Nonlinear Bogolyubov-Valatin transformations

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NIKHEF, September 29, 2011

NIKHEF 04-024
(J. Phys. A 38(2005)10245 [quant-ph/0411170]),
NIKHEF/2010-005
(Annals Phys. (2011), to appear [arXiv:1002.2737]),

with J.-W. van Holten

Overview:

- Introduction
- Algebra
- Strategy for nonlinear Bogolyubov-Valatin transformations (and more algebra)
- Applications (Hamiltonians, $SU(4) \longleftrightarrow SO(6;\mathbb{R})$)

The title:

nonlinear Bogolyubov-Valatin transformations

$$\Downarrow$$

- quantum mechanics, 2nd quantization
- fermions:

$$\hat{a}_k|0\rangle = 0$$

$$\hat{a}_k^\dagger|0\rangle = |1\rangle_k \quad \left(\hat{a}_k^\dagger|1\rangle_k = 0\right)$$

$$\mathrm{CAR:}$$

$$\{\hat{a}_k^\dagger, \hat{a}_l\} = \hat{a}_k^\dagger\ \hat{a}_l + \hat{a}_k\ \hat{a}_l^\dagger = \delta_{kl}$$

$$\{\hat{a}_k, \hat{a}_l\} = \hat{a}_k\ \hat{a}_l + \hat{a}_k\ \hat{a}_l = 0$$

(Grassmann algebra)

finite-dimensional representation for n modes (dimension 2^n)
($\longrightarrow$ (multi-) linear algebra),

one mode:

$$|0\rangle \longrightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad |1\rangle \longrightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$\hat{a} \longrightarrow \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad \hat{a}^\dagger \longrightarrow \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

The title:

nonlinear Bogolyubov-Valatin transformations

$$\Downarrow$$

canonical transformations

- unitary implementation

$$\hat{b}_k = \mathrm{U}\hat{a}_k\mathrm{U}^\dagger \qquad \mathrm{U} \in SU(2^n)$$

CAR :

$$\{\hat{b}_k^\dagger, \hat{b}_l\} = \hat{b}_k^\dagger\ \hat{b}_l + \hat{b}_k\ \hat{b}_l^\dagger = \delta_{kl}$$
$$\{\hat{b}_k, \hat{b}_l\} = \hat{b}_k\ \hat{b}_l + \hat{b}_k\ \hat{b}_l = 0$$

The title:

<u>nonlinear</u> Bogolyubov-Valatin transformations

$$\Downarrow$$

- linear Bogolyubov-Valatin transformations

$$\hat{b}_1 = u\ \hat{a}_1 - v\ \hat{a}_2^\dagger$$
$$\hat{b}_2 = u\ \hat{a}_2 + v\ \hat{a}_1^\dagger$$

$$\longrightarrow \quad \boxed{u^2 + v^2 = 1}$$ (Bogolyubov, Valatin 1958)

- nonlinear Bogolyubov-Valatin transformations: $\longrightarrow$

here: 2 modes ($k = 1, 2$)

$$
\begin{aligned}
\hat{b}_k &= \mathrm{U}\hat{a}_k\mathrm{U}^\dagger \\
&= \lambda_k^{(0|0)}\,\mathbf{1}_4 + \lambda_k^{(1|0)}\,\hat{a}_1^\dagger + \lambda_k^{(2|0)}\,\hat{a}_2^\dagger + \lambda_k^{(0|1)}\,\hat{a}_1 + \lambda_k^{(0|2)}\,\hat{a}_2 \\
&\quad + \lambda_k^{(1,2|0)}\,\hat{a}_1^\dagger\hat{a}_2^\dagger + \lambda_k^{(1|1)}\,\hat{a}_1^\dagger\hat{a}_1 + \lambda_k^{(1|2)}\,\hat{a}_1^\dagger\hat{a}_2 \\
&\quad + \lambda_k^{(2|1)}\,\hat{a}_2^\dagger\hat{a}_1 + \lambda_k^{(2|2)}\,\hat{a}_2^\dagger\hat{a}_2 + \lambda_k^{(0|1,2)}\,\hat{a}_1\hat{a}_2 \\
&\quad + \lambda_k^{(1,2|1)}\,\hat{a}_1^\dagger\hat{a}_2^\dagger\hat{a}_1 + \lambda_k^{(1,2|2)}\,\hat{a}_1^\dagger\hat{a}_2^\dagger\hat{a}_2 \\
&\quad + \lambda_k^{(1|1,2)}\,\hat{a}_1^\dagger\hat{a}_1\hat{a}_2 + \lambda_k^{(2|1,2)}\,\hat{a}_2^\dagger\hat{a}_1\hat{a}_2 \\
&\quad + \lambda_k^{(1,2|1,2)}\,\hat{a}_1^\dagger\hat{a}_2^\dagger\hat{a}_1\hat{a}_2
\end{aligned}
$$

(“supersymmetric” transformation)

Motivation:

- field theory:
  Grassmann integrals $\longrightarrow$ nonlinear transformations
- condensed matter theory: Hubbard model
- quantum information theory:
  - tensor product structures for fermionic systems
  - $n = 2$: two-spin-$\frac{1}{2}$-systems (two-qubit systems),
    one-spin-$\frac{3}{2}$-systems
- mathematics:
  - Clifford algebras ($n = 2$: $\longrightarrow$ Dirac matrices)
  - $n = 2$: $SU(4)$, $SO(6;\mathbb{R})$

Algebra

Grassmann $\longrightarrow$ Clifford algebra:

$$\hat{a}_k^{[1]} = -\hat{a}_k^{[1]\dagger} = i\left(\hat{a}_k + \hat{a}_k^\dagger\right)$$
$$\hat{a}_k^{[2]} = -\hat{a}_k^{[2]\dagger} = \hat{a}_k - \hat{a}_k^\dagger$$

$$\{\hat{a}_k^{[p]}, \hat{a}_l^{[q]}\} = -2\,\delta_{pq}\,\delta_{kl}, \qquad p,q = 1,2$$

rename:

$$\hat{c}_{2k-1} = \hat{a}_k^{[1]}, \quad k = 1,\ldots,n$$
$$\hat{c}_{2k} = \hat{a}_k^{[2]}, \quad k = 1,\ldots,n$$

$$\{\hat{c}_k, \hat{c}_l\} = 2\,\hat{c}_k \cdot \hat{c}_l = 2\,g_{kl} = -2\,\delta_{kl}$$

($n = 1$: "Pauli matrices", $n = 2$: "Dirac matrices")

One fermion mode: quaternions

$C(0,2) \simeq \mathbb{H}$

$$\begin{aligned}
\mathbf{i} &= \hat{a}^{[1]} = i\left(\hat{a}+\hat{a}^\dagger\right) \\
\mathbf{j} &= \hat{a}^{[2]} = \hat{a}-\hat{a}^\dagger \\
\mathbf{k} &= \hat{a}^{[3]} = \hat{a}^{[1]}\,\hat{a}^{[2]} = 2i\left(\hat{a}^\dagger\hat{a}-\frac{1}{2}\right) = i\left(\hat{a}^\dagger\hat{a}-\hat{a}\hat{a}^\dagger\right)
\end{aligned}$$

(Jordan, Wigner 1927/28)

Excursion to the result:

$$\begin{pmatrix} \mathbf{i}' \\ \mathbf{j}' \\ \mathbf{k}' \end{pmatrix} = A \begin{pmatrix} \mathbf{i} \\ \mathbf{j} \\ \mathbf{k} \end{pmatrix} = \mathrm{U} \begin{pmatrix} \mathbf{i} \\ \mathbf{j} \\ \mathbf{k} \end{pmatrix} \mathrm{U}^\dagger$$

$$A \in SO(3;\mathbb{R}),\ \mathrm{U} \in SU(2) \quad \left(A^T = \mathrm{Ad}(\mathrm{U})\right)$$

Two fermion modes: biparavectors

Define:

$$\hat{\hat{c}}_k = \hat{c}_k \ , \qquad k = 1, \ldots, 4$$
$$\hat{\hat{c}}_{(-1)} = i\hat{c}_1\hat{c}_2\hat{c}_3\hat{c}_4$$

(non-universal) Clifford algebra $C(0,5)$ $(k, l = -1, 1, \ldots, 4)$:

$$\left\{\hat{\hat{c}}_k, \hat{\hat{c}}_l\right\} = -2\ \delta_{kl}\ \mathbf{1}_4$$

$$\hat{\hat{\hat{c}}}_k = \hat{\hat{c}}_k \ , \qquad k = -1, 1, \ldots, 4$$
$$\hat{\hat{\hat{c}}}_0 = -\mathbf{1}_4$$

The operators $\hat{\hat{\hat{c}}}_k$ span the **paravector** space V_6.

$$\hat{\hat{\hat{c}}} = \begin{pmatrix} \hat{\hat{\hat{c}}}_{-1} \\ \hat{\hat{\hat{c}}}_0 \\ \hat{\hat{\hat{c}}}_1 \\ \hat{\hat{\hat{c}}}_2 \\ \hat{\hat{\hat{c}}}_3 \\ \hat{\hat{\hat{c}}}_4 \end{pmatrix} = \begin{pmatrix} i\hat{c}_1\hat{c}_2\hat{c}_3\hat{c}_4 \\ -\mathbf{1}_4 \\ \hat{c}_1 \\ \hat{c}_2 \\ \hat{c}_3 \\ \hat{c}_4 \end{pmatrix}$$

$$\hat{\hat{\hat{c}}}_k^{\dagger}\hat{\hat{\hat{c}}}_l + \hat{\hat{\hat{c}}}_l^{\dagger}\hat{\hat{\hat{c}}}_k = \hat{\hat{\hat{c}}}_k\,\hat{\hat{\hat{c}}}_l^{\dagger} + \hat{\hat{\hat{c}}}_l\,\hat{\hat{\hat{c}}}_k^{\dagger} = \underline{\underline{\;2\;\delta_{kl}\;\mathbf{1}_4\;}} \qquad (k, l = -1, \ldots, 4)$$

$$\hat{\hat{\hat{c}}}_m = -\frac{i}{5!}\,\epsilon_m{}^{npqrs}\,\hat{\hat{\hat{c}}}_n\,\hat{\hat{\hat{c}}}_p^{\dagger}\,\hat{\hat{\hat{c}}}_q\,\hat{\hat{\hat{c}}}_r^{\dagger}\,\hat{\hat{\hat{c}}}_s, \qquad ('' \;\gamma_5^2 = \pm\mathbf{1}_4\; '')$$

$$\hat{\hat{\hat{c}}}_n\,\hat{\hat{\hat{c}}}_p^{\dagger}\,\hat{\hat{\hat{c}}}_q\,\hat{\hat{\hat{c}}}_r^{\dagger}\,\hat{\hat{\hat{c}}}_s = -i\;\epsilon_{npqrs}{}^{m}\;\hat{\hat{\hat{c}}}_m$$

Biparavectors:

$M = (m_1, m_2),\ m_1 < m_2$

$$\hat{\hat{\hat{c}}}_M = \hat{\hat{\hat{c}}}_{m_1 m_2} = -\hat{\hat{\hat{c}}}_{m_2 m_1} = \frac{1}{2}\left(\hat{\hat{\hat{c}}}^{\dagger}_{m_1}\hat{\hat{\hat{c}}}_{m_2} - \hat{\hat{\hat{c}}}^{\dagger}_{m_2}\hat{\hat{\hat{c}}}_{m_1}\right) = -\hat{\hat{\hat{c}}}_{m_1 m_2}{}^{\dagger}$$

(Eddington 1928: E-numbers)

- span the 15-dimensional biparavector space $\bigwedge^2(V_6)$
  (= space of non-trivial operators in Fock space)

$$\begin{aligned}[\hat{\hat{\hat{c}}}_{p_1 p_2}, \hat{\hat{\hat{c}}}_{q_1 q_2}] &= \hat{\hat{\hat{c}}}_{p_1 p_2}\,\hat{\hat{\hat{c}}}_{q_1 q_2} - \hat{\hat{\hat{c}}}_{q_1 q_2}\,\hat{\hat{\hat{c}}}_{p_1 p_2} \\ &= -2\,\delta_{p_1 q_1}\,\hat{\hat{\hat{c}}}_{(p_2 q_2)} - 2\,\delta_{p_2 q_2}\,\hat{\hat{\hat{c}}}_{(p_1 q_1)} \\ &\quad + 2\,\delta_{p_1 q_2}\,\hat{\hat{\hat{c}}}_{(p_2 q_1)} + 2\,\delta_{p_2 q_1}\,\hat{\hat{\hat{c}}}_{(p_1 q_2)}\end{aligned}$$

Lie algebra $\mathrm{su}(4) \simeq \mathrm{so}(6)$

(particle physics: $\mathrm{su}(2,2) \simeq \mathrm{so}(4,2) \longrightarrow$ spin representation of conformal group)

Strategy for nonlinear Bogolyubov-Valatin transformations

2 modes ($n = 2,\ k = 1, 2$)

$$
\begin{aligned}
\hat{b}_k &= \mathrm{U}\hat{a}_k\mathrm{U}^\dagger \\
&= \lambda_k^{(0|0)}\,\mathbf{1}_4 + \lambda_k^{(1|0)}\,\hat{a}_1^\dagger + \lambda_k^{(2|0)}\,\hat{a}_2^\dagger + \lambda_k^{(0|1)}\,\hat{a}_1 + \lambda_k^{(0|2)}\,\hat{a}_2 \\
&\quad + \lambda_k^{(1,2|0)}\,\hat{a}_1^\dagger\hat{a}_2^\dagger + \lambda_k^{(1|1)}\,\hat{a}_1^\dagger\hat{a}_1 + \lambda_k^{(1|2)}\,\hat{a}_1^\dagger\hat{a}_2 \\
&\quad + \lambda_k^{(2|1)}\,\hat{a}_2^\dagger\hat{a}_1 + \lambda_k^{(2|2)}\,\hat{a}_2^\dagger\hat{a}_2 + \lambda_k^{(0|1,2)}\,\hat{a}_1\hat{a}_2 \\
&\quad + \lambda_k^{(1,2|1)}\,\hat{a}_1^\dagger\hat{a}_2^\dagger\hat{a}_1 + \lambda_k^{(1,2|2)}\,\hat{a}_1^\dagger\hat{a}_2^\dagger\hat{a}_2 \\
&\quad + \lambda_k^{(1|1,2)}\,\hat{a}_1^\dagger\hat{a}_1\hat{a}_2 + \lambda_k^{(2|1,2)}\,\hat{a}_2^\dagger\hat{a}_1\hat{a}_2 \\
&\quad + \lambda_k^{(1,2|1,2)}\,\hat{a}_1^\dagger\hat{a}_2^\dagger\hat{a}_1\hat{a}_2
\end{aligned}
$$

Transition to Clifford algebra problem:

$$\hat{a}_k^{[1]} = -\hat{a}_k^{[1]\dagger} = i\left(\hat{a}_k + \hat{a}_k^\dagger\right), \qquad \hat{b}_k^{[1]} = -\hat{b}_k^{[1]\dagger} = i\left(\hat{b}_k + \hat{b}_k^\dagger\right)$$

$$\hat{a}_k^{[2]} = -\hat{a}_k^{[2]\dagger} = \hat{a}_k - \hat{a}_k^\dagger, \qquad \hat{b}_k^{[2]} = -\hat{b}_k^{[2]\dagger} = \hat{b}_k - \hat{b}_k^\dagger$$

rename:

$$\hat{c}_{2k-1} = \hat{a}_k^{[1]}, \qquad \hat{d}_{2k-1} = \hat{b}_k^{[1]}$$

$$\hat{c}_{2k} = \hat{a}_k^{[2]}, \qquad \hat{d}_{2k} = \hat{b}_k^{[2]}$$

$$\begin{aligned} \hat{d}_k &= \mathrm{U}\hat{c}_k\mathrm{U}^\dagger \\ &= \chi_k^{[1]\ (m)}\ \hat{c}_m + \frac{1}{2!}\chi_k^{[2]\ (m,n)}\ \hat{c}_m\hat{c}_n + \frac{1}{3!}\chi_k^{[3]\ (m,n,p)}\ i\ \hat{c}_m\hat{c}_n\hat{c}_p \\ &\quad + \frac{1}{4!}\chi_k^{[4]\ (m,n,p,q)}\ i\ \hat{c}_m\hat{c}_n\hat{c}_p\hat{c}_q \end{aligned}$$

$$\begin{aligned}\hat{d}_k \;=\; & \chi_k^{[1]\,(1)}\,\hat{c}_1 + \chi_k^{[1]\,(2)}\,\hat{c}_2 + \chi_k^{[1]\,(3)}\,\hat{c}_3 + \chi_k^{[1]\,(4)}\,\hat{c}_4 \\ & + \chi_k^{[2]\,(1,2)}\,\hat{c}_1\hat{c}_2 + \chi_k^{[2]\,(1,3)}\,\hat{c}_1\hat{c}_3 + \chi_k^{[2]\,(1,4)}\,\hat{c}_1\hat{c}_4 \\ & + \chi_k^{[2]\,(2,3)}\,\hat{c}_2\hat{c}_3 + \chi_k^{[2]\,(2,4)}\,\hat{c}_2\hat{c}_4 + \chi_k^{[2]\,(3,4)}\,\hat{c}_3\hat{c}_4 \\ & + \chi_k^{[3]\,(1,2,3)}\,i\,\hat{c}_1\hat{c}_2\hat{c}_3 + \chi_k^{[3]\,(1,2,4)}\,i\,\hat{c}_1\hat{c}_2\hat{c}_4 \\ & + \chi_k^{[3]\,(1,3,4)}\,i\,\hat{c}_1\hat{c}_3\hat{c}_4 + \chi_k^{[3]\,(2,3,4)}\,i\,\hat{c}_2\hat{c}_3\hat{c}_4 \\ & + \chi_k^{[4]\,(1,2,3,4)}\,i\,\hat{c}_1\hat{c}_2\hat{c}_3\hat{c}_4\end{aligned}$$

nonlinear basis transformation in the Clifford algebra $C(0,4)$!

$$\left\{\hat{d}_k, \hat{d}_l\right\} = \hat{d}_k\hat{d}_l + \hat{d}_l\hat{d}_k \overset{\text{def}}{=} 2\,\hat{d}_k \cdot \hat{d}_l \overset{!}{=} 2\,g_{kl}\,\mathbf{1}_4 = -2\,\delta_{kl}\,\mathbf{1}_4$$

Clifford algebra analogue of CAR

$$\left\{\hat{d}_k, \hat{d}_l\right\} \overset{!}{=} -2\,\delta_{kl}\,\mathbf{1}_4$$

$$= -2\left[\chi^{[1]}_{k\,(m)}\,\chi^{[1]\,(m)}_{l} + \frac{1}{2!}\,\chi^{[2]}_{k\,(m,n)}\,\chi^{[2]\,(m,n)}_{l} + \frac{1}{3!}\,\chi^{[3]}_{k\,(m,n,p)}\,\chi^{[3]\,(m,n,p)}_{l} + \frac{1}{4!}\,\chi^{[4]}_{k\,(m,n,p,q)}\,\chi^{[4]\,(m,n,p,q)}_{l}\right]\underline{\underline{\mathbf{1}_4}}$$

$$- i\left[\chi^{[2]}_{k\,(m,n)}\,\chi^{[3]\,(m,n,q)}_{l} + \chi^{[3]\,(m,n,q)}_{k}\,\chi^{[2]}_{l\,(m,n)}\right]\underline{\underline{\hat{c}_q}}$$

$$- i\left[\chi^{[1]}_{k\,(m)}\,\chi^{[3]\,(m,p,q)}_{l} + \chi^{[3]\,(m,p,q)}_{k}\,\chi^{[1]}_{l\,(m)} + \frac{1}{2}\,\chi^{[2]}_{k\,(m,n)}\,\chi^{[4]\,(m,n,p,q)}_{l} + \frac{1}{2}\,\chi^{[4]\,(m,n,p,q)}_{k}\,\chi^{[2]}_{l\,(m,n)}\right]\underline{\underline{\hat{c}_p \wedge \hat{c}_q}}$$

$$+ \left[\chi^{[1]\,(n)}_{k}\,\chi^{[2]\,(p,q)}_{l} + \chi^{[2]\,(p,q)}_{k}\,\chi^{[1]\,(n)}_{l}\right]\underline{\underline{\hat{c}_n \wedge \hat{c}_p \wedge \hat{c}_q}}$$

$$+ \frac{1}{2}\,\chi^{[2]\,(m,n)}_{k}\,\chi^{[2]\,(p,q)}_{l}\,\underline{\underline{\hat{c}_m \wedge \hat{c}_n \wedge \hat{c}_p \wedge \hat{c}_q}}$$

Calculational background:

$$
\begin{aligned}
&\hat{c}_q \hat{d}_l \\
&= \; -\chi_{l\,(q)}^{[1]}\, \mathbf{1}_4 \;-\; \chi_{l\,(q,s)}^{[2]}\, \hat{c}^{\,s} \\
&\quad + \chi_l^{[1]\,(r)}\, \hat{c}_q \wedge \hat{c}_r \;-\; \frac{1}{2!}\, \chi_{l\,(q,s,t)}^{[3]}\, i\; \hat{c}^s \wedge \hat{c}^{\,t} \\
&\quad + \frac{1}{2!}\, \chi_l^{[2]\,(r,s)}\, \hat{c}_q \wedge \hat{c}_r \wedge \hat{c}_s \;-\; \frac{1}{3!}\, \chi_{l\,(q,s,t,u)}^{[4]}\;\; i\; \hat{c}^{\,s} \wedge \hat{c}^{\,t} \wedge \hat{c}^{\,u} \\
&\quad + \frac{1}{3!}\, \chi_l^{[3]\,(r,s,t)}\; i\; \hat{c}_q \wedge \hat{c}_r \wedge \hat{c}_s \wedge \hat{c}_t
\end{aligned}
$$

- $\mathbf{x}\; A \;=\; \mathbf{x} \rfloor A \;+\; \mathbf{x} \wedge A,$ $\qquad x, y$: vectors, A: multivector
- $\mathbf{x} \rfloor \left(\mathbf{y}_1 \wedge \ldots \wedge \mathbf{y}_p\right) \;=\; \sum_{k=1}^{p} (-1)^{k+1}\, (\mathbf{x} \cdot \mathbf{y}_k)\, (\mathbf{y}_1 \wedge \ldots \wedge \check{\mathbf{y}}_k \wedge \ldots \wedge \mathbf{y}_p)$

  $\check{\mathbf{y}}_k$: vector to be omitted

Example: quadrivector part

$$\chi_k^{[2]\ (m,n)}\ \chi_l^{[2]\ (p,q)}\ \hat{c}_m \wedge \hat{c}_n \wedge \hat{c}_p \wedge \hat{c}_q = 0$$

$$\hat{\chi}_k^{[2]} \wedge \hat{\chi}_l^{[2]} = 0$$

$$\hat{\chi}_k^{[2]} = \chi_k^{[2]\ (m,n)}\ \hat{c}_m \wedge \hat{c}_n$$

$k = l$: decomposability condition for bivectors

Solution: $\hat{\chi}_k^{[2]} = 2\ \hat{L}_0 \wedge \hat{L}_k,$ $\qquad \hat{L}_k = L_k{}^m\ \hat{c}_m$

Final answer (2 fermion modes):

$$\hat{\hat{\hat{d}}}_M = \mathrm{U}\, \hat{\hat{\hat{c}}}_M\, \mathrm{U}^\dagger = \chi_M{}^N\, \hat{\hat{\hat{c}}}_N = C_2(L)_M{}^N\, \hat{\hat{\hat{c}}}_N$$

$$\chi_{M\,N} = C_2(L)_{M\,N} = L_{m_1 n_1} L_{m_2 n_2} - L_{m_1 n_2} L_{m_2 n_1}$$

$$L \in SO(6;\mathbb{R})$$

different formulation: $\hat{\hat{\hat{\mathcal{D}}}} = L\, \hat{\hat{\hat{\mathcal{C}}}} L^T$

(Hristev 1964, ten Kate 1968, Buchdahl 1968)

group of nonlinear Bogolyubov-Valatin transformations:

$$SO(6;\mathbb{R})/\mathbb{Z}_2$$

More algebra. Question:

Which six-dimensional space does $L \in SO(6;\mathbb{R})$ operate in?

Paravector space V_6?:

$$\hat{\hat{\hat{c}}} = \begin{pmatrix} \hat{\hat{\hat{c}}}_{-1} \\ \hat{\hat{\hat{c}}}_0 \\ \hat{\hat{\hat{c}}}_1 \\ \hat{\hat{\hat{c}}}_2 \\ \hat{\hat{\hat{c}}}_3 \\ \hat{\hat{\hat{c}}}_4 \end{pmatrix} = \begin{pmatrix} i\hat{c}_1\hat{c}_2\hat{c}_3\hat{c}_4 \\ -\mathbf{1}_4 \\ \hat{c}_1 \\ \hat{c}_2 \\ \hat{c}_3 \\ \hat{c}_4 \end{pmatrix} \qquad \underline{\underline{\text{No!}}}$$

More algebra. Answer:

$L \in SO(6;\mathbb{R})$ operates in the space of spin(or) (space) bivectors: $\bigwedge^2(\mathbb{C}_4)$

spanned by antisymmetric complex 4×4 matrices:

$$\overset{+}{\Gamma}_k = -\overset{+}{\Gamma}_k{}^T, \qquad k = -1, \ldots, 4$$

Choose:

$$\frac{1}{8}\,\epsilon^{abcd}\left(\overset{+}{\Gamma}_k\right)_{ab}\left(\overset{+}{\Gamma}_l\right)_{cd} = \delta_{kl} = \frac{1}{4}\,\mathrm{tr}\left(\overset{+}{\Gamma}_k\overset{-}{\Gamma}_l\right)$$

$$\left(\overset{-}{\Gamma}_k\right)^{ab} = -\,\frac{1}{2}\,\epsilon^{abcd}\left(\overset{+}{\Gamma}_k\right)_{cd}$$

$$\boxed{\overset{+}{\Gamma}_k\overset{-}{\Gamma}_l + \overset{+}{\Gamma}_l\overset{-}{\Gamma}_k = \overset{-}{\Gamma}_k\overset{+}{\Gamma}_l + \overset{-}{\Gamma}_l\overset{+}{\Gamma}_k = 2\,\delta_{kl}\,\mathbf{1}_4}$$

Impose further condition: $\overset{+}{\Gamma}_k{}^{\dagger} = \overset{-}{\Gamma}_k$ (selects subspace in $\bigwedge^2(\mathbb{C}_4)$: $\mathbb{R}_6$)

$$\overset{+}{\Gamma}_k \overset{-}{\Gamma}_l + \overset{+}{\Gamma}_l \overset{-}{\Gamma}_k = 2\,\delta_{kl}\,\mathbf{1}_4$$

$$\overset{+}{\Gamma}_k \mathbf{1}_4 \overset{-}{\Gamma}_l + \overset{+}{\Gamma}_l \mathbf{1}_4 \overset{-}{\Gamma}_k = 2\,\delta_{kl}\,\mathbf{1}_4$$

$$\overset{+}{\Gamma}_k \overset{-}{\Gamma}_0 \overset{+}{\Gamma}_0 \overset{-}{\Gamma}_l + \overset{+}{\Gamma}_l \overset{-}{\Gamma}_0 \overset{+}{\Gamma}_0 \overset{-}{\Gamma}_k = 2\,\delta_{kl}\,\mathbf{1}_4$$

$$\hat{\hat{\hat{c}}}_k^{\dagger} \hat{\hat{\hat{c}}}_l + \hat{\hat{\hat{c}}}_l^{\dagger} \hat{\hat{\hat{c}}}_k = 2\,\delta_{kl}\,\mathbf{1}_4$$

with: $\hat{\hat{\hat{c}}}_k = -\overset{+}{\Gamma}_0 \overset{-}{\Gamma}_k$, $\hat{\hat{\hat{c}}}_k^{\dagger} = -\overset{+}{\Gamma}_k \overset{-}{\Gamma}_0$

(Schouten 1933, Struik 1934, Haantjes 1936)

$$
\begin{aligned}
\hat{\hat{\hat{c}}}_{m_1 m_2} &= \frac{1}{2}\left(\hat{\hat{\hat{c}}}^{\dagger}_{m_1}\hat{\hat{\hat{c}}}_{m_2} - \hat{\hat{\hat{c}}}^{\dagger}_{m_2}\hat{\hat{\hat{c}}}_{m_1}\right) = \hat{\hat{\hat{c}}}^{\dagger}_{m_1}\hat{\hat{\hat{c}}}_{m_2} \\
&= \frac{1}{2}\left(\overset{+}{\Gamma}_{m_1}\overset{-}{\Gamma}_{m_2} - \overset{+}{\Gamma}_{m_2}\overset{-}{\Gamma}_{m_1}\right) = \overset{+}{\Gamma}_{m_1}\overset{-}{\Gamma}_{m_2} \\
&= \frac{1}{2}\left(\overset{-}{\Gamma}{}^{\dagger}_{m_1}\overset{-}{\Gamma}_{m_2} - \overset{-}{\Gamma}{}^{\dagger}_{m_2}\overset{-}{\Gamma}_{m_1}\right) = \overset{-}{\Gamma}{}^{\dagger}_{m_1}\overset{-}{\Gamma}_{m_2} \\
&= \frac{1}{2}\left(\overset{+}{\Gamma}_{m_1}\overset{+}{\Gamma}{}^{\dagger}_{m_2} - \overset{+}{\Gamma}_{m_2}\overset{+}{\Gamma}{}^{\dagger}_{m_1}\right) = \overset{+}{\Gamma}_{m_1}\overset{+}{\Gamma}{}^{\dagger}_{m_2}
\end{aligned}
$$

Biparavectors are bi(para)vectors of spin bivector space:

$$
\bigwedge^2(V_6) = \bigwedge^2\left(\bigwedge^2_{\mathbb{R}_6}(\mathbb{C}_4)\right) = \bigwedge^2(\mathbb{R}_6)
$$

but $V_6 \neq \mathbb{R}_6$

Hamiltonians: fermions

$$H' = H'^{\dagger}$$

$$\begin{aligned}
H' \;=\;& h^{(0|0)}\,\mathbf{1}_4 + h^{(1|0)}\,\hat{a}_1^{\dagger} + h^{(2|0)}\,\hat{a}_2^{\dagger} + h^{(0|1)}\,\hat{a}_1 + h^{(0|2)}\,\hat{a}_2 \\
& + h^{(1,2|0)}\,\hat{a}_1^{\dagger}\hat{a}_2^{\dagger} + h^{(1|1)}\,\hat{a}_1^{\dagger}\hat{a}_1 + h^{(1|2)}\,\hat{a}_1^{\dagger}\hat{a}_2 \\
& + h^{(2|1)}\,\hat{a}_2^{\dagger}\hat{a}_1 + h^{(2|2)}\,\hat{a}_2^{\dagger}\hat{a}_2 + h^{(0|1,2)}\,\hat{a}_1\hat{a}_2 \\
& + h^{(1,2|1)}\,\hat{a}_1^{\dagger}\hat{a}_2^{\dagger}\hat{a}_1 + h^{(1,2|2)}\,\hat{a}_1^{\dagger}\hat{a}_2^{\dagger}\hat{a}_2 \\
& + h^{(1|1,2)}\,\hat{a}_1^{\dagger}\hat{a}_1\hat{a}_2 + h^{(2|1,2)}\,\hat{a}_2^{\dagger}\hat{a}_1\hat{a}_2 + h^{(1,2|1,2)}\,\hat{a}_1^{\dagger}\hat{a}_2^{\dagger}\hat{a}_1\hat{a}_2 \\
=\;& \left(h^{(0|0)} + \frac{1}{2}\,h^{(1|1)} + \frac{1}{2}\,h^{(2|2)} - \frac{1}{4}\,h^{(1,2|1,2)}\right)\mathbf{1}_4 \\
& - \frac{i}{4}\,\mathrm{tr}_{V_6}\left(Y\hat{\hat{\mathcal{C}}}\right) \qquad \longleftarrow \text{traceless part}
\end{aligned}$$

$$Y = -Y^T =$$

$$\begin{pmatrix} 0 & -\frac{h^{(1,2|1,2)}}{2} & -\mathrm{Im}\ h^{(1,2|2)} & \mathrm{Re}\ h^{(1,2|2)} & \mathrm{Im}\ h^{(1,2|1)} & -\mathrm{Re}\ h^{(1,2|1)} \\ . & 0 & \mathrm{Re}\left(2h^{(1|0)} + h^{(1,2|2)}\right) & \mathrm{Im}\left(2h^{(1|0)} + h^{(1,2|2)}\right) & \mathrm{Re}\left(2h^{(2|0)} - h^{(1,2|1)}\right) & \mathrm{Im}\left(2h^{(2|0)} - h^{(1,2|1)}\right) \\ . & . & 0 & h^{(1|1)} & \mathrm{Im}\left(h^{(1,2|0)} + h^{(1|2)}\right) & -\mathrm{Re}\left(h^{(1,2|0)} - h^{(1|2)}\right) \\ . & . & . & 0 & -\mathrm{Re}\left(h^{(1,2|0)} + h^{(1|2)}\right) & -\mathrm{Im}\left(h^{(1,2|0)} - h^{(1|2)}\right) \\ . & . & . & . & 0 & h^{(2|2)} \\ . & . & . & . & . & 0 \end{pmatrix}$$

Block-diagonalize Y by means of $L \in SO(6;\mathbb{R})$: $Z = L\ Y\ L^T$

$$Z \;=\; -\,Z^T \;=\; \begin{pmatrix} 0 & \nu_{(-1)0} & 0 & 0 & 0 & 0 \\ -\nu_{(-1)0} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \nu_{12} & 0 & 0 \\ 0 & 0 & -\nu_{12} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \nu_{34} \\ 0 & 0 & 0 & 0 & -\nu_{34} & 0 \end{pmatrix}$$

$$H' \;=\; \left(h^{(0|0)} \;+\; \frac{1}{2}\; h^{(1|1)} \;+\; \frac{1}{2}\; h^{(2|2)} \;-\; \frac{1}{4}\; h^{(1,2|1,2)} \right)\; \mathbf{1}_4 \;-\; \frac{i}{4}\; \mathrm{tr}_{\,V_6} \left(Z \hat{\hat{\hat{\mathcal{D}}}} \right)$$

$$H' = \left[h^{(0|0)} + \frac{1}{2} \left(h^{(1|1)} - \nu_{12} \right) + \frac{1}{2} \left(h^{(2|2)} - \nu_{34} \right) - \frac{1}{4} \left(h^{(1,2|1,2)} + 2\nu_{(-1)0} \right) \right] \mathbf{1}_4 + \nu_{12} \, \hat{b}_1^\dagger \hat{b}_1 + \nu_{34} \, \hat{b}_2^\dagger \hat{b}_2 - 2 \, \nu_{(-1)0} \, \hat{b}_1^\dagger \hat{b}_2^\dagger \hat{b}_1 \hat{b}_2$$

Spin Hamiltonians

spin Hamiltonian (two-spin-$\frac{1}{2}$, single-spin-$\frac{3}{2}$)

$\Downarrow$

Jordan-Wigner transformation

$\Downarrow$

two-fermion Hamiltonian

$\Downarrow$

diagonalize (nonlinear Bogolyubov-Valatin transformation)

$\Downarrow$

Jordan-Wigner transformation

$\Downarrow$

diagonalized spin Hamiltonian

$SU(4) \longleftrightarrow SO(6;\mathbb{R})$

$\mathrm{U} \longleftrightarrow L$?

$$\overset{+}{\Gamma}{}'_k = L_{kl}\,\overset{+}{\Gamma}_l = \mathrm{U}\,\overset{+}{\Gamma}_k\,\mathrm{U}^T$$

$$\overset{-}{\Gamma}{}'_k = L_{kl}\,\overset{-}{\Gamma}_l = \overline{\mathrm{U}}\,\overset{-}{\Gamma}_k\,\mathrm{U}^\dagger$$

$$\longrightarrow \quad L_{kl} = \frac{1}{4}\,\mathrm{tr}\left(\overset{+}{\Gamma}_k\,\mathrm{U}^T\,\overset{-}{\Gamma}_l\,\mathrm{U}\right)$$

(Stepanovskiĭ 1966)

adjoint representation:

$$\mathrm{U}\,\hat{\hat{\hat{c}}}_M\,\mathrm{U}^\dagger = \chi_M{}^N\,\hat{\hat{\hat{c}}}_N \longrightarrow \boxed{\mathrm{Ad}\,(\mathrm{U}) = \chi^T = C_2(L^T)}$$

Conclusions

- a strategy for the study of nonlinear Bogolyubov-Valatin transformations has been developed
- the strategy has successfully been applied to one and two fermion modes
- results obtained range from theoretical physics (new method for the diagonalization of 4×4 matrix Hamiltonians) to mathematics (nonlinear basis transformations in a Clifford algebra, expression for the adjoint representation of $SU(4)$ matrices)
- the future study of more than two fermion modes will reveal new structures