 6, we find

$$e^{-i\omega' f(v)} = e^{-i\omega'\int\frac{1}{\kappa(v_0-v)}dv} \tag{A.9}$$
$$e^{-i\omega p(u)} = e^{-i\omega\int e^{-\kappa u}du}$$

The condition (A.6) becomes in our case

$$\kappa \ll \omega' \tag{A.10}$$

Note that $\omega' \approx \omega e^{-\kappa u}$.

In path integral language the transport of a wave function is described as

$$\psi(t) = \int dr_0 \psi(r_0)\Big[\int \mathcal{D}\mathbf{r}\, e^{\frac{i}{\hbar}S(\mathbf{r})}\Big] \tag{A.11}$$

with $\psi(0) = \rho(r_0)e^{i/\hbar S(r_0)}$ and S the action of the path $\mathbf{r}$. For small $\hbar$, thus in the classical limit, the extrema of the action dominate the path integral and the integral can be approximated by the saddlepoint phase

$$\psi(r,t) \approx e^{\frac{i}{\hbar}S_0(r_0)+\int_{r_0}^{r}\frac{\partial S_0}{\partial r_0}dr - Et} \tag{A.12}$$

This formula means that the energy at r_0 is transported along the *classical* trajectory to $r(t)$. However, the frequency of the wave function must vary sufficiently slowly. Imposing that E is constant giving an energy eigenstate, yields the WKB expression for eigenstates in one dimensional quantum mechanics.

Appendix B

Thermal Spectrum

In this appendix I will calculate the expectation value of the number operator in an ideal gas and give the expression for the Planck spectrum of a black body.

The partition function Z is given by

$$Z = \mathrm{Tr} e^{-\beta(H-\mu N)} \tag{B.1}$$

where the number operator and the hamilton operator are given by

$$N = \sum_p a_p^\dagger a_p \qquad H = \sum_p \epsilon_p a_p^\dagger a_p.$$

Hence we find for the expectation value

$$\langle a_p^\dagger a_q \rangle = \frac{1}{Z}\mathrm{Tr}(e^{-\beta \sum_i (\epsilon_i - \mu) N_i} a_p^\dagger a_q). \tag{B.2}$$

An important relation is

$$[A, B] = \gamma B \quad \Longleftrightarrow \quad e^A B e^{-A} = e^\gamma B$$

applied to

$$[N_p, a_p^\dagger] = a_p^\dagger. \tag{B.3}$$

Using this in the first step:

$$\mathrm{Tr}(e^{-\beta \sum_i (\epsilon_i-\mu) N_i} a_p^\dagger a_q) = e^{-\beta(\epsilon_p - \mu)} \mathrm{Tr}(e^{-\beta \sum_i (\epsilon_i-\mu) N_i} a_q a_p^\dagger) = \tag{B.4}$$

$$= e^{-\beta(\epsilon_p-\mu)}[\delta_{pq}\mathrm{Tr}(e^{-\beta \sum_i (\epsilon_i-\mu) N_i}) \pm \frac{1}{Z}\mathrm{Tr}(e^{-\beta \sum_i (\epsilon_i-\mu) N_i} a_p^\dagger a_q)$$

gives us the final result

$$\langle a_p^\dagger a_q \rangle = \frac{\delta_{pq}}{e^{\beta(\epsilon_p - \mu)} \mp 1} \tag{B.5}$$

where the +sign is for fermions and the - sign for bosons and with β the inverse temperature 1/kT.

The precise Planck expression for the density of radiation

$$E(\omega') = \frac{2\omega'^2}{\pi c^3}\frac{\hbar\omega'}{e^{\hbar\omega'/kT} - 1}.$$

With $\hbar = 1$, the maximum of this function lies at $\omega' \approx 3kT$, so the statement that the typical frequency of the out-going radiation is of the order κ, in the case of the moving mirrors, is justified.

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