“Elementary Particles”
Lecture 2

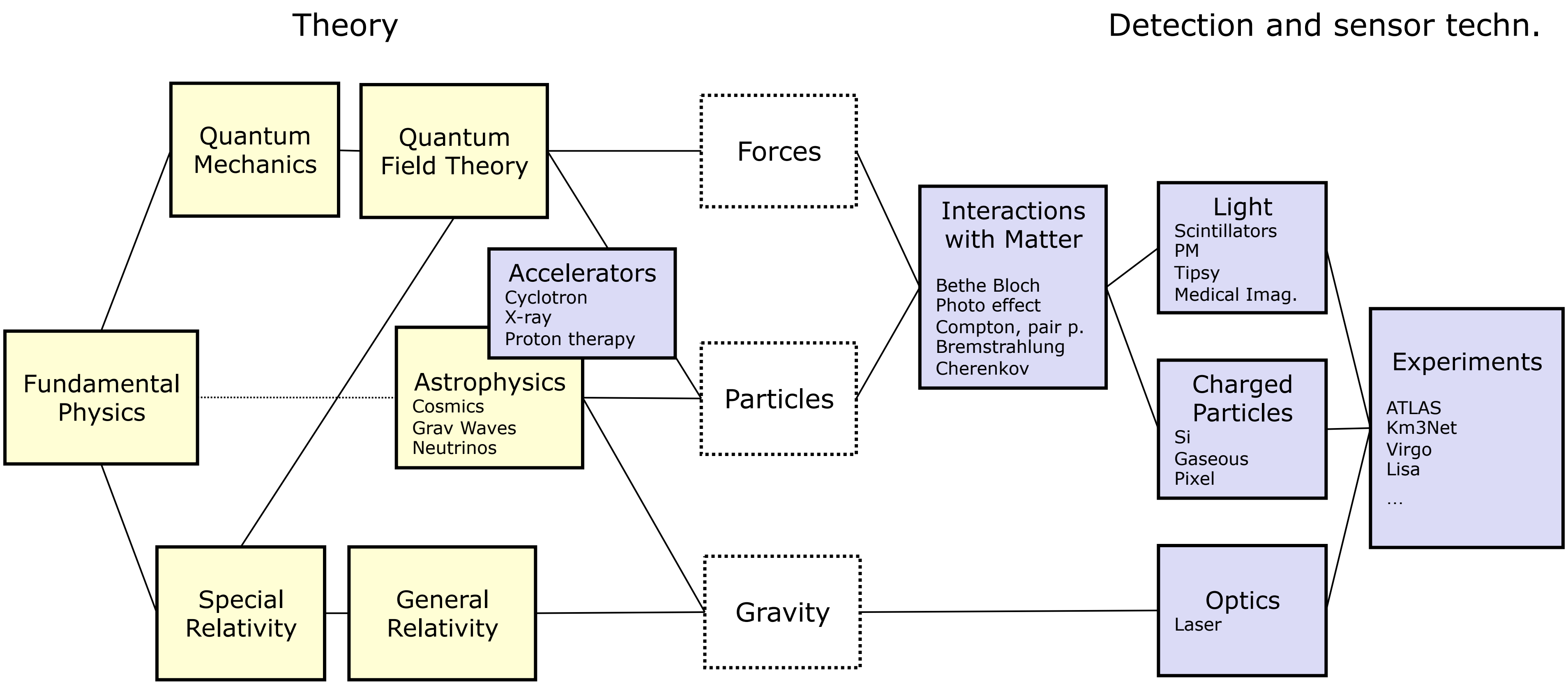
Niels Tuning

Harry van der Graaf

Ernst-Jan Buis

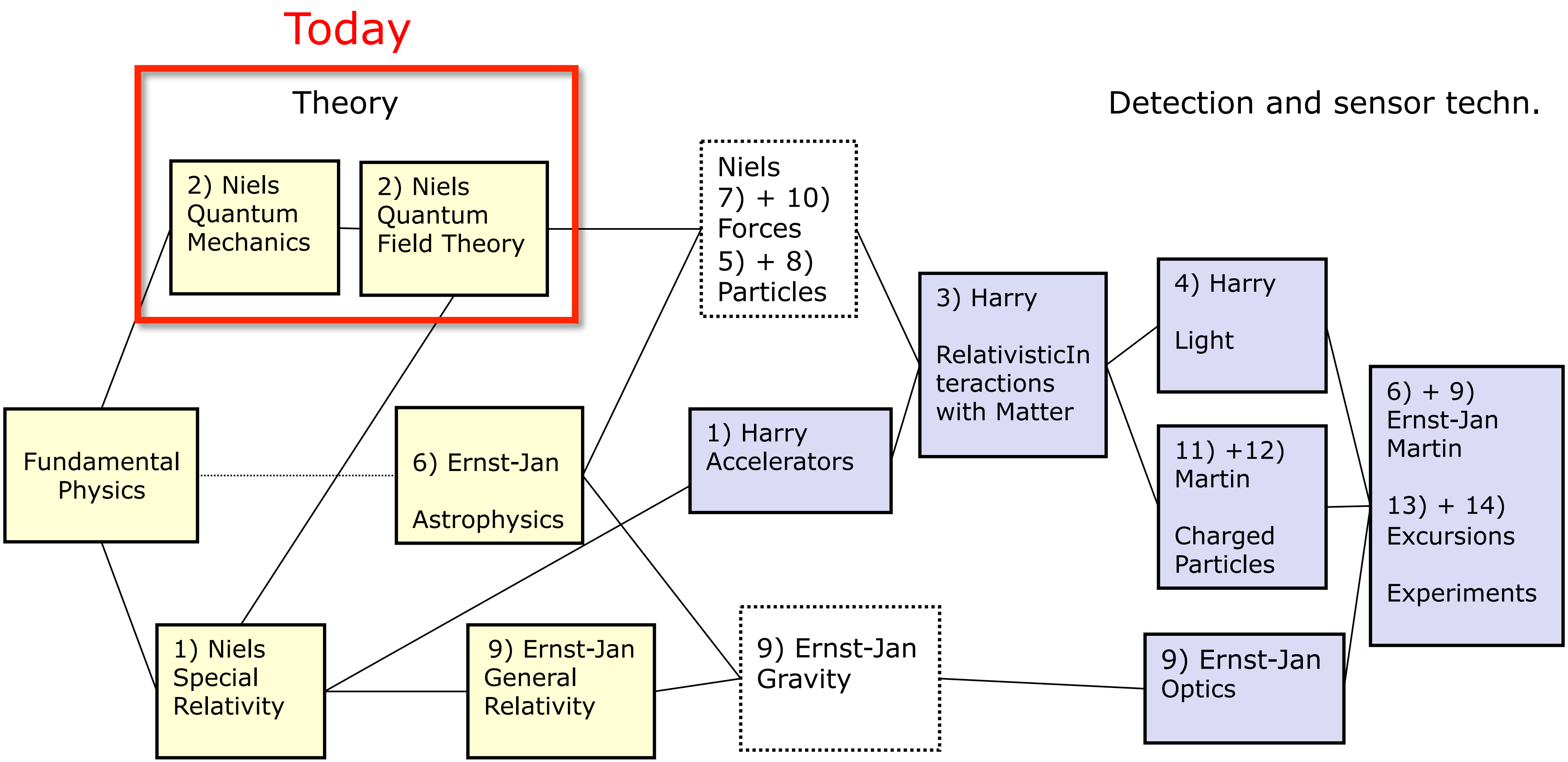
Martin Fransen

Plan



Theory
Detection and sensor techn.
Quantum Mechanics
Quantum Field Theory
Forces
Fundamental Physics
Accelerators
Cyclotron
X-ray
Proton therapy
Astrophysics
Cosmics
Grav Waves
Neutrinos
Particles
Interactions with Matter
Bethe Bloch
Photo effect
Compton, pair p.
Bremstrahlung
Cherenkov
Light
Scintillators
PM
Tipsy
Medical Imag.
Charged Particles
Si
Gaseous
Pixel
Experiments
ATLAS
Km3Net
Virgo
Lisa
...
Special Relativity
General Relativity
Gravity
Optics
Laser

Plan


Today
Theory
Detection and sensor techn.
2) Niels Quantum Mechanics
2) Niels Quantum Field Theory
Niels 7) + 10) Forces 5) + 8) Particles
3) Harry RelativisticInteractions with Matter
4) Harry Light
6) + 9) Ernst-Jan Martin 13) + 14) Excursions Experiments
Fundamental Physics
6) Ernst-Jan Astrophysics
1) Harry Accelerators
11) +12) Martin Charged Particles
1) Niels Special Relativity
9) Ernst-Jan General Relativity
9) Ernst-Jan Gravity
9) Ernst-Jan Optics

Schedule

1) 11 Feb: Accelerators (Harry vd Graaf) + Special relativity (Niels Tuning)
2) 18 Feb: Quantum Mechanics (Niels Tuning)
3) 25 Feb: Interactions with Matter (Harry vd Graaf)
4) 3 Mar: Light detection (Harry vd Graaf)
5) 10 Mar: Particles and cosmics (Niels Tuning)
6) 17 Mar: Astrophysics and Dark Matter (Ernst-Jan Buis)
7) 24 Mar: Forces (Niels Tuning)

break

8) 21 Apr: e^+e^- and ep scattering (Niels Tuning)
9) 28 Apr: Gravitational Waves (Ernst-Jan Buis)
10) 12 May: Higgs and big picture (Niels Tuning)
11) 19 May: Charged particle detection (Martin Franse)
12) 26 May: Applications: experiments and medical (Martin Franse)

13) 2 Jun: Nikhef excursie
14) 8 Jun: CERN excursie

Thanks

- Ik ben schatplichtig aan:
  - Dr. Ivo van Vulpen (UvA)
  - Prof. dr. ir. Bob van Eijk (UT)
  - Prof. dr. M. Merk (VU)

Plan

1) Intro: Standard Model & Relativity — ***11 Feb***

1900-1940 2) Basis — ***18 Feb***
   1) Atom model, strong and weak force
   2) Scattering theory

1945-1965 3) Hadrons — ***10 Mar***
   1) Isospin, strangeness
   2) Quark model, GIM

1965-1975 4) Standard Model — ***24 Mar***
   1) QED
   2) Parity, neutrinos, weak inteaction
   3) QCD

1975-2000 5) e^+e^- and DIS — ***21 Apr***

2000-2015 6) Higgs and CKM — ***12 May***

Exercises Lecture 1: Special Relativity

1 Lorentz transformation

a) The Galilean transformation of the space coordinate, from coordinate system S to system S', with relative velocity v, is given by $x' = x - vt$. What is the Galilean transformation of the time coordinate, between two inertial observers?

b) The Galilean transformation of the space coordinate x, from system S' to S, is given by $x = x' - vt$. Let's find the corresponding transformation if we assume that the speed of light is equal in systems S and S', ie. $x' = ct'$ and $x = ct$. We modify the Galilean transformation rules, by $x' = \gamma(x - vt)$ and find the expression for γ :

$$x' = \gamma(x - vt) \stackrel{x=ct}{=} \gamma(ct - vt) \quad (1)$$

$$x = \gamma(x' - vt') \stackrel{x'=ct'}{=} \gamma(ct' - vt') \quad (2)$$

This leads to:

$$\frac{x'}{\gamma} = \frac{ct'}{\gamma} = \frac{\gamma(ct - vt)}{\gamma} = (ct - vt) \quad (3)$$

Eliminate t in the above expression, and give the expression for γ.

b) $t' = t$

a) Find the expression for γ :

$$\left.\begin{array}{l} x' = \gamma(x - vt) \stackrel{\text{x=ct}}{=} \gamma(ct - vt) \\ x' = ct' \end{array}\right\}$$

This leads to:

$$\frac{x'}{\gamma} = \frac{ct'}{\gamma} = (ct - vt)$$

Eliminate t in the above expression, and give expression for γ:

$$\frac{ct'}{\gamma} = (ct - vt) \stackrel{ct=\gamma(ct'+vt')}{=} \gamma(ct' + vt') - \frac{v}{c}\gamma(ct' + vt')$$

$$\Rightarrow \frac{1}{\gamma^2} = (1 - v^2/c^2)$$

Exercises Lecture 1: Special Relativity

c) Rewrite the Lorentztransformation,

$$x' = \gamma(x - vt) \tag{4}$$

$$t' = \gamma(t - \frac{v}{c^2}x), \tag{5}$$

expressing the velocity as a fraction of the speed of light, $\beta = v/c$, and the time-coordinate as $x^0 \equiv ct$.

c)

$$x' = \gamma(x - vt)$$

$$t' = \gamma(t - \frac{v}{c^2}x),$$

$$x' = \gamma(x^1 - \frac{v}{c}ct) \overset{\beta=v/c)}{=} \gamma(x^1 - \beta x^0)$$

$$ct' = \gamma(ct - \frac{v}{c}x) \overset{x^0=ct)}{=} \gamma(x^0 - \beta x^1)$$

Exercises Lecture 1: Special Relativity

d) The time-coordinate, and three space coordinates can be expressed as 4-vectors $x^\mu = (t/c, x, y, z)$. Show that the quantity $I = \Sigma_{\mu=0,3}\Sigma_{\nu=0,3} g_{\mu\nu} x^\mu x^\nu = x_\mu x^\mu$ is invariant, ie. that $I = I'$. (Apply a boost in the direction of x^1.)

e) Suppose you want to build a muon collider, and you want to keep your muons about 30 minutes in your accelerator before they decay. What boost (ie. value for γ) is then needed for the muons? (The lifetime of muons is 2.2 μs.) To what beam energy does this correspond? (The mass of the muon is 106 MeV/c^2.)

d)

$$\begin{aligned} I' &= (x'^0)^2 - (x'^1)^2 - (x'^2)^2 - (x'^3)^2 && (10) \\ &= (\gamma(x^0 - \beta x^1))^2 - (\gamma(x^1 - \beta x^0))^2 - (x'^2)^2 - (x'^3)^2 && (11) \\ &= \gamma^2((x^0)^2(1-\beta^2) - (x^1)^2)(1-\beta^2)) - (x'^2)^2 - (x'^3)^2 && (12) \\ &= \gamma^2(1-\beta^2)((x^0)^2 - (x^1)^2) - (x'^2)^2 - (x'^3)^2 && (13) \\ &= (x^0)^2 - (x^1)^2 - (x'^2)^2 - (x'^3)^2 = I && (14) \end{aligned}$$

e)

$$\begin{aligned} \gamma &= \Delta t'/\Delta t = 1800/2.2 \times 10^{-6} = 8 \times 10^8 && (15) \\ E &= \gamma m_0 = 8 \times 10^8 \times 0.106 = 8 \times 10^7 \mathrm{GeV} && (16) \end{aligned}$$

2 Relativistic momentum

Given 4-vector calculus, we know that $p_\mu p^\mu = E^2/c^2 - \vec{p}^2 = m_0^2 c^2$.

a) Show that you get in trouble when you use $E = mc^2$ and $\vec{p} = m\vec{v}$.

b) Show that $E = \gamma m_0 c^2$ and $\vec{p} = \gamma m_0 \vec{v}$ obey $E^2/c^2 - \vec{p}^2 = m_0^2 c^2$.

a) Using $E = mc^2$ and $\vec{p} = m\vec{v}$, one finds:
$E^2/c^2 - \vec{p}^2 = m^2c^2 - m^2v^2 = m^2(c^2 - v^2) \neq m^2c^2$

b) Using $E = \gamma m_0 c^2$ and $\vec{p} = \gamma m_0 \vec{v}$, one finds:
$E^2/c^2 - \vec{p}^2 = \gamma^2(m^2(c^2 - v^2)) = m^2c^2 \frac{1-v^2/c^2}{1-v^2/c^2} = m^2c^2$.

Exercises Lecture 1: Special Relativity

3 Center-of-mass energy

a) Not only the space and time can be expressed as a 4-vector, but also energy and momentum can be expressed as 4-vectors, $p^\mu = (E/c, p_x, p_y, p_z)$. Because $p_\mu p^\mu$ is invariant, this means that the rest-mass m_0 of a particle does not change under Lorentz transformations. Show that $p_\mu p^\mu = m_0^2 c^2$.

b) Let's consider two colliding particles a and b, with 4-momenta p_a^μ and p_b^μ. We will use natural units, with $c = 1$ and $\hbar = 1$, so $p_a^\mu = (E_a, \vec{p}_a)$. We take the masses of the two colliding particles equal, $m_a = m_b = m$, and we sit in the center-of-mass frame of the system, $\vec{p}_a = -\vec{p}_b$. What are the four components of the sum of the two 4-vectors, $p_{tot}^\mu = (p_a^\mu + p_b^\mu)$?

c) The 'invariant mass' of the combined system, is often called the 'center-of-mass energy' of the collision. If the energy of both particles a and b is 4 TeV, what is then the center-of-mass energy, $\sqrt{s} \equiv \sqrt{p_{tot}^\mu p_{\mu,tot}}$?

d) Let's consider a fixed-target collision of two protons. One proton has an energy of 4 TeV, and 4-vector p_a^μ, whereas the other proton is at rest, with 4-vector p_b^μ. What are the four components of the sum of the two 4-vectors, $p_{tot}^\mu = (p_a^\mu + p_b^\mu)$? Give the expression for the center-of-mass energy of this system.

a)

$$p_\mu p^\mu = (E/c)^2 - |\vec{p}|^2 = (E^2 - c^2|\vec{p}|^2)/c^2 = (m_0 c^4)/c^2$$

b)

$$p_{tot}^\mu = (p_a^\mu + p_b^\mu) = (E_a, \vec{p}_a) + (E_b, \vec{p}_b) = (E_a + E_b, 0) = (2E, 0)$$

c)

$$\begin{aligned} (E, 0, 0, \sqrt{E^2 - m^2}) + (E, 0, 0, -\sqrt{E^2 - m^2}) &= (2E, 0, 0, 0) \\ \Rightarrow s &= 4E^2 \Rightarrow \sqrt{s} = 2E = 8\text{TeV} \end{aligned}$$

d)

$$\begin{aligned} (E, 0, 0, \sqrt{E^2 - m^2}) + (m, 0, 0, 0) &= (E + m, 0, 0, \sqrt{E^2 - m^2}) \\ \Rightarrow s = (E+m)^2 - (E^2 - m^2) &= 2m^2 + 2Em \\ \Rightarrow \sqrt{s} \sim \sqrt{2Em} &= 89\text{GeV} \end{aligned}$$

Exercises Lecture 1: Special Relativity

e) People were afraid that the earth would be destroyed at the start of the LHC, planning for collisions with beams of 7 TeV each. The earth has been bombarded for billions of years with cosmic rays. What is the center-of-mass energy of the highest energetic cosmic rays (10^{21} eV) hitting the atmosphere? Was the fear justified?

f) What is the energy of a cosmic ray hitting the atmosphere, that corresponds to the center-of-mass energy of collisions of two lead-ions ^{208}Pb with energies of 2.24 TeV per nucleon?

g) Consider relatively low-energy proton-proton collisions, with opposite and equal momenta (ie. the center-of-mass system is at rest). In the process $p+p \rightarrow p+p+p+\bar{p}$ an extra proton-antiproton pair is created. What is the minimum energy of the protons to create two extra (anti)protons?

e)

$$\sqrt{s} \sim \sqrt{2Em} = \sqrt{2 \times 10^{12} \times 1\mathrm{GeV}^2} \sim 10^6\mathrm{GeV} = 10^3\mathrm{TeV} \quad (24)$$

$$\sqrt{s} \sim \sqrt{2Em} = 14\ \mathrm{TeV} \Rightarrow E = \frac{1}{2m}(14 \times 10^3)^2 = 10^{17}\mathrm{eV} \quad (25)$$

f) (Scaled the 1.38 TeV/nucleon from 4 TeV to the expected 6.5 TeV in 2015.)

$$\sqrt{s} \sim \sqrt{2Em} = 2 \times 208 \times 2.24\ \mathrm{TeV} \sim 10^3\ \mathrm{TeV} \Rightarrow E = \frac{1}{2m}(10^6\ \mathrm{GeV})^2 = 5 \times 10^{20}\mathrm{eV} \quad (26)$$

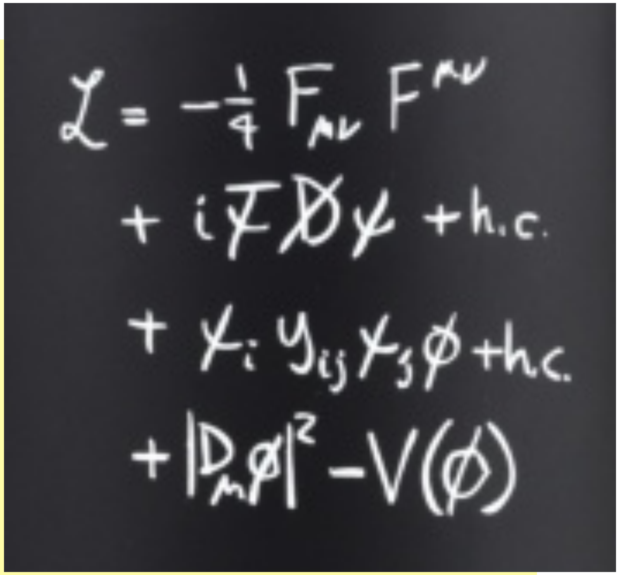
g)

$$\mathrm{Before}: \quad s = 4E^2 \quad (27)$$

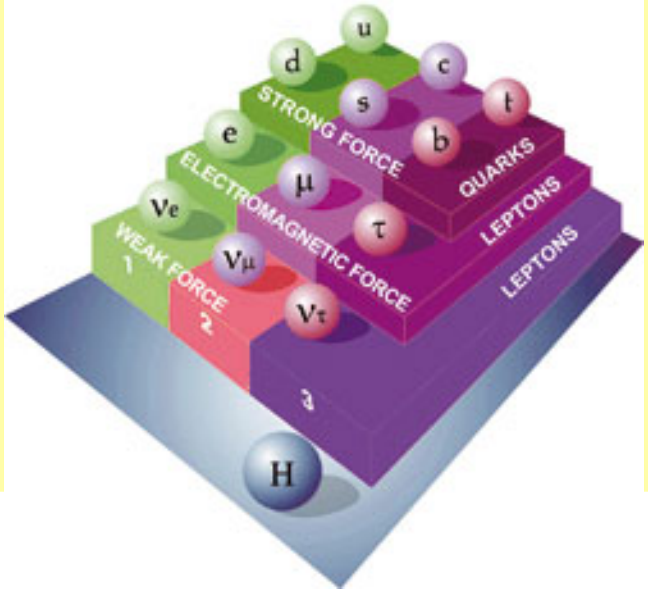
$$\mathrm{After}_{\mathrm{min}}: \quad s = (4m)^2 \Rightarrow E_{min} = 2m = 2\mathrm{GeV} \quad (28)$$

Lecture 1: Standard Model & Relativity

- Standard Model Lagrangian



- Standard Model Particles



Lecture 1: Standard Model & Relativity

- Theory of relativity
  - Lorentz transformations ("boost")
  - Calculate energy in colissions

$$x'^0 = \gamma\left(x^0 - \beta x^1\right)$$
$$x'^1 = \gamma\left(x^1 - \beta x^0\right)$$
$$x'^2 = x^2$$
$$x'^3 = x^3$$

met

$$\beta \equiv \frac{v}{c}$$
$$\gamma \equiv \frac{1}{\sqrt{1-\beta^2}}$$

- 4-vector calculus

$$x^\mu = \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix}, \quad \left(\mu = 0, 1, 2, 3\right)$$

$$p_\mu p^\mu = (E/c)^2 - |\vec{p}|^2 = (E^2 - c^2|\vec{p}|^2)/c^2 = (m_0 c^4)/c^2$$

- High energies needed to make (new) particles

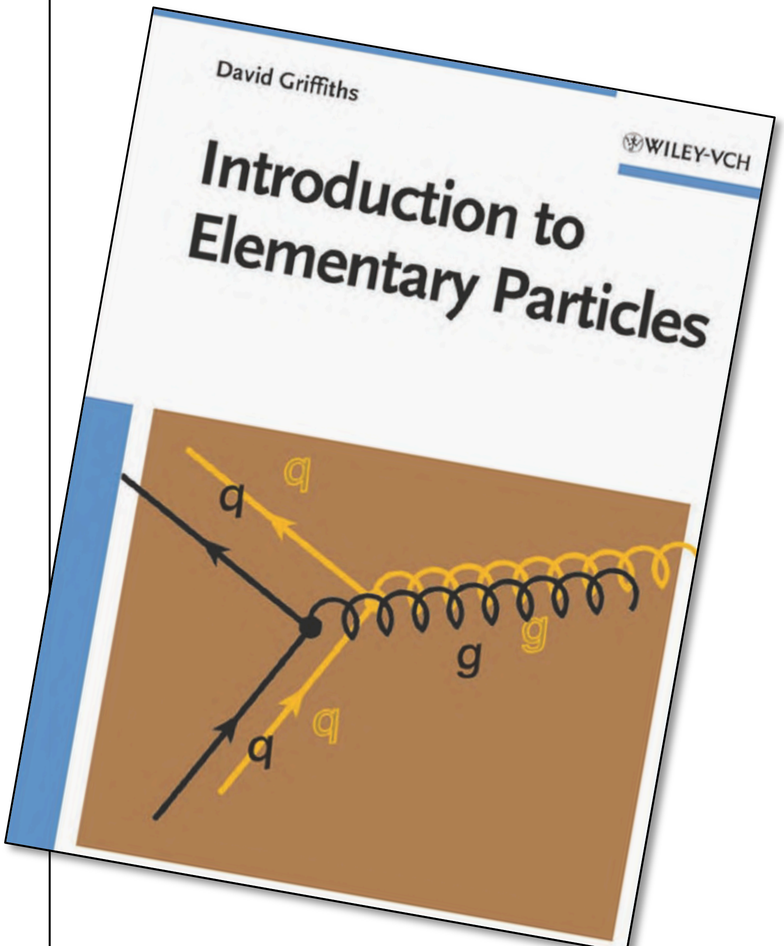
$$s = \left(p_1 + p_2\right)^2 = 2m^2 + 2\left(E^2 + \vec{p}^2\right)$$
$$= 2m^2 + 2E^2 + 2\left(E^2 - m^2\right) = 4E^2$$

Outline for today

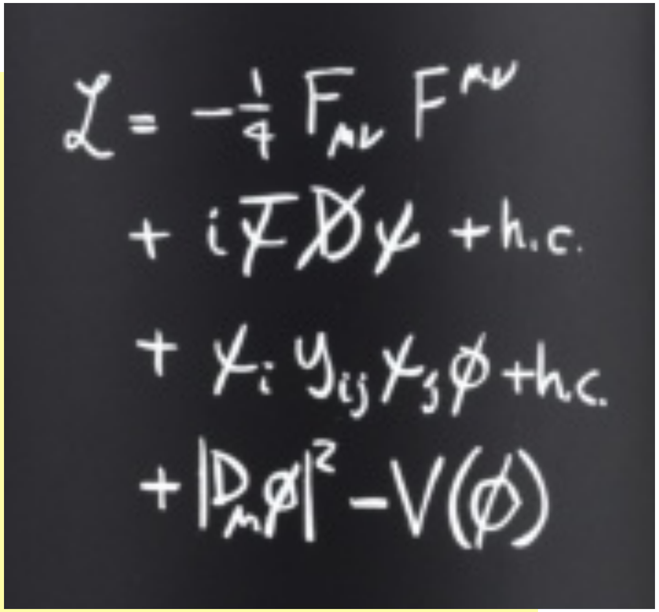
- Quantum mechanics: equations of motions of wave functions
  - Schrodinger, Klein Gordon, Dirac
- Forces
  - Strong force, pion exchange
  - Weak nuclear force, decay
- Scattering Theory
  - Rutherford (classic) and QM
  - “Cross section”
  - Coulomb potential
  - Yukawa potential
  - Resonances

D. Griffiths "Introduction to Elementary Particles"

- Lecture 1:
  - ch.3 Relativistic kinematics
- Lecture 2:
  - ch.5.1 Schrodinger equation
  - ch.7.1 Dirac equation
  - ch.6.5 Scattering
- Lecture 3:
  - ch.1.7 Quarkmodel
  - ch.4 Symmetry/spin
- Lecture 4:
  - ch.7.4 QED
  - ch 11.3 Gauge theories
- Lecture 5:
  - ch.8.2 e+e-
  - ch.8.5 e+p
- Lecture 6:
  - ch.11.8 Higgs mechanism



Lecture 2: QM, Dirac and Scattering

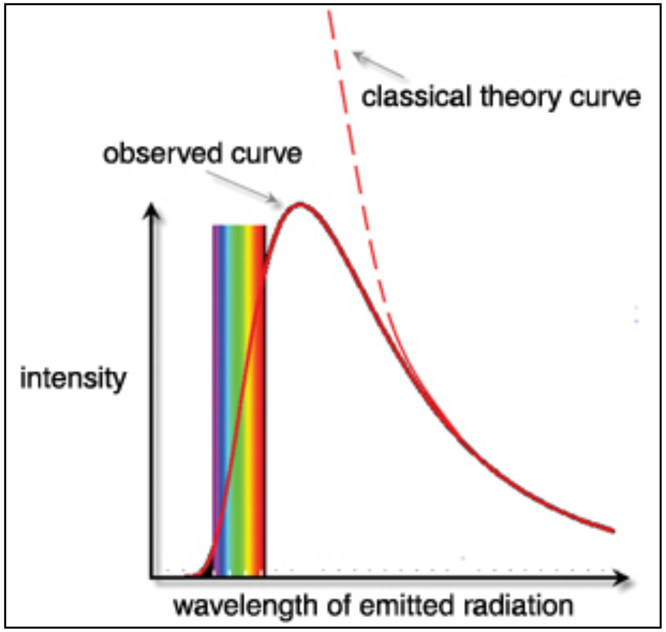


- Introduce "matter particles"
  - spinor ψ from Dirac equation
- Introduce "force particles"
- Introduce basic concepts of scattering processes

Quantum mechanics

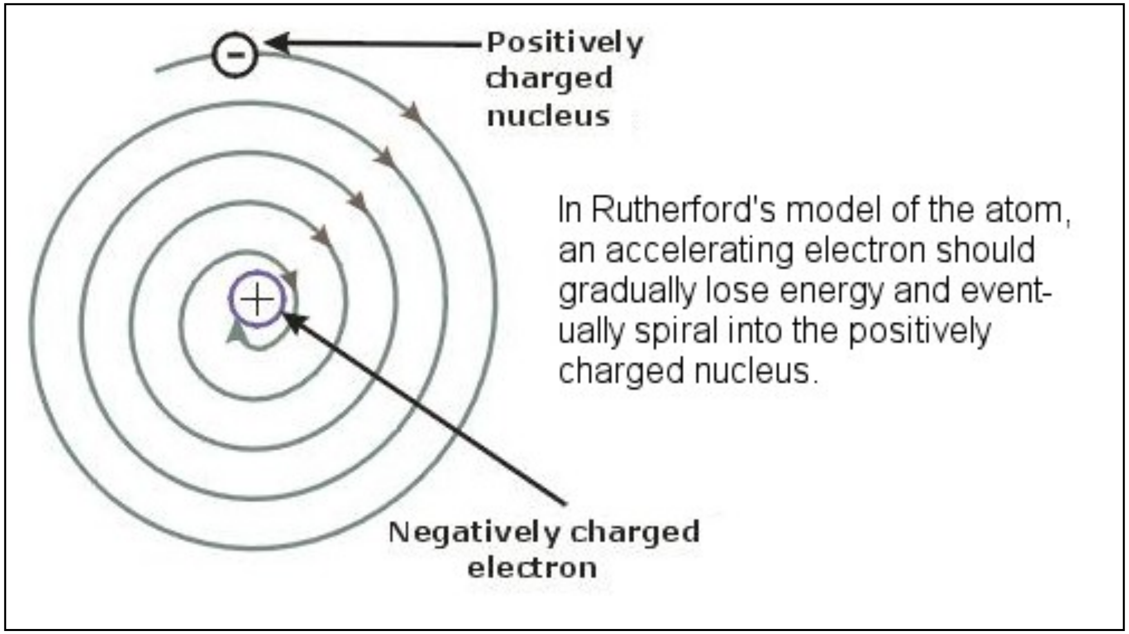
From classic to quantum

Why does the black body spectrum look like it does?



→ *Finite* number of wavelengths ($E=h\nu$)

Why does the electron not fall onto the nucleus?



→ *Finite* number of nuclear orbits

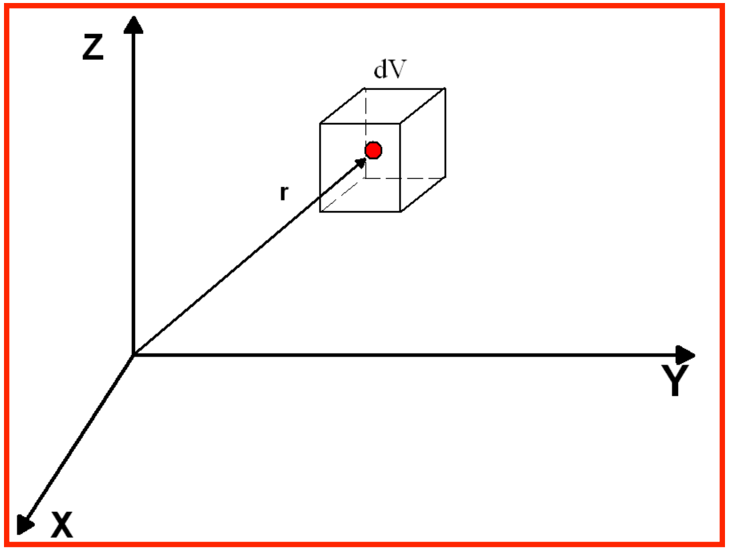
- The *wavefunction* ψ describes a system (eg. particle)
- Physical quantities are given by *operators*

Wavefunction

Each particle may be described by a wave function ψ(*x*,*y*,*z*,*t*), real or complex, having a single value for a given position (*x*,*y*,*z*) and time *t*

- In QM a particle is not *localized*
- Probability of finding a particle somewhere in a volume *V* of space:

$$P(\mathbf{r},t)dV = |\Psi(\mathbf{r},t)|^2 dV$$



- Probability to find particle anywhere in space = 1
  - ➢ condition of normalization:

$$\int_{all\ space} |\Psi(\mathbf{r},t)|^2 dV = 1$$

Operator

Any physical quantity is associated with an operator

- An operator O: the "recipe" to transform $\boldsymbol{\psi}$ into $\boldsymbol{\psi}$'
  - We write: $O\boldsymbol{\psi} = \boldsymbol{\psi}'$
- If $O\boldsymbol{\psi} = o\boldsymbol{\psi}$ then
  - $\boldsymbol{\psi}$ is an eigenfunction of O and
  - o is the eigenvalue.

We have "solved" the wave equation $O\boldsymbol{\psi} = o\boldsymbol{\psi}$ by finding simultaneously $\boldsymbol{\psi}$ and o that satisfy the equation.

➢ o is the measure of O for the particle in the state described by $\boldsymbol{\psi}$

Correspondence?

- What operator belongs to which physical quantity?

Classical quantity		QM operator
$f(x)$	Any function of position, such as x, or potential V(x)	$f(x)$
p_x	x component of momentum (y and z same form)	$\frac{\hbar}{i}\frac{\partial}{\partial x}$
E	Hamiltonian (time independent)	$\frac{p_{op}^2}{2m}+V(x)$
E	Hamiltonian (time dependent)	$i\hbar\frac{\partial}{\partial t}$
E_{kin}	Kinetic energy	$\frac{-\hbar^2}{2m}\frac{\partial^2}{\partial x^2}$
L_z	z component of angular momentum	$-i\hbar\frac{\partial}{\partial \phi}$

Example

Let's try operating:

- Wavefunction:

$$\Psi(x,t) = A\left(\cos[kx-\omega t] + i\sin[kx-\omega t]\right) = Ae^{ikx-\omega t}$$

- Momentum operator :

$$\hat{p}_x \Psi(x,t) = \frac{\hbar}{i}\frac{\partial}{\partial x} Ae^{i(kx-\omega t)} = \frac{\hbar}{i} ikAe^{i(kx-\omega t)} = \hbar k Ae^{i(kx-\omega t)} = \hbar k \Psi(x,t)$$

- Or energy operator:

$$\hat{E}\Psi\left(x,t\right) = i\hbar\frac{\partial}{\partial t} Ae^{i(kx-\omega t)} = i\hbar(-i\omega)Ae^{i(kx-\omega t)} = \hbar\omega Ae^{i(kx-\omega t)} = E\Psi\left(x,t\right)$$

➢ Ψ is indeed eigenfunction ($\hbar k$ and $\hbar\omega$ are the eigenvalues for $\hat{p}$ and $\hat{E}$)

Expectation value

Average value of physical quantity: *expectation value*

Think of the Staatsloterij:

x_i : prize

$p(x_i)$: probability to win that prize

$$E(X) = \sum_i x_i p(x_i) = 0.697 \times 13.50 = 9.41\,\text{EUR}$$

$$\langle W \rangle = \int_{-\infty}^{+\infty} \Psi^*(x,t) \left[\hat{W} \Psi(x,t) \right] dx$$

Example:

$$\psi(x) = Ae^{ikx} \quad \text{with} \quad \int_{-\infty}^{+\infty} |\psi(x)|^2 dx = \int_{-\infty}^{+\infty} \left[Ae^{ikx} \right]^* \left[Ae^{ikx} \right] dx = 1,$$

where $A \to 0$ as limits of integration $\to \infty$

$$\langle p \rangle = \int_{-\infty}^{+\infty} \left[Ae^{ikx} \right]^* \left[\frac{\hbar}{i} \frac{\partial}{\partial x} Ae^{ikx} \right] dx$$

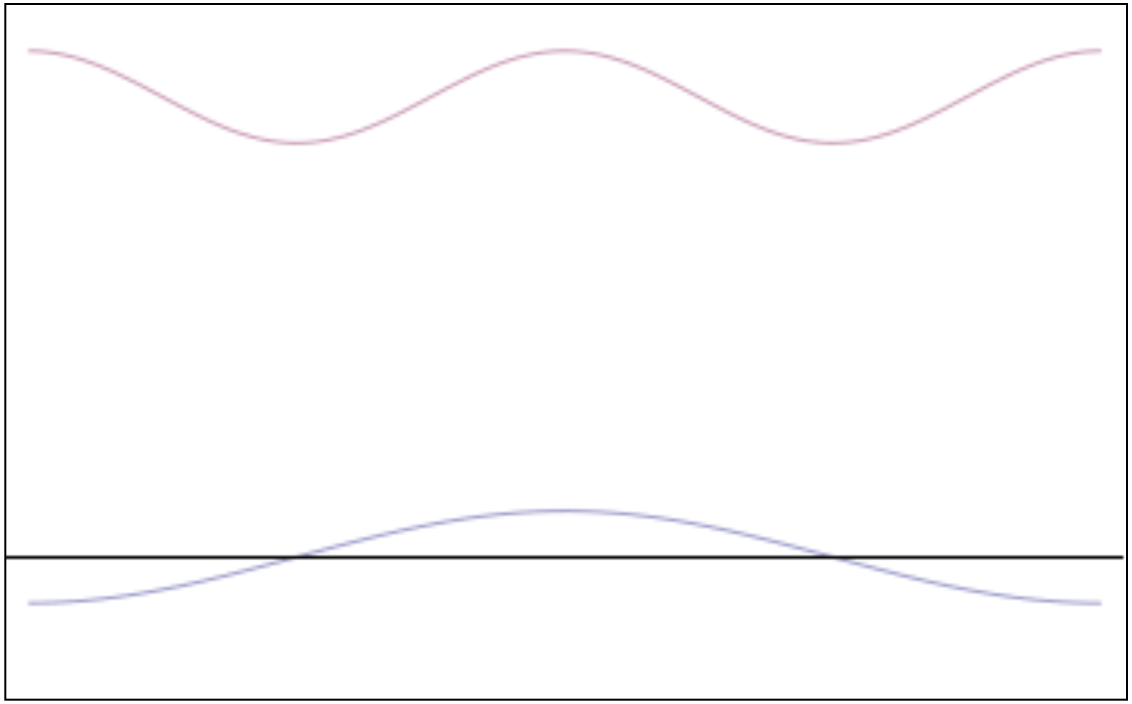
$$\langle p \rangle = \int_{-\infty}^{+\infty} \left[Ae^{ikx} \right]^* \frac{\hbar}{i} ik \left[Ae^{ikx} \right] dx = \hbar k \underbrace{\int_{-\infty}^{+\infty} \left[Ae^{ikx} \right]^* \left[Ae^{ikx} \right] dx}_{\equiv 1} = \hbar k = p$$

Heisenberg

How to describe a particle that is "localized" somewhere, but which is also "wave-like" ?

- k can be any value: $$\psi = Ae^{ikx}$$
- Fourier decomposition of many frequencies
  - The more frequencies you add, the more it gets localized
  - The worse you know p, the better you know x !

$$\Delta x \approx \frac{h}{\Delta p_x}$$



Heisenberg

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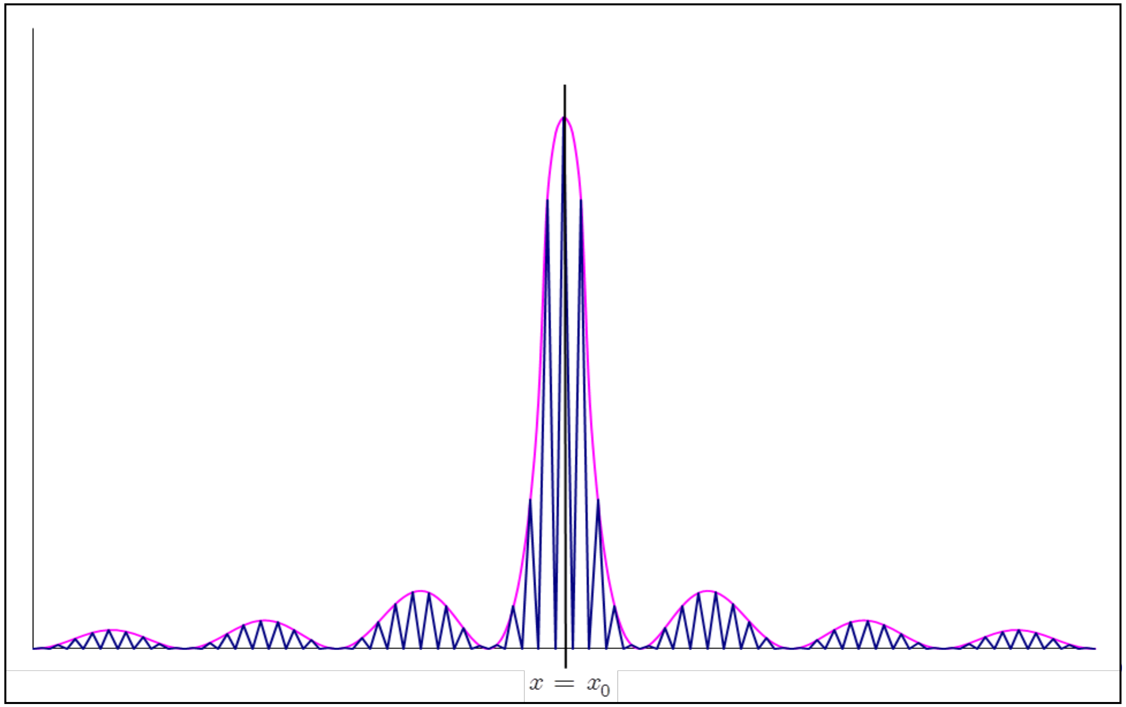

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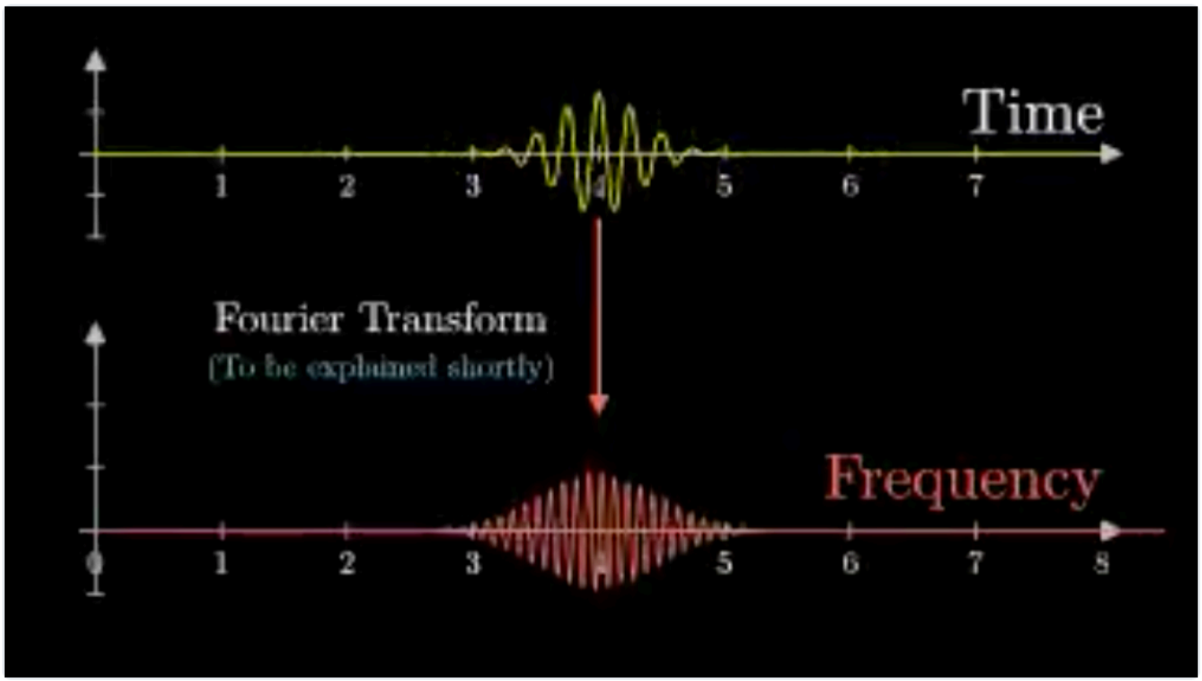
Grant Sanderson @3Blue1Brown · Feb 24

New video!

Uncertainty principle? It's not about quantum.

youtu.be/MBnnXbOM5S4



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Heisenberg

- Uncertainty relation → Commutation relation
  - ➢ A wave function cannot be simultaneously an eigenstate of position and momentum

$$P_x X\Psi = \frac{\hbar}{i}\frac{\partial}{\partial x}\left(x\Psi\right) = \frac{\hbar}{i}\left(\Psi + x\frac{\partial\Psi}{\partial x}\right)$$

$$XP_x\Psi = x\frac{\hbar}{i}\frac{\partial\Psi}{\partial x}$$

$$\left(P_x X - XP_x\right)\Psi = \left[P_x, X\right]\Psi = \frac{\hbar}{i}\Psi$$

- Suppose it *was*, what then??
  - ➢ Then the operators would commute:

$$\left(P_x X - XP_x\right)\Psi = \left(P_x x_0 - Xp_0\right)\Psi = \left(x_0 p_0 - x_0 p_0\right)\Psi = 0$$

Schrödinger

Classic relation between E and p:

$$E = \frac{\vec{p}^2}{2m}$$

Quantum mechanical substitution:

(operator acting on wave function ψ)

$$E \rightarrow i\frac{\partial}{\partial t} \quad \text{and} \quad \vec{p} \rightarrow -i\vec{\nabla}$$

Schrodinger equation:

$$i\frac{\partial}{\partial t}\psi = \frac{-1}{2m}\nabla^2\psi$$

Solution:

$$\psi = N\, e^{i(\vec{p}\vec{x} - Et)}$$

(show it is a solution)

Intermezzo: “radial Schrödinger equation”

- Polar coordinates

$$\Psi(x, y, z, t) = \psi(x, y, z)e^{-iEt/\hbar}$$

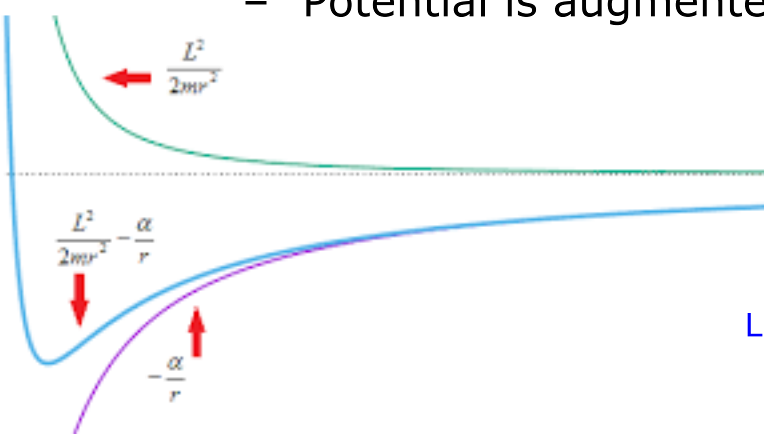
- Separate variables

$$\psi(r, \theta, \phi) = R(r)\Theta(\theta)\Phi(\phi)$$

- Three differential equations for R, φ, θ:

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) = \left[\frac{l(l+1)}{r^2} + \frac{2m}{\hbar^2}(V(r) - E)\right]R$$

- Radial Schrödinger equation
  - Potential is augmented by “centrifugal barrier” : (apparent centrifugal force)



$$-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} + \left[V(r) + \frac{\hbar^2}{2m}\frac{l(l+1)}{r^2}\right]u = Eu$$

Linear momentum Potential energy Angular momentum

Klein-Gordon

Relativistic relation between E and p: $E^2 = \vec{p}^2 + m^2$

Quantum mechanical substitution:
(operator acting on wave function ψ)

$$E \rightarrow i\frac{\partial}{\partial t} \quad \text{and} \quad \vec{p} \rightarrow -i\vec{\nabla}$$

Klein-Gordon equation:

$$-\frac{\partial^2}{\partial t^2}\phi = -\nabla^2\phi + m^2\,\phi$$

$$\text{or}\ : \quad \left(\Box + m^2\right)\,\phi(x) = 0$$

$$\text{or}\ : \quad \left(\partial_\mu\partial^\mu + m^2\right)\,\phi(x) = 0$$

Solution: $\phi(x) = N\,e^{-ip_\mu x^\mu}$ with eigenvalues: $E^2 = \vec{p}^2 + m^2$

But! Negative energy solution?

$$E = \pm\sqrt{\vec{p}^2 + m^2}$$

Dirac

Paul Dirac tried to find an equation that was

- relativistically correct,
- but *linear* in d/dt to avoid negative energies
- (and linear in d/dx (or ∇) for Lorentz covariance)

He found an equation that

- turned out to describe spin-1/2 particles and
- predicted the existence of anti-particles

Dirac

- How to find that relativistic, linear equation ??

Write Hamiltonian in general form, $H\psi = (\vec{\alpha}\cdot\vec{p} + \beta m)\ \psi$

but when squared, it must satisfy: $H^2\psi = \left(\vec{p}^2 + m^2\right)\ \psi$

Let's find α_i and β !

$$
\begin{aligned}
H^2\psi &= (\alpha_i p_i + \beta m)^2\ \psi \qquad \text{with: } i = 1,2,3 \\
&= \left(\underbrace{\alpha_i^2}_{=1}\ p_i^2 + \underbrace{(\alpha_i\alpha_j + \alpha_j\alpha_i)}_{=0 \quad i>j} p_i p_j + \underbrace{(\alpha_i\beta + \beta\alpha_i)}_{=0} p_i m + \underbrace{\beta^2}_{=1}\ m^2\right)\ \psi
\end{aligned}
$$

So, α_i and β must satisfy:

- $\alpha_1{}^2 = \alpha_2{}^2 = \alpha_3{}^2 = \beta^2$
- $\alpha_1, \alpha_2, \alpha_3, \beta$ anti-commute with each other
- (not a unique choice!)

Dirac

$$H\psi = (\vec{\alpha}\cdot\vec{p} + \beta m)\ \psi$$

- What are α and β ??

The lowest dimensional matrix that has the desired behaviour is **4x4** !?

Often used

Pauli-Dirac representation:

$$\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix} \quad ; \quad \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

with:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad ; \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad ; \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

So, α_i and β must satisfy:

- $\alpha_1{}^2 = \alpha_2{}^2 = \alpha_3{}^2 = \beta^2$
- $\alpha_1, \alpha_2, \alpha_3, \beta$ anti-commute with each other
- (not a unique choice!)

Dirac

$$H\psi = (\vec{\alpha}\cdot\vec{p} + \beta m)\ \psi$$

Usual substitution: $H \rightarrow i\frac{\partial}{\partial t},\ \vec{p} \rightarrow -i\vec{\nabla}$

Leads to: $$i\frac{\partial}{\partial t}\psi = \left(-i\vec{\alpha}\cdot\vec{\nabla} + \beta m\right)\ \psi$$

Multiply by β: $$\left(i\beta\frac{\partial}{\partial t}\psi \quad + \quad i\beta\alpha_1\frac{\partial}{\partial x} + i\beta\alpha_2\frac{\partial}{\partial y} + i\beta\alpha_3\frac{\partial}{\partial z}\right)\ \psi - m\psi = 0$$

($\beta^2=\mathbf{1}$)

Gives the famous Dirac equation:

$$\begin{aligned} (i\gamma^\mu\partial_\mu - m)\ \psi &= 0 \\ \text{with}:\ \gamma^\mu = (\beta, \beta\vec{\alpha}) &\equiv \text{Dirac } \gamma\text{−matrices} \end{aligned}$$

for each j=1,2,3,4 : $$\sum_{k=1}^{4}\left[\sum_{\mu=0}^{3} i\,(\gamma^\mu)_{jk}\ \partial_\mu - m\delta_{jk}\right](\psi_k) = 0$$

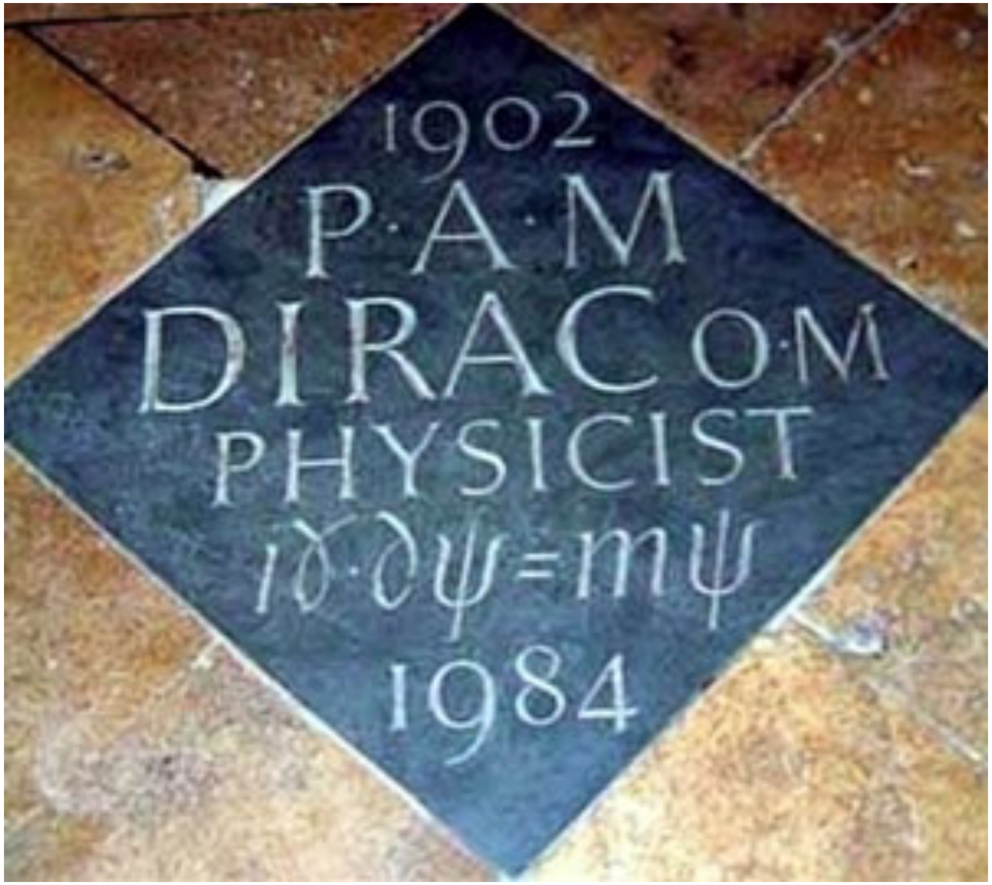
Dirac

$$H\psi = (\vec{\alpha}\cdot\vec{p} + \beta m)\ \psi$$

The famous Dirac equation:

$$\begin{aligned} (i\gamma^\mu \partial_\mu - m)\ \psi &= 0 \\ \text{with}: \ \gamma^\mu = (\beta, \beta\vec{\alpha}) &\equiv \text{Dirac } \gamma\text{-matrices} \end{aligned}$$

R.I.P. :



Dirac

$$H\psi = (\vec{\alpha}\cdot\vec{p} + \beta m)\ \psi$$

The famous Dirac equation:

$$\begin{aligned}(i\gamma^\mu\partial_\mu - m)\ \psi &= 0\\ \text{with}:\ \gamma^\mu = (\beta, \beta\vec{\alpha}) &\equiv \text{Dirac } \gamma\text{−matrices}\end{aligned}$$

Remember!

- μ : Lorentz index
- 4x4 γ matrix: Dirac index

Less compact notation:

$$\text{for each } j=1,2,3,4 \quad : \quad \sum_{k=1}^{4}\left[\sum_{\mu=0}^{3} i\,(\gamma^\mu)_{jk}\ \partial_\mu - m\delta_{jk}\right](\psi_k) = 0$$

Even less compact... :

$$\left[\begin{pmatrix}1\!\!1 & 0\\ 0 & -1\!\!1\end{pmatrix}\frac{i\partial}{\partial t} + \begin{pmatrix}0 & \sigma_1\\ -\sigma_1 & 0\end{pmatrix}\frac{i\partial}{\partial x} + \begin{pmatrix}0 & \sigma_2\\ -\sigma_2 & 0\end{pmatrix}\frac{i\partial}{\partial y} + \begin{pmatrix}0 & \sigma_3\\ -\sigma_3 & 0\end{pmatrix}\frac{i\partial}{\partial z} - \begin{pmatrix}1\!\!1 & 0\\ 0 & 1\!\!1\end{pmatrix} m\right]\begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix}$$

➢ What are the solutions for ψ ??

Intermezzo: The "Four-derivative"

- Transformation of contravariant 4-vector:

$$x^\mu: \quad \begin{aligned} x'^0 &= \gamma\left(x^0 - \beta x^1\right) \\ x'^1 &= \gamma\left(x^1 - \beta x^0\right) \\ x'^2 &= x^2 \\ x'^3 &= x^3 \end{aligned}$$

- Lowering the index, costs minus-sign:

$$x_\mu = g_{\mu\nu} x^\nu$$

$$x_0 = x^0,\ x_1 = -x^1,\ x_2 = -x^2,\ x_3 = -x^3$$

$$g_{\mu\nu} \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

- Transformation of covariant 4-vec:

$$x_\mu: \quad \begin{aligned} x'_0 &= \gamma(x_0 + \beta x_1) \\ -x'_1 &= \gamma(-x_1 - \beta x_0) \Rightarrow x'_1 = \gamma(x_1 + \beta x_0) \\ x'_2 &= x_2 \\ x'_3 &= x_3. \end{aligned}$$

- Derivative $\partial_\mu = \partial/\partial x^\mu$ transforms as covariant 4-vec (consistent with index):

(Sum over index υ)

$$(\partial_\mu \phi)' = \frac{\partial\phi}{\partial x^{\mu\prime}} = \frac{\partial\phi}{\partial x^\nu}\frac{\partial x^\nu}{\partial x^{\mu\prime}} = \frac{\partial x^\nu}{\partial x^{\mu\prime}}(\partial_\nu \phi)$$

$$\frac{\partial x^0}{\partial x^{0\prime}} = \gamma, \quad \frac{\partial x^0}{\partial x^{1\prime}} = \gamma\beta, \quad \frac{\partial x^1}{\partial x^{0\prime}} = \gamma\beta, \quad \frac{\partial x^1}{\partial x^{1\prime}} = \gamma$$

$$(\partial_0\phi)' = (\partial_0\phi)\frac{\partial x^0}{\partial x^{0\prime}} + (\partial_1\phi)\frac{\partial x^1}{\partial x^{0\prime}} = \gamma\left[(\partial_0\phi) + \beta(\partial_1\phi)\right]$$

$$(\partial_1\phi)' = (\partial_0\phi)\frac{\partial x^0}{\partial x^{1\prime}} + (\partial_1\phi)\frac{\partial x^1}{\partial x^{1\prime}} = \gamma\left[(\partial_1\phi) + \beta(\partial_0\phi)\right]$$

- And:

$$\partial^\mu = \left(\frac{1}{c}\frac{\partial}{\partial t}, -\nabla\right) \quad \text{and} \quad \partial_\mu = \left(\frac{1}{c}\frac{\partial}{\partial t}, \nabla\right)$$

Griffiths, p.214:

* The gradient with respect to a *contravariant* position-time four-vector x^μ is itself a *covariant* four-vector, hence the placement of the index. Written out in full, equation (7.5) says $(E/c, -\mathbf{p}) \to i\hbar\left(\frac{1}{c}\frac{\partial}{\partial t}, \nabla\right)$.

$(E, -\mathbf{p}) \to i\hbar(\partial/\partial t, \nabla)$

Dirac

$$H\psi = (\vec{\alpha}\cdot\vec{p} + \beta m)\ \psi$$

The famous Dirac equation:

$$\begin{array}{rcl} (i\gamma^\mu\partial_\mu - m)\ \psi & = & 0 \\ \text{with}:\ \gamma^\mu = (\beta, \beta\vec{\alpha}) & \equiv & \text{Dirac } \gamma\text{−matrices} \end{array}$$

Solutions to the Dirac equation?

Try plane wave: $\psi(x) = u(p)\, e^{-ipx}$ →

$$\begin{array}{rcl} (\gamma^\mu p_\mu - m)\ u(p) & = & 0 \\ \text{or}:\ (\not{p} - m)\ u(p) & = & 0 \end{array}$$

Linear set of eq:

$$\left[\begin{pmatrix} 1\!\!1 & 0 \\ 0 & -1\!\!1 \end{pmatrix} E - \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} p^i - \begin{pmatrix} 1\!\!1 & 0 \\ 0 & 1\!\!1 \end{pmatrix} m\right] \begin{pmatrix} u_A \\ u_B \end{pmatrix} = 0$$

- 2 coupled equations:

$$\left\{\begin{array}{rcl} (\vec{\sigma}\cdot\vec{p})\ u_B & = & (E-m)\, u_A \\ (\vec{\sigma}\cdot\vec{p})\ u_A & = & (E+m)\, u_B \end{array}\right.$$

If p=0:

$$u^{(1)} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \ ,\quad u^{(2)} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \ ,\quad u^{(3)} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \ ,\quad u^{(4)} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Dirac

$$H\psi = (\vec{\alpha}\cdot\vec{p} + \beta m)\ \psi$$

The famous Dirac equation:

$$\begin{aligned} (i\gamma^\mu \partial_\mu - m)\ \psi &= 0 \\ \text{with}:\ \ \gamma^\mu = (\beta, \beta\vec{\alpha}) &\equiv \text{Dirac } \gamma\text{−matrices} \end{aligned}$$

Solutions to the Dirac equation?

Try plane wave: $\psi(x) = u(p)\, e^{-ipx}$ →

$$\begin{aligned} (\gamma^\mu p_\mu - m)\ u(p) &= 0 \\ \text{or}:\ \ (\not{p} - m)\ u(p) &= 0 \end{aligned}$$

- 2 coupled equations:

$$\begin{cases} (\vec{\sigma}\cdot\vec{p})\ u_B = (E-m)\, u_A \\ (\vec{\sigma}\cdot\vec{p})\ u_A = (E+m)\, u_B \end{cases}$$

If p≠0:

Two solutions for E>0:

(and two for E<0)

$$u^{(1)} = \begin{pmatrix} u_A^{(1)} \\ u_B^{(1)} \end{pmatrix} \quad , \quad u^{(2)} = \begin{pmatrix} u_A^{(2)} \\ u_B^{(2)} \end{pmatrix}$$

with:

$$u_A^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad u_A^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$u_B^{(1)} = \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\, u_A^{(1)} = \frac{\vec{\sigma}\cdot\vec{p}}{E+m} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad u_B^{(2)} = \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\, u_A^{(2)} = \frac{\vec{\sigma}\cdot\vec{p}}{E+m} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Dirac

$$H\psi = (\vec{\alpha}\cdot\vec{p} + \beta m)\ \psi$$

The famous Dirac equation:

$$\begin{aligned}(i\gamma^\mu\partial_\mu - m)\ \psi &= 0 \\ \text{with}:\ \gamma^\mu = (\beta, \beta\vec{\alpha}) &\equiv \text{Dirac } \gamma\text{−matrices}\end{aligned}$$

Solutions to the Dirac equation?

Try plane wave: $\psi(x) = u(p)\, e^{-ipx}$ →

$$\begin{aligned}(\gamma^\mu p_\mu - m)\ u(p) &= 0 \\ \text{or}:\ (\not{p} - m)\ u(p) &= 0\end{aligned}$$

- 2 coupled equations:

$$\begin{cases}(\vec{\sigma}\cdot\vec{p})\ u_B = (E-m)\, u_A \\ (\vec{\sigma}\cdot\vec{p})\ u_A = (E+m)\, u_B\end{cases}$$

If $p \neq 0$:

Two solutions for E>0:

(and two for E<0)

$$u^{(1)} = \begin{pmatrix} u_A^{(1)} \\ u_B^{(1)} \end{pmatrix} \quad , \quad u^{(2)} = \begin{pmatrix} u_A^{(2)} \\ u_B^{(2)} \end{pmatrix}$$

$$u^{(1)} = \begin{pmatrix} 1 \\ 0 \\ \vec{\sigma}\bullet\vec{p}/(E+m) \\ 0 \end{pmatrix} \qquad u^{(2)} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vec{\sigma}\bullet\vec{p}/(E+m) \end{pmatrix}$$

Dirac

$$H\psi = (\vec{\alpha} \cdot \vec{p} + \beta m)\ \psi$$

The famous Dirac equation:

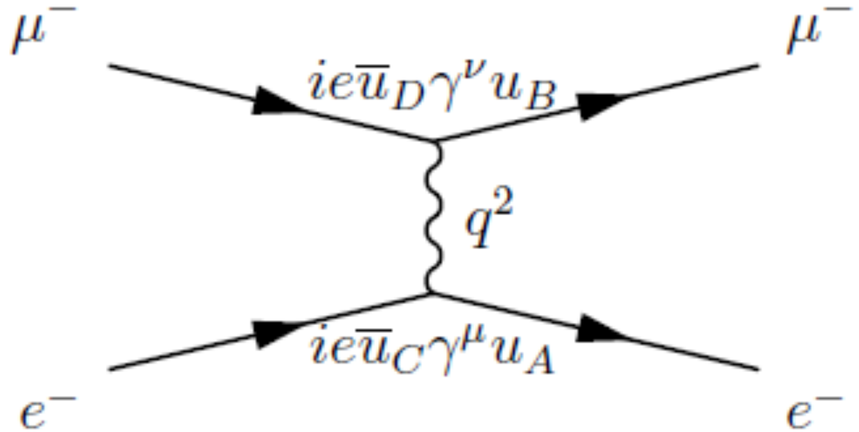
$$\begin{aligned} (i\gamma^{\mu}\partial_{\mu} - m)\ \psi &= 0 \\ \text{with}: \ \gamma^{\mu} = (\beta, \beta\vec{\alpha}) &\equiv \text{Dirac } \gamma\text{-matrices} \end{aligned}$$

ψ is 4-component spinor

4 solutions correspond to fermions and anti-fermions with spin+1/2 and -1/2

➢ What do we need this for ??

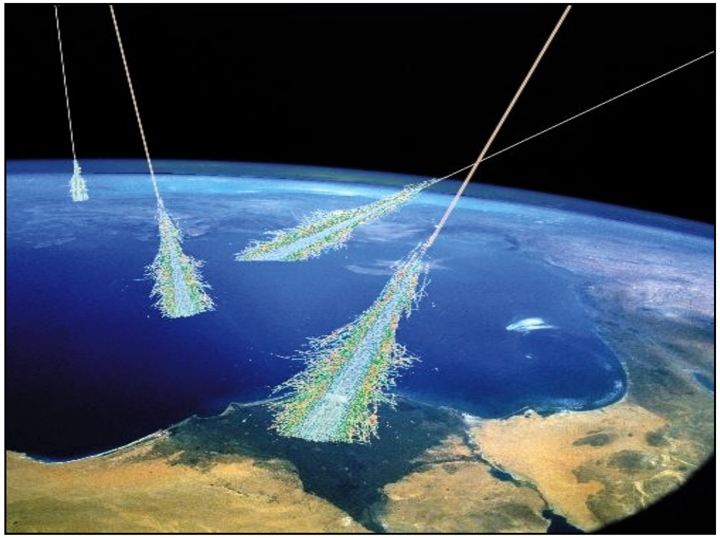
Needed e.g. to calculate the probability for a scattering process like:

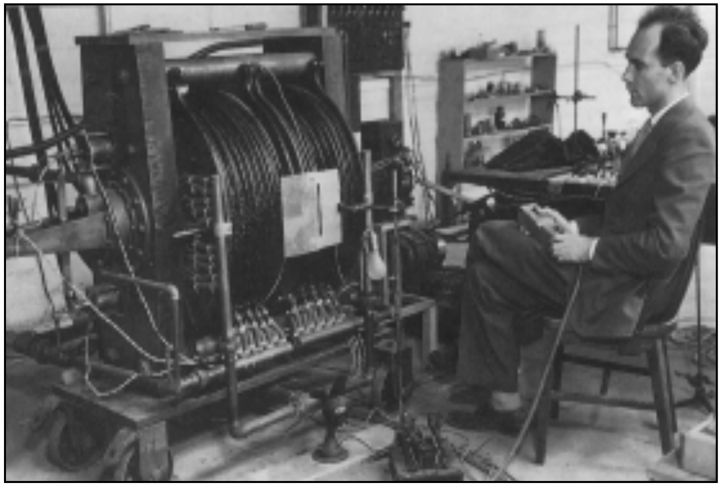


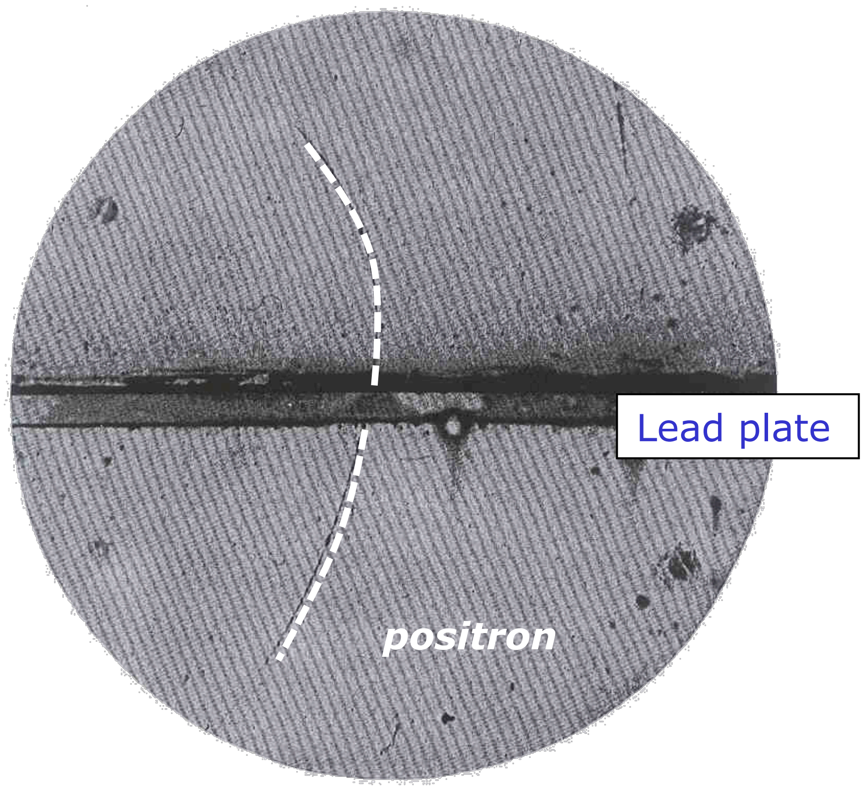
Prediction of anti-matter

- Dirac found his equation in 1928
- The existence of anti-matter was not taken serious until 1932, when Anderson discovered the anti-electron: the positron

Discovery of anti-matter







Nobelprize 1936

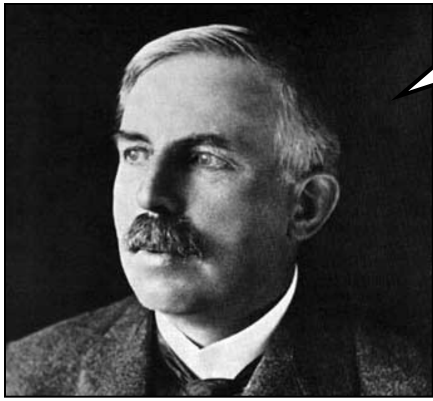
What else happened in 1932 ?

- Discovery of the neutron, by J. Chadwick

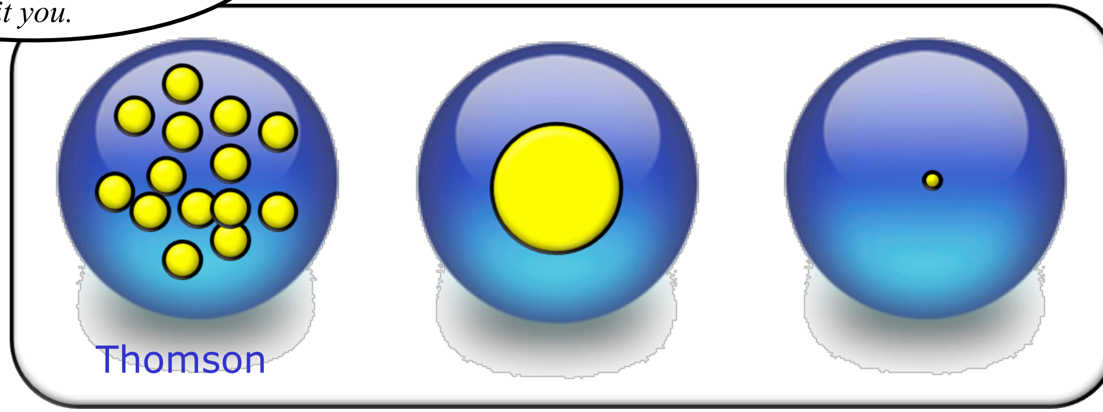
➢ What was known at that time about the nucleus?

Nobelprize 1935

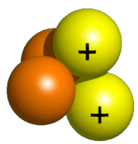
Rutherford



It was as if you fired a 15-inch shell at a sheet of tissue paper and it came back to hit you.

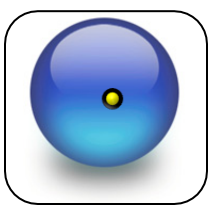


'Bullet':



6 MeV alpha particle

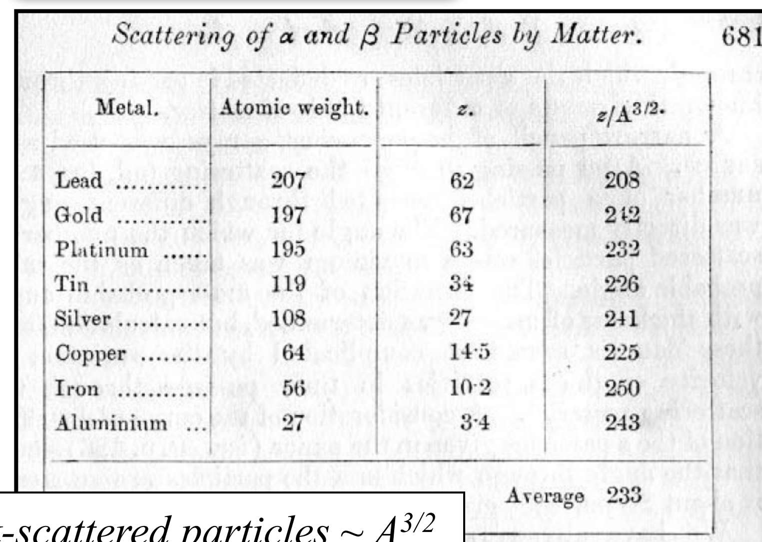
Hypothesis:



$$V \propto \frac{1}{r}$$

$$\frac{d\sigma}{d\Omega}(\theta) = \left(\frac{ZZ'}{4E}\right)^2 \frac{1}{\sin^4(\theta/2)}$$

Measurement:



Scattering of α and β Particles by Matter. 681

Metal.	Atomic weight.	z.	$z/A^{3/2}$.
Lead	207	62	208
Gold	197	67	242
Platinum	195	63	232
Tin	119	34	226
Silver	108	27	241
Copper	64	14·5	225
Iron	56	10·2	250
Aluminium ...	27	3·4	243
		Average	233

Number of back-scattered particles ~ $A^{3/2}$

The Number of Possible Elements and Mendeléeff's "Cubic" Periodic System.

According to Rutherford's theory of "single scattering" ("On the Scattering of α and β Particles by Matter and the Structure of the Atom," *Phil. Mag.*, May, 1911), and to Barkla's "Note on the Energy of Scattered X-Radiation" (*ibid.*), the numbers of electrons per atom is half the atomic weight; thus, for U, about 120. Now, a reconstruction of Mendeléeff's "cubic" periodic system, as suggested in his famous paper "Die Beziehungen zwischen den Eigenschaften der Elemente und ihren Atomgewichten" (*Ostw. Klass.*, No. 68, pp. 32, 36, 37, and 74), gives a constant mean difference between consecutive atomic weights =2, and thus, from H to U, 120 as the number of possible elements (van den Broek, "Das Mendelejeff'sche 'Kubische' Periodische System der Elemente und die Einordnung der Radioelemente in dieses System," *Physik. Zeitschr.* 12, p. 490). Hence, if this cubic periodic system should prove to be correct, then the number of possible elements is equal to the number of possible permanent charges of each sign per atom, or to each possible permanent charge (of both signs) per atom belongs a possible element.

A. van den Broek.

Noordwijk-Zee, June 23.

78 NATURE [July 20, 1911

NO. 2177, VOL. 87]

Antonius van den Broek

July 1911

'Hey, that is funny… looking at Rutherford's results, one notices that the number of electrons per atom is precisely halve the atom mass.'
(Note: Proton only proposed in 1920)

What else happened in 1932 :

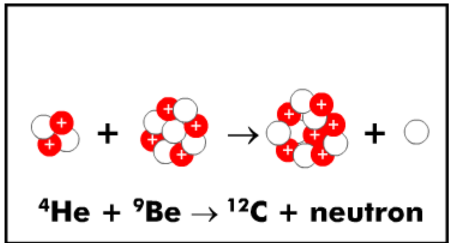
- Discovery of the neutron, by J. Chadwick

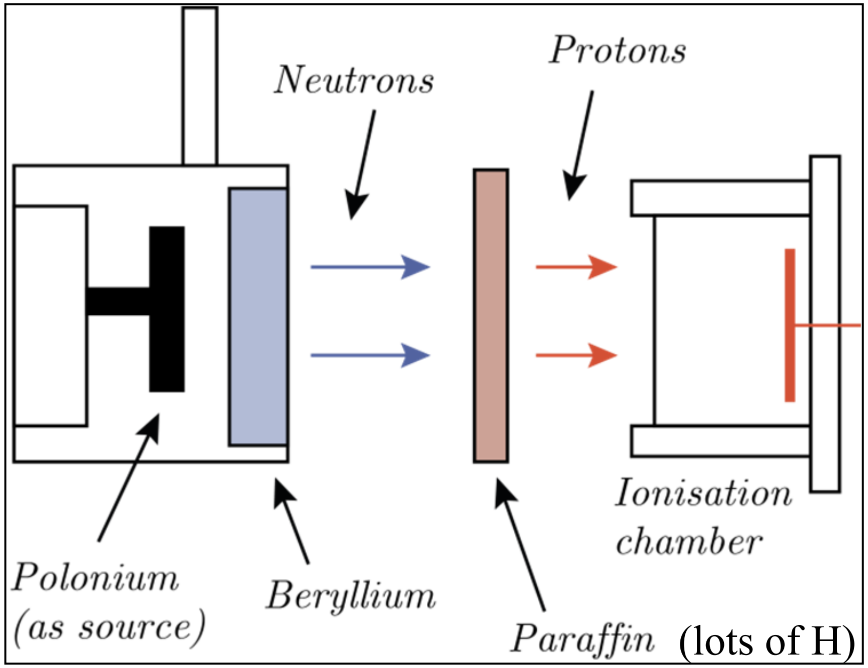
$$\alpha + Be \rightarrow \text{non-ionizing radiation}$$

1) Neutral
   - Gamma? No! protons too energetic
2) $m_n \sim m_p$

➢ Interpretation:

$${}^{4}_{2}He + {}^{9}_{4}Be \rightarrow {}^{1}_{0}n + {}^{12}_{6}C$$





Nobelprize 1935

Towards massive force carriers

Strong interaction

- 1932: discovery of neutron
  - $\alpha + {}^{9}\mathrm{Be} \rightarrow \mathrm{n} + {}^{12}\mathrm{C}$
- Nuclear effect only → short range

- What can you then deduce about:
  - Energy scale
  - Potential

Yukawa

- 1935: Introduced *strong* carriers on *small* distances
- Massive particle, that exists only shortly
  - 'virtual' particle

<u>Compare:</u>

- Electro-magnetism
  - Infinite range
  - Transmitted by massless photon
  - ➢ <u>Coulomb</u> potential

$$V(r) = -e^2 \frac{1}{r}$$

$R \rightarrow \infty$

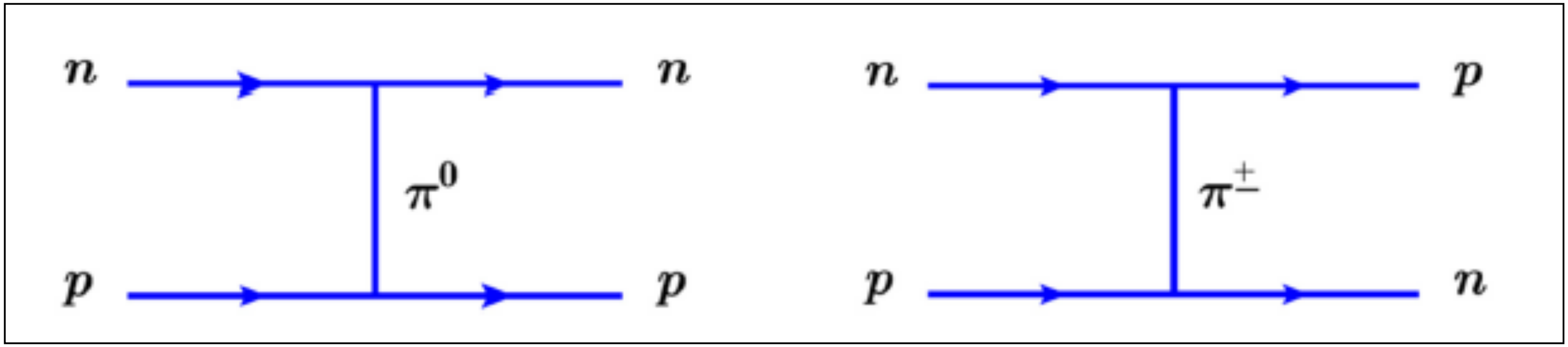
- Strong force
  - Finite range
  - Transmitted by massive pion
  - ➢ <u>Yukawa</u> potential

$$U(r) = -g^2 \frac{e^{-r/R}}{r}$$

R: range

Yukawa

- 1935: Introduced *strong* carriers on *small* distances
- Massive particle, that exists only shortly
  - 'virtual' particle



- Strong force
  - Finite range
  - Transmitted by massive pion
  - ➢ Yukawa potential

$$U(r) = -g^2 \frac{e^{-r/R}}{r}$$

R: range

Yukawa's strong nuclear force

Strength: ***coupling constant***

$$\text{Coulomb}: V \propto -\alpha \frac{1}{r}$$

$$\alpha = \frac{e^2}{4\pi\varepsilon_0 \hbar c} = \frac{1}{137.04}$$

$$\text{Yukawa}: V \propto -g^2 \frac{e^{-r/R}}{r}$$

Coupling $g^2 > \alpha$

Short range: ***massive quanta***

What is the typical mass that limits the range to the proton radius (10^{-15} m) ?

Homework

Prediction!

Yukawa's pions – pictures

Powell used a new detection technique

Photographic emulsion:

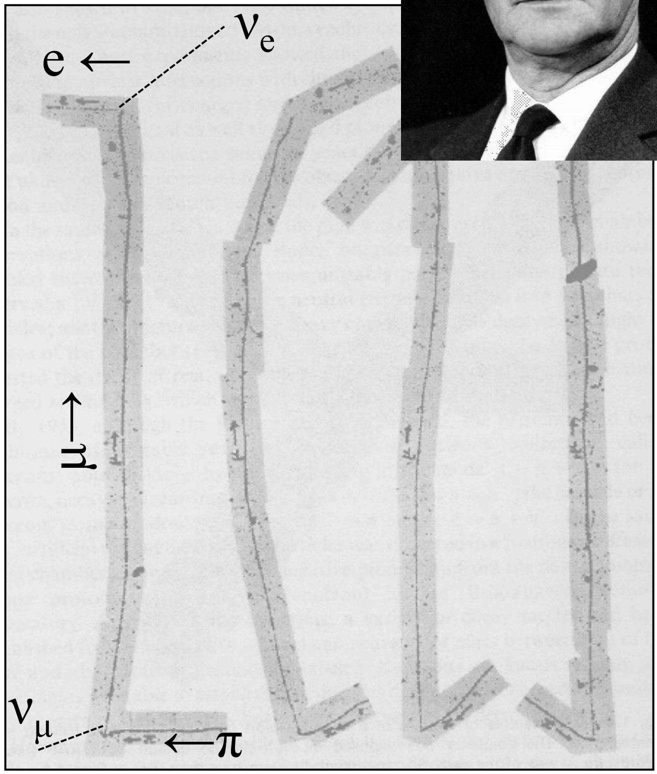
- Thick photosensitive film
- Charged particles leave tracks

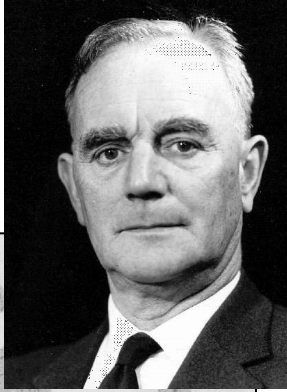
Results: two particles (pion and muon)

$$\pi^+ \rightarrow \mu^+ \nu_\mu$$

$$\hookrightarrow \mu^+ \rightarrow e^+ + \nu_e + \bar{\nu}_\mu$$

- 1947 Discovery of pion (Powell): Nobelprijs 1950
- 1935 Prediction of pion (Yukawa): Nobelprijs 1949





- π-meson, m=140 MeV, short lifetime

Produced high in atmosphere and decays before reaching sealevel.

- muon (μ), m=105 MeV, long lifetime

Reaches sea-level and weakly interacts with matter

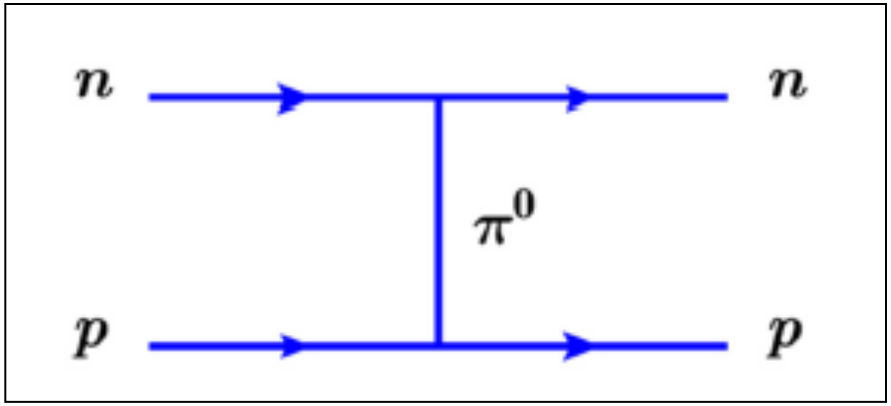
Intermezzo: Strong force nowadays:

- Yukawa:

"Effective" description

Still useful to describe some features!

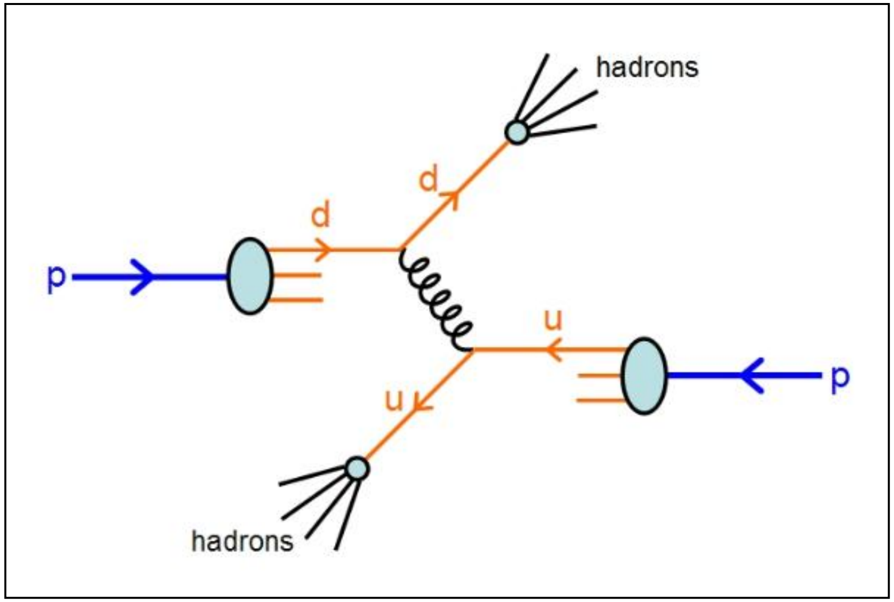
In particular at "lower energies"



- Gluons:

More fundamental description

But fails at low energies…



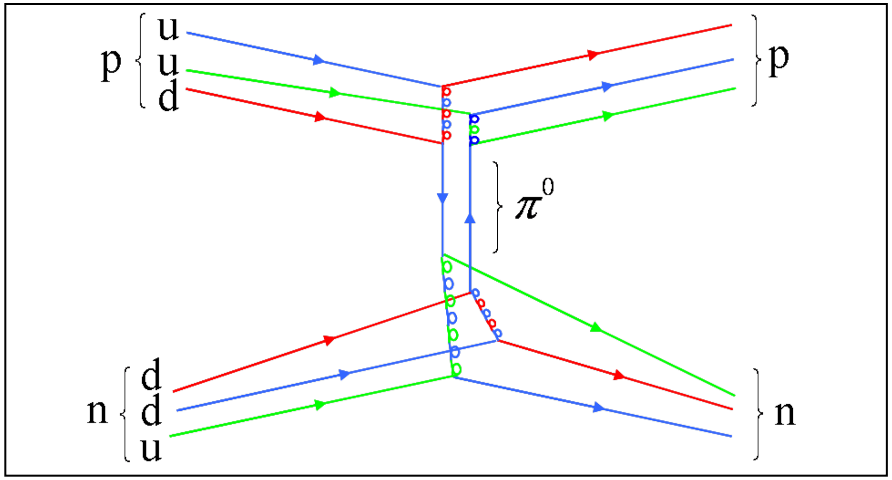
Intermezzo: Strong force nowadays:

- Yukawa:

“Effective” description (≠ wrong!)

Still useful to describe some features!

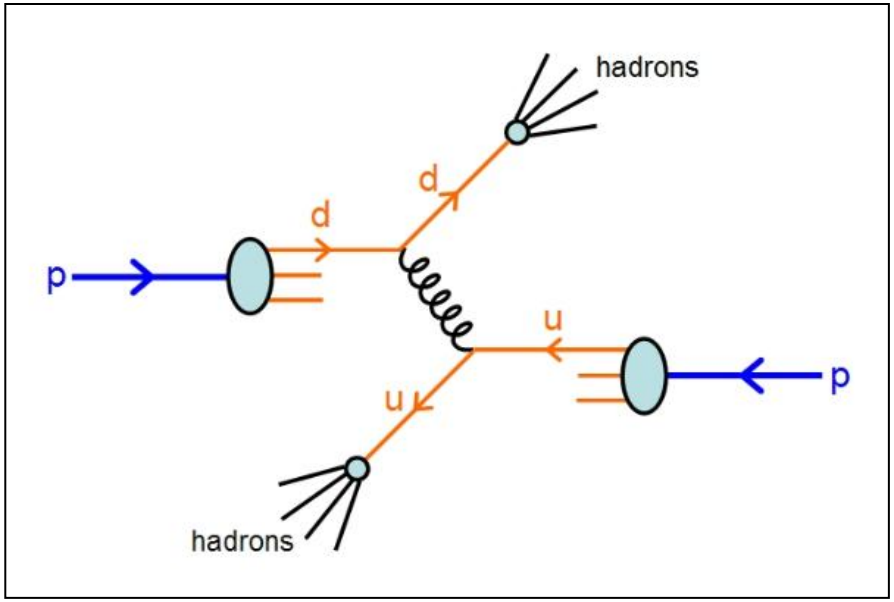
In particular at “lower energies”



- Gluons:

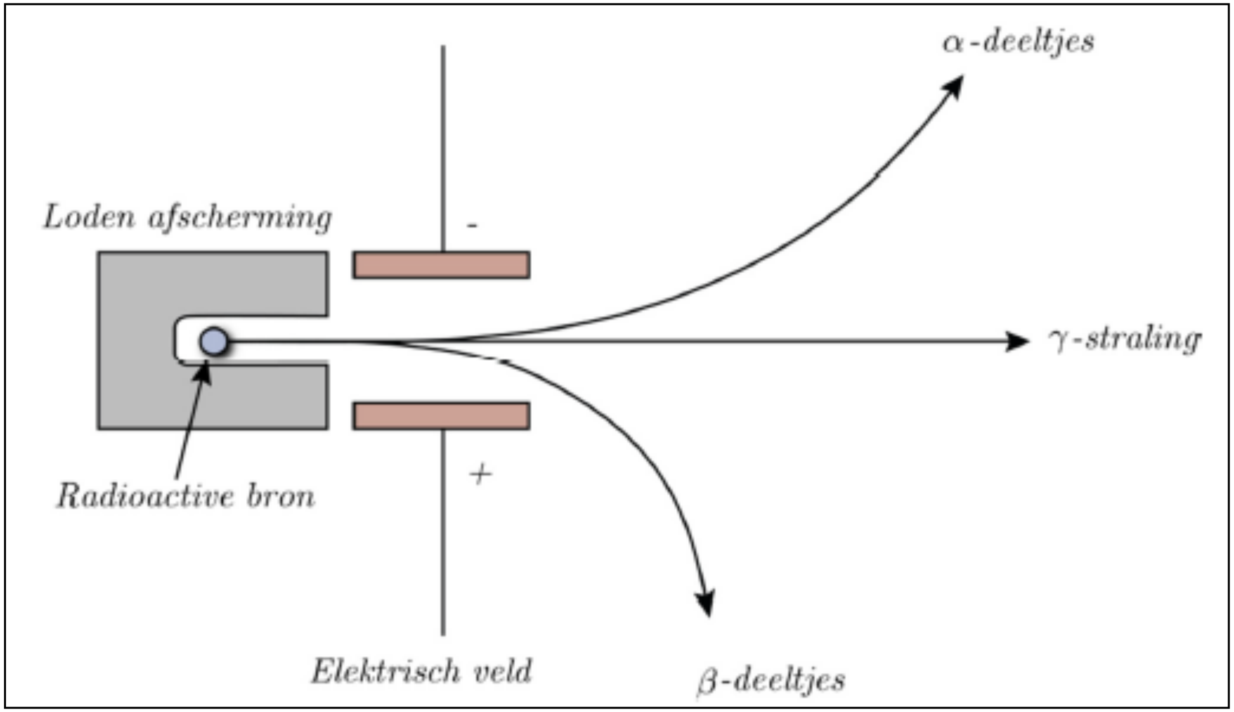
More fundamental description

But fails at low energies…



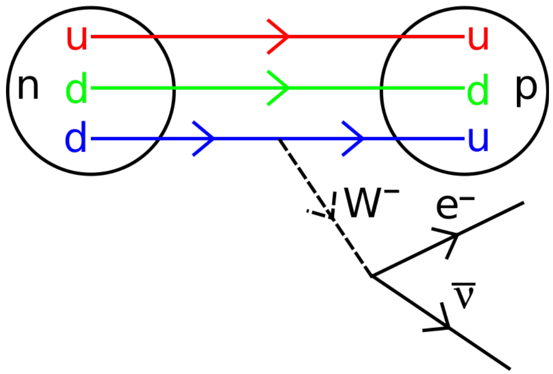
Radioactive decay

- 1895: Röntgen discovered radiation from vaccum tubes (**γ**)
- 1895: Bequerel measured radiation from ^{238}U (**n**)
- 1898: Curie measured radiation from ^{232}Th (**α**)
- 1899: Rutherford concluded $\alpha \neq$ **β**
- 1914: Rutherford determined wavelength of γ (scattering of crystals)

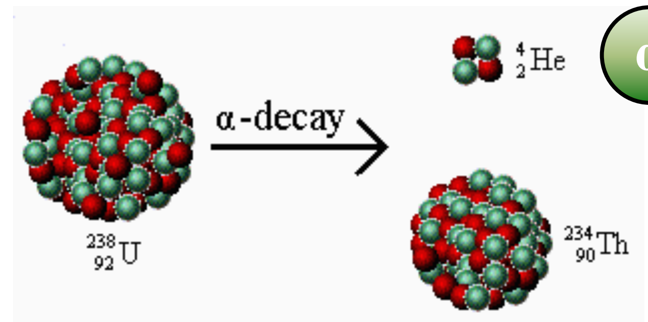


Link with Modern physics

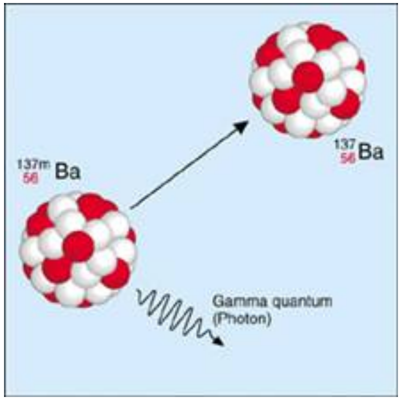
- β-decay: weak interaction
  - W-exchange



- α-decay: strong interaction
  - Pions (gluons?!) keeps nucleus together



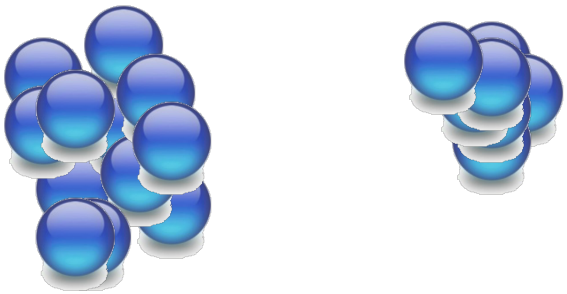
- γ-decay: electro-magnetic interaction
  - Excited states



Particle Decay

1) Number of decayed particles, dN, is proportional to: N, dt and constant Γ:

t ***t+dt***



$$dN = -N\,\Gamma\,dt$$

$$N = N_0 e^{-\Gamma t} = N_0 e^{-t/\tau}$$

lifetime $\tau = \frac{1}{T_{fi}} = \frac{1}{\Gamma}$

2) States that decay, do not correspond to one specific energy level, but have a "width" ΔE:

Heisenberg: $\Delta E \Delta t \sim \hbar$

$$\Delta E\,\tau \sim \hbar$$

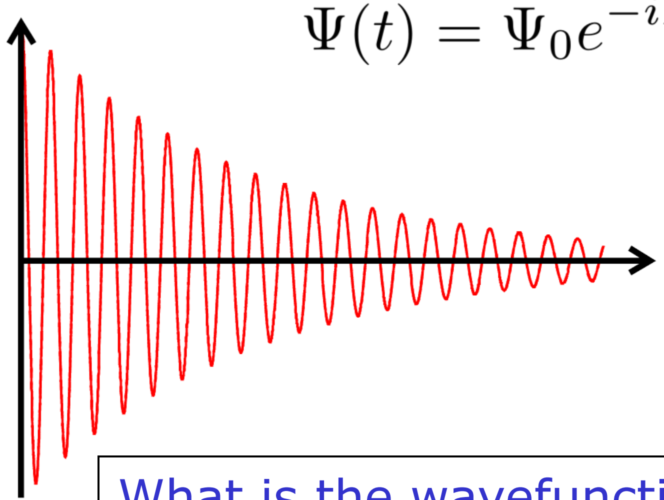
$$\Delta E \sim \hbar/\tau = \hbar\,\Gamma$$

➢ The width of a particle is inverse proportional to its lifetime!

Quantum mechanical description of decay

State with energy E_0 ($\hbar\omega$) and lifetime τ

To allow for decay, we need to change the time-dependence:



$$\Psi(t) = \Psi_0 e^{-iE_0 t} \Longrightarrow \Psi(t) = \Psi_0 e^{-iE_0 t} e^{-\frac{t}{2\tau}}$$

$$\Psi^*\Psi = \Psi_0^*\Psi_0 e^{-\frac{t}{\tau}}$$

What is the wavefunction in terms of energy (instead of time) ?

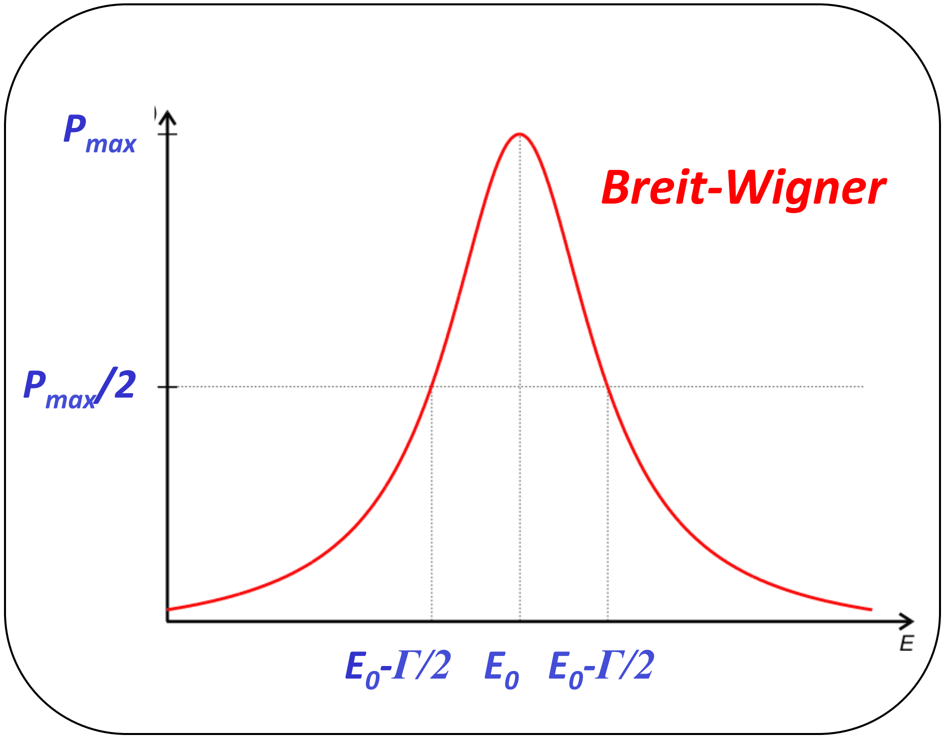
- Infinite sum of flat waves, each with own energy
- Fourier transformation:

$$f(\omega) = f(E) = \int_0^\infty \Psi_0 e^{-t(iE_0 + \frac{1}{2\tau})} e^{iEt} dt = \Psi_0 \frac{1}{i\left((E_0 - E) - \frac{i}{2}\Gamma\right)}$$

Resonance

Probability to find particle with energy E:

$$f(E)^* f(E) = \Psi_0^* \Psi_0 \frac{1}{(E_0 - E)^2 + \frac{1}{4}\Gamma^2}$$



Resonance-structure contains information on:

- Mass
- Lifetime
- Decay possibilities

Scattering

Outline for today

- Quantum mechanics: equations of motions of wave functions
  - Schrodinger, Klein Gordon, Dirac
- Forces
  - Strong force, pion exchange
  - Weak nuclear force, decay
- Scattering Theory
  - Rutherford (classic) and QM
  - "Cross section"
  - Coulomb potential
  - Yukawa potential
  - Resonance

Decay and Scattering: *decay width* and *cross section*

- Decay $$a \to b + c$$
  - *Decay width* is reciprocal of decay time: $\tau = 1/\Gamma$
  - Total width is sum of *partial widths*: $\Gamma_{tot} = \Sigma_i \Gamma_i$
  - *Branching fraction* for certain decay mode: $BR = \Gamma_i / \Gamma_{tot}$
  - Unit: inverse seconds

- Scattering $$a + b \to c + d$$
  - Parameter of interest is "size of target", *cross section* σ
  - Total cross section is sum of possible processes: $\sigma_{tot} = \Sigma_i \sigma_i$
  - Unit: surface

➢ Golden rule:

$$\text{transition rate} = \frac{2\pi}{\hbar} |\mathcal{M}|^2 \times (\text{phase space})$$

Fermi's Golden Rule

$$\Gamma = \frac{2\pi}{\hbar}|\langle f|H^{'}|i\rangle|^2\rho(E_f)$$

$\rho(E_f)$ density of final states

$\langle f|H^{'}|i\rangle$ Matrix element

Fermi's "golden rule" gives:
The transition probability
to go from initial state i to final state f

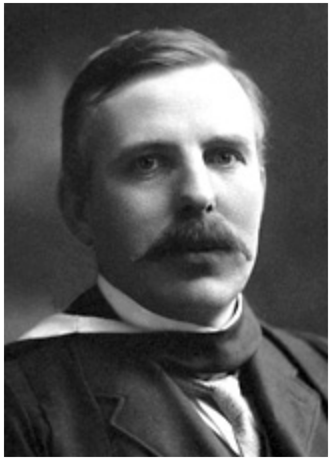
Amplitude $\mathcal{M}$:
contains dynamical information
fundamental physics

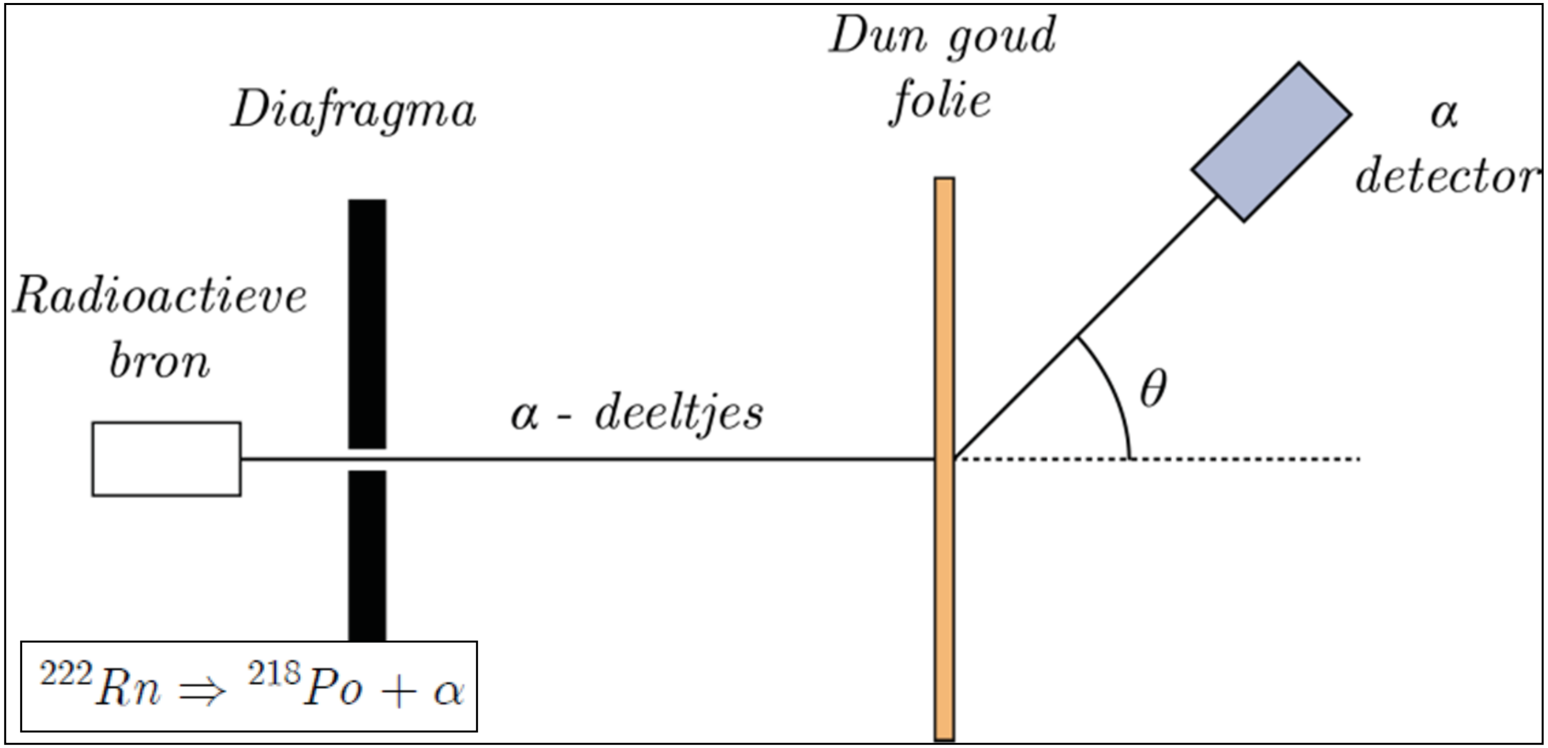
Phase space φ
contains kinematic information
masses, momenta

$$\text{transition rate} = \frac{2\pi}{\hbar}|\mathcal{M}|^2 \times (\text{phase space})$$

Rutherford

- Classical calculation of cross section of a scattering process

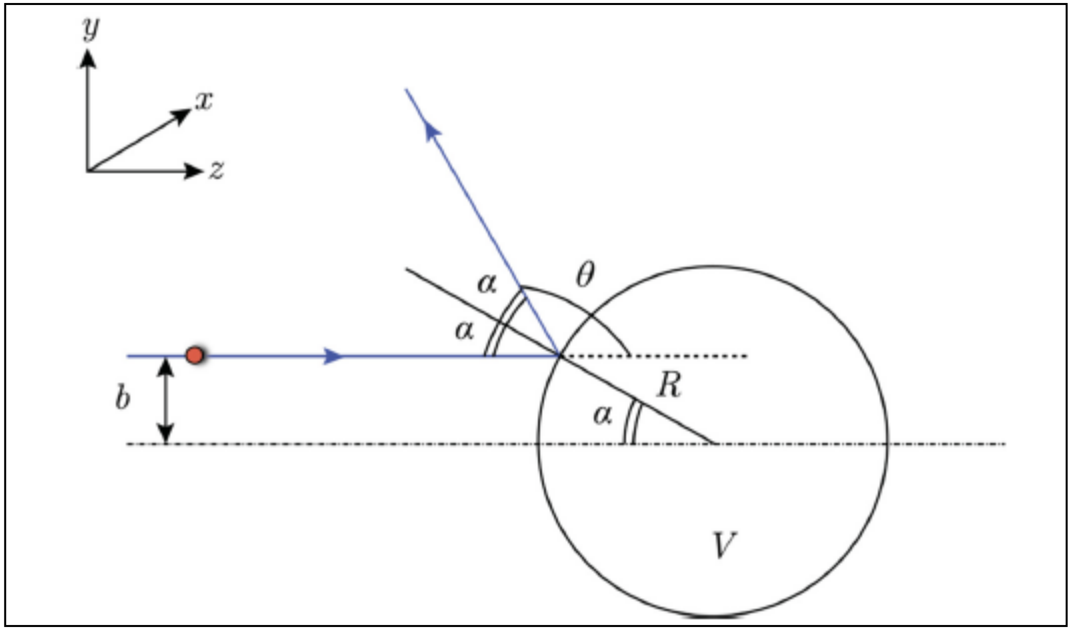




Rutherford

- Scatter from spherical potential
- Incoming: impact parameter between b and db
- Outgoing: scattering angle between θ and dθ

➢ 3d: incoming particle “sees” surface dσ, and scatters off solid angle dΩ

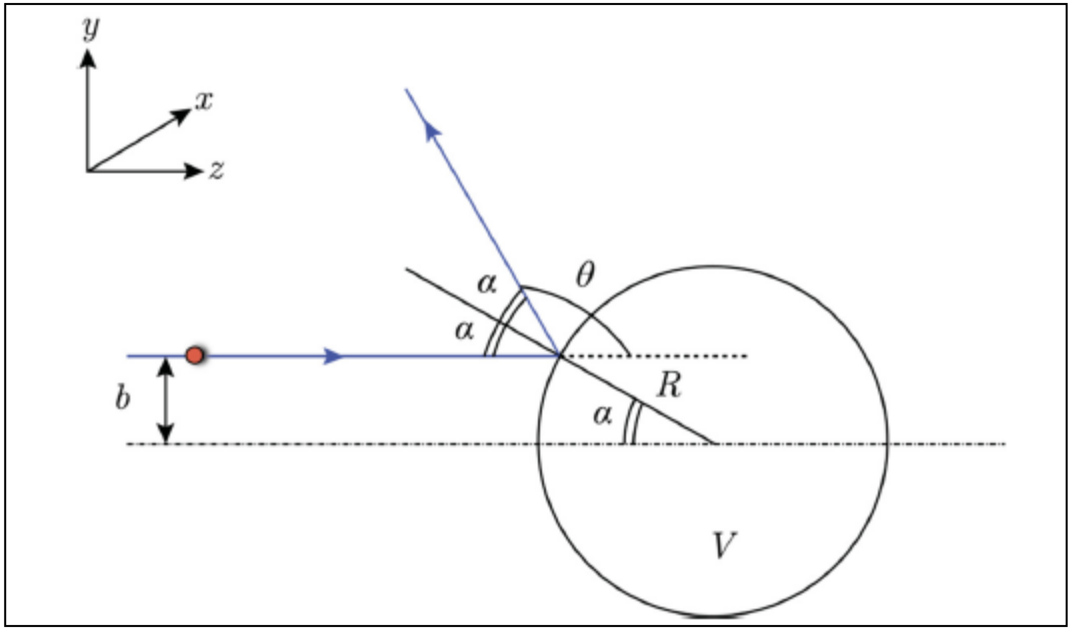


Rutherford

- 3d: incoming particle "sees" surface dσ, and scatters off solid angle dΩ
- Calculate:

$$\frac{d\sigma}{d\Omega} = D(\theta, \varphi)$$

$$d\sigma = |b\ db\ d\varphi|$$
$$d\Omega = |\sin\theta\ d\theta\ d\varphi|$$



Rutherford

- 3d: incoming particle “sees” surface dσ, and scatters off solid angle dΩ

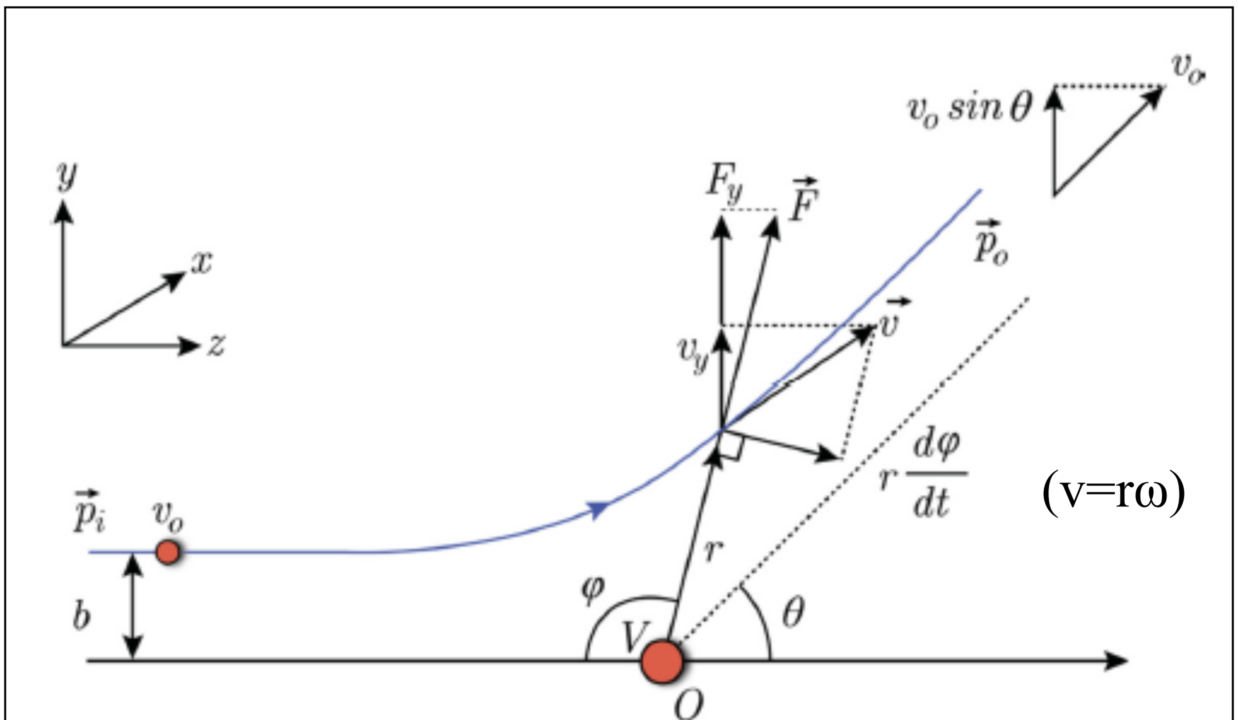
- Conservation of angular momentum:

Before $L = mv_0 b$

After: $L = mr\frac{d\phi}{dt}r$

- Force:

$$F(r) = \frac{Z_1 Z_2 \alpha}{r^2}$$



Rutherford

- 3d: incoming particle "sees" surface dσ, and scatters off solid angle dΩ
  - Conservation of angular momentum:
  - Force:

Before $L = m v_0 b$

After: $L = m r \frac{d\phi}{dt} r$

$$F(r) = \frac{Z_1 Z_2 \alpha}{r^2}$$

$$m \frac{dv_y}{dt} = F_y = F \sin\phi = \frac{Z_1 Z_2 \alpha}{r^2} \sin\phi$$

Replace r by b, using L conservation

$$\frac{dv_y}{dt} = \frac{Z_1 Z_2 \alpha}{m v_0 b} \sin\phi \frac{d\phi}{dt}$$

$$\int_0^{v_0 \sin\theta} dv_y = \frac{Z_1 Z_2 \alpha}{m v_0 b} \int_{\cos\pi}^{\cos\theta} d\cos\phi$$

=(cosθ + 1)

$$\cot\frac{\theta}{2} = \frac{m v_0^2}{Z_1 Z_2 \alpha} b$$

Use expression $b = \ldots$

$$\frac{d\sigma}{d\Omega} = \left| \frac{b}{\sin\theta} \frac{db}{d\theta} \right|$$

$$= \left| \left(\frac{Z_1 Z_2 \alpha}{m v_0^2} \right)^2 \frac{\cot\frac{\theta}{2}}{\sin\theta} \frac{d\cot\frac{\theta}{2}}{d\theta} \right|$$

$$= \left(\frac{Z_1 Z_2 \alpha}{m v_0^2} \right)^2 \frac{1}{4 \sin^4 \frac{\theta}{2}}$$

Rutherford scattering → Cross section

- Differential cross section

$$\frac{d\sigma}{d\Omega} = \left(\frac{Z_1 Z_2 \alpha}{m v_0^2}\right)^2 \frac{1}{4\sin^4\frac{\theta}{2}}$$

$$p_i = (E, 0, 0, mv_0)$$
$$p_o = (E, 0, mv_0 \sin\theta, mv_0 \cos\theta)$$
$$q \equiv p_i - p_o$$

$$\frac{d\sigma}{d\Omega} = \left(\frac{2mZ_1 Z_2 \alpha}{q^2}\right)^2$$

- Luminosity $\mathcal{L}$
  - $\mathcal{L} = dN/d\sigma$
  - Number of incoming particles per unit surface

$$dN = \mathcal{L}\, d\sigma = \mathcal{L} \frac{d\sigma}{d\Omega} d\Omega$$

Scattering Theory: QM

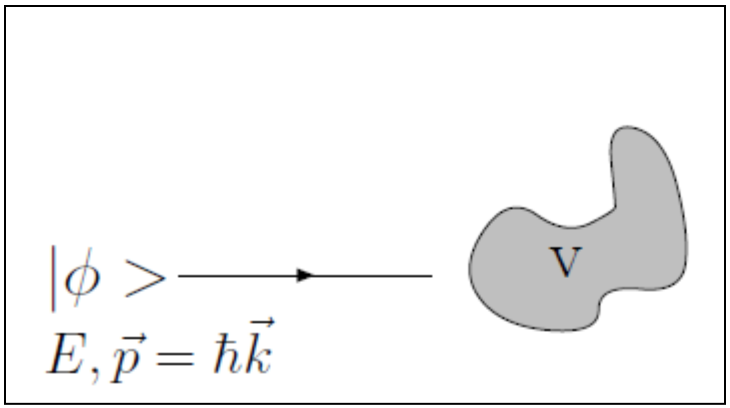
$$-\frac{\hbar^2}{2m}\nabla^2\psi(\vec{r},t) = i\hbar\frac{\partial}{\partial t}\psi(\vec{r},t)$$

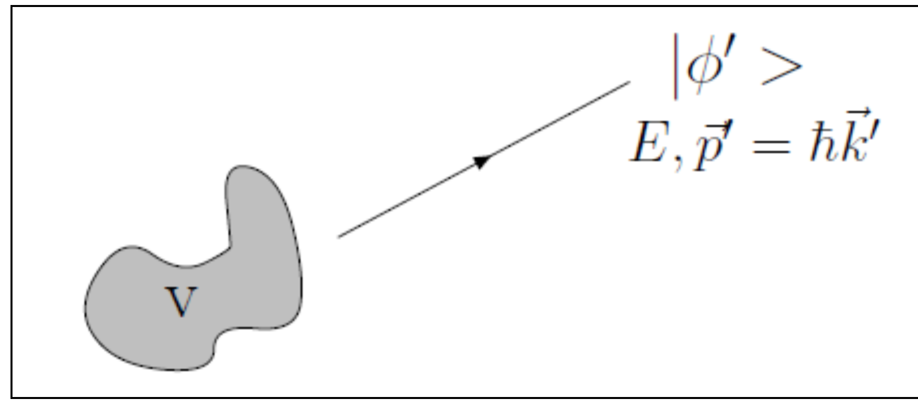
- Describe a ***stationary*** state, that satisfies the incoming and outgoing wave

$$\left(\nabla^2 + k^2\right)\psi(\vec{\mathbf{r}}) = \frac{2m}{\hbar^2}V(\vec{\mathbf{r}})\psi(\vec{\mathbf{r}})$$

V≠0

k^2=2mE



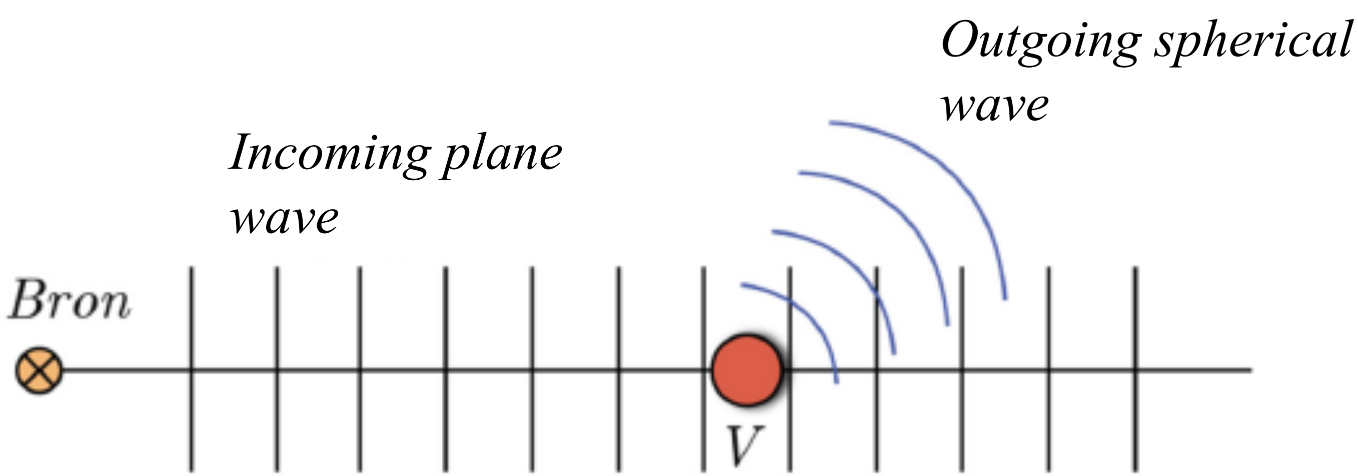


Scattering Theory: QM

- Describe a *stationary* state, that satisfies the incoming and outgoing wave

$$\left(\nabla^2 + k^2\right)\psi(\vec{r}) = \frac{2m}{\hbar^2}V(\vec{r})\psi(\vec{r})$$

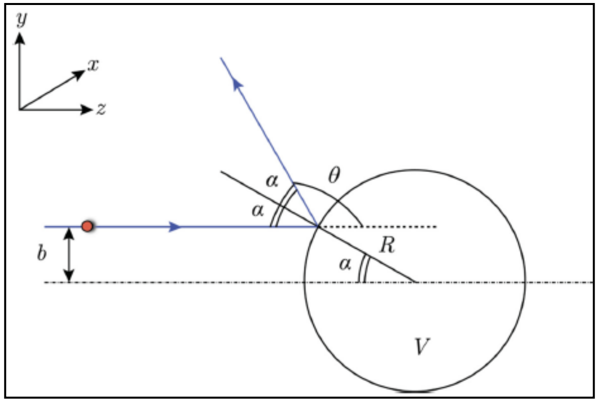
- Find a solution which is a superposition of the incoming wave, and the outgoing waves



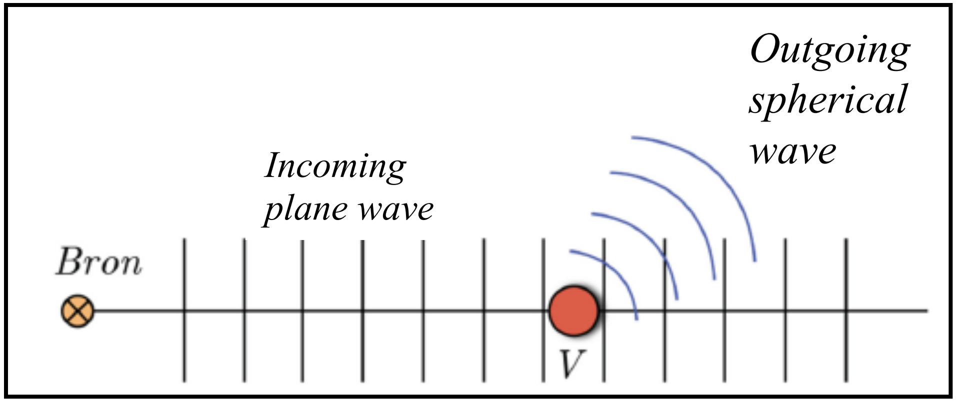
$$\Phi(\vec{r}) = \underbrace{\phi_a(\vec{r})}_{\text{ingoing}} + \underbrace{f(\vec{k_a}, \vec{k_b})\frac{e^{ikr}}{r}}_{\text{outgoing}}$$

Scattering Theory: Quantum mechanics

Classical:



QM:



- Superposition of incoming wave and outgoing waves
- Scattering amplitude f calculated from potential V
  - Fourier transform of potential:

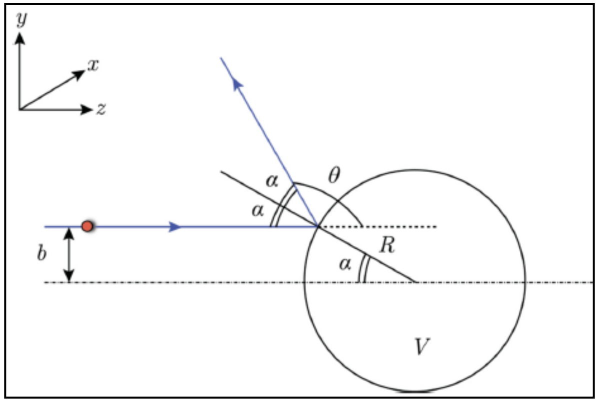
$$\Phi(\vec{r}) = \underbrace{\phi_a(\vec{r})}_{\text{ingoing}} + \underbrace{f(\vec{k_a}, \vec{k_b})\frac{e^{ikr}}{r}}_{\text{outgoing}}$$

$$f(\vec{k_a}, \vec{k_b}) = -\frac{m}{2\pi\hbar^2}\int e^{-i\vec{k_b}\cdot\vec{r}'} V(\vec{r}')\Phi(\vec{r}')\, d^3r$$

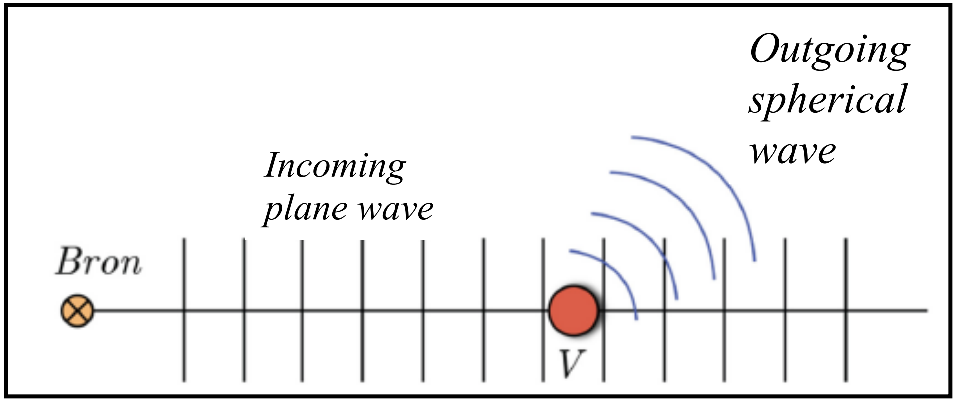
$$\frac{d\sigma}{d\Omega} = |f(\vec{k_a}, \vec{k_b})|^2$$

Scattering Theory: Quantum mechanics

Classical:



QM:



- Yukawa:

$$V(r) = \frac{Z_1 Z_2 e^2}{r} e^{-ar}$$

$$\frac{d\sigma}{d\Omega} = |f|^2 = \frac{m^2}{(2\pi\hbar^2)^2}\left[\frac{4\pi Z_1 Z_2 e^2}{q^2 + a^2}\right]^2$$

- Coulomb:

$$V(r) = \frac{Z_1 Z_2 e^2}{r}$$

$$\frac{d\sigma}{d\Omega} = \frac{m^2}{(2\pi\hbar^2)^2}\left[\frac{4\pi Z_1 Z_2 e^2}{q^2}\right]^2 = \left[\frac{Z_1 Z_2 e^2}{2mv^2 \sin^2\frac{\theta}{2}}\right]^2$$

- Centrifugal Barier:

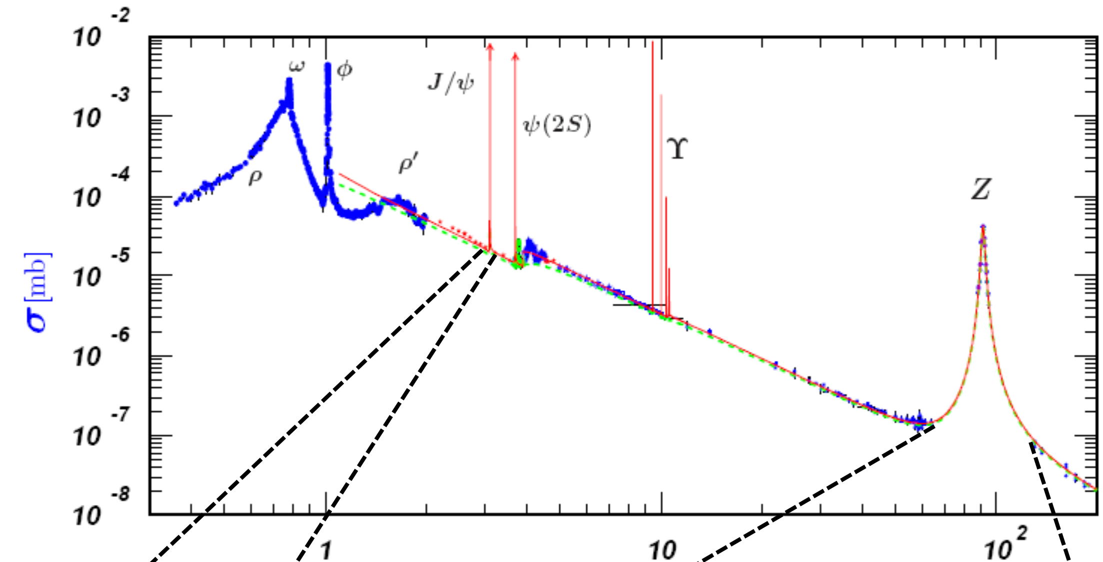
$$V_{eff} = V(r) + \frac{\hbar^2 l(l+1)}{2mr^2}$$

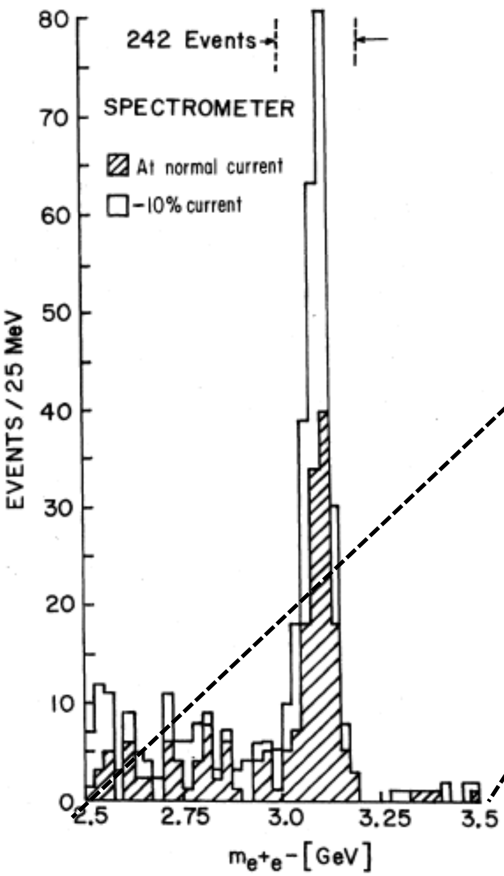
$$\frac{d\sigma}{d\Omega} = |f^{res}(\theta)|^2 = \frac{(2l+1)^2}{k^2}\frac{\frac{\Gamma^2}{4}}{(E_r - E)^2 + \frac{\Gamma^2}{4}}|P_l(\cos\theta)|^2$$

Well-known resonances

$$e^+e^- \rightarrow R \rightarrow e^+e^-$$

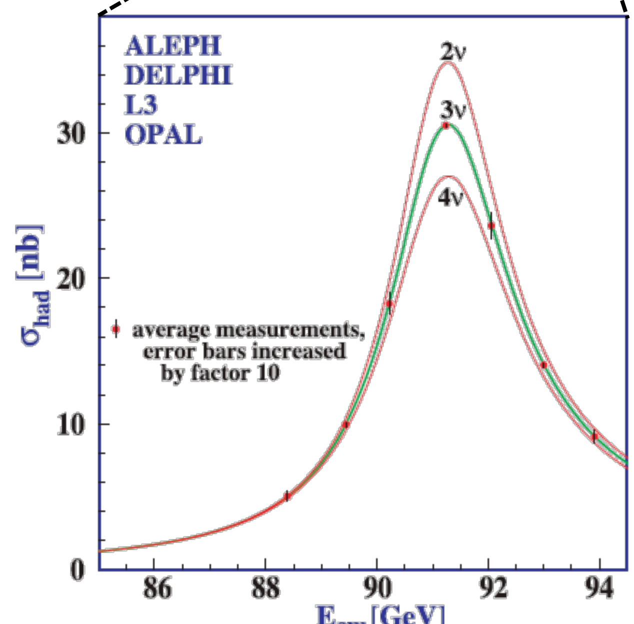
e$^+$e$^-$ cross-section





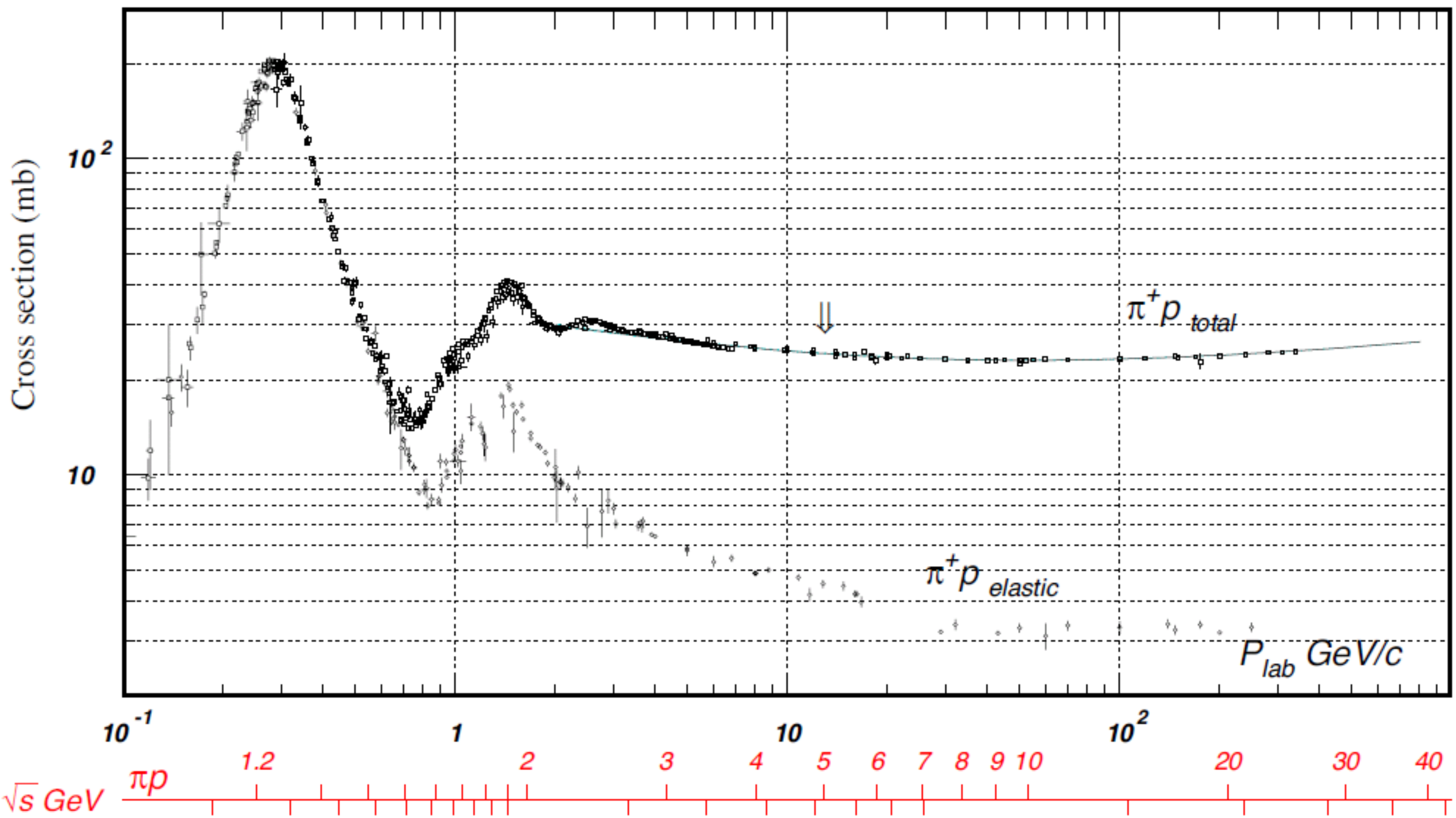
J/ψ

Z-boson



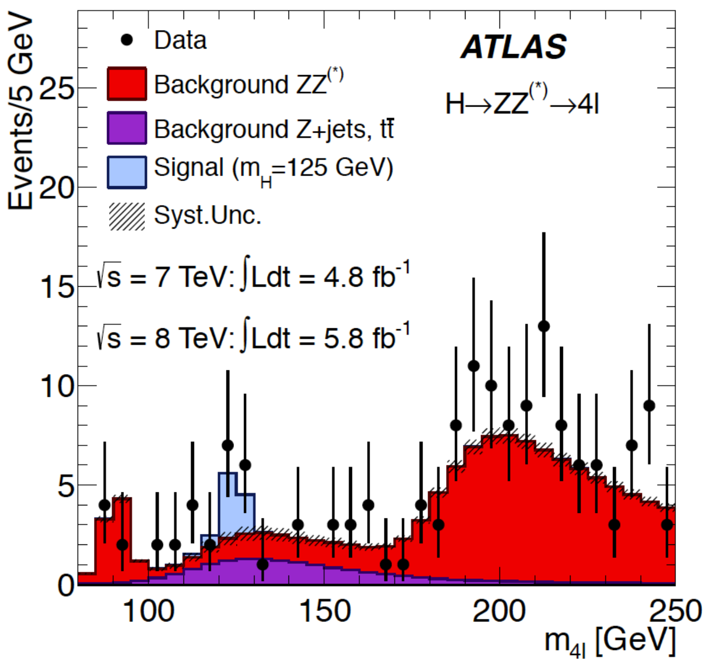
More resonances

$$\pi^+ p \rightarrow R \rightarrow \pi^+ p$$



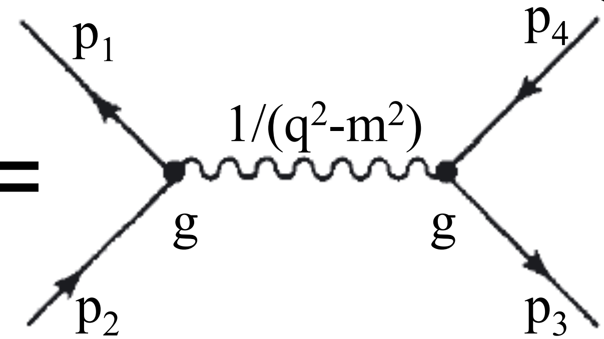
Why did we need this mathematical trickery?

- That is how we see and discover particles!
- As <u>*resonances*</u>!



Feynman Rules

$M =$



- How to calculate amplitude M ?
- The 'drawing' is a mathematical object!

D.1 External Lines

Spin 0: (nothing)

Spin f:
- Incoming particle: u
- Incoming antiparticle: $\bar{v}$
- Outgoing particle: $\bar{u}$
- Outgoing antiparticle: v

Spin 1:
- Incoming: ϵ^{μ}
- Outgoing: $\epsilon^{\mu *}$

D.2 Propagators

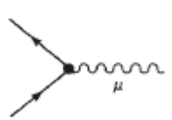
Spin 0: $\dfrac{i}{q^2-(mc)^2}$

Spin f: $\dfrac{i(\not q + mc)}{q^2-(mc)^2}$

Spin 1:
- Massless: $\dfrac{-ig_{\mu\nu}}{q^2}$
- Massive: $\dfrac{-i[g_{\mu\nu}-q_{\mu}q_{\nu}/(mc)^2]}{q^2-(mc)^2}$

D.3 Vertex Factors

QED:



$ig_e\gamma^{\mu}$ $\quad (g_e = \sqrt{4\pi\alpha})$

$$\frac{1}{(2\pi)^4}\int d^4q(-ig)^2\frac{i}{q^2-m_C^2}(2\pi)^4\delta(q-p_1-p_2)(2\pi)^4\delta(q-p_3-p_4)$$

$$= (-ig)^2\frac{i}{(p_1+p_2)^2-m_C^2}(2\pi)^4\delta(p_1+p_2-p_3-p_4) \Rightarrow$$

$$-i\mathcal{M}_b = (-ig)^2\frac{i}{(p_1+p_2)^2-m_C^2} \Rightarrow \mathcal{M}_b = \frac{g^2}{(p_1+p_2)^2-m_C^2}$$

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Those lines in the Feynman diagrams? They do not depict particle paths. No they don't. They're visual aids that encode long formulas by help of which you can calculate the outcome of certain experiments. Yes, it's all abstract math. Not, they do not depict particle paths.

20 113 386

Plan

1) Intro: Standard Model & Relativity — ***11 Feb***

1900-1940 — 2) Basis — ***18 Feb***
   1) Atom model, strong and weak force
   2) Scattering theory

1945-1965 — 3) Hadrons — ***10 Mar***
   1) Isospin, strangeness
   2) Quark model, GIM

1965-1975 — 4) Standard Model — ***24 Mar***
   1) QED
   2) Parity, neutrinos, weak inteaction
   3) QCD

1975-2000 — 5) e^+e^- and DIS — ***21 Apr***

2000-2015 — 6) Higgs and CKM — ***12 May***

Extra: derivation of scattering amplitude f

Scattering Theory

- Describe a ***stationary*** state, that satisfies the incoming and outgoing wave

$$\left(\nabla^2 + k^2\right)\psi(\vec{\mathbf{r}}) = \frac{2m}{\hbar^2}V(\vec{\mathbf{r}})\psi(\vec{\mathbf{r}})$$

- Incoming particle (almost) free: $\left(\nabla^2 + k^2\right)\phi(\vec{\mathbf{r}}) = 0$
- Introduce Green function such: $\left(\nabla^2 + k^2\right)G(\vec{\mathbf{r}}|\vec{\mathbf{r}}') = \delta(\vec{\mathbf{r}} - \vec{\mathbf{r}}')$
- If we know G, then the solution for $\left(\nabla^2 + k^2\right)\phi(\vec{\mathbf{r}}) = A(\vec{\mathbf{r}})$
- is indeed a sum of the 2 waves: $\Phi(\vec{\mathbf{r}}) = \phi_a(\vec{\mathbf{r}}) + \int G(\vec{\mathbf{r}}|\vec{\mathbf{r}}')A(\vec{\mathbf{r}}')\,\mathrm{d}^3r' = \phi_a(\vec{\mathbf{r}}) + \phi_{sc}(\vec{\mathbf{r}})$

- With: $G_{(+)} = -\frac{e^{ik|\vec{\mathbf{r}}-\vec{\mathbf{r}}'|}}{4\pi|\vec{\mathbf{r}}-\vec{\mathbf{r}}'|}$

$$\Phi(\vec{\mathbf{r}}) = \phi_a(\vec{\mathbf{r}}) - \frac{m}{2\pi\hbar^2}\int \frac{e^{ik|\vec{\mathbf{r}}-\vec{\mathbf{r}}'|}}{|\vec{\mathbf{r}}-\vec{\mathbf{r}}'|}V(\vec{\mathbf{r}}')\Phi(\vec{\mathbf{r}}')\,\mathrm{d}^3r'$$

Scattering Theory

- Describe a ***stationary*** state, that satisfies the incoming and outgoing wave

$$\left(\nabla^2 + k^2\right)\psi(\vec{r}) = \frac{2m}{\hbar^2}V(\vec{r})\psi(\vec{r})$$

$$\Phi(\vec{r}) = \phi_a(\vec{r}) - \frac{m}{2\pi\hbar^2}\int \frac{e^{ik|\vec{r}-\vec{r}'|}}{|\vec{r}-\vec{r}'|}V(\vec{r}')\Phi(\vec{r}')\,\mathrm{d}^3r'$$

  - Large r:

$$k|\vec{r}-\vec{r}'| \approx kr - (\vec{k_b}\cdot\vec{r}')$$

$$\Phi(\vec{r}) = \phi_a(\vec{r}) + f(\vec{k_a},\vec{k_b})\frac{e^{ikr}}{r}$$

  - f: "scattering amplitude":

$$f(\vec{k_a},\vec{k_b}) = -\frac{m}{2\pi\hbar^2}\int e^{-i\vec{k_b}\cdot\vec{r}'}V(\vec{r}')\Phi(\vec{r}')\,\mathrm{d}^3r$$

  - which we will use for:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = |f(\vec{k_a},\vec{k_b})|^2$$

➢ Differential equation became integral equation, but how do we solve it??

Scattering Theory

- Describe a *stationary* state, that satisfies the incoming and outgoing wave

$$\left(\nabla^2 + k^2\right)\psi(\vec{\mathrm{r}}) = \frac{2m}{\hbar^2}V(\vec{\mathrm{r}})\psi(\vec{\mathrm{r}})$$

  - f: "scattering amplitude":

$$\Phi(\vec{\mathrm{r}}) = \phi_a(\vec{\mathrm{r}}) + f(\vec{\mathrm{k}_\mathrm{a}}, \vec{\mathrm{k}_\mathrm{b}})\frac{e^{ikr}}{r}$$

  - which we will use for:

$$f(\vec{\mathrm{k}_\mathrm{a}}, \vec{\mathrm{k}_\mathrm{b}}) = -\frac{m}{2\pi\hbar^2}\int e^{-i\vec{\mathrm{k}_\mathrm{b}}\cdot\vec{\mathrm{r}}'}V(\vec{\mathrm{r}}')\Phi(\vec{\mathrm{r}}')\ \mathrm{d}^3r$$

➢ How do we solve it?? Not analytic… ➔ Perturbation series!

  - 1st approximation:

$$\Phi(\vec{\mathrm{r}}) = \phi_a(\vec{\mathrm{r}}) - \frac{m}{2\pi\hbar^2}\int \frac{e^{ik|\vec{\mathrm{r}}-\vec{\mathrm{r}}'|}}{|\vec{\mathrm{r}}-\vec{\mathrm{r}}'|}V(\vec{\mathrm{r}}')\phi_a(\vec{\mathrm{r}}')\ \mathrm{d}^3r'$$

$$f^{[1]}(\vec{\mathrm{k}_\mathrm{a}}, \vec{\mathrm{k}_\mathrm{b}}) = \frac{m}{2\pi\hbar^2}\int e^{i(\vec{\mathrm{k}_\mathrm{a}}-\vec{\mathrm{k}_\mathrm{b}})\cdot\vec{\mathrm{r}}'}V(\vec{\mathrm{r}}')\ \mathrm{d}^3r'$$

➢ Scattered wave is described by Fourier transform of the potential

Scattering Theory

- Let's try the Yukawa potential

$$\Phi(\vec{\mathrm{r}}) = \phi_a(\vec{\mathrm{r}}) - \frac{m}{2\pi\hbar^2}\int \frac{e^{ik|\vec{\mathrm{r}}-\vec{\mathrm{r}}'|}}{|\vec{\mathrm{r}}-\vec{\mathrm{r}}'|} V(\vec{\mathrm{r}}')\phi_a(\vec{\mathrm{r}}')\ \mathrm{d}^3r'$$

$$f^{[1]}(\vec{\mathrm{k}}_\mathrm{a}, \vec{\mathrm{k}}_\mathrm{b}) = \frac{m}{2\pi\hbar^2}\int e^{i(\vec{\mathrm{k}}_\mathrm{a}-\vec{\mathrm{k}}_\mathrm{b})\cdot\vec{\mathrm{r}}'} V(\vec{\mathrm{r}}')\ \mathrm{d}^3r'$$

$$V(r) = \frac{Z_1 Z_2 e^2}{r} e^{-ar}$$

$$f(\vec{\mathrm{k}}_\mathrm{a}, \vec{\mathrm{k}}_\mathrm{b}) = -\frac{m}{2\pi\hbar^2} Z_1 Z_2 e^2 \int \frac{e^{-ar'}}{r'} e^{i(\vec{\mathrm{k}}_\mathrm{a}-\vec{\mathrm{k}}_\mathrm{b})\cdot\vec{\mathrm{r}}'}\ \mathrm{d}^3r' = -\frac{m}{2\pi\hbar^2}\frac{4\pi Z_1 Z_2 e^2}{q^2+a^2}$$

- Yukawa:

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = |f|^2 = \frac{m^2}{(2\pi\hbar^2)^2}\left[\frac{4\pi Z_1 Z_2 e^2}{q^2+a^2}\right]^2$$

- Coulomb (a→0) :

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{m^2}{(2\pi\hbar^2)^2}\left[\frac{4\pi Z_1 Z_2 e^2}{q^2}\right]^2 = \left[\frac{Z_1 Z_2 e^2}{2mv^2\sin^2\frac{\theta}{2}}\right]^2$$

➢ We found back the classical solution from Rutherford

Interpretation

$$\Phi(\vec{\mathbf{r}}) = \phi_a(\vec{\mathbf{r}}) + f(\vec{\mathbf{k}}_\mathbf{a}, \vec{\mathbf{k}}_\mathbf{b})\frac{e^{ikr}}{r}$$

ingoing outgoing

- Consider again the amplitude:

(Fourier transform of potential)

$$f(\vec{k_b}, \vec{k_a}) = \frac{-m}{2\pi\hbar^2}\int d\vec{r'} e^{i(\vec{k_a}-\vec{k_b})\cdot\vec{r'}} V(\vec{r'})$$

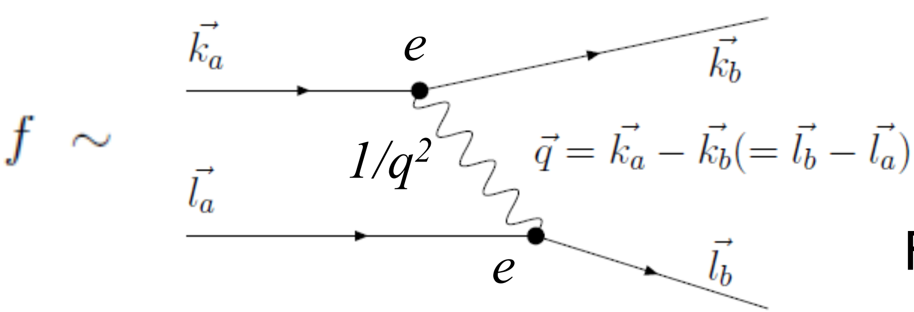
$$V(r) = -\frac{\alpha}{r}$$

$$f(\vec{k_b}, \vec{k_a}) = -\frac{2m\alpha}{q^2} \qquad q = |\vec{k_b} - \vec{k_a}|$$

$$\frac{d\sigma}{d\Omega} = \frac{4m^2\alpha^2}{q^4}$$

- We used quantum mechanics,

- but with relativistic quantum field theory, the concept is similar:



$$f \sim e \cdot e \cdot propagator$$

Feynman diagram

Scattering Theory

- Let's try an effective potential:

$$V_{eff} = V(r) + \frac{\hbar^2 l(l+1)}{2mr^2}$$

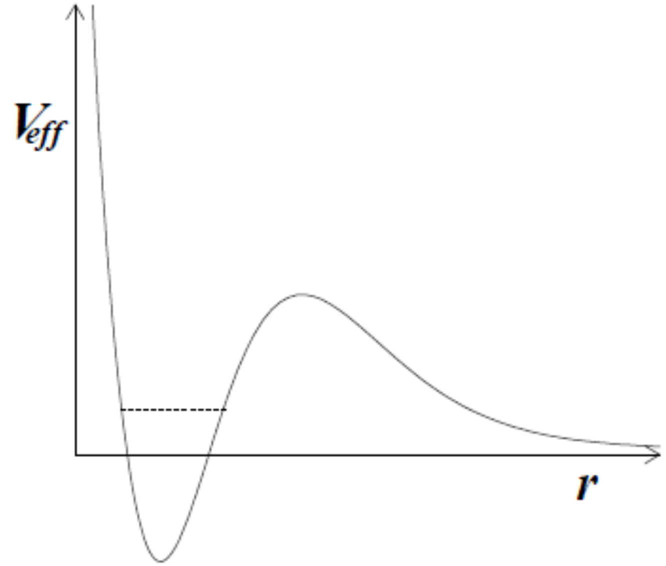
$$\Phi(\vec{\mathrm{r}}) = \phi_a(\vec{\mathrm{r}}) - \frac{m}{2\pi\hbar^2}\int \frac{e^{ik|\vec{\mathrm{r}}-\vec{\mathrm{r}}'|}}{|\vec{\mathrm{r}}-\vec{\mathrm{r}}'|} V(\vec{\mathrm{r}}')\phi_a(\vec{\mathrm{r}}')\, \mathrm{d}^3r'$$

$$f^{[1]}(\vec{\mathrm{k}}_\mathrm{a}, \vec{\mathrm{k}}_\mathrm{b}) = \frac{m}{2\pi\hbar^2}\int e^{i(\vec{\mathrm{k}}_\mathrm{a}-\vec{\mathrm{k}}_\mathrm{b})\cdot\vec{\mathrm{r}}'} V(\vec{\mathrm{r}}')\, \mathrm{d}^3r'$$

$$f^{res}(\theta) = \frac{(2l+1)}{k}\frac{\Gamma/2}{(E_r - E) - i\Gamma/2}P_l(\cos\theta)$$

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = |f^{res}(\theta)|^2 = \frac{(2l+1)^2}{k^2}\frac{\frac{\Gamma^2}{4}}{(E_r - E)^2 + \frac{\Gamma^2}{4}}|P_l(\cos\theta)|^2$$

$$\sigma_l = \frac{4\pi(2l+1)}{k^2}\frac{\frac{\Gamma^2}{4}}{(E_r - E)^2 + \frac{\Gamma^2}{4}}$$

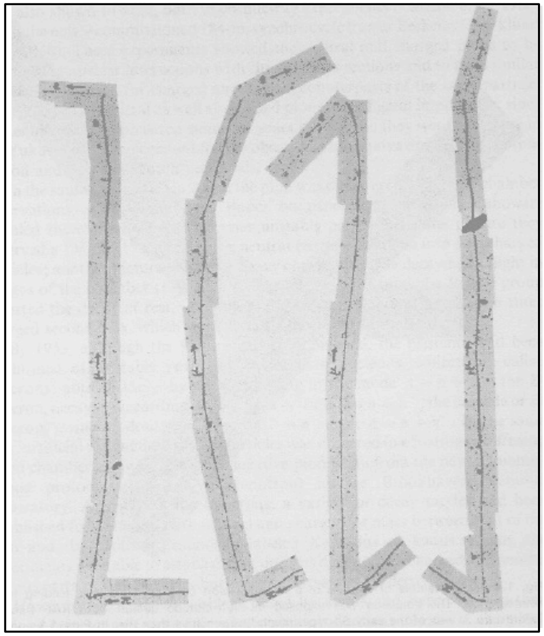


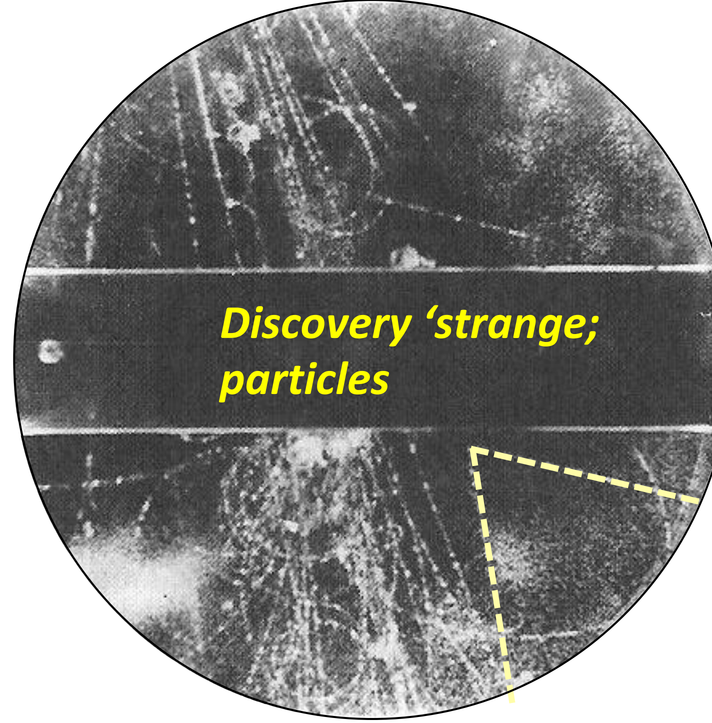
Scattering to this potential can lead to a bound system, that can then "tunnel away"

➢ We found the non-relativistic Breit-Wigner resonance formula!

Use cosmic rays

Discovery pions / muons





Discovery anti-matter