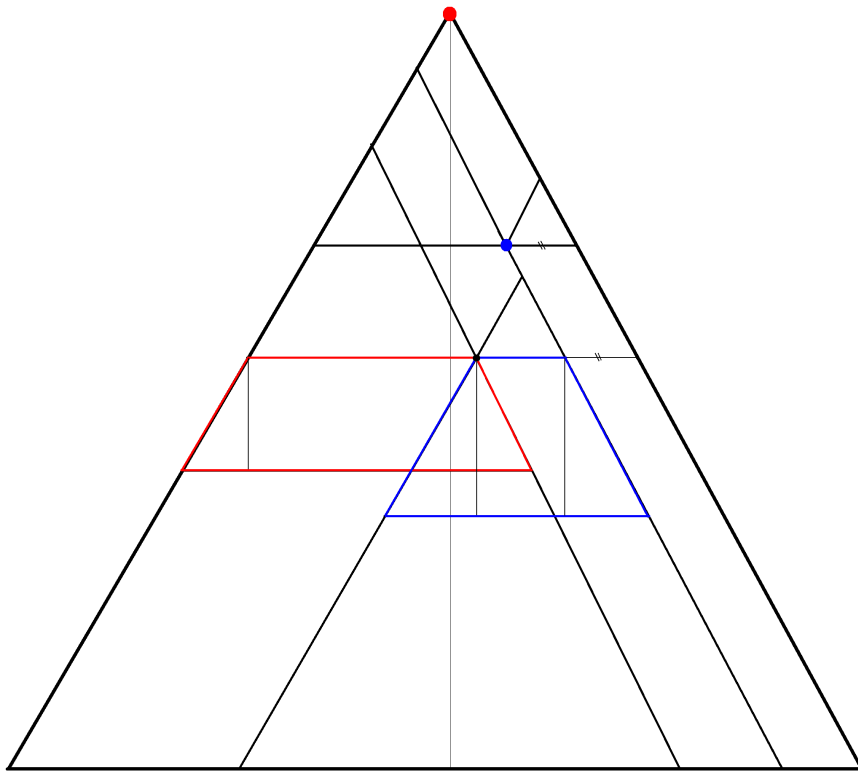


Parton Showers in the Colour Dipole Model and the Webs Model

Master Thesis Physics, track Theoretical Physics

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Abstract

In this thesis two models for parton showering are discussed. To this end first Deep Inelastic Scattering (DIS) is discussed, followed by some aspects of QCD branching: The parton splitting functions and DGLAP equations are derived and the concept of angular ordering is explained. After a short discussion on sampling distributions, the Colour Dipole Model (CDM) and the Webs model (an implication of the eikonal approximation) are introduced. Finally, these two models are compared. Two programs were written, one to test the theory of sampling distributions and the other as a toy shower model for the CDM.

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Chapter 1

Introduction

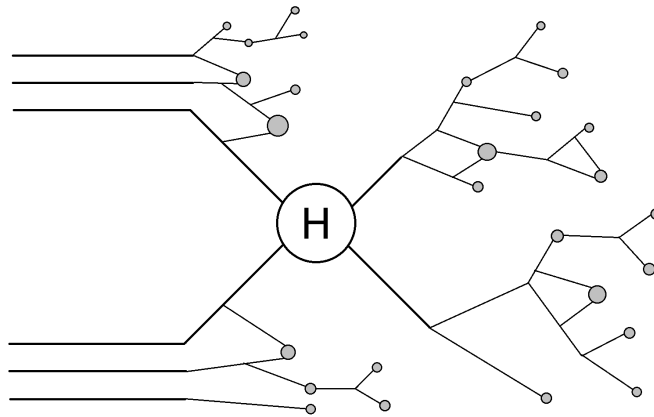


Figure 1.1: A schematic picture of a hadron collider event with initial and final state radiation, hard interaction H , beam remnants, hadronisation and decays.

Now that the Large Hadron Collider is up and running, a lot of people feel that we are entering a new era of physics. But this new era also gives new challenges, especially computational ones. At LHC we collide protons instead of annihilating electrons with positrons. Since protons are really very complicated bags of particles, these reactions are far more difficult to understand. The energy scale at which we work, the multiple TeV-scale, has not been accessed before. This means we can see all kinds of exciting new physics, but it also means we should understand the vast background of the known Standard Model very well. It is like searching for a needle in a haystack: The signals that interest us will be only a tiny fraction of the events. Apart from that, with the high luminosity LHC gives us, there is a tremendous amount of data that needs processing. These facts all in the end makes it necessary for us to really understand the simulations and be able to use all the computing power we can get to the largest possible extend.

A hadron collider event gives a complicated picture (see figure 1.1). There are at least two partons, either quarks or gluons from the protons that participated in the hard interaction. Even if there is no second hard interaction, the remains of the protons give showers of particles of their own, these are the beam remnants. Both before and after the hard interaction, the partons can radiate off more partons, which accounts for the name parton showers. At a certain point in energy scale, the partons stop showering and begin hadronising, that is, forming colourless mesons and baryons.

These need not be stable, so these hadrons could also decay into more stable particles. Only at or after this point is the detector, those are the particles we can measure.

Fortunately, not all these steps need to be considered simultaneously when building models, since they all have their own energy scale. In fact, this whole classification is based on this fact. The hard interaction can in general be calculated perturbatively, while for the parton showers one needs to follow a more numerical approach.

In this thesis we will compare two parton shower models, the Colour Dipole Model (CDM) and the Webs model. To do this, we will first need an introduction in Deep Inelastic Scattering (DIS) and QCD parton branching. There will also be a chapter devoted to the art of sampling distributions, something that will be quite useful in the end to make a numerical toy model of the CDM.

Chapter 2

Relevant matrix elements

In this chapter we will first calculate some matrix elements and cross sections that will prove to be useful in later chapters. These are the matrix elements for the Born diagram $\gamma \rightarrow q + \bar{q}$ and for its first order correction $\gamma \rightarrow (q \rightarrow q + \gamma)\bar{q}$ and $\gamma \rightarrow q(\bar{q} \rightarrow \bar{q} + \gamma)$. We will need these later on in chapters 6 and 7.

2.1 No emission, the Born diagram

Let us first calculate the amplitude squared for the process $\gamma \rightarrow q\bar{q}$, as reference to further, more elaborate calculations

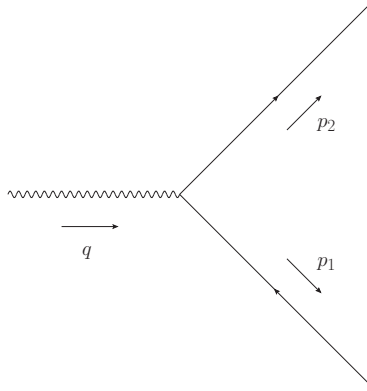


Figure 2.1: Born

$$\begin{aligned}
 |\bar{\mathcal{M}}|^2 &= \sum_{s,\lambda} \bar{v}(p_1)(iee_q\gamma^\mu)u(p_2)\bar{u}(p_2)(-iee_q\gamma^\nu)v(p_1)\epsilon_\mu\epsilon_\nu^* \\
 &= -e^2e_q^2\text{Tr}[\not{p}_2\gamma^\mu\not{p}_1\gamma_\mu] \\
 &= 2e^2e_q^2\text{Tr}[\not{p}_2\not{p}_1] = 8e^2e_q^2(p_1 \cdot p_2) \\
 &= 4e^2e_q^2Q^2
 \end{aligned} \tag{2.1}$$

Here we used $s = Q^2 = (p_1 + p_2)^2 = 2p_1 \cdot p_2$, because $p_1^2 = p_2^2 = 0$. The phase space factor is

$$\frac{1}{2Q} \int \frac{d^3p_1}{(2\pi)^3 2E_1} \int \frac{d^3p_2}{(2\pi)^3 2E_2} (2\pi)^4 \delta^{(4)}(q - p_1 - p_2) \tag{2.2}$$

We will use the integral over p_2 to reduce the four-dimensional delta-function to the energy variant. In the mean time, we use the fact that we are working in the relativistic limit $|\vec{p}_i| = E_i$:

$$\begin{aligned} & \frac{1}{2Q} \frac{1}{(2\pi)^2} \int \frac{|\vec{p}_1|^2 d|\vec{p}_1| d\Omega}{2E_1} \frac{1}{2E_2} \delta(Q - E_1 - E_2) \\ &= \frac{1}{2Q} \frac{1}{4(2\pi)^2} \int d\Omega dE_1 \frac{E_1}{E_2} \delta(Q - E_1 - E_2) \end{aligned} \quad (2.3)$$

p_1 and p_2 are back-to-back due to momentum conservation, hence we can say $E_1 = E_2 = E$. We use the relation

$$\delta(\alpha x) = \frac{1}{|\alpha|} \delta(x) \quad (2.4)$$

to adjust the delta function to the integration variable and perform the angular integration

$$\begin{aligned} & \frac{1}{2Q} \frac{4\pi}{4(2\pi)^2} \int dE \frac{E}{E} \delta(Q - 2E) \\ &= \frac{1}{2Q} \frac{1}{4\pi} \int dE \delta(2(\frac{Q}{2} - E)) \\ &= \frac{1}{8\pi Q} \int dE \frac{1}{2} \delta(E - \frac{Q}{2}) \\ &= \frac{1}{16\pi Q} \end{aligned} \quad (2.5)$$

Therefore, for the whole decay rate we get:

$$\sigma_0 = N_C |\mathcal{M}|^2 d\Phi = N_C \frac{1}{16\pi Q} (4e^2 e_q^2 Q^2) = 3\alpha e_q^2 Q \quad (2.6)$$

Here $N_C = 3$ is the number of colours and $4\pi\alpha = e^2$

2.2 The emission of one gluon

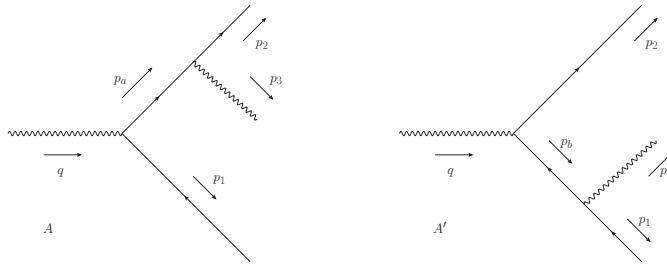


Figure 2.2: A and A'

The emission of a gluon from a quark-antiquark pair has two contributing diagrams: A: $\gamma \rightarrow \bar{q}(\rightarrow \bar{q}g)q$ and A': $\gamma \rightarrow \bar{q}q(\rightarrow qg)$

$$A = \bar{u}(p_2) (-ig_s \gamma^\nu T_{ij}^a) \left(\frac{-i\not{p}_a}{p_a^2} \right) (-iee_q \gamma^\mu) v(p_1) \epsilon_\mu^{\lambda_1} \epsilon_\nu^{\lambda_2*} \quad (2.7)$$

$$A' = \bar{u}(p_2) (-iee_q \gamma^\mu) \left(\frac{i\not{p}_b}{p_b^2} \right) (-ig_s \gamma^\nu T_{ij}^a) v(p_1) \epsilon_\mu^{\lambda_1} \epsilon_\nu^{\lambda_2*} \quad (2.8)$$

Here we have used for the internal momenta

$$\begin{aligned} p_a &= p_2 + p_3 \\ p_b &= p_1 + p_3 \end{aligned} \quad (2.9)$$

For the matrix element we need to square the sum of these two diagrams so we have four terms to calculate:

$$|\mathcal{M}|^2 = |A + A'|^2 = |A|^2 + |A'|^2 + A^* A' + A'^* A \quad (2.10)$$

Now using standard Mandelstam variables, and assuming the external lines to be on-shell ($p^2 = 0$) we will have the follow quantities:

$$\begin{aligned} s &= (p_1 + p_3)^2 = 2p_1 \cdot p_3 = Q^2(1 - x_2) \\ t &= (p_2 + p_3)^2 = 2p_2 \cdot p_3 = Q^2(1 - x_1) \\ u &= (p_1 + p_2)^2 = 2p_1 \cdot p_2 = Q^2(1 - x_3) \\ p_a &= p_2 + p_3 \Rightarrow p_a^2 = t \\ p_b &= p_1 + p_3 \Rightarrow p_b^2 = s \end{aligned} \quad (2.11)$$

In the calculation that follows we will be using the following identities:

$$\begin{aligned} \sum_s u_\alpha \bar{u}_\beta &= (\not{p} + m)_{\alpha\beta} \\ Tr[\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma] &= 4(g^{\mu\nu} g^{\rho\sigma} - g^{\mu\rho} g^{\nu\sigma} + g^{\mu\sigma} g^{\nu\rho}) \\ \gamma^\alpha \gamma^\mu \gamma_\alpha &= (2 - N)\gamma^\mu \\ Tr[\not{a}\not{b}] &= 4a \cdot b \\ \sum_\lambda \epsilon_\mu \epsilon_\nu^* &= -g_{\mu\nu} \\ Tr[T^a T^a] &= 4 \end{aligned} \quad (2.12)$$

First, let us calculate $|A|^2$:

$$\begin{aligned} |A|^2 &= \sum_{\lambda, s} \epsilon_\nu^{\lambda_2} \epsilon_\mu^{\lambda_1*} \bar{v}(p_1) (ie e_q \gamma^{\mu'}) \left(\frac{i\not{p}_a}{p_a^2}\right) (igs \gamma^{\nu'} T_{ji}^a) u(p_2) \times \bar{u}(p_2) (-igs \gamma^\nu T_{ij}^a) \left(\frac{-i\not{p}_a}{p_a^2}\right) (-ie e_q \gamma^\mu) v(p_1) \epsilon_\mu^{\lambda_1} \epsilon_\nu^{\lambda_2*} \\ &= e^2 e_q^2 g_S^2 Tr[T^a T^a] \bar{v}(p_1) \gamma_\mu \frac{\not{p}_a}{p_a^2} \gamma_\nu u(p_2) \bar{u}(p_2) \gamma^\nu \frac{\not{p}_a}{p_a^2} \gamma^\mu v(p_1) \\ &= \frac{4e^2 e_q^2 g_S^2}{p_a^4} Tr \left[\gamma_\mu \not{p}_a \gamma_\nu \not{p}_2 \gamma^\nu \not{p}_a \gamma^\mu \not{p}_1 \right] \\ &= (2 - N)^2 \frac{4e^2 e_q^2 g_S^2}{p_a^4} Tr \left[\not{p}_a \not{p}_2 \not{p}_a \not{p}_1 \right] \\ &= 4(2 - N)^2 \frac{4e^2 e_q^2 g_S^2}{p_a^4} \{ (p_a \cdot p_2)(p_a \cdot p_1) - p_a^2(p_2 \cdot p_1) + (p_a \cdot p_1)(p_2 \cdot p_a) \} \\ &= 4(2 - N)^2 \frac{4e^2 e_q^2 g_S^2}{t^2} \{ (p_3 \cdot p_2)((p_2 + p_3) \cdot p_1) - t(p_2 \cdot p_1) + ((p_2 + p_3) \cdot p_1)(p_2 \cdot p_3) \} \\ &= 4(2 - N)^2 \frac{4e^2 e_q^2 g_S^2}{t^2} \left\{ \frac{t}{2} \left(\frac{u}{2} + \frac{s}{2} \right) - t \frac{u}{2} + \left(\frac{u}{2} + \frac{s}{2} \right) \frac{t}{2} \right\} \\ &= 4(2 - N)^2 e^2 e_q^2 g_S^2 \frac{2s}{t} \end{aligned} \quad (2.13)$$

This becomes in $N = 4$:

$$|A|^2 = 32e^2 e_q^2 g_S^2 \frac{s}{t} \quad (2.14)$$

The result for $|A'|^2$ can be obtained through crossing $p_1 \leftrightarrow p_2$ and therefore $s \leftrightarrow t$, so:

$$|A'|^2 = 32e^2 e_q^2 g_S^2 \frac{t}{s} \quad (2.15)$$

Next we calculate $A^* A'$. For this we need the extra identities:

$$\gamma^\alpha \gamma^\mu \gamma^\nu \gamma_\alpha = 4g^{\mu\nu} 1_N + (N-4)\gamma^\mu \gamma^\nu \quad (2.16)$$

$$\gamma^\alpha \gamma^\mu \gamma^\nu \gamma^\rho \gamma_\alpha = -2\gamma^\rho \gamma^\nu \gamma^\mu - (N-4)\gamma^\mu \gamma^\nu \gamma^\rho \quad (2.17)$$

We will work in $N = 4$ straight away, to simplify the calculation.

$$\begin{aligned} \frac{A^* A'}{\text{Tr}[T^a T^a] e^2 e_q^2 g_S^2} &= - \sum_s \bar{v}(p_1) \gamma_\mu \frac{\not{p}_a}{p_a^2} \gamma_\nu u(p_2) \bar{u}(p_2) \gamma^\mu \frac{\not{p}_b}{p_b^2} \gamma^\nu v(p_1) \\ &= - \frac{1}{p_a^2 p_b^2} \text{Tr} \left[\not{p}_1 \gamma_\mu \not{p}_a \gamma_\nu \not{p}_2 \gamma^\mu \not{p}_b \gamma^\nu \right] \\ &= 2 \frac{1}{p_a^2 p_b^2} \text{Tr} \left[\not{p}_1 \not{p}_2 \gamma_\nu \not{p}_a \not{p}_b \gamma^\nu \right] = 8 \frac{p_a \cdot p_b}{p_a^2 p_b^2} \text{Tr} \left[\not{p}_1 \not{p}_2 \right] = 32 \frac{p_a \cdot p_b p_1 \cdot p_2}{p_a^2 p_b^2} \\ &= 8 \frac{Q^2 u}{st} \end{aligned} \quad (2.18)$$

So we end up with:

$$A^* A' = 32e^2 e_q^2 g_S^2 \frac{Q^2 u}{st} \quad (2.19)$$

This is equal to $A'^* A$, since the expression is symmetric in $s \leftrightarrow t$. So in total we get:

$$\begin{aligned} |A + A'|^2 &= |A|^2 + |A'|^2 + A^* A' + A'^* A \\ &= 32e^2 e_q^2 g_S^2 \frac{s^2 + t^2 + 2Q^2 u}{st} \\ &= 32e^2 e_q^2 g_S^2 \frac{Q^4(1-x_2)^2 + Q^4(1-x_1)^2 + 2Q^4(1-x_3)}{Q^4(1-x_2)(1-x_1)} \\ &= 32e^2 e_q^2 g_S^2 \frac{x_1^2 + x_2^2}{(1-x_1)(1-x_2)} \end{aligned} \quad (2.20)$$

Here we introduced the x_i -variable as

$$x_i = \frac{2E_i}{\sqrt{s}}, \sum_{i=1}^3 x_i = 2 \quad (2.21)$$

2.2.1 Phase space factor for three body decay

The phase space factor of this process is a little more difficult than the previous one. If a single particle decays into three particles, these particles have to be in a plane. This means the degrees of freedom are limited in number: It suffices to consider two particles' energies and the angle between them. This fixes the energy and direction of the third particle. The phase space factor is:

$$d\Phi_3 = \frac{1}{2Q} \int \frac{d^3 p_1}{(2\pi)^3 2E_1} \int \frac{d^3 p_2}{(2\pi)^3 2E_2} \int \frac{d^3 p_3}{(2\pi)^3 2E_3} (2\pi)^4 \delta^{(4)}(q - p_1 - p_2 - p_3) \quad (2.22)$$

If we take as the degrees of freedom apart from those for p_1 also the angle θ between p_1 and p_2 and the rotational symmetry of p_2 around the p_1 -axis, the phase space factor becomes:

$$d\Phi_3 = \frac{1}{2Q} \frac{1}{8(2\pi)^5} \int d\Omega \int_0^{2\pi} d\phi \int dE_1 E_1 \int dE_2 E_2 \int_0^\pi \frac{\sin \theta}{E_3} \delta(Q - E_1 - E_2 - E_3) \quad (2.23)$$

Here $E_3 = \sqrt{E_1^2 + E_2^2 + 2E_1E_2 \cos \theta}$ Consider now the delta-function as function of $\cos \theta$ We can then use the relation

$$\delta(f(x)) = \sum_i \frac{\delta(x - x_i)}{\left. \frac{d}{dx} f(x) \right|_{x=x_i}} \quad (2.24)$$

where $f(x_i) = 0$

to rewrite the delta-function:

$$\delta(Q - E_1 - E_2 - E_3(\cos \theta)) = \left| \frac{\partial E_3}{\partial \cos \theta} \right|^{-1} \delta(\cos \theta - \alpha) \quad (2.25)$$

where α is an unimportant constant which is the solution to

$$\sqrt{E_1^2 + E_2^2 + 2E_1E_2\alpha} + E_1 + E_2 = Q \quad (2.26)$$

Since

$$\frac{dE_3}{d \cos \theta} = \frac{E_1E_2}{E_3} \quad (2.27)$$

we get:

$$\begin{aligned} d\Phi_3 &= \frac{1}{2Q} \frac{(4\pi)(2\pi)}{8(2\pi)^5} \int dE_1 E_1 \int dE_2 E_2 \int_{-1}^1 \frac{d \cos \theta}{E_3} \frac{E_3}{E_1 E_2} \delta(\cos \theta - \alpha) \\ &= \frac{1}{2Q} \frac{1}{4(2\pi)^3} \int dE_1 \int dE_2 \end{aligned} \quad (2.28)$$

(if $\alpha \in [-1, 1]$, which is true because those are the boundaries of the cosine) Use the definition of x_i to rewrite this to

$$d\Phi_3 = \frac{1}{2Q} \frac{1}{4(2\pi)^3} \int \frac{Q dx_1}{2} \frac{Q dx_3}{2} = \frac{Q}{32(2\pi)^3} dx_1 dx_3 \quad (2.29)$$

Combine this with the matrix element squared to get the final result

$$\begin{aligned} d\sigma &= |\mathcal{M}|^2 d\Phi_3 \\ &= 32e^2 e_q^2 g_S^2 \frac{x_1^2 + x_2^2}{(1-x_1)(1-x_2)} \frac{Q}{32(2\pi)^3} dx_1 dx_3 \\ &= \frac{2e_q^2 \alpha \alpha_S}{\pi} Q \frac{x_1^2 + x_2^2}{(1-x_1)(1-x_2)} dx_1 dx_2 \end{aligned} \quad (2.30)$$

Using the cross section from the previous calculation, we can write this as:

$$\frac{1}{\sigma_0} \frac{d^2\sigma}{dx_1 dx_2} = \frac{2e_q^2 \alpha \alpha_S Q}{\pi} \frac{1}{3\alpha e_q^2 Q} = \frac{2\alpha_S}{3\pi} \frac{x_1^2 + x_2^2}{(1-x_1)(1-x_2)} \quad (2.31)$$

2.2.2 Splitting cross sections

The whole purpose of this section was to show this last result, the differential cross section of a $q\bar{q}$ pair radiating off a gluon. In QCD, however, being a non-Abelian gauge theory, there are two other radiating possibilities: a qg or a $\bar{q}g$ pair radiation off a second gluon or a gg pair radiating off a third gluon. These cross sections are increasingly complicated, but not impossible, to calculate. We will refrain from doing so, and instead give the final results for all three possibilities here for further reference. Unfortunately, different reference sources use different conventions for labeling the partons. In deriving the cross sections, it is commonplace to first talk about p_1 and p_2 and therefore about x_1 and x_2 and then introduce the first emission as x_3 . However, when we will talk about dipoles (Chapter 6), the original partons will be on the 'outside' with the new emitted

parton 'in between' them, so then it makes more sense to talk about x_1 and x_3 as the original partons and x_2 as the new one 'in between'. To avoid confusion when cross checking references, I will from here on use this last convention.

$$\begin{aligned}\frac{1}{\sigma_0} \frac{d^2\sigma_{qq}}{dx_1 dx_3} &= \frac{2\alpha_S}{3\pi} \frac{x_1^2 + x_3^2}{(1-x_1)(1-x_3)} \\ \frac{1}{\sigma_0} \frac{d^2\sigma_{qg}}{dx_1 dx_3} &= \frac{3\alpha_S}{4\pi} \frac{x_1^3 + x_3^3}{(1-x_1)(1-x_3)} \\ \frac{1}{\sigma_0} \frac{d^2\sigma_{gg}}{dx_1 dx_3} &= \frac{3\alpha_S}{4\pi} \frac{x_1^3 + x_3^3}{(1-x_1)(1-x_3)}\end{aligned}\tag{2.32}$$

2.3 Transverse momentum

In the next section, we will want to make a transformation to light cone variables. In order to do this, we need an expression of the transverse momentum squared (k_\perp^2) in terms of x_i .

Consider $p_A + p_B \rightarrow k + k'$. p_A and p_B are along the beam line. We have the Mandellstam variables:

$$s = (p_A + p_B)^2 \approx 2p_A \cdot p_B \tag{2.33}$$

$$t = (p_A + k)^2 \approx 2p_A \cdot k \tag{2.34}$$

$$u = (p_B + k)^2 \approx 2p_B \cdot k \tag{2.35}$$

and therefore we have:

$$p_A = \frac{\sqrt{s}}{2}(1, 0, 0, 1) \tag{2.36}$$

$$p_B = \frac{\sqrt{s}}{2}(1, 0, 0, -1) \tag{2.37}$$

$$k = (\omega, k_T, k_z) \tag{2.38}$$

For the t and u Mandellstam variables we then get:

$$t \approx 2p_A \cdot k = \sqrt{s}(\omega - k_z) \tag{2.39}$$

$$u \approx 2p_B \cdot k = \sqrt{s}(\omega + k_z) \tag{2.40}$$

Since k is considered on-shell $k^2 = \omega^2 - k_T^2 - k_z^2 = 0$ and therefore $k_T^2 = \omega^2 - k_z^2$, we conclude that

$$k_T^2 = \frac{tu}{s} \tag{2.41}$$

Now instead consider the process $p_A + p_B \rightarrow p_1 + p_3 + k$. If we assume that $k \ll p_1, p_3$, so a soft emission, then we can assume that p_1 and p_3 are almost back-to-back. In this way we can redefine k_T as being transverse to this axis and defining k as p_2 and

$$s_{ij} = (p_i + p_j)^2 \tag{2.42}$$

$$s = s_{ijk} = (p_1 + p_2 + k)^2 \tag{2.43}$$

we can therefore write this as:

$$k_T^2 = \frac{s_{12}s_{23}}{s} \tag{2.44}$$

Use the x_i -variable from the previous sections:

$$\begin{aligned}x_i &= \frac{2E_i}{\sqrt{s}}, \\ \sum_{i=1}^3 x_i &= 2\end{aligned}\tag{2.45}$$

This way we can write:

$$s_{ij} = (p_i + p_j)^2 = ((p_1 + p_2 + p_3) - p_k)^2 = s - 2(p_1 + p_2 + p_3) \cdot p_k \quad (2.46)$$

In the CM frame $\vec{p}_1 + \vec{p}_3 + \vec{k} = 0$, so $(p_1 + p_2 + p_3) \cdot p_k = \sqrt{s}E_k$, so

$$s_{ij} = s - 2\sqrt{s}E_k = s\left(1 - \frac{2E_k}{\sqrt{s}}\right) = s(1 - x_k) \quad (2.47)$$

This means we can write the transverse momentum as:

$$k_T^2 = \frac{s_{12}s_{23}}{s} = s(1 - x_1)(1 - x_3) \quad (2.48)$$

Which will be useful later on.

2.4 Coordinate transformation to light cone variables

The differential cross section for the emission of a soft gluon can also be written as:

$$d\sigma \approx \frac{\alpha_S}{4\pi^2} N_C \frac{dk_{\perp}^2}{k_{\perp}^2} dy d\phi \quad (2.49)$$

To show this, we want to do a coordinate transformation to transverse momentum and rapidity. In the previous chapter, transverse momentum was proven to be equation (2.48):

$$k_{\perp}^2 = S_{\text{dip}}(1 - x_1)(1 - x_3) \quad (2.50)$$

Rapidity is defined as

$$y = \frac{1}{2} \ln \left(\frac{1 - x_1}{1 - x_3} \right) \quad (2.51)$$

To obtain the Jacobian for this transformation, we need the derivatives:

$$\frac{\partial}{\partial x_1} k_{\perp}^2 = -S(1 - x_3), \quad \frac{\partial}{\partial x_1} y = \frac{1}{2} \frac{-1}{1 - x_1} \quad (2.52)$$

$$\frac{\partial}{\partial x_3} k_{\perp}^2 = -S(1 - x_1), \quad \frac{\partial}{\partial x_3} y = \frac{1}{2} \frac{-1}{1 - x_3} \quad (2.53)$$

Which gives for the Jacobian:

$$\frac{\partial(k_{\perp}^2, y)}{\partial(x_1, x_3)} = \left| -S(1 - x_3) \frac{1}{2} \frac{1}{1 - x_3} - \left(-S(1 - x_1) \frac{1}{2} \frac{-1}{1 - x_1} \right) \right| = S \quad (2.54)$$

This S cancels the S from the $\frac{1}{k_{\perp}^2}$ factor. Integration over ϕ gives an extra factor 2π . Hence we end up with:

$$d\sigma \approx \frac{\alpha_S}{2\pi} N_C \frac{dx_1 dx_3}{(1 - x_1)(1 - x_3)} \quad (2.55)$$

In the limit that $E_2 \ll E_1, E_3$, we recover the Born diagram situation in which the two original partons are back-to-back. This means that $E_1 \approx E_3$. The relation $x_1 + x_2 + x_3 = 2$ is equal to $\frac{2}{\sqrt{s}}(E_1 + E_2 + E_3) = 2$ and reduces in this limit to $2E = \sqrt{s}$ and therefore $E^2 = \frac{s}{4}$. Now we can say that $x_1^2 + x_3^2 = \frac{4}{s}(E_1^2 + E_3^2) = \frac{4}{s} \frac{s}{2} = 2$. Therefore we conclude that in the soft limit, the numerator is a constant numerical value. Although we will not prove it here, it turns out that all three cross sections given in (2.32) in this limit reduce to a constant. Therefore, in this limit, we can approximate the differential cross sections for all processes given by (2.32) as:

$$\frac{1}{\sigma_0} \frac{d^2\sigma}{dk_{\perp}^2 dy} \propto \frac{1}{k_{\perp}^2} \quad (2.56)$$

Chapter 3

Deep inelastic scattering

This chapter and the next one follow several chapters from *QCD and Collider Physics* by R.K. Ellis, W.J. Stirling and B.R. Webber[4]. Up until now, we have assumed the quarks to be free particles, not so different from leptons, and for all calculations we used perturbative QCD. But quarks are not free, they are confined to hadrons and mesons. Most relevant for us in the LHC era, they can be found in protons. We need to probe the protons at a higher energy level to be able to see the protons, in other words we have to look 'deep' into the proton. Since only one constituent of the proton takes part in the reaction, the proton is destroyed in the process, something we call inelastic. Hence the name Deep inelastic scattering (DIS).

3.1 Deep inelastic scattering

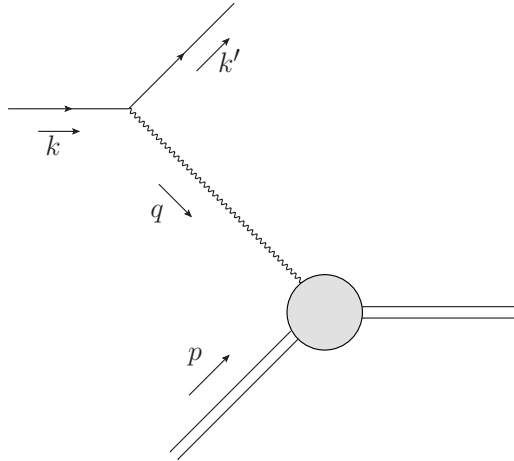


Figure 3.1: Deep Inelastic Scattering variables.

Consider an electron interacting with a quark from a proton, like in figure 3.1. The Deep

Inelastic Scattering variables are then defined as:

$$M^2 = p^2 \quad (3.1)$$

$$\nu = p \cdot q = M(E' - E) \quad (3.2)$$

$$x = \frac{Q^2}{2\nu} = \frac{Q^2}{2M(E' - E)} \quad (3.3)$$

$$y = \frac{q \cdot p}{k \cdot p} = 1 - \frac{E'}{E} \quad (3.4)$$

The differential scattering cross sections, which we will derive in the next section for this process are.

$$\frac{d^2\sigma^{em}}{dxdy} = \frac{8\pi\alpha^2 ME}{Q^4} \left[\left(\frac{1 + (1-y)^2}{2} \right) 2xF_1^{em} + (1-y)(F_2^{em} - 2xF_1^{em}) - \frac{M}{2E}xyF_2^{em} \right] \quad (3.5)$$

and

$$\frac{d^2\sigma^{\nu(\bar{\nu})}}{dxdy} = \frac{G_F^2 ME}{\pi} \left[\left(1 - y - \frac{M}{2E}xy\right)F_2^{\nu(\bar{\nu})} + y^2xF_1^{\nu(\bar{\nu})} + (-)y\left(1 - \frac{1}{2}y\right)xF_3^{\nu(\bar{\nu})} \right]. \quad (3.6)$$

Let us rewrite equation (3.5) as a differential to Q^2 instead of y . For this purpose, take equations (3.3) and (3.4) to express Q^2 as a function of y :

$$Q^2 = 2EMxy \quad (3.7)$$

The chain rule gives

$$\frac{d^2\sigma^{em}}{dxdy} = \frac{d^2\sigma^{em}}{dxdQ^2} \frac{dQ^2}{dy} = \frac{d^2\sigma^{em}}{dxdQ^2} 2EMx \quad (3.8)$$

Therefore, divide equation (3.5) by $2EMx$ to get the following equation, plus a term proportional to M , which we neglect because $M \ll P$

$$\frac{d^2\sigma^{em}}{dxdQ^2} = \frac{4\pi\alpha^2}{Q^4} \left[[1 + (1-y)^2]F_1 + \frac{(1-y)}{x}(F_2 - 2xF_1) \right] \quad (3.9)$$

The spin averaged matrix element squared for massless $e^-(k) + q(p_q) \rightarrow e^-(k') + q(p'_q)$ is equal to that of $e^-(k) + q(p_q) \rightarrow e^-(k') + q(p'_q)$ through crossing. Define Mandelstam variables as usual:

$$\hat{s} = (k + p_q)^2 \quad (3.10)$$

$$\hat{t} = (k - k')^2 \quad (3.11)$$

$$\hat{u} = (p_q - k')^2 \quad (3.12)$$

Now we want to express this into DIS variables. Obviously, \hat{t} equals $-Q^2$. \hat{s} and \hat{u} are a little less trivial:

$$\hat{s} = (k + p_q)^2 = k^2 + p_q^2 + 2k \cdot p_q \quad (3.13)$$

In the limit $M \ll P$ the first two terms, masses squared, are zero. Now using equation (3.2) and (3.3) and defining $p_q = \xi p$ we get:

$$\hat{s} = 2k \cdot p_q = 2\xi k \cdot p = \frac{2\xi q \cdot p}{y} = \frac{2\xi Q^2 q \cdot p}{2xy\nu} = \frac{\xi Q^2}{xy} \quad (3.14)$$

Now for \hat{u} , again omitting the mass squared terms:

$$\begin{aligned} \hat{u} &= (p_q - k')^2 = -2p_q \cdot k' = 2\xi p \cdot (q - k) = \xi [2p \cdot q - 2p \cdot k] \\ &= \xi \left[\frac{Q^2}{x} - \frac{Q^2}{xy} \right] = \frac{Q^2}{xy} (y - 1) = \hat{s}(y - 1) \end{aligned} \quad (3.15)$$

The amplitude for this process is

$$\sum_{\bar{}} |M|^2 = 2e_q^2 e^4 \frac{\hat{s}^2 + \hat{u}^2}{\hat{t}^2} \quad (3.16)$$

The differential cross section for massless $2 \rightarrow 2$ scattering is:

$$\frac{d\hat{\sigma}}{d\hat{t}} = \frac{1}{16\pi\hat{s}^2} \sum_{\bar{}} |M|^2 = \frac{e_q^2 e^4}{8\pi\hat{s}^2} \frac{\hat{s}^2 + \hat{u}^2}{\hat{t}^2} = \frac{e_q^2 e^4}{8\pi} \frac{1 + (y-1)^2}{(Q^2)^2}. \quad (3.17)$$

Therefore, using $e^2 = 4\pi\alpha$, we end up with:

$$\frac{d\hat{\sigma}}{dQ^2} = \frac{2\pi\alpha^2 e_q^2}{Q^4} [1 + (1-y)^2] \quad (3.18)$$

For the outgoing quark momentum we have:

$$p'_q = (p_q + q)^2 = q^2 + 2p_q \cdot q = -2p \cdot q \left(\frac{-q^2}{2p \cdot q} - \xi \right) = -2p \cdot q(x - \xi). \quad (3.19)$$

Constraining this quark to its mass shell, $(p'_q)^2 = 0$, it follows that x equals ξ , a relation which we can write like $\int_0^1 dx \delta(x - \xi)$ to easily incorporate into the differential cross section given in (3.18), which we can now write as:

$$\frac{d^2\hat{\sigma}}{dx dQ^2} = \frac{4\pi\alpha^2}{Q^4} [1 + (1-y)^2] \frac{1}{2} e_q^2 \delta(x - \xi) \quad (3.20)$$

Comparing this with (3.9), we can conclude that in this model the structure functions are related via:

$$\hat{F}_2 = x e_q^2 \delta(x - \xi) = 2x \hat{F}_1 \quad (3.21)$$

Experimental data shows that rather than a delta function, the structure function \hat{F}_2 is a distribution in x . This means that the constituents carry a distribution of momenta. This leads to the 'naive parton model', which assumes a probability distribution $q(\xi)$. The quark structure functions are now functions of x and weighted by this distribution:

$$F_2(x) = 2xF_1(x) = \sum_{q,\bar{q}} \int_0^1 d\xi q(\xi) x e_q^2 \delta(x - \xi) = \sum_{q,\bar{q}} e_q^2 x q(x) \quad (3.22)$$

With four quark flavours in the case of a charged lepton scattering off a proton ($lp \rightarrow lX$), the expression for the structure function thus becomes:

$$F_2^{em}(x) = x \left[\frac{4}{9} (u(x) + \bar{u}(x) + c(x) + \bar{c}(x)) + \frac{1}{9} (d(x) + \bar{d}(x) + s(x) + \bar{s}(x)) \right] \quad (3.23)$$

Including the Z-boson generalises this picture to the complete neutral current exchange for $e^- p \rightarrow e^- X$. In the end, this leads to the idea of the proton consisting of valence quarks and an infinite sea of light $q\bar{q}$ pairs. Probing a proton with an energy Q^2 allows for sea quark flavours with masses $m_q \ll Q$. For example, around 1 GeV, there would be:

$$u(x) = u_V(x) + S(x) \quad (3.24)$$

$$d(x) = d_V(x) + S(x) \quad (3.25)$$

$$S(x) = \bar{u}(x) = \bar{d}(x) = s(x) = \bar{s}(x) \quad (3.26)$$

This is of course a simplified picture, in which the sea $S(x)$ is SU(3) flavour symmetric. There are three valence quarks in a proton, two up and one down type. Therefore, the integral of the

respective distribution functions should equal 2 and 1.

$$\int_0^1 dx u_V(x) = 2 \quad (3.27)$$

$$\int_0^1 dx d_V(x) = 1 \quad (3.28)$$

Integrating over the possible momentum fractions weighted with the sum of all quark flavour distribution functions (the two valence quark distributions and the six possible sea quark flavours) gives a measurable quantity, which turns out to be around 0.5:

$$\sum_q \int_0^1 x [q(x) + \bar{q}(x)] = \int_0^1 x [u_V(x) + d_V(x) + 6S(x)] \approx 0.5 \quad (3.29)$$

This means that the quarks carry about 50% of the momentum of the proton. The other half is carried by the gluons.

3.2 Parton model from field theory

The scattering proces we discussed so far has the following field theory amplitude:

$$A = e\bar{u}(k')\gamma^\alpha u(k) \frac{1}{q^2} \langle X | j_\alpha(0) | P \rangle \quad (3.30)$$

This amplitude can be factorized into a leptonic and a hadronic part:

$$\frac{d^2\sigma}{dx dy} \propto L_{\alpha\beta} W^{\alpha\beta} \quad (3.31)$$

The leptonic part is given by:

$$L_{\alpha\beta} = e^2 \text{Tr}[\not{k}' \gamma_\alpha \not{k} \gamma_\beta] = 4e^2 (k_\alpha k'_\beta + k_\beta k'_\alpha + g_{\alpha\beta} k \cdot k') \quad (3.32)$$

The hadronic tensor is given by:

$$\begin{aligned} W_{\alpha\beta}(p, q) &= \frac{1}{4\pi} \sum_X \langle P | j_\beta^\dagger | X \rangle \langle X | j_\alpha(0) | P \rangle (2\pi)^4 \delta^4(q + p - p_X) \\ &= \frac{1}{4\pi} \int dz e^{iz \cdot (q + p - p_X)} \sum_X \langle P | j_\beta^\dagger | X \rangle \langle X | j_\alpha(0) | P \rangle \\ &= \frac{1}{4\pi} \int dz e^{iz \cdot q} \sum_X \langle P | e^{iz \cdot p} j_\beta^\dagger e^{-iz \cdot p_X} | X \rangle \langle X | j_\alpha(0) | P \rangle \\ &= \frac{1}{4\pi} \int dz e^{iz \cdot q} \sum_X \langle P | e^{iz \cdot \hat{p}} j_\beta^\dagger (e^{iz \cdot \hat{p}})^\dagger | X \rangle \langle X | j_\alpha(0) | P \rangle \\ &= \frac{1}{4\pi} \int d^4 z e^{iq \cdot z} \langle P | j_\beta^\dagger j_\alpha(0) | P \rangle \\ &= \frac{1}{4\pi} \int d^4 z e^{iq \cdot z} \langle P | [j_\beta^\dagger(z), j_\alpha(0)] | P \rangle \end{aligned} \quad (3.33)$$

In the second line we replaced the delta function by its integral form. In the fourth line we used the fact that the eigenvalues of \hat{p} acting on the states $|P\rangle$ and $|X\rangle$ are p and p_X respectively. In the fifth line we used the completeness of X . In the last line the second term of the commutator is zero, since j_β^\dagger is a lowering operator which yields zero when acting on $|P\rangle$ because this is already the ground state of the proton.

$W^{\alpha\beta}$ is symmetric in its indices. We therefore write down the most general form of a symmetric tensor with p and q as variables.

$$W^{\alpha\beta} = Ag^{\alpha\beta} + Bq^\alpha q^\beta + C(q^\alpha p^\beta + p^\alpha q^\beta) + Dp^\alpha p^\beta \quad (3.34)$$

We can use the fact that the current is conserved, $q \cdot W = 0$, to reduce the number of coefficients:

$$W^{\alpha\beta} q_\beta = Aq^\alpha + Bq^2 q^\alpha + C(q^\alpha(p \cdot q) + p^\alpha q^2) + D(p \cdot q)p^\alpha = 0 \quad (3.35)$$

Now since p and q are independent variables, we get two homogeneous equations, which can be combined to be left with only two unknown coefficients:

$$\begin{aligned} A + Bq^2 + C(q \cdot p) &= 0 \\ Cq^2 + D(p \cdot q) &= 0 \Rightarrow C = -D \frac{p \cdot q}{q^2} \\ \Rightarrow A + Bq^2 - D \left(\frac{p \cdot q}{q^2} \right)^2 &= 0 \\ \Rightarrow B &= \frac{1}{q^2} \left[D \left(\frac{p \cdot q}{q} \right)^2 - A \right] \end{aligned} \quad (3.36)$$

So $W^{\alpha\beta}$ becomes, using relation (3.3) to substitute x in the third line:

$$\begin{aligned} Ag^{\alpha\beta} + \frac{1}{q^2} \left[D \left(\frac{p \cdot q}{q^2} \right)^2 - A \right] q^\alpha q^\beta - D \frac{p \cdot q}{q^2} (q^\alpha p^\beta + p^\alpha q^\beta) + Dp^\alpha p^\beta = \\ A \left[g^{\alpha\beta} - \frac{q^\alpha q^\beta}{q^2} \right] + D \left[\left(\frac{p \cdot q}{q^2} \right)^2 q^\alpha q^\beta - \left(\frac{p \cdot q}{q^2} \right) (q^\alpha p^\beta + p^\alpha q^\beta) + p^\alpha p^\beta \right] = \\ A \left[g^{\alpha\beta} - \frac{q^\alpha q^\beta}{q^2} \right] + D \left[\left(\frac{1}{2x} \right)^2 q^\alpha q^\beta + \left(\frac{1}{2x} \right) (q^\alpha p^\beta + p^\alpha q^\beta) + p^\alpha p^\beta \right] \end{aligned} \quad (3.37)$$

Renaming A and D as W_1 and W_2 we end up with

$$W^{\alpha\beta}(p, q) = \left(g^{\alpha\beta} - \frac{q^\alpha q^\beta}{q^2} \right) W_1(x, Q^2) + \left(p^\alpha + \frac{1}{2x} q^\alpha \right) \left(p^\beta + \frac{1}{2x} q^\beta \right) W_2(x, Q^2) \quad (3.38)$$

Filling this in into equation (3.31), we get

$$L_{\alpha\beta} W^{\alpha\beta} = 4e^2 (k_\alpha k'_\beta + k_\beta k'_\alpha - g_{\alpha\beta} k \cdot k') \times \quad (3.39)$$

$$\times \left[\left(g^{\alpha\beta} - \frac{q^\alpha q^\beta}{q^2} \right) W_1(x, Q^2) + \left(p^\alpha + \frac{1}{2x} q^\alpha \right) \left(p^\beta + \frac{1}{2x} q^\beta \right) W_2(x, Q^2) \right] \quad (3.40)$$

$$= W_1 \left[-k \cdot k' - \frac{2(q \cdot k)(q \cdot k')}{q^2} \right] + \quad (3.41)$$

$$+ W_2 \left[2 \left(p \cdot k + \frac{1}{2x} q \cdot k \right) \left(p \cdot k' + \frac{1}{2x} q \cdot k' \right) + k \cdot k' \left(p + \frac{1}{2x} q \right)^2 \right] \quad (3.42)$$

Which can be worked out to give equation (3.5). Introduce two lightlike vectors p and n with $n \cdot p = 1$

$$k^\mu = ap^\mu + bn^\mu + k_T^\mu \quad (3.43)$$

Now we have:

$$p^2 = n^2 = n \cdot k_T = p \cdot k_T = 0 \quad (3.44)$$

$$q^\mu = \nu n^\mu + q_T^\mu \Rightarrow q \cdot n = \nu n^2 + q_T \cdot n = 0 \quad (3.45)$$

Hence the contraction of $W_{\alpha\beta}$ with n^α and n^β projects out W_2 :

$$\begin{aligned} n^\alpha n^\beta W_{\alpha\beta} &= n^\alpha \left[(n_\alpha - q_\alpha) W_1 + \left(p_\alpha + \frac{1}{2x} q_\alpha \right) \left(p \cdot n + \frac{1}{2x} q \cdot n \right) \right] \\ &= (n^2 - q \cdot n) W_1 + \left(p \cdot n + \frac{1}{2x} q \cdot n \right)^2 W_2 \\ &= W_2 \end{aligned} \quad (3.46)$$

Contracting with $\frac{4x^2}{\nu} p^\alpha p^\beta$ gives F_L :

$$\begin{aligned} \frac{4x^2}{\nu} p^\alpha p^\beta W_{\alpha\beta} &= \frac{4x^2}{\nu} p^\alpha p^\beta \left(p^2 - \frac{(q \cdot p)^2}{q^2} \right) W_1 + \left(p^2 + \frac{1}{2x} q \cdot p \right)^2 W_2 \\ &= \frac{4x^2}{\nu} p^\alpha p^\beta \left(-\frac{\nu^2}{q^2} W_1 + \frac{\nu^2}{4x^2} W_2 \right) \\ &= \nu W_2 - 2x W_1 = F_2 - 2x F_1 = F_L \end{aligned} \quad (3.47)$$

The assumption is now that photons scatter incoherently off the partons. The hadronic tensor W is in that case obtained via the so-called 'handbag diagram':

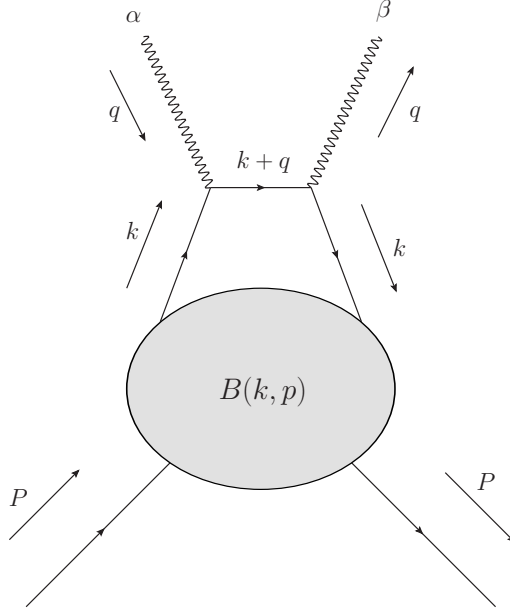


Figure 3.2: Handbag diagram.

$$W^{\alpha\beta}(p, q) = e_q^2 \int \frac{d^4 k}{(2\pi)^4} [\gamma^\alpha (\not{k} + \not{q}) \gamma^\beta]_{ij} [B(k, p)]_{ji} \delta((k+q)^2) \quad (3.48)$$

In the naive parton model, the quarks carry a fraction ξ of the proton momentum. It would therefore not make sense for the transverse momenta to be large, therefore the quark momentum should more or less equal this fraction, $k^\mu \approx \xi p^\mu$. Equation (3.43) therefore becomes:

$$k^\mu = \xi p^\mu + b n^\mu + k_T^\mu \quad (3.49)$$

Hence

$$\begin{aligned}
k^2 &= \xi^2 p^2 + b^2 n^2 + k_T^2 + 2\xi b p \cdot n + 2\xi p \cdot k_T + 2bn \cdot k_T = k_T^2 + 2\xi b \Rightarrow \\
b &= \frac{k^2 - k_T^2}{2\xi}
\end{aligned} \tag{3.50}$$

This gives for the decomposition of the quark four-momentum k :

$$k^\mu = \xi p^\mu + \frac{k^2 + k_T^2}{2\xi} n^\mu + k_T^\mu \tag{3.51}$$

The delta-function in (3.48) can be simplified when considering the assumption of the parton model that the amplitude $B(k, p)$ is strongly damped when the transverse momentum k_T^2 and virtuality k^2 are large. In that case we get:

$$\begin{aligned}
&\delta((k+q)^2) \\
&= \delta(k^2 + q^2 + 2k \cdot q) \\
&= \delta(k^2 + q^2 + 2\xi p \cdot q + 2\frac{k^2 + k_T^2}{2\xi} n \cdot q + 2k_T \cdot q) \\
&= \delta(k^2 + 2\xi \nu - 2k_T \cdot q_T) \\
&\approx \delta(2\xi \nu - Q^2) \\
&= \frac{1}{2\nu} \delta(\xi - x)
\end{aligned} \tag{3.52}$$

Some arithmetic:

$$\begin{aligned}
\not{q} \not{b} \not{q} &= a_\mu b_\nu a_\rho \gamma^\mu \gamma^\nu \gamma^\rho \\
&= a_\mu b_\nu a_\rho (2g^{\mu\nu} - \gamma^\nu \gamma^\mu) \gamma^\rho \\
&= 2(a \cdot b) \not{q} - a_\mu b_\nu a_\rho \gamma^\nu \frac{1}{2} (\gamma^\mu \gamma^\rho + \gamma^\rho \gamma^\mu) \\
&= 2(a \cdot b) \not{q} - a_\mu b_\nu a_\rho \gamma^\nu g^{\mu\rho} \\
&= 2(a \cdot b) \not{q} - a^2 \not{b}
\end{aligned} \tag{3.53}$$

Using (3.52) and (3.53), we can simplify expression (3.48):

$$\begin{aligned}
\nu W_2 &= \nu n_\alpha n_\beta W^{\alpha\beta} = \frac{e_q^2}{2} \int \frac{d^4 k}{(2\pi)^4} [\not{n}(\not{k} + \not{q})\not{n}]_{ij} B_{ji}(k, p) \delta(\xi - x) \\
&= \frac{e_q^2}{2} \int \frac{d^4 k}{(2\pi)^4} [2\not{n}(k \cdot n + q \cdot n) + n^2(\not{k} + \not{q})]_{ij} B_{ji}(k, p) \delta(\xi - x) \\
&= \frac{e_q^2}{2} (2\xi) \int \frac{d^4 k}{(2\pi)^4} Tr [\not{n} B(k, p)] \delta(\xi - x) \\
&= e_q^2 x q(x)
\end{aligned} \tag{3.54}$$

Here $q(x)$ is the quark distribution,

$$q(x) = \int \frac{d^4 k}{(2\pi)^4} Tr [\not{n} B(k, p)] \delta(n \cdot k - x) \tag{3.55}$$

There is no dependence on Q^2 anymore, the structure function only depends on the dimensionless variable x . This means that the structure function scales:

$$F_2(x, Q^2) \rightarrow F_2(x) \tag{3.56}$$

Now project out F_L :

$$\begin{aligned}
\frac{4x^2}{\nu} p_\alpha p_\beta W^{\alpha\beta} &= \frac{4x^2}{\nu} e_q^2 \int \frac{d^4 k}{(2\pi)^4} \text{Tr} [\{\not{p}(\not{k} + \not{q})\not{p}\} B(p, q)] \frac{1}{2\nu} \delta(\xi - x) \\
&= \frac{2x^2}{\nu^2} e_q^2 \int \frac{d^4 k}{(2\pi)^4} \text{Tr} [\{2\not{p}(p \cdot k + p \cdot q) + p^2(\not{k} + \not{q})\} B(p, q)] \delta(\xi - x) \\
&= \frac{4x^2}{\nu^2} e_q^2 \int \frac{d^4 k}{(2\pi)^4} \text{Tr} [\{\not{p}(0 + \nu) + 0\} B(p, q)] \delta(\xi - x) \\
&= \frac{4x^2}{\nu} e_q^2 \int \frac{d^4 k}{(2\pi)^4} \text{Tr} [\not{p} B(p, q)] \delta(\xi - x)
\end{aligned} \tag{3.57}$$

This shows that $F_L = O(\nu^{-1})$, which becomes zero in the Bjorken limit. This means we still end up with the Callan-Gross relation: $F_2 = 2xF_1$.

3.3 The parton model and QCD

3.3.1 Leading order

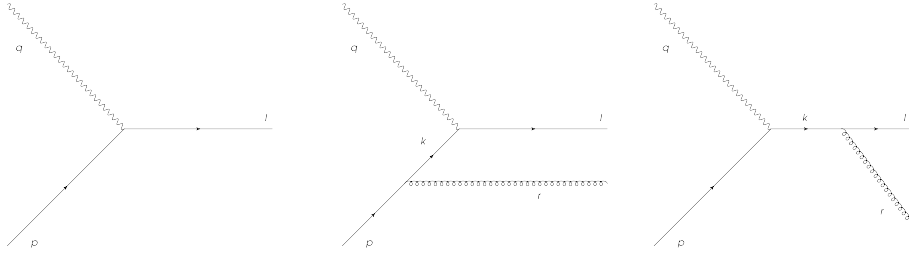


Figure 3.3: The leading order (left) and the two next-to-leading order diagrams of the scattering of a virtual photon off a free quark.

Let us first consider the scattering of a virtual photon off a free quark with momentum p , the left diagram of figure 3.3. This means setting ξ to one in our previous formulas.

$$\gamma^* + q(p) \rightarrow q(l) \tag{3.58}$$

The invariant matrix element of this process is:

$$M^\alpha = -ie_q \bar{u}(l) \gamma^\alpha u(p) \tag{3.59}$$

Now squaring, summing and averaging over spins and colours and project out again F_2 , using the identity from (3.53):

$$\begin{aligned}
n^\alpha n^\beta \sum_{\text{spins, colours}} |M|_{\alpha\beta}^2 &= n^\alpha n^\beta \sum_{\text{spins, colours}} e_q^2 \bar{u}(p) \gamma_\beta u(l) \bar{u}(l) \gamma_\alpha u(p) \\
&= \frac{1}{2} e_q^2 n^\alpha n^\beta \text{Tr} [\gamma_\beta \not{l} \gamma_\alpha \not{p}] \\
&= \frac{e_q^2}{2} \text{Tr} [l \not{\eta} \not{p} \not{\eta}] \\
&= \frac{e_q^2}{2} \text{Tr} [l (2(n \cdot p) \not{\eta} + n^2 \not{p})] \\
&= e_q^2 \text{Tr} [l \not{\eta}] \\
&= 4e_q^2 l \cdot n \\
&= 4e_q^2
\end{aligned} \tag{3.60}$$

The phase space is, using (3.52) with $\xi = 1$ since we are now dealing with a single quark (indicated by the hat):

$$d\Phi_1 = 2\pi\delta((p+q)^2) = \frac{\pi}{\nu}\delta(1-x) \quad (3.61)$$

This combined together with the factor $\frac{1}{4\pi}$ that we got as flux factor from equation (3.33) gives:

$$\hat{F}_2(x) = e_0^2\delta(1-\xi) \quad (3.62)$$

3.3.2 Next-to-leading order

Now consider the process in which the quark emits a gluon, the next-to-leading order (NLO) diagram:

$$\gamma^*(q) + q(p) \rightarrow g(r) + q(l) \quad (3.63)$$

The two NLO diagrams are the two on the right of figure 3.3. Let us first consider the one in which it is the incoming quark that emits the gluon. We assign the incoming quark the momentum p , the incoming photon the momentum q , the outgoing quark line l , the gluon line the momentum r and the internal line the momentum k . In this case we have the relations $r = p - k$ and $l = q + k$. To get the phase space factor, we have to integrate over all internal momenta. Obviously, k, r and l are not independent, so it will suffice to just integrate over r and l : $\int \frac{d^4r}{(2\pi)^4}$ and $\int \frac{d^4l}{(2\pi)^4}$. Since they are assigned to external lines in the original $\gamma^*(q) + q(p) \rightarrow q(l) + g(r)$ process, r and l should be on shell. This introduces two delta functions in the phase space factor: $(2\pi)\delta^+(r^2)$ and $(2\pi)\delta^+(l^2)$. Finally, there should be momentum conservation, $p + k = r + l$, or in a delta function: $(2\pi)^4\delta(p + k - r - l)$. Putting this all together we get:

$$d\Phi_2 = \int \frac{d^4r}{(2\pi)^3} \frac{d^4l}{(2\pi)^3} \delta^+(r^2)\delta^+(l^2)(2\pi)^4\delta^4(p + q - r - l) \quad (3.64)$$

Performing one of the integrals and noting that, again because of the dependence between the internal momenta, an integral over r can be replaced by an integral over k , we end up with:

$$\begin{aligned} d\Phi_2 &= \frac{1}{(2\pi)^2} \int d^4r \delta^+(r^2)\delta^+((p+q-r)^2) \\ &= \frac{1}{4\pi^2} \int d^4k \delta^+((p-k)^2)\delta^+((k+q)^2) \end{aligned} \quad (3.65)$$

We again want to write the four-vector k^μ in terms of n , p and k_T^μ . Identifying $k^2 = -|k^2|$, we write:

$$k^\mu = \xi p^\mu + \frac{k_T^2 - |k^2|}{2\xi} n^\mu + k_T^\mu \quad (3.66)$$

A simple base in this case is the choice of p and n as four-momenta along the z-axis:

$$p = (p, 0, 0, p) \quad (3.67)$$

$$n = \left(\frac{1}{2p}, 0, 0, \frac{1}{2p} \right) \quad (3.68)$$

In that case the transverse momentum k_T^μ has only a x and y component:

$$k_T^\mu = \left(0, k_T^{(1)}, k_T^{(2)}, 0 \right) \quad (3.69)$$

The total expression for k therefore becomes:

$$k = \left(\xi p + \frac{k_T^2 + k^2}{2\xi} \frac{1}{2p}, k_T^{(1)}, k_T^{(2)}, \xi p - \frac{k_T^2 + k^2}{2\xi} \frac{1}{2p} \right) \quad (3.70)$$

To do a coordinate transformation from the four components of k^μ to the four variables k^2 , ξ , $k_T^{(1)}$ and $k_T^{(2)}$, we need the Jacobian of this transformation. For this purpose, we note that $k_T^2 = k_T^\mu k_{T\mu} = -(k_T^{(1)})^2 - (k_T^{(2)})^2$

$$\begin{aligned}
J &= \det\left(\frac{\partial}{\partial x_i} k^\mu\right) \text{ with } x_i \in \left\{k^2, \xi, k_T^{(1)}, k_T^{(2)}\right\} \\
&= \begin{vmatrix} \frac{1}{2\xi} \frac{1}{2p} & 0 & 0 & -\frac{1}{2\xi} \frac{1}{2p} \\ p - \frac{k_T^2 + k^2}{2\xi^2} \frac{1}{2p} & 0 & 0 & p + \frac{k_T^2 + k^2}{2\xi^2} \frac{1}{2p} \\ \frac{-2k_T^{(1)}}{2\xi} \frac{1}{2p} & 1 & 0 & \frac{2k_T^{(1)}}{2\xi} \frac{1}{2p} \\ \frac{-2k_T^{(2)}}{2\xi} \frac{1}{2p} & 0 & 1 & \frac{2k_T^{(2)}}{2\xi} \frac{1}{2p} \end{vmatrix} \\
&= \frac{1}{2\xi} \frac{1}{2p} \left(p + \frac{k_T^2 + k^2}{2\xi^2} \frac{1}{2p}\right) + \frac{1}{2\xi} \frac{1}{2p} \left(p - \frac{k_T^2 + k^2}{2\xi^2} \frac{1}{2p}\right) = \frac{1}{2\xi}
\end{aligned} \tag{3.71}$$

Now for the delta functions, we note that

$$(p - k)^2 = k^2 - 2p \cdot k = -|k^2| - 2 \left(\frac{k_T^2 - |k^2|}{2\xi} \right) = (1 - \xi) \frac{|k^2|}{\xi} - \frac{k_T^2}{\xi} \tag{3.72}$$

and that

$$\begin{aligned}
(k + q)^2 &= k^2 + q^2 + 2k \cdot q = -Q^2 - |k^2| + 2 \left(\xi p + \frac{k_T^2 - |k^2|}{2\xi} n + k_T \right) \cdot (\nu n + q_T) \\
&= 2\xi\nu - Q^2 - |k^2| - 2q_T \cdot k_T
\end{aligned} \tag{3.73}$$

This allows us to rewrite the phase space factor:

$$\begin{aligned}
d\Phi_2 &= \frac{1}{4\pi^2} \int d\xi dk^2 d^2 k_T \delta^+ \left((1 - \xi) \frac{|k^2|}{\xi} - \frac{k_T^2}{\xi} \right) \delta^+ (2\xi\nu - Q^2 - |k^2| - 2q_T \cdot k_T) \frac{1}{2\xi} \\
&= \frac{1}{8\pi^2} \int d\xi dk^2 d^2 k_T \delta^+ \left((1 - \xi) |k^2| - k_T^2 \right) \delta^+ \left(2\nu \left[\xi - \frac{Q^2}{2\nu} - \frac{|k^2| - 2q_T \cdot k_T}{2\nu} \right] \right) \\
&= \frac{1}{16\nu\pi^2} \int d\xi dk^2 d^2 k_T \delta^+ \left((1 - \xi) |k^2| - k_T^2 \right) \delta^+ \left(\xi - x - \frac{|k^2| - 2q_T \cdot k_T}{2\nu} \right)
\end{aligned} \tag{3.74}$$

The matrix element of this process is

$$M^\alpha = -ig e_q \bar{u}(l) \gamma^\alpha \frac{1}{\not{k}} \not{\epsilon} t^A u(p) \tag{3.75}$$

Squaring this matrix and averaging over colours and spins we get

$$\sum_{\text{pol}} |M|_{\alpha\beta}^2 = \frac{1}{2} e_q^2 g^2 \sum_{\text{pol}} C_F \text{Tr} \left[\gamma^\beta (\not{k} + \not{q}) \gamma^\alpha \not{k} \not{\epsilon} \not{p} \not{\epsilon}^* \not{k} \right] \frac{1}{k^4} \tag{3.76}$$

Now remember that: $n \cdot q = 0$, $n \cdot p = 1$, $n^2 = 0$ so $\not{n}(\not{p} + \not{q})\not{n} = 2\not{n}$

$r = p - k$, so $n \cdot r = n \cdot p - n \cdot k = 1 - \xi$

$\not{p}\not{p} = a_\mu a_\nu \gamma^\mu \gamma^\nu = a_\mu a_\nu \frac{1}{2} (\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu) = a_\mu a_\nu g^{\mu\nu} = a^2$, so $\not{p}\not{p} = p^2 = 0$ Using this and the projection identity,

$$\sum_{\text{pol}} \epsilon_\mu \epsilon_\nu = -g_{\mu\nu} + \frac{n_\mu r_\nu + n_\nu r_\mu}{n \cdot r}, \tag{3.77}$$

we can calculate \hat{F}^2 , by projecting it out in the usual way:

$$\begin{aligned}
\frac{1}{4\pi} n^\alpha n^\beta \sum_{\bar{}} |M|_{\alpha\beta}^2 &= \frac{1}{4\pi} n^\alpha n^\beta \frac{1}{2} e_q^2 g^2 \sum_{\text{pol}} C_F \text{Tr} [\gamma^\beta (\not{k} + \not{q}) \gamma^\alpha \not{k} \not{\epsilon} \not{p} \not{\epsilon}^* \not{k}] \frac{1}{k^4} \\
&= \frac{e_q^2 g^2 C_F}{8\pi k^4} \sum_{\text{pol}} \text{Tr} [\not{n} (\not{k} + \not{q}) \not{n} \not{k} \not{\epsilon} \not{p} \not{\epsilon}^* \not{k}] \\
&= \frac{e_q^2 g^2 C_F \xi}{4\pi k^4} \sum_{\text{pol}} \text{Tr} [\not{n} \not{k} \not{\epsilon} \not{p} \not{\epsilon}^* \not{k}] \\
&= \frac{e_q^2 g^2 C_F \xi}{4\pi k^4} \left(-g_{\mu\nu} + \frac{n_\mu r_\nu + n_\nu r_\mu}{n \cdot r} \right) \text{Tr} [\not{n} \not{k} \gamma^\mu \not{p} \gamma^\nu \not{k}] \\
&= \frac{e_q^2 g^2 C_F \xi}{4\pi k^4} \left(2\text{Tr} [\not{n} \not{k} \not{p} \not{k}] + \frac{1}{1-\xi} (\text{Tr} [\not{n} \not{k} \not{n} \not{p} \not{r} \not{k}] + \text{Tr} [\not{n} \not{k} \not{r} \not{p} \not{n} \not{k}]) \right) \\
&= \frac{e_q^2 g^2 C_F \xi}{4\pi k^4} \left(2\text{Tr} [\not{n} \not{k} \not{p} \not{k}] + \frac{2\xi}{1-\xi} (\text{Tr} [\not{n} \not{p} \not{r} \not{k}] + \text{Tr} [\not{n} \not{k} \not{r} \not{p}]) \right) \\
&= \frac{e_q^2 g^2 C_F \xi}{4\pi k^4} \left(2\text{Tr} [\not{n} \not{k} \not{p} \not{k}] + \frac{2\xi}{1-\xi} (\text{Tr} [\not{n} \not{p} \not{p} \not{k}] - \text{Tr} [\not{n} \not{p} \not{k} \not{k}] + \text{Tr} [\not{n} \not{k} \not{p} \not{p}] - \text{Tr} [\not{n} \not{k} \not{k} \not{p}]) \right) \\
&= \frac{e_q^2 g^2 C_F \xi}{4\pi k^4} \left(2\text{Tr} [\not{n} \not{k} \not{p} \not{k}] - \frac{4\xi k^2}{1-\xi} \text{Tr} [\not{n} \not{p}] \right) \\
&= \frac{e_q^2 g^2 C_F \xi}{4\pi k^4} \left(2\text{Tr} [\not{n} (2(k \cdot p) \not{k} - k^2 \not{p})] - \frac{16\xi k^2}{1-\xi} \right) \\
&= \frac{e_q^2 g^2 C_F \xi}{4\pi k^4} \left(4 \frac{k_T^2 - |k^2|}{2\xi} \text{Tr} [\not{n} \not{k}] - 2k^2 \text{Tr} [\not{n} \not{p}] - \frac{16\xi k^2}{1-\xi} \right) \\
&= \frac{e_q^2 g^2 C_F \xi}{4\pi k^4} \left(8(k_T^2 - |k^2|) - 8k^2 - \frac{16\xi k^2}{1-\xi} \right) \\
&= \frac{\alpha_S e_q^2 C_F \xi}{|k^2|^2} \left(8k_T^2 - \frac{16\xi k^2}{1-\xi} \right) \\
&= \frac{\alpha_S e_q^2 C_F \xi}{|k^2|^2} \left(8(1-\xi)|k^2| + \frac{16\xi |k^2|}{1-\xi} \right) \\
&= \frac{8\alpha_S e_q^2 C_F \xi}{|k^2|} \left(\frac{(1-\xi)^2 + 2\xi}{1-\xi} \right) \\
&= \frac{8\alpha_S e_q^2 C_F \xi}{|k^2|} \left(\frac{1+\xi^2}{1-\xi} \right)
\end{aligned} \tag{3.78}$$

Here we used the fact that the delta functions in the phase space tell us that $k_T^2 = (1-\xi)|k^2|$ and $k^2 = -|k^2|$.

In the end we are left with:

$$\frac{1}{4\pi} n^\alpha n^\beta \sum_{\bar{}} |M|_{\alpha\beta}^2 = \frac{8e_q^2 \alpha_S}{|k^2|} \xi P(\xi) \tag{3.79}$$

With the splitting function $P(\xi)$ given by:

$$P(\xi) = C_F \frac{1+\xi^2}{1-\xi} \tag{3.80}$$

Putting everything together, we get for \hat{F}_2

$$\hat{F}_2 = e_q^2 \frac{\alpha_S}{2\pi^2} \int_0^{2\nu} \frac{d|k^2|}{|k^2|} \int_{\xi_-}^{\xi_+} d\xi \frac{\xi P(\xi)}{\sqrt{(\xi_+ - \xi)(\xi - \xi_-)}} \tag{3.81}$$

with

$$\xi_{\pm}(z, x) = x + z - 2zx \pm \sqrt{4x(1-x)z(1-z)} \quad (3.82)$$

Here we introduced the variable $z = \frac{|k^2|}{2\nu}$. Note that $\xi_{\pm} \rightarrow x$ as $z \rightarrow 0$. Now using that:

$$\int_{\xi_-}^{\xi_+} d\xi \frac{1}{\sqrt{(\xi_+ - \xi)(\xi - \xi_-)}} = \pi \quad (3.83)$$

and introducing a small cut off value κ^2 in the integral of $|k^2|$, we see that the divergent term becomes

$$\hat{F}_2 \Big|_{\text{div}} = e_q^2 \frac{\alpha_S}{2\pi} x P(x) \int_{\kappa^2}^{2\nu} \frac{d|k^2|}{|k^2|} = e_q^2 \frac{\alpha_S}{2\pi} x P(x) \ln \left(\frac{2\nu}{\kappa^2} \right) \quad (3.84)$$

This is all just the calculation for the first diagram of figure 3.4. We considered this one first,

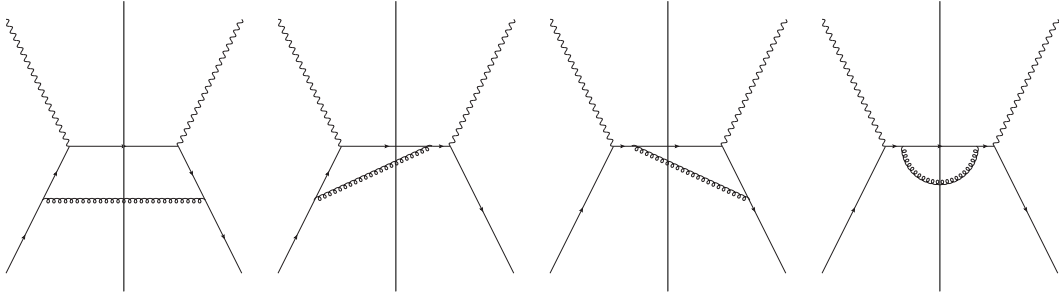


Figure 3.4: The four terms in the NLO matrix element squared.

because it is the only diagram that has two $\frac{1}{|k^2|}$ propagators. All the others have less, for instance, the second diagram has one propagator proportional to $\frac{1}{|k^2|}$ and one to $\frac{1}{(p+q)^2}$. Since there is also a factor $k_{\perp} \propto \sqrt{|k^2|}$ for each $q\bar{q}g$ vertex, all these contributions are finite. If we include the other three and the leading order diagram and use the definition of $2\nu = Q^2/x$ so $\ln(2\nu) = \ln(Q^2) - \ln x$ to rewrite this in terms of Q^2 while incorporating the x dependent part in an overall x dependent part $C(x)$, we end up with

$$\hat{F}_2(x, Q^2) = e_q^2 x \left[\delta(1-x) + \frac{\alpha_S}{2\pi} \left(P(x) \ln \frac{Q^2}{\kappa^2} + C(x) \right) \right] \quad (3.85)$$

This leads to the conclusion that beyond leading order \hat{F}_2 depends on Q^2 and has Bjorken scaling broken by logarithms of Q^2 . Hence, the quark distribution function is given by:

$$q(x, Q^2) = \delta(1-x) + \frac{\alpha_S}{2\pi} \left(P(x) \ln \frac{Q^2}{\kappa^2} + C(x) \right) \quad (3.86)$$

We will now also consider vertex corrections. Since all virtual loops have a delta-function $\delta((p+q)^2)$ in common, all adjustments to the splitting function are proportional to $\delta(1-x)$

$$P(x) \rightarrow P(x) + K\delta(1-x) \quad (3.87)$$

The right hand side should integrate to zero. Use the definition:

$$\int_0^1 \frac{f(x)}{(1-x)_+} = \int_0^1 \frac{f(x) - f(1)}{(1-x)} \quad (3.88)$$

And use as function the given splitting function $P(x) = \frac{1+x^2}{1-x}$

$$\begin{aligned}
& \int_0^1 dx \left(\frac{1+x^2}{(1-x)_+} + K\delta(1-x) \right) \\
&= \int_0^1 dx \left(\frac{(1+x^2)-2}{1-x} \right) + K \\
&= K - \int_0^1 dx \frac{(x-1)(x+1)}{(x-1)} \\
&= K - \int_0^1 dx(x+1) = K - \frac{3}{2}
\end{aligned} \tag{3.89}$$

So we conclude that $K = \frac{3}{2}$ and therefore:

$$P(x) = \frac{1+x^2}{(1-x)_+} + \frac{3}{2}\delta(1-x) \tag{3.90}$$

Now project out \hat{F}_L , using $p \cdot k = \frac{k^2+k_T^2}{2\xi} = b$:

$$\frac{x^2}{\pi\nu} p^\alpha p^\beta \sum_{\bar{}} |M|_{\alpha\beta}^2 = \frac{x^2}{\pi\nu} \frac{1}{2} e_q^2 g^2 C_F \frac{1}{k^4} Tr [\not{p}(\not{k} + \not{q})\not{p}\not{k}\not{p}\not{k}^*] \tag{3.91}$$

The trace is:

$$\begin{aligned}
& \left(-g_{\mu\nu} + \frac{n_\mu r_\nu + n_\nu r_\mu}{n \cdot r} \right) Tr [\not{p}(\not{k} + \not{q})\not{p}\not{k}\gamma^\mu \not{p}\gamma^\nu \not{k}] \\
& 2(b+\nu) \left\{ 2Tr [\not{p}\not{k}\not{p}\not{k}] + \frac{1}{1-\xi} Tr [\not{p}\not{k}\not{q}\not{p}\not{k}] + \frac{1}{1-\xi} Tr [\not{p}\not{k}\not{r}\not{p}\not{k}] \right\} \\
& 2(b+\nu) \left\{ 2Tr [2(p \cdot k)\not{p}\not{k}] + \frac{1}{1-\xi} (Tr [\not{p}\not{k}\not{q}\not{p}\not{k}] - Tr [\not{p}\not{k}\not{q}\not{p}\not{k}] + Tr [\not{p}\not{k}\not{p}\not{p}\not{k}] - Tr [\not{p}\not{k}\not{k}\not{p}\not{q}\not{k}]) \right\} \\
& 2(b+\nu) \left\{ 16b^2 + 0 - 0 + 0 - 0 \right\} = 32(b+\nu)b^2 = 32 \left(-\frac{1}{8}|k^2|^3 + \frac{1}{4}\nu|k^2|^2 \right) \\
& = -4|k^2|^3 + 8\nu|k^2|^2
\end{aligned} \tag{3.92}$$

Here we used the relation $k_T^2 = (1-\xi)|k^2|$, obtained from the phase space, and therefore $b = \frac{k_T^2 - |k^2|}{2\xi} = -\frac{1}{2}|k^2|$. Hence as total we have:

$$\frac{x^2}{\pi\nu} p^\alpha p^\beta \sum_{\bar{}} |M|_{\alpha\beta}^2 = \frac{8\alpha_S x^2}{\nu} e_q^2 C_F (2\nu - |k^2|) \tag{3.93}$$

Combining this with the phase space factor,

$$d\Phi_2 = \frac{1}{16\nu\pi} \int d\xi dk^2 dk_T^2 \delta(k_T - (1-\xi|k^2|)) \delta \left(\xi - x - \frac{|k^2| + 2q_T \cdot k_T}{2\nu} \right), \tag{3.94}$$

where we already used the delta function for k_T and since there is no ξ dependence left, we can leave out the integral over ξ we get:

$$\begin{aligned}
\hat{F}_2(x) &= -\frac{1}{16\nu\pi} \int_0^{2\nu} d|k^2| \frac{8\alpha_S x^2}{\nu} e_q^2 C_F (2\nu - |k^2|) \\
&= -\frac{\alpha_S x^2}{2\pi\nu^2} e_q^2 C_F (0 - \frac{1}{2}(2\nu)^2)
\end{aligned} \tag{3.95}$$

Here we replaced $\int dk^2$ with $-\int d|k^2|$. The final result is therefore:

$$\hat{F}_L(x, Q^2) = \frac{\alpha_S}{\pi} e_q^2 x^2 C_F \tag{3.96}$$

This is conform equation (3.85)

3.4 Refactorisation

Consider now a bare quark with momentum ξ between $x \leq \xi \leq 1$, containing quark with momentum $0 \leq \frac{x}{\xi} \leq 1$. Then we can write $F_2(x, Q^2)$ as a convolution between the bare quark probability function $q_0(\xi)$ and the structure function \hat{F}_2 of the inner quark.

$$\begin{aligned}
F_2(x, Q^2) &= \sum_{q, \bar{q}} \int_x^1 d\xi q_0(\xi) \hat{F}_2\left(\frac{x}{\xi}, Q^2\right) \\
&= \sum_{q, \bar{q}} \int_x^1 d\xi x e_q^2 q_0(\xi) \left[\xi \delta(\xi - x) + \frac{\alpha_S}{2\pi} \left\{ P\left(\frac{x}{\xi}\right) \ln \frac{Q^2}{\kappa^2} + C\left(\frac{x}{\xi}\right) \right\} + O(\alpha_S^2) \right] \quad (3.97) \\
&= x \sum_{q, \bar{q}} e_q^2 \left[q_0(x) + \frac{\alpha_S}{2\pi} \int_x^1 \frac{d\xi}{\xi} q_0(\xi) \left\{ P\left(\frac{x}{\xi}\right) \ln \frac{Q^2}{\kappa^2} + C\left(\frac{x}{\xi}\right) \right\} + O(\alpha_S^2) \right]
\end{aligned}$$

Now define a refactorized distribution function $q(x, \mu^2)$ at factorization scale μ :

$$q(x, \mu^2) = q_0(x) + \frac{\alpha_S}{2\pi} \int_x^1 \frac{d\xi}{\xi} q_0(\xi) \left[P\left(\frac{x}{\xi}\right) \ln \frac{\mu^2}{\kappa^2} + C\left(\frac{x}{\xi}\right) \right] + O(\alpha_S^2) \quad (3.98)$$

Then

$$\begin{aligned}
&\int_x^1 \frac{d\xi}{\xi} q(\xi, \mu^2) \left[\delta\left(1 - \frac{x}{\xi}\right) + \frac{\alpha_S}{2\pi} P\left(\frac{x}{\xi}\right) \ln \frac{Q^2}{\mu^2} + O(\alpha_S^2) \right] \\
&= \int_x^1 \frac{d\xi}{\xi} \left\{ q_0(\xi) + \frac{\alpha_S}{2\pi} \int_x^1 \frac{d\xi'}{\xi'} q_0(\xi') \left[P\left(\frac{\xi'}{\xi}\right) \ln \frac{\mu^2}{\kappa^2} + C\left(\frac{\xi'}{\xi}\right) \right] + O(\alpha_S^2) \right\} \times \\
&\times \left[\delta\left(1 - \frac{x}{\xi}\right) + \frac{\alpha_S}{2\pi} P\left(\frac{x}{\xi}\right) \ln \frac{Q^2}{\mu^2} + O(\alpha_S^2) \right] \quad (3.99) \\
&= q_0(x) + \frac{\alpha_S}{2\pi} \int_x^1 \frac{d\xi}{\xi} q_0(\xi) \left[P\left(\frac{x}{\xi}\right) \ln \frac{Q^2}{\mu^2} + C\left(\frac{x}{\xi}\right) \right] + \\
&+ \frac{\alpha_S}{2\pi} \int_x^1 \frac{d\xi'}{\xi'} q_0(\xi') \left[P\left(\frac{\xi'}{\xi}\right) \ln \frac{\mu^2}{\kappa^2} + C\left(\frac{\xi'}{\xi}\right) \right] + O(\alpha_S^2) \\
&= q_0(x) + \frac{\alpha_S}{2\pi} \int_x^1 \frac{d\xi}{\xi} q_0(\xi) \left[P\left(\frac{x}{\xi}\right) \ln \frac{Q^2}{\kappa^2} + C\left(\frac{x}{\xi}\right) \right] + O(\alpha_S^2)
\end{aligned}$$

These two integrals over ξ and ξ' could be combined because they are just integration variables. Using this expression in combination with (3.97), we conclude that:

$$F_2(x, Q^2) = x \sum_{q, \bar{q}} e_q^2 \int_x^1 \frac{d\xi}{\xi} q(\xi, \mu^2) \left[\delta\left(1 - \frac{x}{\xi}\right) + \frac{\alpha_S}{2\pi} P\left(\frac{x}{\xi}\right) \ln \frac{Q^2}{\mu^2} + O(\alpha_S^2) \right] \quad (3.100)$$

We see that we have factorized out the divergent part (the logarithm of $\frac{1}{\kappa^2}$): it is all hidden in the distribution function. Apart from that we also factorized out the finite term $C(x)$. This however is arbitrary: we could have factored out less of this finite term. This might at first sight seem to leave the theory with no predictive power, but since it is the same everytime, this is not true. The choice to factorize out all of the finite term is called the Deep Inelastic Scattering Scheme (DIS scheme).

3.4.1 Scattering from a gluon

Consider now the situation in which the photon scatters from a initial gluon. The phase space and flux factor remain the same, but in this case we will have a different splitting function:

$$\hat{F}_2^g(x, Q^2) = x \sum_{q, \bar{q}} e_q^2 \frac{\alpha_S}{2\pi} \left(P_{qg}(x) \ln \frac{Q^2}{\kappa^2} + C_g(x) \right) \quad (3.101)$$

To calculate this splitting function, we notice that the diagram is obtain from the previous one by replacing $C_F \rightarrow T_R$, and switching $p \leftrightarrow -r$ through crossing. Hence:

$$\sum_{\text{pol}} |M|_{\alpha\beta}^2 = \frac{1}{2} e_q^2 g^2 T_R \sum_{\text{pol}} \text{tr} [\gamma^\beta (\not{k} + \not{q}) \gamma^\alpha \not{k} \not{\epsilon} \not{\epsilon}^* \not{k}] \frac{1}{k^4} \quad (3.102)$$

The orthogonality relation is now:

$$\sum_{\text{pol}} \epsilon_\mu(-p) \epsilon_\nu^*(-p) = -g_{\mu\nu} + \frac{n_\mu p_\nu + n_\nu p_\mu}{n \cdot p} = -g_{\mu\nu} + n_\mu p_\nu + n_\nu p_\mu \quad (3.103)$$

Let us now again project out F_2 :

$$\begin{aligned} \frac{1}{4\pi} n^\alpha n^\beta \sum |M|_{\alpha\beta}^2 &= -\frac{\frac{1}{2} e_q^2 g^2 T_R}{4\pi k^4} (-g_{\mu\nu} + n_\mu p_\nu + n_\nu p_\mu) \text{Tr} [\not{n} (\not{k} + \not{q}) \not{n} k^\mu \not{\epsilon} \gamma^\nu \not{k}] \\ &= -\frac{\frac{1}{2} e_q^2 \alpha_S T_R}{k^4} (2\xi) (-g_{\mu\nu} + n_\mu p_\nu + n_\nu p_\mu) \text{Tr} [\not{n} k^\mu \not{\epsilon} \gamma^\nu \not{k}] \\ &= -\frac{e_q^2 \alpha_S T_R}{k^4} \xi \{ 2\text{Tr} [\not{n} k \not{\epsilon} k] + \text{Tr} [\not{n} k \not{n} \not{\epsilon} k] + \text{Tr} [\not{n} k \not{p} \not{\epsilon} k] \} \\ &= -\frac{e_q^2 \alpha_S T_R}{k^4} \xi \{ 2\text{Tr} [\not{n} k \not{p} k] - 2\text{Tr} [\not{n} k k k] + \text{Tr} [\not{n} k \not{n} p p k] - \text{Tr} [\not{n} k \not{n} k p k] + \text{Tr} [\not{n} k p p n k] - \text{Tr} [\not{n} k p k n k] \} \\ &= -\frac{e_q^2 \alpha_S T_R}{k^4} \xi \{ 2\text{Tr} [\not{n} k \not{p} k] - 2k^2 \text{Tr} [\not{n} k] - 2\text{Tr} [\not{n} k \not{n} k p k] \} \\ &= -\frac{2e_q^2 \alpha_S T_R}{k^4} \xi \{ \text{Tr} [\not{n} k \not{p} k] - k^2 \text{Tr} [\not{n} k] - 2\xi \text{Tr} [\not{n} k p k] \} \\ &= -\frac{2e_q^2 \alpha_S T_R}{k^4} \xi \{ (1 - 2\xi) \text{Tr} [\not{n} (k(k \cdot p) - k^2 \not{p})] - k^2 \text{Tr} [\not{n} k] \} \\ &= -\frac{8e_q^2 \alpha_S T_R}{k^4} \xi \{ (1 - 2\xi)(2b\xi - k^2) - \xi k^2 \} \\ &= -\frac{8e_q^2 \alpha_S T_R}{k^2} \xi \{ (1 - \xi)(\xi - 1) - \xi \} \\ &= \frac{8e_q^2 \alpha_S T_R}{k^2} \xi \{ \xi^2 + (1 - \xi)^2 \} \end{aligned} \quad (3.104)$$

This way we obtain the splitting function:

$$P_{qq}(x) = T_R [x^2 + (1 - x)^2] \quad (3.105)$$

The factorisation has to be adjusted to incorporate this. The bare quark distribution (3.98) now becomes:

$$\begin{aligned} q(x, \mu^2) &= q_0(x) + \frac{\alpha_S}{2\pi} \int_x^1 \frac{d\xi}{\xi} q_0(\xi) \left[P_{qq} \left(\frac{x}{\xi} \right) \ln \frac{\mu^2}{\xi^2} + C_q \left(\frac{x}{\xi} \right) \right] \\ &\quad + \frac{\alpha_S}{2\pi} \int_x^1 \frac{d\xi}{\xi} g_0(\xi) \left[P_{qg} \left(\frac{x}{\xi} \right) \ln \frac{\mu^2}{\xi^2} + C_g \left(\frac{x}{\xi} \right) \right] + O(\alpha_S^2) \end{aligned} \quad (3.106)$$

We then find for the structure function:

$$F_2(x, Q^2) = x \sum_{q, \bar{q}} e_q^2 \int_x^1 \frac{d\xi}{\xi} \left[q(\xi, Q^2) \left\{ \delta \left(1 - \frac{x}{\xi} \right) + \frac{\alpha_S}{2\pi} C_q' \left(\frac{x}{\xi} \right) \right\} + g(\xi, Q^2) \left\{ \frac{\alpha_S}{2\pi} C_g' \left(\frac{x}{\xi} \right) \right\} \right] \quad (3.107)$$

The coefficient functions $C_q'(x)$ and $C_g'(x)$ can be chose arbitrarily. In the DIS scheme, they are zero.

These parton distribution functions are very hard or impossible to calculate. We can however investigate how they depend on the factorisation scale μ . We first note that the righthandside of (3.100) should be independent of μ^2 . Now define $t = \mu^2$ and take the $\ln t$ partial derivative of (3.100):

$$t \frac{\partial}{\partial t} F_2(x, Q^2) = t \frac{\partial}{\partial t} x \sum_{q, \bar{q}} e_q^2 \left(q(x, t) + \int_x^1 \frac{d\xi}{\xi} q(\xi, t) \frac{\alpha_S}{2\pi} P\left(\frac{x}{\xi}\right) \ln \frac{Q^2}{t} \right) = 0 \quad (3.108)$$

Leaving out the sum over quark flavours, this means that

$$t \frac{\partial}{\partial t} q(x, t) = \frac{\alpha_S}{2\pi} \int_x^1 \frac{d\xi}{\xi} q(\xi, t) P\left(\frac{x}{\xi}\right) - \frac{\alpha_S}{2\pi} \int_x^1 \frac{d\xi}{\xi} t \frac{\partial}{\partial t} q(\xi, t) P\left(\frac{x}{\xi}\right) \ln \frac{Q^2}{t} \quad (3.109)$$

This means that $t \frac{\partial}{\partial t} q(x, t) = O(\alpha_S)$ and therefore that the second term is $O(\alpha_S^2)$, which we can therefore leave out to be left with:

$$t \frac{\partial}{\partial t} q(x, t) = \frac{\alpha_S}{2\pi} \int_x^1 \frac{d\xi}{\xi} q(\xi, t) P\left(\frac{x}{\xi}\right) \quad (3.110)$$

This equation is a simplified version of the Dokshitzer-Gribov-Lipatov-Altarelli-Parisi equation (DGLAP equation). The full version is a matrix equation with the splitting functions specified for quarks and gluons and including the summation over quark flavours previously left out.

$$t \frac{\partial}{\partial t} \begin{pmatrix} q_i(x, t) \\ g(x, t) \end{pmatrix} = \frac{\alpha_S(t)}{2\pi} \sum_{q, \bar{q}} \int_x^1 \frac{d\xi}{\xi} \begin{pmatrix} P_{q_i q_j}\left(\frac{x}{\xi}, \alpha_S(t)\right) & P_{q_i g}\left(\frac{x}{\xi}, \alpha_S(t)\right) \\ P_{g q_j}\left(\frac{x}{\xi}, \alpha_S(t)\right) & P_{g g}\left(\frac{x}{\xi}, \alpha_S(t)\right) \end{pmatrix} \begin{pmatrix} q_i(x, t) \\ g(x, t) \end{pmatrix} \quad (3.111)$$

The DGLAP equations tell us how parton momentum distribution functions change as a function of the energy scale at which they are probed. They can therefore be used to evolve from one parton distribution function to another. This is particularly useful when considering parton branching, which will be the subject of the next chapter.

Chapter 4

Branching

Full perturbative calculation in QCD can only be done to very low orders. The complexity of the calculations roughly increase factorial with each next order one wants to calculate. For most calculations next-to-leading order or next-to-next-to-leading order is as far as we can go. Meanwhile, it is obvious that at certain regions in phase space we want to be able to calculate a lot further than one or two orders. Collinear emission of a parton in deep inelastic scattering is a good example. So instead of trying to calculate everything very precise to a limited order, we now take a different approach in trying to approximate a calculation to all orders. This will lead to the conceptually appealing picture of parton showers. The idea of this chapter is to separate high and low energetic QCD dynamics, represented by momentum transfer squared t . Assuming hadronisation takes place at energies lower than some cut-off value t_0 , we can use Monte Carlo simulation for a perturbative treatment at energies above this value.

4.1 Parton branching

Consider a parton branching $a \rightarrow b + c$ and assume that

$$p_b^2, p_c^2 \ll p_a^2 \equiv t \quad (4.1)$$

Now

$$t = p_a^2 = (p_b + p_c)^2 \approx 2p_b \cdot p_c \quad (4.2)$$

Define energy fractions, using energy conservation ($E_a = E_b + E_c$):

$$z = \frac{E_b}{E_a} = \frac{E_a - E_c}{E_a} = 1 - \frac{E_c}{E_a} \quad (4.3)$$

Which means

$$E_b = zE_a \quad (4.4)$$

$$E_c = (1 - z)E_a \quad (4.5)$$

In the relativistic limit in which $\vec{p} \gg m$ and thus $E^2 = \vec{p}^2 + m^2$ turns into $E = |\vec{p}|$, we therefore get:

$$t = 2p_b \cdot p_c = 2(E_b E_c \vec{p}_b \cdot \vec{p}_c) = 2E_b E_c \left(1 - \frac{|\vec{p}_b| |\vec{p}_c| \cos \theta}{E_b E_c} \right) = 2E_b E_c (1 - \cos \theta) \quad (4.6)$$

Assuming the angle θ is small and thus $\cos \theta \approx 1 - \frac{\theta^2}{2}$, and using (4.4) and (4.5) we end up with

$$t = 2zE_a(1 - z)E_a \left(1 - \left(1 - \frac{\theta^2}{2} \right) \right) = z(1 - z)E_a^2 \theta^2 \quad (4.7)$$

Rewriting this as an expression for the angle, we get:

$$\theta = \frac{1}{E_a} \sqrt{\frac{t}{z(1-z)}} \quad (4.8)$$

Or, using transverse momentum conservation

$$\begin{aligned} |\vec{p}_b| \sin \theta_b &= |\vec{p}_c| \sin \theta_c \\ E_b \sin \theta_b &= E_c \sin \theta_c \\ z E_a \theta_b &= (1-z) E_a \theta_c \\ \Rightarrow \theta_b &= \frac{1-z}{z} \theta_c \Rightarrow \theta = \theta_b + \theta_c = \frac{1}{1-z} \theta_b = \frac{1}{z} \theta_c \end{aligned} \quad (4.9)$$

4.1.1 Three gluon vertex

Firstly, let us consider a three gluon vertex, with the momenta defined as going out ($p_a + p_b + p_c = 0$). Using ϵ_i^μ as the polarisation vector of gluon i , the corresponding factor is:

$$V_{ggg} = igf^{ABC} \epsilon_a^\alpha \epsilon_b^\beta \epsilon_c^\gamma [g_{\alpha\beta}(p_a - p_b)_\gamma + g_{\beta\gamma}(p_b - p_c)_\alpha + g_{\gamma\alpha}(p_c - p_a)_\beta] \quad (4.10)$$

Since $\epsilon_i \cdot p_i = 0$, we can rewrite this as:

$$\begin{aligned} V_{ggg} &= igf^{ABC} [(\epsilon_a \cdot \epsilon_b) \epsilon_c \cdot (-2p_b - p_c) + (\epsilon_a \cdot \epsilon_b) \epsilon_a \cdot (2p_b + p_a) + (\epsilon_a \cdot \epsilon_b) \epsilon_b \cdot (2p_c + p_b)] \\ &= -2igf^{ABC} [(\epsilon_a \cdot \epsilon_b)(\epsilon_c \cdot p_b) - (\epsilon_a \cdot \epsilon_b)(\epsilon_a \cdot p_b) - (\epsilon_a \cdot \epsilon_b)(\epsilon_b \cdot p_c)] \end{aligned} \quad (4.11)$$

To calculate the inner products, take the polarisations in the plane to have a positive angle of $\frac{\pi}{2}$ with respect to their three momenta. Now since ϵ_a^{in} does not have a y - and z -component and of course neither a zeroth component and is normalised to -1 , we can use the fact that $|p_a| = E_a$ to get:

$$\epsilon_a^{\text{in}} \cdot p_b = -\epsilon_a^x |p_b| \cos\left(\frac{\pi}{2} - \theta_b\right) = -E_b \sin(\theta_b) \approx -E_b \theta_b = -z(1-z) E_a \theta \quad (4.12)$$

Similarly,

$$\epsilon_b^{\text{in}} \cdot p_c = -E_c |\cos\left(\theta + \frac{\pi}{2}\right)| \approx E_c \theta = (1-z) E_a \theta \quad (4.13)$$

and

$$\epsilon_c^{\text{in}} \cdot p_b = -E_b |\cos\left(\frac{\pi}{2} - \theta\right)| \approx -E_b \theta = -z E_a \theta \quad (4.14)$$

Combining this with the propagator factor $\frac{1}{t}$ for gluon a and using $f_{ABC} f^{ABC} = C_A$ and expression (4.8) we get when all polarisations are taken in the plane:

$$\begin{aligned} |M_{n+1}|^2 &= \left|\frac{1}{t} V_{ggg}\right|^2 |M_n|^2 = \frac{4g^2 f_{ABC} f^{ABC}}{t^2} (E_a^2 \theta^2) [z - z(1-z) + (1-z)]^2 |M_n|^2 \\ &= \frac{4g^2 C_A}{t^2} (E_a^2 \theta^2) [z^2 - z^2(1-z)^2 + (1-z)^2 - 2z^2(1-z) + 2z(1-z) - 2z(1-z)^2] |M_n|^2 \\ &= \frac{4g^2 C_A}{t^2} (E_a^2 \theta^2) (z(1-z)) \left[\frac{z}{1-z} - z(1-z) + \frac{1-z}{z} - 2z + 2 - 2(1-z) \right] |M_n|^2 \\ &= \frac{4g^2 C_A}{t} \left[\frac{z}{1-z} - z(1-z) + \frac{1-z}{z} \right] |M_n|^2 \end{aligned} \quad (4.15)$$

If we take two polarisation out of the plane, due to $\epsilon_i^{\text{out}} \cdot p_j = 0$ the respective quadratic terms in the second line vanish as do all the cross terms. We would therefore be left with only one term in the final line, corresponding with the nonzero quadratic term. Hence we end up with

$$|M_{n+1}|^2 = \frac{4g^2 C_A}{t} F(z; \epsilon_a, \epsilon_b, \epsilon_c) |M_n|^2 \quad (4.16)$$

ϵ_a	ϵ_b	ϵ_c	$F(z; \epsilon_a, \epsilon_b, \epsilon_c)$
in	in	in	$(1-z)/z + z/(1-z) + z(1-z)$
in	out	out	$z(1-z)$
out	in	out	$(1-z)/z$
out	out	in	$z/(1+z)$

Table 4.1: Polarisation function $F(z; \epsilon_a, \epsilon_b, \epsilon_c)$ for the three gluon vertex $g \rightarrow gg$

Where $F(z; \epsilon_a, \epsilon_b, \epsilon_c)$ is given by table 4.1. Combinations which have one or three polarisations out of the plane are forbidden by momentum conservation.

Now define $\langle F \rangle$ as the average of F with respect to the polarisation of a and summing over the polarisations of b and c. We then find that:

$$C_A \langle F \rangle \equiv \hat{P}_{gg}(z) = C_A \left[\frac{1-z}{z} + \frac{z}{1-z} + z(1-z) \right] \quad (4.17)$$

Now suppose that the polarisation of gluon is not in the plane but at an angle ϕ to the plane. F would then change into (using the goniometric identity $2 \cos^2 \phi = 1 + \cos 2\phi$):

$$\begin{aligned} F_\phi &= \cos \phi |M(\epsilon_a^{\text{in}}, \epsilon_b, \epsilon_c)|^2 + \sin \phi |M(\epsilon_a^{\text{out}}, \epsilon_b, \epsilon_c)|^2 \\ &= \cos^2 \phi \left[\frac{1-z}{z} + \frac{z}{1-z} + 2z(1-z) \right] + \sin^2 \phi \left[\frac{1-z}{z} + \frac{z}{1-z} \right] \\ &= \frac{1-z}{z} + \frac{z}{1-z} + z(1-z) \cos 2\phi \end{aligned} \quad (4.18)$$

The first three terms are exactly the same as the unpolarized result, the rotation out of the plane contributes the last term. Note that it makes the polarisation to favour being in the plane.

4.1.2 Gluon splitting into quark anti-quark vertex

Now consider a gluon going into a quark and an anti-quark. The vertex vector is now:

$$V_{gq\bar{q}} = -igt_{bc}^A \bar{u}^b \gamma_\mu \epsilon_a^\mu v^c \quad (4.19)$$

Where u^b and v^c are the quark and anti-quark spinors. We will use the common convention for the Dirac algebra

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \quad (4.20)$$

For the helicity spinors we use:

$$u_+^b = \sqrt{E_b} \begin{pmatrix} 1 \\ \theta_b/2 \\ 1 \\ \theta_b/2 \end{pmatrix}, u_-^b = \sqrt{E_b} \begin{pmatrix} -\theta_b/2 \\ 1 \\ \theta_b/2 \\ -1 \end{pmatrix}, v_+^c = \sqrt{E_c} \begin{pmatrix} -\theta_c/2 \\ -1 \\ \theta_c/2 \\ 1 \end{pmatrix}, v_-^c = \sqrt{E_c} \begin{pmatrix} -1 \\ \theta_c/2 \\ -1 \\ \theta_c/2 \end{pmatrix}, \quad (4.21)$$

Since $\epsilon_a^{\text{in}} = (0, 1, 0, 0)$, $\gamma_\mu \epsilon^\mu = -\gamma^1$. The adjoint spinor gives a γ^0 , so to calculate the vertex factors we will need the identity

$$\gamma^0 \gamma^1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & \sigma_x \\ -\sigma_x & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad (4.22)$$

Similarly, $\epsilon_a^{\text{out}} = (0, 0, 1, 0)$ so $\gamma_\mu \epsilon^\mu = -\gamma^2$.

$$\gamma^0 \gamma^2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & \sigma_y \\ -\sigma_y & 0 \end{pmatrix} = i \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad (4.23)$$

The quark and anti-quark produced should have opposite helicities. There are therefore four combinations that have to be calculated, two in and two out of the plane. In all cases, we have a factor

$$-ig \cdot \sqrt{E_b} \cdot i\sqrt{E_c} \cdot -1 = -g\sqrt{z(1-z)}E_a \quad (4.24)$$

(The extra factor -1 is from the the product of $\gamma \cdot \epsilon$ as stated above) Now we have:

$$\begin{aligned} & (1 \quad \theta_b/2 \quad 1 \quad \theta_b/2) \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -1 \\ \theta_c/2 \\ -1 \\ \theta_c/2 \end{pmatrix} \\ &= (1 \quad \theta_b/2 \quad 1 \quad \theta_b/2) \begin{pmatrix} \theta_c/2 \\ -1 \\ \theta_c/2 \\ -1 \end{pmatrix} = \theta_c - \theta_b \end{aligned} \quad (4.25)$$

$$\begin{aligned} & (-\theta_b/2 \quad 1 \quad \theta_b/2 \quad -1) \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -\theta_c/2 \\ -1 \\ \theta_c/2 \\ 1 \end{pmatrix} \\ &= (-\theta_b/2 \quad 1 \quad \theta_b/2 \quad -1) \begin{pmatrix} 1 \\ \theta_c/2 \\ -1 \\ -\theta_c/2 \end{pmatrix} = \theta_c - \theta_b \end{aligned} \quad (4.26)$$

$$\begin{aligned} & (1 \quad \theta_b/2 \quad 1 \quad \theta_b/2) \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -1 \\ \theta_c/2 \\ -1 \\ \theta_c/2 \end{pmatrix} \\ &= (1 \quad \theta_b/2 \quad 1 \quad \theta_b/2) \begin{pmatrix} -\theta_c/2 \\ -1 \\ -\theta_c/2 \\ -1 \end{pmatrix} = -\theta_c - \theta_b \end{aligned} \quad (4.27)$$

$$\begin{aligned} & (-\theta_b/2 \quad 1 \quad \theta_b/2 \quad -1) \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -\theta_c/2 \\ -1 \\ \theta_c/2 \\ 1 \end{pmatrix} \\ &= (-\theta_b/2 \quad 1 \quad \theta_b/2 \quad -1) \begin{pmatrix} -1 \\ \theta_c/2 \\ 1 \\ -\theta_c/2 \end{pmatrix} = \theta_b + \theta_c \end{aligned} \quad (4.28)$$

Hence we have:

$$\begin{aligned} -ig\bar{u}_+^b \gamma^\mu \epsilon_\mu^{\text{in}} v_-^c &= -g\sqrt{z(1-z)}E_a(\theta_c - \theta_b) = g\sqrt{z(1-z)}E_a(1-z-z)\theta = g\sqrt{t}\sqrt{1-2z} \\ -ig\bar{u}_-^b \gamma^\mu \epsilon_\mu^{\text{in}} v_+^c &= -g\sqrt{z(1-z)}E_a(\theta_c - \theta_b) = g\sqrt{z(1-z)}E_a(1-z-z)\theta = g\sqrt{t}\sqrt{1-2z} \\ -ig\bar{u}_+^b \gamma^\mu \epsilon_\mu^{\text{out}} v_-^c &= -ig\sqrt{z(1-z)}E_a(-\theta_c - \theta_b) = ig\sqrt{z(1-z)}E_a(-z - (1-z))\theta = ig\sqrt{t} \\ -ig\bar{u}_-^b \gamma^\mu \epsilon_\mu^{\text{out}} v_+^c &= -ig\sqrt{z(1-z)}E_a(\theta_c + \theta_b) = ig\sqrt{z + (1-z)}E_a(z)\theta = ig\sqrt{t} \end{aligned} \quad (4.29)$$

ϵ_a	λ_b	λ_c	$F(z; \epsilon_a, \lambda_b, \lambda_c)$
in	\pm	\mp	$(1-2z)^2$
out	\pm	\mp	1

Table 4.2: Polarisation function $F(z; \epsilon_a, \epsilon_b, \epsilon_c)$ for the gluon splitting vertex $g \rightarrow qq$

The matrix element squared is now, still with propagator $\frac{1}{t}$:

$$|M_{n+1}|^2 \propto \frac{g^2}{t} T_R F(z; \epsilon_a, \lambda_b, \lambda_c) |M_n|^2 \quad (4.30)$$

With $F(z; \epsilon_a, \lambda_b, \lambda_c) |M_n|^2$ given by table 4.2 and $T_R = \frac{1}{8} \text{Tr}(t^A t^A) = \frac{1}{2}$. There are no longer singularities ($z \rightarrow 0$ and $z \rightarrow 1$) here, because those represent the emission of a soft gluon. The spin-average of F is now:

$$\langle F \rangle = \frac{1}{2s+1} \frac{1}{2} [2(1-2z)^2 + 2] = \frac{1}{2} [1 - 4z + 4z^2 + 1] = [z^2 + (1-z)^2] \quad (4.31)$$

So we get for the splitting function:

$$\hat{P}_{gg}(z) = T_R \langle F \rangle = T_R [z^2 + (1-z)^2] \quad (4.32)$$

$$\begin{aligned} F_\phi &\propto \sum_{\lambda_b, \lambda_c} |\cos \phi M(\epsilon_a^{\text{in}}, \lambda_b, \lambda_c) + \sin \phi M(\epsilon_a^{\text{out}}, \lambda_b, \lambda_c)|^2 \\ &= |\cos \phi(1-2z) + i \sin \phi|^2 + |\cos \phi(1-2z) - i \sin \phi|^2 \\ &\propto \cos^2 \phi(1-2z)^2 + \sin^2 \phi + \sin \phi \cos \phi [i(2z-1) - i(2z-1)] \\ &= 1 + (4z^2 + 4z) \cos^2 \phi = 1 + 2(z^2 - z)(1 + \cos 2\phi) \\ &= z^2 + (1-z)^2 - 2z(1-z) \cos 2\phi \end{aligned} \quad (4.33)$$

4.1.3 Quark or anti-quark radiating off a gluon

Now the last branching possibility: a quark or anti-quark radiating off a gluon. The vertex factor for this process is

$$V_{qqg} = -igt_{ab}^A \epsilon_c^\mu \bar{u}_b^{\lambda_b} \gamma_\mu u_a^{\lambda_a} \quad (4.34)$$

The polarisation vector of the gluon still has to be perpendicular to the momentum p_c and is therefore rotated over an angle $-\theta_c$:

$$\epsilon_c^{\text{in}} = (0, \cos \theta_c, 0 \sin \theta_c) \approx (0, 1, 0, \theta_c) \quad (4.35)$$

Where in the last approximation we again assumed θ_c to be small. Now we have:

$$\gamma_0(\gamma \cdot \epsilon_c^{\text{in}}) = -\gamma_0(\gamma_1 + \theta_c \gamma_3) \quad (4.36)$$

This means that in addition to the contribution we already had from $\gamma_0 \gamma_1$, we now also have to add:

$$\theta_c \gamma_0 \gamma_3 = \theta_c \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & \sigma_z \\ -\sigma_z & 0 \end{pmatrix} = \theta_c \begin{pmatrix} 0 & \sigma_z \\ \sigma_z & 0 \end{pmatrix} \quad (4.37)$$

Which means that

$$\gamma_0(\gamma_1 + \theta_c \gamma_3) = \begin{pmatrix} 0 & 0 & \theta_c & 1 \\ 0 & 0 & 1 & -\theta_c \\ \theta_c & 1 & 0 & 0 \\ 1 & -\theta_c & 0 & 0 \end{pmatrix} \quad (4.38)$$

Therefore we now have (note that $\theta_a = 0$):

$$\begin{aligned}
& (1 \quad \theta_b/2 \quad 1 \quad \theta_b/2) \begin{pmatrix} 0 & 0 & \theta_c & 1 \\ 0 & 0 & 1 & -\theta_c \\ \theta_c & 1 & 0 & 0 \\ 1 & -\theta_c & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \\
&= (1 \quad \theta_b/2 \quad 1 \quad \theta_b/2) \begin{pmatrix} \theta_c \\ 1 \\ \theta_c \\ 1 \end{pmatrix} = \theta_b + 2\theta_c = (1+z)\theta
\end{aligned} \tag{4.39}$$

$$\begin{aligned}
& (-\theta_b/2 \quad 1 \quad \theta_b/2 \quad -1) \begin{pmatrix} 0 & 0 & \theta_c & 1 \\ 0 & 0 & 1 & -\theta_c \\ \theta_c & 1 & 0 & 0 \\ 1 & -\theta_c & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} \\
&= (-\theta_b/2 \quad 1 \quad \theta_b/2 \quad -1) \begin{pmatrix} -1 \\ \theta_c \\ 1 \\ -\theta_c/2 \end{pmatrix} = \theta_b + 2\theta_c = (1+z)\theta
\end{aligned} \tag{4.40}$$

For the polarisation out of the plane nothing changed, therefore:

$$\begin{aligned}
& (1 \quad \theta_b/2 \quad 1 \quad \theta_b/2) \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \\
&= (1 \quad \theta_b/2 \quad 1 \quad \theta_b/2) \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} = \theta_b = (1-z)\theta
\end{aligned} \tag{4.41}$$

$$\begin{aligned}
& (-\theta_b/2 \quad 1 \quad \theta_b/2 \quad -1) \begin{pmatrix} 0 & 0 & \theta_c & 1 \\ 0 & 0 & 1 & -\theta_c \\ \theta_c & 1 & 0 & 0 \\ 1 & -\theta_c & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} \\
&= (-\theta_b/2 \quad 1 \quad \theta_b/2 \quad -1) \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} = -\theta_b = -(1-z)\theta
\end{aligned} \tag{4.42}$$

So in the end we get:

$$\begin{aligned}
& -ig\epsilon_\mu^{\text{in}} \bar{u}_b^+ \gamma^\mu u_a^+ = igE_a \theta \sqrt{z}(1+z) = igt \frac{1+z}{\sqrt{1-z}} \\
& -ig\epsilon_\mu^{\text{in}} \bar{u}_b^- \gamma^\mu u_a^- = igE_a \theta \sqrt{z}(1+z) = igt \frac{1+z}{\sqrt{1-z}} \\
& -ig\epsilon_\mu^{\text{out}} \bar{u}_b^+ \gamma^\mu u_a^+ = -gE_a \theta \sqrt{z}(1-z) = igt \frac{1-z}{\sqrt{1-z}} \\
& -ig\epsilon_\mu^{\text{out}} \bar{u}_b^- \gamma^\mu u_a^- = -gE_a \theta \sqrt{z}(1-z) = igt \frac{1-z}{\sqrt{1-z}}
\end{aligned} \tag{4.43}$$

This leads to table 4.3 We now have:

$$\hat{P}_{qq}(z) = C_F \langle F \rangle = C_F \frac{1+z^2}{1-z} \tag{4.44}$$

λ_a	λ_b	ϵ_c	$F(z; \epsilon_a, \lambda_b, \lambda_c)$
\pm	\pm	in	$(1+z)^2/(1-z)$
\pm	\pm	out	1

Table 4.3: Polarisation function $F(z; \epsilon_a, \epsilon_b, \epsilon_c)$ for the quark or anti-quark radiating vertex $q \rightarrow qg$ or $\bar{q} \rightarrow \bar{q}g$

For the angle dependent polarisation function we have:

$$\begin{aligned}
F_\phi &= \sum_{\lambda_b \epsilon_c} |M_{\text{in}} \cos \phi + M_{\text{out}} \sin \phi|^2 \\
&= 2 \left| \cos \phi \frac{1+z}{\sqrt{1-z}} i + \sin \phi (-1) \frac{1-z}{\sqrt{1-z}} \right|^2 \\
&\propto \cos^2 \phi \frac{1+z^2}{1-z} + \sin^2 \phi \frac{(1-z)^2}{1-z} \\
&= \frac{(1-z)^2 + 4z}{1-z} \cos^2 \phi + \sin^2 \phi \\
&= (1-z)(\cos^2 \phi + \sin^2 \phi) + \frac{4z}{1-z} \cos^2 \phi \\
&= 1-z + \frac{2z}{1-z} + \frac{2z}{1-z} \cos 2\phi \\
&= \frac{1+z^2}{1-z} + \frac{2z}{1-z} \cos 2\phi
\end{aligned} \tag{4.45}$$

This means that if the polarisation is completely out of the plane, $\phi = \frac{\pi}{2}$, F_ϕ becomes

$$F_\phi = \frac{1+z^2-2z}{1-z} = \frac{(1-z)^2}{1-z} = 1-z \tag{4.46}$$

Only if the polarisation is partly in the plane there is a singularity. The amplitude for the polarisation out of the plane even completely vanishes as $z \rightarrow 1$. This shows the gluon prefers its polarisation being in the plane.

4.1.4 Phase space

The cross-section,

$$d\sigma_n = \mathcal{F} |M_n|^2 d\Phi_n, \tag{4.47}$$

can now be calculated if we know the phase space. The branching of a parton a into b and c means for the phase space that

$$d\Phi_n = \dots \frac{d^3 \vec{p}_a}{2(2\pi)^3 E_a} \tag{4.48}$$

has to be replaced by

$$d\Phi_{n+1} = \dots \frac{d^3 \vec{p}_b}{2(2\pi)^3 E_b} \frac{d^3 \vec{p}_c}{2(2\pi)^3 E_c} \tag{4.49}$$

Since $p_a = p_b + p_c$, we get $d^3 \vec{p}_c = d^3 \vec{p}_a$ for fixed p_b . This means we can write:

$$d\Phi_{n+1} = \dots \frac{d^3 \vec{p}_a}{2(2\pi)^3 E_c} \frac{E_a}{E_a} \frac{d^3 \vec{p}_b}{2(2\pi)^3 E_b} = d\Phi_n \frac{E_a}{E_c} \frac{d^3 \vec{p}_b}{2(2\pi)^3 E_b} \tag{4.50}$$

Now since we work in the relativistic limit $m \ll p$, we know that $E = |\vec{p}|$ and therefore that in the small angle approach ($\sin \theta_b \approx \theta_b$):

$$d^3 \vec{p}_b = |\vec{p}_b|^2 \sin \theta_b d\theta_b d\phi d|\vec{p}_b| = E_b^2 \theta_b d\theta_b d\phi dE_b \tag{4.51}$$

Also, we can rewrite

$$\frac{E_a}{E_c} = \frac{E_a}{E_a - E_b} = \frac{1}{1 - \frac{E_b}{E_a}} = \int_0^1 \frac{dz}{1-z} \delta\left(z - \frac{E_b}{E_a}\right) \quad (4.52)$$

Combine this with a rewritten form of (4.8),

$$\theta = \frac{1}{E_a} \sqrt{\frac{t}{z(1-z)}} \Rightarrow t = z(1-z)E_a^2\theta^2 = E_b E_c \theta^2 \Rightarrow \int \delta(t - E_b E_c \theta^2) dt \quad (4.53)$$

to get in the end:

$$d\Phi_{n+1} = d\Phi_n \frac{1}{2(2\pi)^3} \int E_b dE_b \theta_b d\theta_b d\phi dt \frac{dz}{1-z} \delta(t - E_b E_c \theta^2) \delta\left(z - \frac{E_b}{E_a}\right) \quad (4.54)$$

The first delta function can be rewritten by $\delta(g(x)) = \frac{1}{|g'(x_0)|} \sum_{x_0} \delta(x-x_0)$ with for each $g(x_0) = 0$. Now since we only integrate over positive angle θ_b , there is only one delta function in this sum, the one with $\theta_b = +(1-z)\sqrt{\frac{t}{E_a E_b}}$. In this case we have:

$$g(\theta_b) = t - E_b E_c \frac{1}{(1-z)^2} \theta_b^2 \Rightarrow g'(\theta_b) = 2E_b E_c \frac{1}{(1-z)^2} \theta_b \quad (4.55)$$

Hence we get:

$$d\Phi_{n+1} = d\Phi_n \frac{1}{2(2\pi)^3} dt dz d\phi \int_0^\pi \theta_b d\theta_b \frac{(1-z)^2}{2E_a E_b \theta_b} \times \\ \times \delta\left(\theta_b - (1-z)\sqrt{\frac{t}{E_b E_c}}\right) \int_0^\infty dE_b \frac{E_a E - b}{1-z} \delta(E_b - E_a z) \quad (4.56)$$

So that in the end we have:

$$d\Phi_{n+1} = \frac{d\Phi_n}{2(2\pi)^3} dt dz d\phi \frac{E_a}{2E_c} \frac{(1-z)^2}{1-z} = \frac{d\Phi_n}{4(2\pi)^3} dt dz d\phi \quad (4.57)$$

Filling this in into equation (4.47), we get:

$$d\sigma_{n+1} = \frac{4g^2}{t} CF \mathcal{F} \frac{d\Phi_n}{4(2\pi)^3} dt dz d\phi = d\sigma_n \frac{dt}{t} dz \frac{d\phi}{2\pi} \frac{\alpha_S}{2\pi} CF \quad (4.58)$$

If the azimuthal angle ϕ does not interest us, we can make the expression for the cross section explicitly dependent on the splitting function, by using:

$$\int \frac{d\phi}{2\pi} CF = \hat{P}_{ba}(z) \quad (4.59)$$

The differential cross section then becomes

$$d\sigma_{n+1} = d\sigma_n \frac{dt}{t} dz \frac{\alpha_S}{2\pi} \hat{P}_{ba}(z) \quad (4.60)$$

4.2 Evolution equations

Consider a process in which a parton branches n times. The parton i has virtual mass-squared t_i and carries a fraction x_i of the hadrons momentum. Consider now figure 4.1. $f(x, t)$ is momentum fraction distribution of partons at energy scale t . To get $f(x, t)$, first consider box in fig 4.1 of

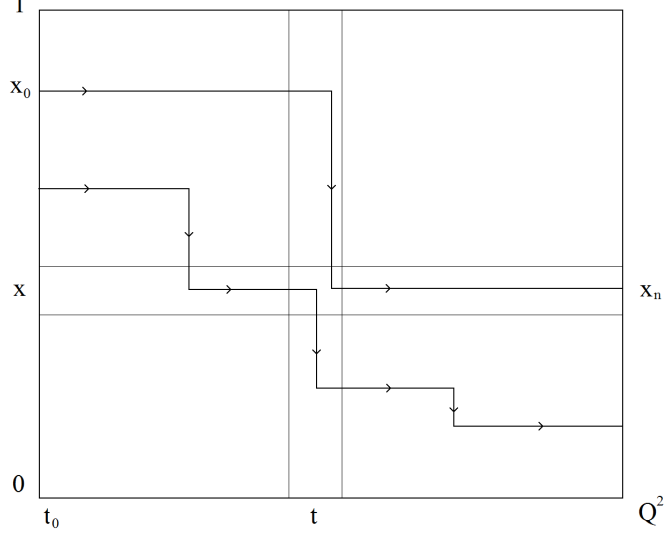


Figure 4.1: Parton branching by paths.

$(\delta t, \delta x)$. The change of $f(x, t)$ when t is increased to $t + \delta t$ is the number of paths arriving minus those leaving, divided by δx to normalize again to an interval of 0 to 1.

Paths arriving in this box can only have come from above, in the interval x to 1. The chance for a path with momentum fraction x' of branching with a momentum fraction z is $P(z)$. But we want only those z that end up in our box, that is obeying $x' = \frac{x}{z}$. Hence the probability to branch is $\int dz P(z) \delta(x - zx')$. To get the total number of branchings from this momentum fraction into the box, we have to multiply with the parton density. To then finally get the total number of branching we have to integrate over the given interval above the box and normalise, to end up with

$$\begin{aligned} \delta f_{\text{in}(x,t)} &= \frac{\delta t}{t} \int_x^1 dx' \int_0^1 dz \frac{\alpha_S}{2\pi} \hat{P}(z) f(x', t) \delta(x - zx') \\ &= \frac{\delta t}{t} \int_0^1 dz \frac{\alpha_S}{z} \frac{1}{2\pi} \hat{P}(z) f\left(\frac{x}{z}, t\right) \end{aligned} \quad (4.61)$$

Since $f(x', t)$ is zero for $x' > 1$, which means given the delta function for $z < x$, we can set the lower limit of the integration over z on either x or 0. For the number of paths leaving the element, we follow a similar procedure. Integrate over all lower momentum fractions $x' = zx$. Only the distribution within the little box (that is, $f(x, t)$) is relevant now, so we get

$$\begin{aligned} \delta f_{\text{out}(x,t)} &= \frac{\delta t}{t} f(x, t) \int_0^x dx' \int_0^1 dz \frac{\alpha_S}{2\pi} \hat{P}(z) \delta(x' - zx) \\ &= \frac{\delta t}{t} f(x, t) \int_0^1 dz \frac{\alpha_S}{2\pi} \hat{P}(z) \end{aligned} \quad (4.62)$$

The net change therefore becomes:

$$\delta f(x, t) = \delta f_{\text{in}} - \delta f_{\text{out}} = \frac{\delta t}{t} \int_0^1 dz \frac{\alpha_S}{2\pi} \hat{P}(z) \left[\frac{1}{z} f\left(\frac{x}{z}, t\right) - f(x, t) \right] \quad (4.63)$$

Using the plus prescription from the previous chapter, equation (3.88), this reduces to:

$$t \frac{\partial}{\partial t} f(x, t) = \int_x^1 \frac{dz}{z} \frac{\alpha_S}{2\pi} P(z) f\left(\frac{x}{z}, t\right) \quad (4.64)$$

This is again the simplified DGLAP equation. To get the full one, we have include all the possible splitting functions:

$$t \frac{\partial}{\partial t} f_i(x, t) = \sum_j \int_x^1 \frac{dz}{z} \frac{\alpha_S}{2\pi} f_j\left(\frac{x}{z}, t\right) \quad (4.65)$$

There are two ways for a quark to enter the element ($i = q$), via $q \rightarrow qq$ or $g \rightarrow q\bar{q}$, but there is only one way to leave the element: $q \rightarrow qq$. Therefore $P(z)$ can either be $\hat{P}_{qq}(z)_+$ or $\hat{P}_{qg}(z)$:

$$P_{qq}(z) = \hat{P}_{qq}(z)_+ = C_F \left(\frac{1+z^2}{1-z} \right)_+ \quad (4.66)$$

$$P_{qg}(z) = \hat{P}_{qg}(z) = T_R [z^2 + (1-z)^2] \quad (4.67)$$

Gluons can enter the element via $q \rightarrow qq$ (or the \bar{q} of course) or in two ways via $g \rightarrow gg$, since either gluon can contribute. Hence we get:

$$\begin{aligned} \delta f_{g,\text{in}}(x, t) = \frac{\delta t}{t} \int_0^1 \frac{dz}{z} \frac{\alpha_S}{2\pi} \left\{ \hat{P}_{gg}(z) \left[f_g\left(\frac{x}{z}, t\right) + f_g\left(\frac{x}{1-z}, t\right) \right] \right. \\ \left. + \hat{P}_{qq}(z) \left[f_q\left(\frac{x}{1-z}, t\right) + f_{\bar{q}}\left(\frac{x}{1-z}, t\right) \right] \right\} \quad (4.68) \end{aligned}$$

Since $\hat{P}_{qg}(z)$ is symmetric under $z \rightarrow 1-z$, we can write this as:

$$\delta f_{g,\text{in}}(x, t) = \frac{\delta t}{t} \int_0^1 \frac{dz}{z} \frac{\alpha_S}{2\pi} \left\{ 2\hat{P}_{gg}(z) f_g\left(\frac{x}{z}, t\right) + \hat{P}_{qq}(1-z) \left[f_q\left(\frac{x}{z}, t\right) + f_{\bar{q}}\left(\frac{x}{z}, t\right) \right] \right\} \quad (4.69)$$

The gluon can leave the element only in two ways, by splitting in two other gluons, $g \rightarrow gg$, or by splitting into a quark antiquark pair, $g \rightarrow q\bar{q}$, with n_f possible flavours.

$$\delta f_{g,\text{out}}(x, t) = \frac{\delta t}{t} f_g(x, t) \int_0^1 dz \frac{\alpha_S}{2\pi} \left[\hat{P}_{gg}(z) + n_f \hat{P}_{qg}(z) dz \right] \quad (4.70)$$

For the same symmetric reason as before, we have:

$$\int_0^1 dz \frac{1-z}{z} = \int_1^0 d(1-z) \frac{z}{1-z} = \int_0^1 dz \frac{z}{1-z} \quad (4.71)$$

This means we can write

$$\int_0^1 \hat{P}_{gg}(z) dz = 2C_A \int_0^1 \left[\frac{z}{1-z} + \frac{1}{2} z(1-z) \right] dz \quad (4.72)$$

And for $\hat{P}_{qg}(z)$:

$$\int_0^1 \hat{P}_{qg}(z) dz = T_R \int_0^1 [z^2 + (1-z)^2] dz = \frac{2}{3} T_R \quad (4.73)$$

Therefore, we get for the kernel of equation (4.65):

$$\begin{aligned} P_{gg}(z) &= 2\hat{P}_{gg}(z) - \hat{P}_{gg}(z) - n_f \hat{P}_{qg}(z) = \\ &= 2C_A \left[\left(\frac{z}{1-z} + \frac{1}{2} z(1-z) \right)_+ + \frac{1-z}{z} + \frac{1}{2} z(1-z) \right] - \frac{2}{3} n_f T_R \delta(1-z) \quad (4.74) \end{aligned}$$

$$P_{q\bar{q}}(z) = P_{g\bar{q}} = \hat{P}_{q\bar{q}}(1-z) = C_F \frac{1+(1-z)^2}{z}$$

Now work out the plus-prescriptions like we did before, when we noted that

$$\left(\frac{1+z^2}{1-z}\right)_+ = \frac{1+z^2}{(1-z)_+} + \frac{3}{2}\delta(1-z) \quad (4.75)$$

In the same way (integrating and equating zero to get the coefficient), we get:

$$\begin{aligned} \int_0^1 dz \left[\left(\frac{z}{1-z} + \frac{1}{2}z(1-z) \right)_+ + K\delta(1-z) \right] &= \int_0^1 dz \frac{z-1}{1-z} + \frac{1}{2}z(1-z) + K \\ \Rightarrow K &= - \int_0^1 dz \left(-1 + \frac{1}{2}z - \frac{1}{2}z^2 \right) = z - \frac{1}{4}z^2 + \frac{1}{6}z^3 \Big|_0^1 = \frac{11}{12} \end{aligned} \quad (4.76)$$

So

$$\left(\frac{z}{1-z} + \frac{1}{2}z(1-z) \right)_+ = \frac{z}{(1-z)_+} + \frac{1}{2}z(1-z) + \frac{11}{12}\delta(1-z) \quad (4.77)$$

This means that for the regularized splitting functions we end up with:

$$P_{qq}(z) = C_F \left[\frac{1+z^2}{(1-z)_+} + \frac{3}{2}\delta(1-z) \right] \quad (4.78)$$

$$P_{gg}(z) = 2C_A \left[\frac{z}{(1-z)_+} + \frac{1-z}{z} + z(1-z) \right] + \frac{1}{6}(11C_A - 4n_f T_R)\delta(1-z) \quad (4.79)$$

Now define the so called Sudakov form factor

$$\Delta(t) \equiv \exp \left[- \int_{t_0}^t \frac{dt'}{t'} \int dz \frac{\alpha_S}{2\pi} \hat{P}(z) \right] \quad (4.80)$$

Since $t\partial/\partial t$ equals $\partial/\partial \ln t$ and

$$\int_{t_0}^t \frac{dt'}{t'} = \ln t - \ln t_0 \quad (4.81)$$

we get that

$$t \frac{\partial}{\partial t} \Delta(t) = \frac{\partial}{\partial \ln t} \exp \left[- (\ln t - \ln t_0) \int dz \frac{\alpha_S}{2\pi} \hat{P}(z) \right] = - \int dz \frac{\alpha_S}{2\pi} \hat{P}(z) \Delta(t) \quad (4.82)$$

This means that we can write equation (4.63) as:

$$t \frac{\partial}{\partial t} f(x, t) = \int \frac{dz}{z} \frac{\alpha_S}{2\pi} \hat{P}(z) f\left(\frac{x}{z}, t\right) + \frac{f(x, t)}{\Delta(t)} t \frac{\partial}{\partial t} \Delta(t) \quad (4.83)$$

And using the product rule,

$$t \partial_t \left(\frac{f}{\Delta} \right) = \frac{\Delta t \partial_t f - f t \partial_t \Delta}{\Delta^2} = \frac{1}{\Delta} \left(t \partial_t f - \frac{f}{\Delta} t \partial_t \Delta \right), \quad (4.84)$$

we can also write this as:

$$t \frac{\partial}{\partial t} \left(\frac{f}{\Delta} \right) = \frac{1}{\Delta} \int \frac{dz}{z} \frac{\alpha_S}{2\pi} \hat{P}(z) f\left(\frac{x}{z}, t\right) \quad (4.85)$$

To integrate this, notice that, since $\Delta(t_0) = 1$:

$$\int_0^t \frac{dt'}{t'} t' \partial_{t'} \frac{f(x, t')}{\Delta(t')} = \frac{f(x, t)}{\Delta(t)} - \frac{f(x, t_0)}{\Delta(t_0)} = \frac{f(x, t)}{\Delta(t)} - f(x, t_0) \quad (4.86)$$

Hence we end up with:

$$f(x, t) = \Delta(t) f(x, t_0) + \int_{t_0}^t \frac{dt'}{t'} \frac{\Delta(t)}{\Delta(t')} \int \frac{dz}{z} \hat{P}(z) f\left(\frac{x}{z}, t\right) \quad (4.87)$$

The first term is the contribution from paths that do not branch between t_0 and t . The Sudakov form factor thus gives the probability of evolving without branching. The second term gives the contribution from the paths that have their last branching at t' . The factor $\frac{\Delta(t)}{\Delta(t')}$ is the probability of evolving from t' to t without branching.

Of course we can generalise by specialising for different kinds of partons. We therefore get:

$$\Delta_i(t) \equiv \exp \left[- \sum_j \int_{t_0}^t \frac{dt'}{t'} \int dz \frac{\alpha_S}{2\pi} \hat{P}_{ji}(z) \right] \quad (4.88)$$

And equation (4.85) becomes:

$$t \frac{\partial}{\partial t} \left(\frac{f_i}{\Delta_i} \right) = \frac{1}{\Delta_i} \sum_j \int \frac{dz}{z} \frac{\alpha_S}{2\pi} \hat{P}_{ij}(z) f_j \left(\frac{x}{z}, t \right) \quad (4.89)$$

4.3 Coherent Branching

So far in our process of developing a parton branching formalism, we have only considered collinear enhancements. But there are also enhancements due to soft gluon emission. We will consider them in this section, eventually leading to the concept of angular ordering.

4.3.1 Soft gluon emission



Figure 4.2: An external line having emitted or about to emit a gluon.

Consider a process with an external line representing a parton with momentum p and mass m , having emitted or about to emit a gluon with soft (negligible) momentum q (figure 4.2). This means that there is an internal line with momentum $p \pm q$. There is a propagator factor $\frac{1}{(p \pm q)^2 - m^2}$. Since $q^2 = 0$ and $p^2 = m^2$, we can write this as (using $E = \gamma m$ and $|\vec{p}| = \gamma m |\vec{v}|$)

$$\frac{1}{(p \pm q)^2 - m^2} = \frac{\pm 1}{2p \cdot q} = \frac{\pm 1}{2(E\omega - |\vec{q}||\vec{p}| \cos \theta)} = \frac{\pm 1}{2\omega E(1 - v \cos \theta)} \quad (4.90)$$

Where $v = |\vec{v}|$ and $p_0 = E$ and $q_0 = |\vec{q}| = \omega$. Consider for now the case with the minus sign: an incoming quark with momentum p emitting a gluon with momentum q . The full matrix element for this process is

$$V = gC \epsilon_\mu^*(q) \frac{-1}{2p \cdot q} (\not{p} - \not{q} + m) \gamma^\mu u(p) \quad (4.91)$$

with C being some colour factor. Now using the Dirac equation for the spinor that is on shell ($(\not{p} - m)u(p) = 0$), we note that:

$$\begin{aligned} & \epsilon_\mu^*(q) (p_\nu \gamma^\nu + m) \gamma^\mu u(p) \\ &= \epsilon_\mu^*(q) (p_\nu (2g^{\mu\nu} - \gamma^\mu \gamma^\nu) + m \gamma^\mu) u(p) \\ &= 2\epsilon^* \cdot p - \not{\epsilon} (\not{p} - m) u(p) = 2\epsilon^* \cdot p \end{aligned} \quad (4.92)$$

Since $q_0 = |\vec{q}| = \omega$, the term with q instead of p vanishes in the soft limit $\omega \rightarrow 0$ and hence we end up with

$$V = gF_{\text{soft}}u(p) \quad (4.93)$$

with

$$F_{\text{soft}} = \frac{\epsilon^* \cdot p}{p \cdot q} \quad (4.94)$$

The extra phase space coming from this enhancement is

$$d\Phi_1 = \frac{d^3q}{(2\pi)^3 2q_0} = \frac{|\vec{q}|^2 d|\vec{q}| d\Omega}{(2\pi)^3 2q_0} = \frac{\omega^2 d\omega d\Omega}{(2\pi)^3 2\omega} \quad (4.95)$$

The enhanced differential cross section therefore becomes:

$$d\sigma_{n+1} = d\sigma_n |V|^2 d\Phi_1 = d\sigma_n g^2 \left(\frac{\epsilon \cdot p}{p \cdot q} \right) \left(\frac{\epsilon^* \cdot p}{p \cdot q} \right) \quad (4.96)$$

We can project out the epsilons as follows:

$$(\epsilon^* \cdot p_i)(\epsilon \cdot p_j) = \epsilon_\mu^*(q) \epsilon_\nu(q) p_i^\mu p_j^\nu = \left(-g_{\mu\nu} + \frac{n_\mu q_\nu + n_\nu q_\mu}{n \cdot q} \right) p_i^\mu p_j^\nu \quad (4.97)$$

Noting that $n \cdot q = 1$ and $n \cdot p = 0$ (the symbols have switched meaning compared to the previous chapter), this equals $-p_i \cdot p_j$. Hence we get:

$$d\sigma_{n+1} = d\sigma_n \frac{d\omega}{\omega} \frac{d\Omega}{2\pi} \frac{\alpha_S}{2\pi} \sum_{i,j} C_{ij} W_{ij} \quad (4.98)$$

With $C_{i,j}$ being a colour factor to be computed and:

$$\begin{aligned} W_{ij} &= \frac{\omega^2 p_i \cdot p_j}{(p_i \cdot q)(p_j \cdot q)} = \frac{\omega^2 E_i E_j (1 - v_i v_j \cos \theta_{ij})}{E_i E_j \omega^2 (1 - v_i \cos \theta_{iq})(1 - v_j \cos \theta_{jq})} \\ &= \frac{1 - v_i v_j \cos \theta_{ij}}{(1 - v_i \cos \theta_{iq})(1 - v_j \cos \theta_{jq})} \end{aligned} \quad (4.99)$$

4.3.2 Angular ordering

$$W_{ij} = \frac{\omega^2 p_i \cdot p_j}{p_i \cdot q \cdot j \cdot q} = \frac{1 - \cos \theta_{ij}}{(1 - \cos \theta_{iq})(1 - \cos \theta_{jq})} \quad (4.100)$$

Splitting this in $W_{ij} = U_{ij} + U_{ji}$ with

$$U_{ij} = \frac{1}{2} \left(W_{ij} + \frac{1}{\cos \theta_{iq}} - \frac{1}{\cos \theta_{jq}} \right) \quad (4.101)$$

we want to calculate

$$\int_0^{2\pi} \frac{d\phi_{iq}}{2\pi} U_{ij} \quad (4.102)$$

To calculate this, let's first look at

$$1 - \cos \theta_{jq} \quad (4.103)$$

Consider a unit sphere with polar coordinates (r, θ, ϕ) defined relative to parton i . This means that the \hat{i} direction is $(1, 0, 0)$. To get an orthonormal basis, we can then define the other unit basis vectors as $\hat{k} = (1, \frac{\pi}{2}, 0)$ and $\hat{l} = (1, \frac{\pi}{2}, \frac{\pi}{2})$. Now we have

$$\begin{aligned} \cos \theta_{jq} &= \hat{j} \cdot \hat{q} = (\hat{j} \cdot \hat{i})(\hat{q} \cdot \hat{i}) + (\hat{j} \cdot \hat{k})(\hat{q} \cdot \hat{k}) + (\hat{j} \cdot \hat{l})(\hat{q} \cdot \hat{l}) \\ &= \cos \theta_{ij} \cos \theta_{iq} + \sin \theta_{ij} \cos \phi_{ij} \sin \theta_{iq} \cos \phi_{iq} + \sin \theta_{ij} \sin \phi_{ij} \sin \theta_{iq} \sin \phi_{iq} \end{aligned} \quad (4.104)$$

If we take j in the (\hat{i}, \hat{k}) -plane, that is, setting ϕ_{ij} to zero, this gives

$$1 - \cos \theta_{jq} = 1 - \cos(\theta_{iq} - \theta_{ij}) = 1 - \cos \theta_{iq} \cos \theta_{ij} - \sin \theta_{iq} \sin \theta_{ij} = a - b \cos \phi_{iq} \quad (4.105)$$

with

$$\begin{aligned} a &= 1 - \cos \theta_{ij} \cos \theta_{iq}, \\ b &= \sin \theta_{ij} \sin \theta_{iq} \end{aligned} \quad (4.106)$$

Now define $z = e^{-i\phi_{iq}}$. Then

$$d\phi_{iq} = \frac{dz}{-iz} \quad (4.107)$$

and

$$\cos(\phi_{iq}) = \frac{1}{2}(e^{i\phi_{iq}} + e^{-i\phi_{iq}}) = \frac{1}{2}\left(z + \frac{1}{z}\right) \quad (4.108)$$

So we get, using Cauchy's theorem:

$$\int_0^{2\pi} \frac{d\phi_{iq}}{2\pi} \frac{1}{1 - \cos \theta_{jq}} = \frac{1}{2\pi} \oint \frac{dz}{-iz} \frac{1}{a - \frac{b}{2}\left(z + \frac{1}{z}\right)} = \frac{1}{i\pi b} \oint \frac{dz}{z^2 - \frac{2za}{b} - 1} \quad (4.109)$$

Now if we define

$$z_{\pm} = \frac{a}{b} \pm \sqrt{\frac{a^2}{b^2} - 1} \quad (4.110)$$

Then

$$\begin{aligned} (z - z_+)(z - z_-) &= \left(z - \frac{a}{b} - \sqrt{\frac{a^2}{b^2} - 1}\right) \left(z - \frac{a}{b} + \sqrt{\frac{a^2}{b^2} - 1}\right) \\ &= z^2 + \frac{a^2}{b^2} - \frac{a^2}{b^2} + 1 - 2\frac{az}{b} = z^2 - \frac{2az}{b} + 1 \end{aligned} \quad (4.111)$$

Now only z_- is within the integration area, so the residue theorem tells us that

$$\oint \frac{dz}{(z - z_+)(z - z_-)} = 2\pi i \text{Res}(z = z_-) = 2\pi i \frac{1}{z_- - z_+} = 2\pi i \frac{1}{2\sqrt{\frac{a^2}{b^2} - 1}} \quad (4.112)$$

So we get for the whole integral

$$\int_0^{2\pi} \frac{d\phi_{iq}}{2\pi} \frac{1}{1 - \cos \theta_{jq}} = \frac{1}{i\pi b} \frac{2\pi i}{2\sqrt{\frac{a^2}{b^2} - 1}} = \frac{1}{\sqrt{a^2 - b^2}} \quad (4.113)$$

Since

$$\begin{aligned} a^2 - b^2 &= (1 - \cos \theta_{ij} \cos \theta_{iq})^2 - (\sin \theta_{ij} \sin \theta_{iq})^2 \\ &= 1 - 2 \cos \theta_{ij} \cos \theta_{iq} + \cos^2 \theta_{ij} \cos^2 \theta_{iq} - \sin^2 \theta_{ij} \sin^2 \theta_{iq} \\ &= 1 - 2 \cos \theta_{ij} \cos \theta_{iq} + (1 - \sin^2 \theta_{ij}) \cos^2 \theta_{iq} - \sin^2 \theta_{ij} (1 - \cos^2 \theta_{iq}) \\ &= 1 - 2 \cos \theta_{ij} \cos \theta_{iq} + \cos^2 \theta_{iq} - \sin^2 \theta_{ij} \cos^2 \theta_{iq} - \sin^2 \theta_{ij} + \sin^2 \theta_{ij} \cos \theta_{iq} \\ &= -2 \cos \theta_{ij} \cos \theta_{iq} + \cos^2 \theta_{ij} + \cos^2 \theta_{iq} \\ &= (\cos \theta_{ij} - \cos \theta_{iq})^2 \end{aligned} \quad (4.114)$$

we get

$$\int_0^{2\pi} \frac{d\phi_{iq}}{2\pi} \frac{1}{1 - \cos \theta_{jq}} = \frac{1}{\sqrt{a^2 - b^2}} = \frac{1}{|\cos \theta_{ij} - \cos \theta_{iq}|} \quad (4.115)$$

Now we have

$$\begin{aligned}
\int_0^{2\pi} \frac{d\phi_{iq}}{2\pi} U_{ij} &= \int_0^{2\pi} \frac{d\phi_{iq}}{2\pi} \left(\frac{1 - \cos \theta_{ij}}{(1 - \cos \theta_{iq})(1 - \cos \theta_{jq})} + \frac{1}{1 - \cos \theta_{iq}} + \frac{1}{1 - \cos \theta_{jq}} \right) \\
&= \int_0^{2\pi} \frac{d\phi_{iq}}{2\pi} \frac{1}{2(1 - \cos \theta_{iq})} \left(1 + \frac{\cos \theta_{iq} - \cos \theta_{ij}}{1 - \cos \theta_{jq}} \right) \\
&= \frac{1}{2(1 - \cos \theta_{iq})} \left(1 + \frac{\cos \theta_{iq} - \cos \theta_{ij}}{|\cos \theta_{iq} - \cos \theta_{ij}|} \right)
\end{aligned} \tag{4.116}$$

The second term in the brackets is +1 if θ_{iq} is smaller than θ_{ij} and -1 if it is smaller.

$$\int_0^{2\pi} \frac{d\phi_{iq}}{2\pi} U_{ij} = \begin{cases} \frac{1}{(1 - \cos \theta_{iq})} & \text{for } \theta_{iq} < \theta_{ij}, \\ 0 & \text{otherwise} \end{cases} \tag{4.117}$$

Therefore, the whole expression is zero for $\theta_{iq} > \theta_{ij}$. This means the parton associated with q can only be within a cone around parton i . Hence we get angular ordering: A daughter parton can only be emitted within a cone around the mother parton. See figure 4.3

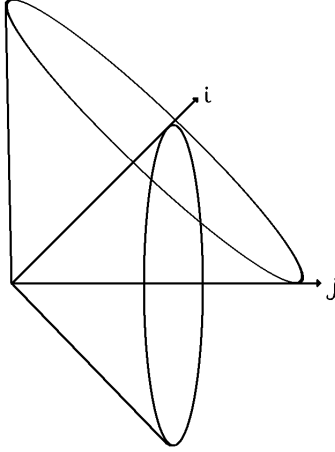


Figure 4.3: Parton branching by paths.

Chapter 5

Sampling distributions

Distributions give a chance of something happening at a certain value of the variable one is considering. Usually this evolving parameter is some energy scale, but it could also be time (f.e. in radioactive decay). In Monte Carlo simulation, one often rather wants to select a random value with a chance given by the distribution. This is called sampling a distribution. In this chapter we will show how to correctly do this, both for simple distributions and more complicated ones. For the latter we will formulate a numerical sampling technique called the Veto algorithm.

5.1 Sampling a simple distribution

We want to generate a branching of energy scale t given a probability distribution $f(t)$. The chance that the first branching happens at t is the chance that a branching happens at t times the chance that it did not happen before t :

$$P(t) = -\frac{d\Delta(t)}{dt} = f(t)\Delta(t) \quad (5.1)$$

where $\Delta(t)$ is the Sudakov form factor. Hence we can solve this for $\Delta(t)$:

$$\Delta(t) = \Delta(0) \exp \left\{ -\int_0^t dt' f(t') \right\} \quad (5.2)$$

Since we need to normalise anyway in the end, we can set $\Delta(0) = 1$ for simplicity. Filling this in into equation (5.1), we get:

$$P(t) = f(t) \exp \left\{ -\int_0^t dt' f(t') \right\} \quad (5.3)$$

We now have $P(t)$ as the distribution in events. We will use the Monte Carlo procedure:

$$\int_0^t dt' P(t') = \mathcal{R} \int_0^\infty dt' P(t') \quad (5.4)$$

Where \mathcal{R} is a random number between 0 and 1. For the left hand side we can write

$$\int_0^t dt' P(t') = \Delta(0) - \Delta(t) = 1 - \exp \left\{ -\int_0^t dt' f(t') \right\} \quad (5.5)$$

and for the right hand side:

$$\mathcal{R} \int_0^\infty dt' P(t') = \mathcal{R}(\Delta(0) - \Delta(\infty)) = \mathcal{R} \quad (5.6)$$

Using the fact that we can replace $\mathcal{R} \leftrightarrow 1 - \mathcal{R}$, we can write equation (5.4) as:

$$\exp \left\{ - \int_0^t dt' f(t') \right\} = \mathcal{R} \quad (5.7)$$

If $f(t)$ has a primitive $F(t)$, we can solve this for t as:

$$t = F^{-1}(F(0) - \ln \mathcal{R}) \quad (5.8)$$

5.1.1 Unitarity

We will show that our definition for $P(t)$ obeys unitarity: The chance of not branching between 0 and ∞ is:

$$P_0 = \exp \left\{ - \int_0^\infty dt' f(t') \right\} \quad (5.9)$$

The chance of branching exactly once is the chance of not branching until t , then branching at t and then not branching between t and ∞ and this integrated over all possible values of t .

$$\begin{aligned} P_1 &= \int_0^\infty dt \left[\exp \left\{ - \int_0^t dt' f(t') \right\} f(t) \exp \left\{ - \int_t^\infty dt' f(t') \right\} \right] \\ &= \int_0^\infty dt \left[f(t) \exp \left\{ - \int_0^\infty dt' f(t') \right\} \right] \end{aligned} \quad (5.10)$$

The chance of branching twice can be constructed in a similar way:

$$\begin{aligned} P_2 &= \int_0^\infty dt_1 \int_{t_1}^\infty dt_2 \left[\exp \left\{ - \int_0^{t_1} dt' f(t') \right\} f(t_1) \exp \left\{ - \int_{t_1}^{t_2} dt' f(t') \right\} f(t_2) \exp \left\{ - \int_{t_2}^\infty dt' f(t') \right\} \right] \\ &= \int_0^\infty dt_1 \int_{t_1}^\infty dt_2 \left[f(t_1) f(t_2) \exp \left\{ - \int_0^\infty dt' f(t') \right\} \right] \end{aligned} \quad (5.11)$$

This can also be written as:

$$\int_0^\infty dt_2 \int_0^{t_2} dt_1 \left[f(t_1) f(t_2) \exp \left\{ - \int_0^\infty dt' f(t') \right\} \right] \quad (5.12)$$

We can now interchange the indices (which are only integration variables) to write P_2 as the sum of those two, while compensating by a factor $\frac{1}{2!}$ to avoid overcounting:

$$\begin{aligned} P_2 &= \frac{1}{2!} \left[\int_0^\infty dt_1 \int_{t_1}^\infty dt_2 f(t_1) f(t_2) + \int_0^\infty dt_1 \int_0^{t_1} dt_2 f(t_1) f(t_2) \right] \exp \left\{ - \int_0^\infty dt' f(t') \right\} \\ &= \frac{1}{2!} \left(\int_0^\infty dt f(t) \right)^2 \exp \left\{ - \int_0^\infty dt' f(t') \right\} \end{aligned} \quad (5.13)$$

Generalising to n branchings, this becomes:

$$P_n = \frac{1}{n!} \left(\int_0^\infty dt f(t) \right)^n \exp \left\{ - \int_0^\infty dt' f(t') \right\} \quad (5.14)$$

So if we sum all possible branchings we get:

$$\begin{aligned} \sum_{n=0}^\infty P_n &= \sum_{n=0}^\infty \frac{1}{n!} \left(\int_0^\infty dt f(t) \right)^n \exp \left\{ - \int_0^\infty dt' f(t') \right\} \\ &= \exp \left\{ \int_0^\infty dt f(t) \right\} \times \exp \left\{ - \int_0^\infty dt' f(t') \right\} = 1 \end{aligned} \quad (5.15)$$

Which proves unitarity.

5.2 Veto algorithm

Equation (5.8) can only be used if $f(t)$ has a primitive $F(t)$ that has inverse $F^{-1}(t)$. If it does not have such a function, it becomes necessary to use the Veto algorithm. For this purpose, one first needs a function $g(t)$ that does have a primitive $G(t)$ with an inverse $G^{-1}(t)$ and that is required to obey $g(t) \geq f(t)$ for all $t \geq 0$. The Veto algorithm is now:

1. Start with $i = 0$ and $t_i = 0$
2. $i \rightarrow i + 1$
3. Generate Random \mathcal{R}
4. $t_i = G^{-1}(G(t_{i-1}) - \ln \mathcal{R})$
5. If $t_i \leq t_{i-1}$, return to step 3
6. If $\frac{f(t_i)}{g(t_i)} \leq \mathcal{R}$, return to step 2
7. t_i is accepted as final t

We will prove this gives the correct distribution in t by a reasoning very similar to the prove of unitarity, but now instead of systematically going through the number of branchings, we will go through the different values of i . The probability that the first try (t_1) immediately is accepted is the chance to pick t_1 in the distribution $P^g(t)$ times the chance that it passed step 6:

$$P_0(t) = P^g(t) \frac{f(t)}{g(t)} = g(t) \exp \left\{ - \int_0^t dt' g(t') \right\} \frac{f(t)}{g(t)} = f(t) \exp \left\{ - \int_0^t dt' g(t') \right\} \quad (5.16)$$

The chance that t_1 is rejected but t accepted is

$$P^g(t_1) \times P(t_1 \text{ is rejected}) \times P^g(t) \times P(t \text{ is accepted}) \quad (5.17)$$

Since t_1 can have any value smaller than t , we have to integrate over this interval:

$$\begin{aligned} P_1(t) &= g(t_1) \int_0^t dt_1 \exp \left\{ - \int_0^{t_1} dt' g(t') \right\} \left[1 - \frac{f(t_1)}{g(t_1)} \right] g(t) \exp \left\{ - \int_{t_1}^t dt' g(t') \right\} \frac{f(t)}{g(t)} \\ &= f(t) \exp \left\{ - \int_0^t dt' g(t') \right\} \int_0^t dt_1 [g(t_1) - f(t_1)] \\ &= P_0(t) \int_0^t dt_1 [g(t_1) - f(t_1)] \end{aligned} \quad (5.18)$$

If two rejected steps are necessary: $0 \leq t_1 \leq t_2 \leq t$, we get in a similar way:

$$P_2(t) = P_0(t) \int_0^t dt_1 [g(t_1) - f(t_2)] \int_{t_1}^t dt_2 [g(t_1) - f(t_2)] = \frac{1}{2!} \left(\int_0^t dt' [g(t') - f(t')] \right)^2 \quad (5.19)$$

Where we used the same symmetrisation trick as in the unitarity calculation. Generalising this and summing it gives the total probability of selecting t with any number of steps:

$$\begin{aligned} P(t) &= \sum_{n=0}^{\infty} P_n(t) = \sum_{n=0}^{\infty} P_0(t) \frac{1}{n!} \left(\int_0^t dt' [g(t') - f(t')] \right)^n \\ &= f(t) \exp \left\{ - \int_0^t dt' g(t') \right\} \times \exp \left\{ \int_0^t dt' [g(t') - f(t')] \right\} \\ &= f(t) \exp \left\{ - \int_0^t dt' f(t') \right\} \end{aligned} \quad (5.20)$$

Which gives the desired distribution as given in equation (5.3).

5.2.1 Two examples

As an example, let's consider two choices for $f(t)$. First, let us choose the simple

$$f(t) = t \tag{5.21}$$

which gives the distribution:

$$P(t) = f(t) \exp \left\{ - \int_0^t dt' f(t') \right\} = t e^{-\frac{1}{2}t^2} \tag{5.22}$$

$f(t)$ is simple enough to have a primitive which has an inverse:

$$F(t) = \frac{1}{2}t^2 \tag{5.23}$$

$$F^{-1}(x) = \sqrt{2x} \tag{5.24}$$

This means we can sample the distribution using equation (5.8): Doing this n times and binning this in a histogram eventually gives back the $P(t)$ of equation (5.22).

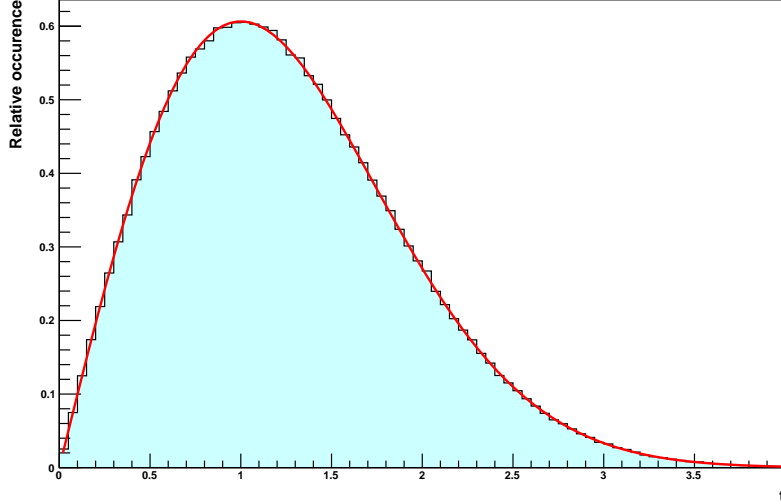


Figure 5.1: The red line is the function $P(t) = t e^{\frac{1}{2}t^2}$. The histogram is obtained by using equation (5.8) with the $F(t)$ from equation (5.23). Refer to appendix A for the code of the generator.

Now let's try the Veto algorithm for this function. The function

$$g(t) = t^2 + a \tag{5.25}$$

is always greater than $f(t)$ provided that $a > \frac{1}{4}$. It has an primitive that has an inverse, although the latter requires solving a third degree polynomial.

$$G(t) = \frac{1}{3}t^3 + at \tag{5.26}$$

$$G^{-1}(t) = \sqrt[3]{\frac{2a^3}{(3x + \sqrt{4a^3 + 9x^2})}} + \sqrt[3]{\frac{(3x + \sqrt{4a^3 + 9x^2})}{2}} \tag{5.27}$$

We can use this to test the Veto algorithm. The result is shown in figure 5.2.

With $f(t) = t$ there was no real need to use the Veto algorithm, so let us look at an example in which it is necessary. Take for $f(t)$

$$f(t) = \ln(t + 1) \tag{5.28}$$

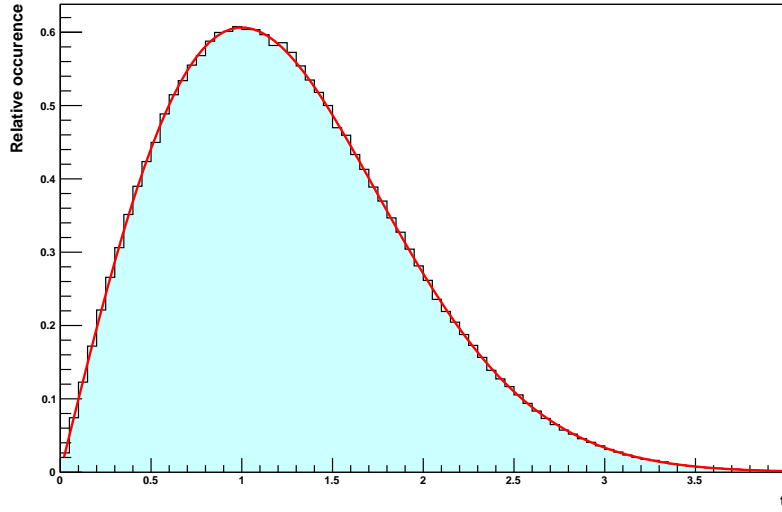


Figure 5.2: The red line is the function $P(t) = te^{\frac{1}{2}t^2}$. The histogram is obtained by using the Veto algorithm with the $g(t)$ and $G(t)$ from equation (5.25) and (5.26). Refer to appendix A for the code of the generator.

This function does have a primitive, but this primitive does not have an inverse:

$$F(t) = (t + 1) \ln(t + 1) - t \quad (5.29)$$

It does however allow us to calculate the expected function $P(t)$

$$P(t) = \ln(t + 1) \exp \left\{ - \int_0^t dt' \ln(t' + 1) \right\} = \ln(t + 1) \exp \{ -(t + 1) \ln(t + 1) + t \} \quad (5.30)$$

Using for $g(t)$ a simple function with a simple primitive:

$$g(t) = t + 1 \quad (5.31)$$

$$G(t) = \frac{1}{2}t^2 + t \quad (5.32)$$

$$G^{-1} = \sqrt{1 + 2x} - 1 \quad (5.33)$$

we can again use the veto algorithm. The results are shown in figure 5.3.

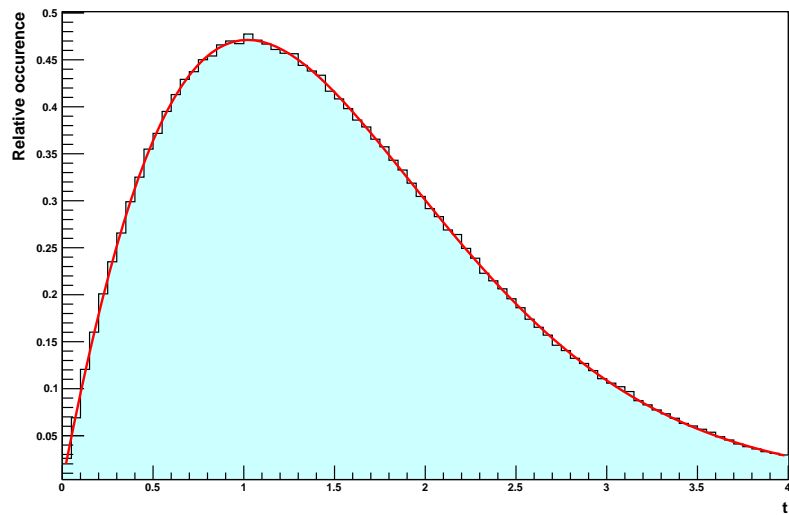


Figure 5.3: The red line is the function $P(t) = \ln(t+1) \exp \{-(t+1) \ln(t+1) + t\}$. The histogram is obtained by using the Veto algorithm with the $g(t)$ from equation (5.31). Refer to appendix A for the code of the generator.

Chapter 6

Colour Dipole Model

This chapter follows several chapters from *The Lund Model* by B. Andersson[1]. The colour dipole model (CDM) is a model for parton showering that takes the colour of the partons into account. It was originally developed by Gustafson and Andersson in a series of articles in the 1980s[5][6][2]. It is implemented in the QCD parton shower simulation program Ariadne [7]. Consider a quark anti-quark pair, like in the first chapter. The anti-quark carries the anti-colour of the quark, f.e. if the quark is red, the anti-quark is anti-red. So even after the initial splitting, the quarks are in a sense connected by a colour dipole. If either of the two particles now emits a gluon, the particle that emitted it changes colour with its anti-colour being carried by the gluon, together with the original colour. In our previous example, for instance, the red quark could split in a green quark and a red-anti-green gluon. Now there are two dipoles, one 'green' one spanning between the quark and the gluon, and one 'red' one spanning between the gluon and the anti-quark. This picture is exactly the same if it was the anti-quark which emitted the gluon: Also in this case the result would be a 'green' and a 'red' dipole. Therefore, the colour dipole model does not consider

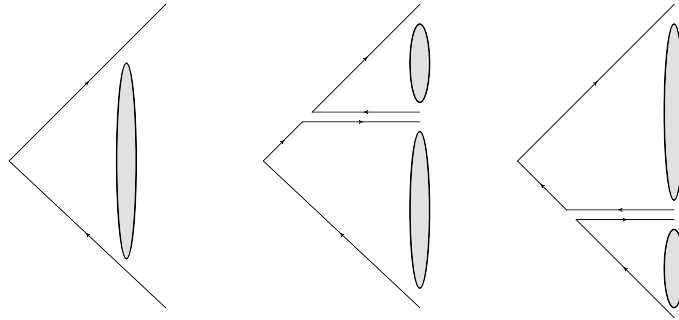


Figure 6.1: The lines represent the colour flow. Irrespective of which parton emits the gluon, the same dipoles are created.

partons radiating of more partons, but instead considers a shower to be dipoles splitting into more dipoles. The cross sections remain the same as in the original treatment in chapter 2 (equations

(2.32)) and we will also keep using the light cone variables and other variables from that chapter:

$$\begin{aligned}
s &= (p_1 + p_3)^2 = 2p_1 \cdot p_3 \\
x_i &= \frac{2E_i}{\sqrt{s}} \\
\sum x_i &= 2 \\
s_{ij} &= s(1 - x_k) \\
k_\perp^2 &= s(1 - x_1)(1 - x_3) \\
y &= \frac{1}{2} \ln \left(\frac{1 - x_1}{1 - x_3} \right)
\end{aligned} \tag{6.1}$$

We will only consider soft emissions, i.e. $x_2 \ll x_1, x_3$

6.1 Phase space triangle

We want to establish what the allowed phase space is for a new emission. Since $s = s_{12} + s_{13} + s_{23}$, and $x_2 < 1$, we can conclude that

$$s \geq s_{12} + s_{23} \tag{6.2}$$

Now using the definitions of the transverse momentum k_\perp^2 and the rapidity y , we can see that this equals $2k_\perp \sqrt{s} \cosh y$:

$$\begin{aligned}
2k_\perp \sqrt{s} \cosh y &= 2\sqrt{s(1 - x_1)(1 - x_3)} \sqrt{s} \left(\frac{e^y + e^{-y}}{2} \right) \\
&= s\sqrt{(1 - x_1)(1 - x_3)} \left[\left(\frac{1 - x_1}{1 - x_3} \right)^{1/2} + \left(\frac{1 - x_3}{1 - x_1} \right)^{1/2} \right] \\
&= s[(1 - x_1) + (1 - x_3)] = s_{12} + s_{23}
\end{aligned} \tag{6.3}$$

We can rewrite this as:

$$\frac{\sqrt{s}}{k_\perp} \geq e^y + e^{-y} \tag{6.4}$$

We will now show that this condition can be approximated in the limit $x_2 \ll 1$ as $|y| \leq \frac{L - \kappa}{2}$ where

$$\kappa(k_\perp^2) = \ln \left(\frac{k_\perp^2}{s_0} \right) \tag{6.5}$$

and $L = \kappa(\sqrt{s})$, s_0 being a not important cut off value.

So,

$$|y| \leq \frac{L - \kappa}{2} = \frac{1}{2} \left(\ln \frac{s}{s_0} \frac{s_0}{k_\perp^2} \right) = \ln \frac{\sqrt{s}}{k_\perp} \tag{6.6}$$

This means that y can take values between $\pm \ln \frac{\sqrt{s}}{k_\perp}$. On this interval $e^y + e^{-y} \leq \frac{\sqrt{s}}{k_\perp} + \frac{k_\perp}{\sqrt{s}}$. Filling in the definition of k_\perp^2 , this last term becomes $\frac{k_\perp}{\sqrt{s}} = (1 - x_1)(1 - x_3)$. Since k_\perp is the norm of the transverse momentum and cannot become negative, we get the condition that $x_1, x_3 \leq 1$. In the limit under consideration this means that both x_1 and x_3 are very close to one, which means that the second term can be neglected. Hence we end up with:

$$e^y + e^{-y} \leq \frac{\sqrt{s}}{k_\perp} \tag{6.7}$$

which was indeed what we were looking for. Rewriting the inequality from equation (6.6) as

$$\kappa \leq L - 2|y| \tag{6.8}$$

we get figure 6.2: a triangle with the endpoints at $\kappa = \ln s$ and $y = \pm \ln \sqrt{s}$. All the values within the triangle are possible. Note that the bottom of the triangle lies at $\ln(k_{\perp}^2 = 0) = -\infty$. This needs not to be a problem in our calculations, since we will always calculate heights as differences of logarithms.

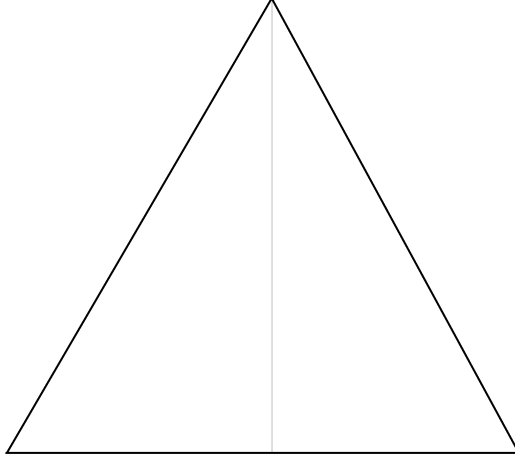


Figure 6.2: The domain of $\kappa(k_{\perp}^2)$ is $(\infty, \ln s)$. At $\kappa = -\infty$, y ranges from $-\frac{1}{2} \ln s$ to $+\frac{1}{2} \ln s$. This means that the triangle is equilateral.

6.2 Second emission

Using the expressions for the transverse momentum squared and the rapidity, we can rewrite the expressions for s_{12} and s_{23} .

$$y_1 = \frac{1}{2} \ln \left(\frac{1-x_1}{1-x_3} \right) = \frac{1}{2} \ln \left(\frac{k_{\perp 1}^2}{s(1-x_3)} \right) = \ln \left(\frac{k_{\perp 1} \sqrt{s}}{s_{12}} \right) \quad (6.9)$$

$$\Rightarrow s_{12} = k_{\perp 1} \sqrt{s} e^{-y_1}$$

Where the subscript 1 indicates that we are talking about the first emission. And similarly

$$s_{23} = k_{\perp 1} \sqrt{s} e^{y_1} \quad (6.10)$$

Taking the logarithm of these expressions, we get:

$$\ln(s_{12}/s_0) = \ln(k_{\perp 1}/\sqrt{s_0}) + \ln(\sqrt{s}/\sqrt{s_0}) + \ln(e^{y_1}) = (\kappa_1 + L)/2 + y_1 \quad (6.11)$$

Calling this expression L_{12} , we have the equivalent of expression (6.5)

$$L_{12} \equiv \ln(s_{12}/s_0) = (\kappa_1 + L)/2 + y_1 \quad (6.12)$$

$$L_{23} \equiv \ln(s_{23}/s_0) = (\kappa_1 + L)/2 - y_1 \quad (6.13)$$

The original triangle had a base line with length L , so we can picture this increase in phase space by an extra fold with height κ_1 and the tip at a length of $\kappa_1/2$. L_{12} is then the piece along the

original triangle ($L/2 + y_1$) plus the length to the tip $\kappa_1/2$ and L_{23} is similarly the piece back from the tip to the original triangle ($\kappa_1/2$) and the rest of the latter's base line ($L/2 - y_1$). The total length of the base line therefore has increased to $L_{12} + L_{23} = L + \kappa_1$.

Angular ordering implies that there can be no gluons with a higher transverse momentum than the first one, this means that there are no gluons in top of the triangle (above κ_1). Apart from

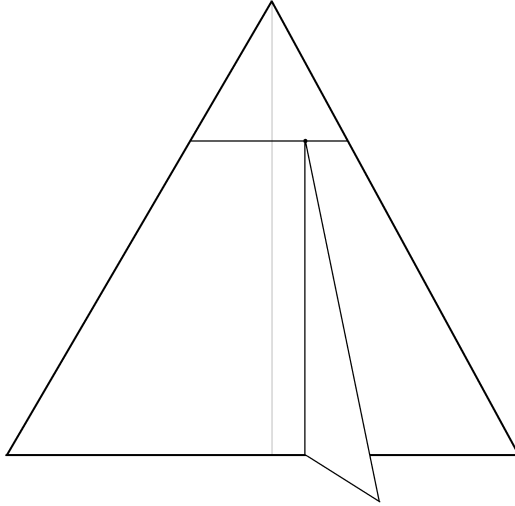


Figure 6.3: With the first emission the phase space increases. This can be pictured as a fold coming out of the plane.

this region there are two other regions, two cutoff triangles. The left one (which is best seen if you imagine you turn the tip to the right) corresponds to the new qg -dipole s_{12} . The right one (turn the tip to the left) corresponds to the $\bar{q}g$ -dipole s_{23} (see figure 6.4). This process can be repeated in one of the new cutoff triangles. The new fold that then appears can be either in the plane of the original triangle or in the previous fold (figure 6.5).

6.3 Numerically evolving down the triangle

In chapter 2 we saw that all cross sections could be approximated by

$$\frac{d\sigma}{dk_{\perp}^2 dy} \propto \frac{1}{k_{\perp}^2} \quad (6.14)$$

Now in chapter 5 we considered branching probabilities $P(t)$ using an evolving parameter t that started at 0 and went on to infinity. In this case, our evolving parameter is k_{\perp}^2 , but this evolves from a maximum value, $k_{\perp}^2 = s$, down to some cut off value in general or to $k_{\perp}^2 = 0$ at most. To be able to still use our description from chapter 5, we can define an evolving parameter t as:

$$t = \ln s - \ln k_{\perp}^2 \quad (6.15)$$

At $k_{\perp}^2 = s$, $t = 0$ and as $k_{\perp}^2 \rightarrow 0$, $t \rightarrow \infty$. We now get the differential probability of emitting a gluon as:

$$\frac{dP}{dk_{\perp}^2 dy} = \frac{d\sigma}{dk_{\perp}^2 dy} \exp \left\{ - \int_{k_{\perp}^2}^s dq_{\perp}^2 \int_{y_{\min}(q_{\perp}^2)}^{y_{\max}(q_{\perp}^2)} \frac{d\sigma}{dq_{\perp}^2 dy} \right\} \quad (6.16)$$

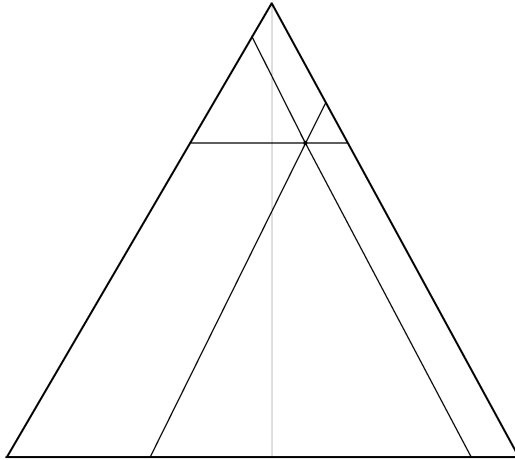


Figure 6.4: The first emission creates two new triangles. Angular ordering implies that there can be no gluons in the triangle above the first emission.

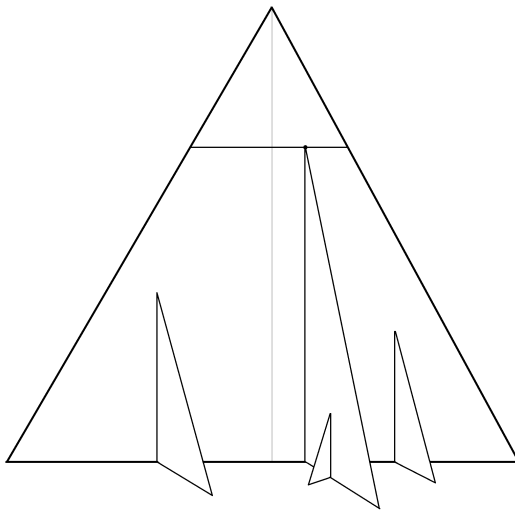


Figure 6.5: On both sides of the fold, the process of showering can be repeated. The new folds can appear both in the original plane and in the old folds.

A problem here, is that the equivalent of $f(t)$, being the cross section, is to be integrated over two variables which are not independent. At a certain k_{\perp}^2 , the height of the triangle is $L - \kappa$ and the width is $2\frac{L-\kappa}{2}$ is also $L - \kappa$. The surface of the triangle, which is the equivalent of $F(t)$ in chapter

5, is

$$F(t) = \frac{1}{2}(\ln s - \ln k_{\perp}^2)^2 = \frac{1}{2}t^2 \quad (6.17)$$

which has an inverse:

$$F^{-1}(x) = \sqrt{2x} \quad (6.18)$$

This means we will not need the Veto algorithm, but can use the simple relation:

$$t = F^{-1}(F(0) - \ln(R)) = \sqrt{-2 \ln R} \quad (6.19)$$

Or, to directly generate the new κ value

$$\kappa = \ln s - \sqrt{-2 \ln R} \quad (6.20)$$

To get the distribution in k_{\perp}^2 only,

$$\frac{dP}{dk_{\perp}^2} = \frac{d\sigma}{dk_{\perp}^2} e^{-F} \quad (6.21)$$

we need

$$\frac{d\sigma}{dk_{\perp}^2} = \frac{dF(k_{\perp}^2)}{dk_{\perp}^2} = \frac{d}{dk_{\perp}^2} \left[\frac{1}{2} \ln^2 \left(\frac{s}{k_{\perp}^2} \right) \right] = \ln \left(\frac{s}{k_{\perp}^2} \right) \frac{k_{\perp}^2}{s} \left(-\frac{s}{(k_{\perp}^2)^2} \right) = \frac{1}{k_{\perp}^2} \ln \left(\frac{k_{\perp}^2}{s} \right) \quad (6.22)$$

So we expect the differential distribution that is generated by choosing κ according to (6.20) to be

$$\frac{dP}{dk_{\perp}^2} = \frac{1}{k_{\perp}^2} \ln \left(\frac{k_{\perp}^2}{s} \right) \exp \left\{ -\frac{1}{2} \ln^2 \left(\frac{s}{k_{\perp}^2} \right) \right\} \quad (6.23)$$

as indeed it does, see figure 6.6

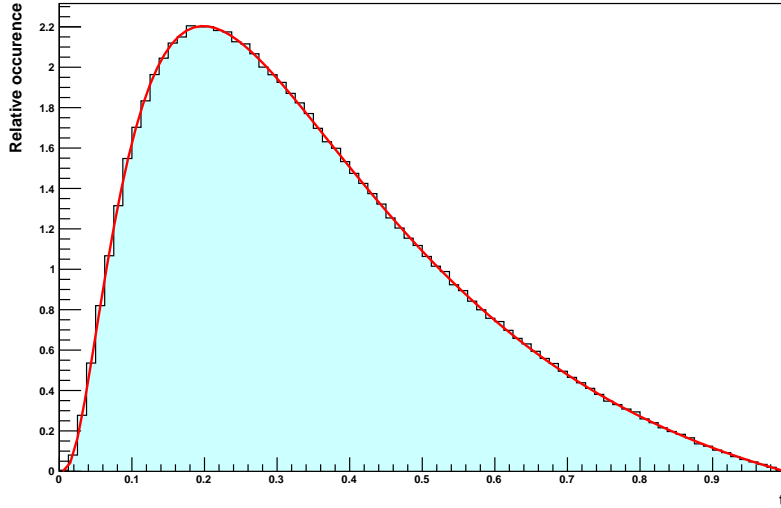


Figure 6.6: The red line is the function $\frac{dP}{dk_{\perp}^2} = \frac{1}{k_{\perp}^2} \ln \left(\frac{k_{\perp}^2}{s} \right) \exp \left\{ -\frac{1}{2} \ln^2 \left(\frac{s}{k_{\perp}^2} \right) \right\}$. The histogram is obtained by using equation (5.8) with the $F(t)$ from equation (6.17). Refer to appendix A for the code of the generator.

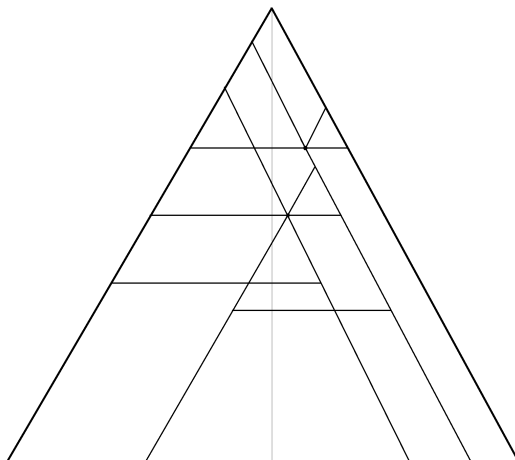


Figure 6.7: The phase space for the second emission is bounded by the triangle it is in (in this case, the left one) and, because of angular ordering, by the κ of the last emission. Therefore one gets a trapezium. This emission in turn generates two subsequent trapezia as phase spaces for the following emissions. Note that the dipoles shower independent of each other: A emission in a left triangle does not affect the probability of emission in the corresponding right one, nor in any other triangle before.

6.3.1 Two emissions

But this is only the first emission. For the second emission, which cannot have a $\kappa_2 \geq \kappa_1$, the equivalent of $F(t)$ is the trapezium bounded like in figure 6.7. More general, for an emission $i + 1$, we have as evolution variable:

$$t = \ln(k_{i\perp}^2) - \ln(k_{i+1\perp}^2) \quad (6.24)$$

But the surface of the trapezium becomes increasingly complicated, because it depends on whether you choose the left or right triangle. The trapezium between κ_i and κ_{i+1} has two oblique sides, one of which is defined by y_i , the other by the last time the choice between the left and right triangle has changed. This can be traced back by looking at the top of the largest triangle that has this one side as part of its own side. Therefore, call the variables of this last change $(\kappa_{\text{top}}, y_{\text{top}})$, and define $H = \{-1, 0, 1\}$ as the handedness of a triangle with $H = 0$ for the original triangle, $H = 1$ for a left triangle and $H = -1$ of a right triangle. The length of the upper piece of the trapezium then becomes

$$\frac{\kappa_{\text{top}} - \kappa_i}{2} + H_i y_i + H_{\text{top}} y_{\text{top}} \quad (6.25)$$

And the surface of the trapezium is

$$\text{Surface} = \left(\frac{\kappa_{\text{top}} - \kappa_i}{2} + H_i y_i + H_{\text{top}} y_{\text{top}} \right) (\kappa_i - \kappa_{i+1}) + \frac{1}{2} (\kappa_i - \kappa_{i+1})^2 \quad (6.26)$$

And therefore $F(t)$ becomes

$$F(t) = At + t^2 \quad (6.27)$$

with $A = \frac{\kappa_{\text{top}} - \kappa_i}{2} + H_i y_i + H_{\text{top}} y_{\text{top}}$, which is a numerical factor since it does not depend on $k_{i+1\perp}^2$, which means it does not depend on t .

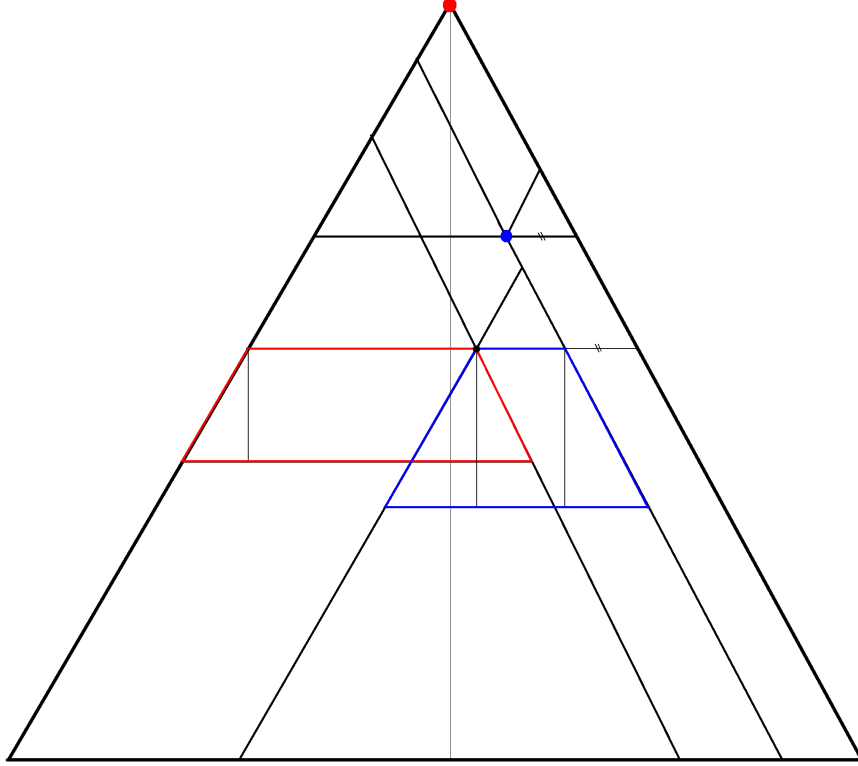


Figure 6.8: The variables that are needed for the calculation of the surface of a trapezium somewhere down the shower depends on the last time that the choice between left and right changed. Or, put differently, one has to consider the largest triangle that shares a side with the trapezium. For instance, the red trapezium (lower left) shares a side with the largest triangle, therefore it needs the values κ_0 and y_0 to calculate the surface. The length of its upper side is $\frac{\kappa_0 - \kappa_3}{2} + y_0 + y_3$. The blue trapezium on the other hand, shares only a side with the triangle formed by the variables κ_1 and y_1 . The length of its upper side is $\frac{\kappa_0 - \kappa_3}{2} - y_3 - (\frac{\kappa_0 - \kappa_1}{2} - y_1) = \frac{\kappa_1 - \kappa_3}{2} + y_1 - y_3$. In general, calling the variables of the top κ_{top} and y_{top} , the length of the upper side is $\frac{\kappa_{\text{top}} - \kappa_i}{2} + H_{\text{top}} y_{\text{top}} + H_i y_i$, where $H = \pm 1$ depending on whether it is a left (+1) or right (-1) triangle.

Note that $F(t = 0) = 0$ and that if we take $\kappa_0 = \ln s = L$ and $y_0 = 0$, we recover the triangle phase space we used in the previous section. $F(t)$ does have an inverse, which can be obtained by solving the simple second degree polynomial:

$$F^{-1}(x) = -\frac{A}{2} + \frac{1}{2}\sqrt{A^2 + 4x} \quad (6.28)$$

This in the end means we should select our t by

$$\begin{aligned}
t &= F^{-1}(F(0) - \ln R) = F^{-1}(-\ln R) = -\frac{A}{2} + \frac{1}{2}\sqrt{A^2 - 4\ln R} \\
&= -\frac{1}{2}\left(\frac{\kappa_{\text{top}} - \kappa_i}{2} + H_i y_i + H_{\text{top}} y_{\text{top}}\right) + \frac{1}{2}\sqrt{\left(\frac{\kappa_{\text{top}} - \kappa_i}{2} + H_i y_i + H_{\text{top}} y_{\text{top}}\right)^2 - 4\ln R} \quad (6.29)
\end{aligned}$$

A Markovian process is a process in which each step only depends on the previous step, not of any step before that. Equation (6.29) seems to suggest remarkably that the colour dipole model is not Markovian, since it does not only depend on the previous step, but also on the step which corresponds to the top of the triangle.

6.3.2 The end of the shower

It is not to any use to really take t to go all the way to infinity (that is, k_{\perp}^2 to zero), since before that energy scale is reached, other processes take over. To be precise: The partons start hadronising and the shower stops. Even when not taking hadronisation into account, a more practical view is that detectors have a limited energy resolution and are therefore not able to detect particles under a certain energy value. In general therefore one defines a cut off value after which the parton shower stops.

The distribution in y and k_{\perp}^2 of the last particles before the cut off value cannot - or at least not easily - be calculated or approximated algebraically, because of the dependence of t on the kinetic values that generated the top triangle. However, we know that the distribution in y should be symmetric around zero, since there is no process favouring one direction or the other and the original phase space is symmetric in y . One also expects that the distribution in y should give a hint towards the shape of the phase space. Since all particles have to end up between $-\frac{1}{2}\ln s$ and $+\frac{1}{2}\ln s$ and will have more chance of ending up around $y = 0$, we expect to see some shape in the distribution that resembles a triangle.

It's obvious that the chance of only branching once in a high value (small upper surface, therefore low chance) and not branching in the whole energy range until the cut off value (which is a large surface, therefore very low chance of not branching), is extremely small. This chance increases as we approach the cut off value: The surface of the trapezia does not necessary increase drammatially, but the surface that a trapezium could have above the cut off value does become increasingly small. One therefore would expect a distribution in k_{\perp}^2 to asymptotically grow down to where k_{\perp}^2 equals the chosen cut off value.

6.3.3 My own toy model

I programmed everything stated above in a small toy model, the code of which can be found in Appendix B. The program can be initialised with certain parameter values (s , cut off value, the scaling parameter s_0) and then generates your desired number of events. Within each event, subsequent triangles are generated, until the cut off value has been reached. The rapidity and transverse momentum squared of the last parton before the cut off value is then binned. The resulting histogram is normalised to give a distribution function which integrates to unity. The toy model does not take the different splitting functions $P_{ij}(z)$ from chapter 4 into account, i.e. it does not discriminate the different splitting possibilities.

The resulting distributions are shown in figures 6.9 and 6.10. Obviously they obey the criteria we stated in the previous subsection. But there does seem to be a small number of events that does not shower at all and therefore stays at $p_{\perp}^2 = 1$ and $y = 0$.

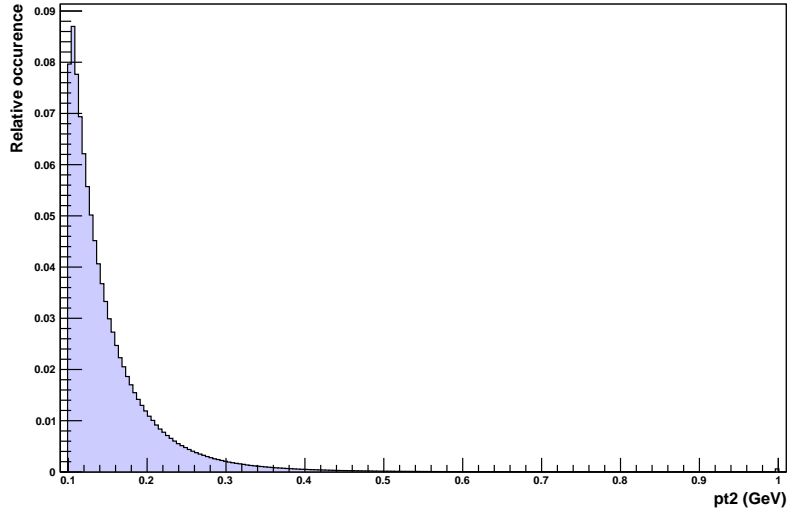


Figure 6.9: The transverse momentum squared p_{\perp}^2 of the last parton before the cut off value. This plot was generated using the code in appendix B, with $S = 0$ and cut off value $c = 0.10$.

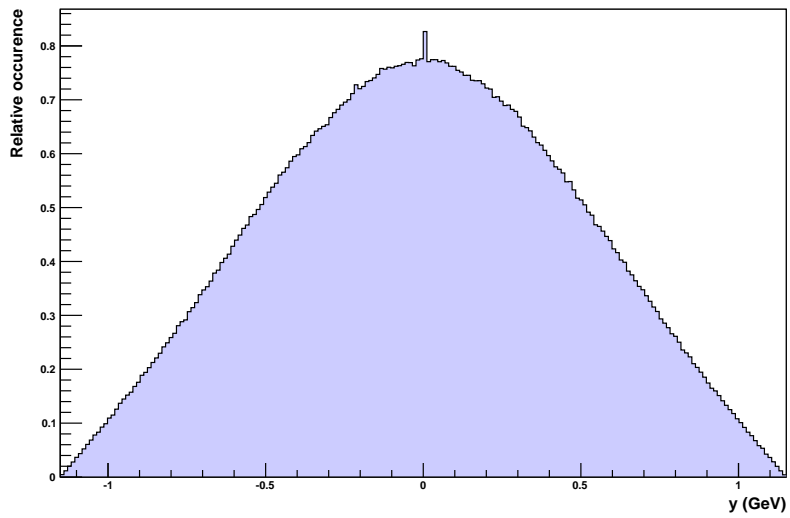


Figure 6.10: The rapidity y of the last parton before the cut off value. This plot was generated using the code in appendix B, with $S = 0$ and cut off value $c = 0.10$.

Chapter 7

Webs

The Webs model was developed in part by Carola Berger[3]. Consider the diagram on the left

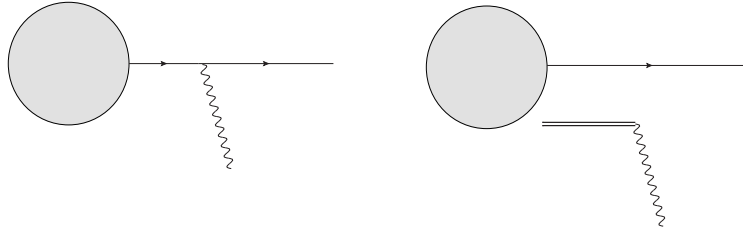


Figure 7.1: A soft emitted gluon can be cut of the rest of the diagram. The double line is the eikonal propagator.

in figure 7.1. Denoting the blob, the rest of the diagram, with \mathcal{B} , we have as matrix element

$$\mathcal{M} = g_S \bar{u}(p) \not{\epsilon} \frac{\not{p} + \not{k}}{(p+k)^2} \mathcal{B} \quad (7.1)$$

Make approximation: $k \ll p$ and $p^2 = k^2 = 0$

$$\mathcal{M} = g_S \bar{u}(p) \not{\epsilon} \frac{\not{p}}{2p \cdot k} \mathcal{B} \quad (7.2)$$

Now since:

$$\bar{u}(p) \not{\epsilon} \not{p} = \bar{u}(p) (2\epsilon \cdot p - \not{\epsilon} \not{p}) = 2\bar{u}(p) \epsilon \cdot p \quad (7.3)$$

where we used the massless Dirac equation, $\bar{u}(p) \not{p} = 0$, we get, denoting $p^\mu = |p|n^\mu$:

$$\mathcal{M} = g_S \bar{u}(p) \frac{\epsilon \cdot n}{n \cdot k} \mathcal{B} \quad (7.4)$$

This is the eikonal approximation: In the soft limit, a soft line can be 'cut off' the diagram. The diagram in that case factorises in the non-mission Born diagram \mathcal{B} times an eikonal part, $n^\mu/n \cdot k$, which does no longer depend on the momenta (p in this case) in the other part of the diagram.

7.1 Virtual Gluons

Now we will first look at virtual gluons.

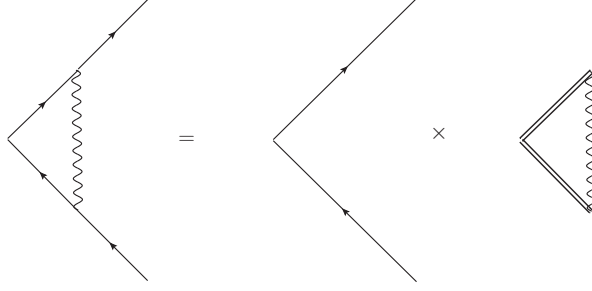


Figure 7.2: The one loop diagram can be written as the Born diagram times the eikonal loop diagram.

7.1.1 One virtual gluon

We first consider the one loop:

$$\mathcal{M} = g_S^2 \text{Tr}[T^a T^a] \bar{u}(p_1) \gamma^\mu \frac{(\not{p}_1 + \not{k})}{(q_1 + k)^2} \frac{(\not{q} - \not{p}_2)}{(q - p_2)^2} \gamma^\nu \frac{\eta_{\mu\nu}}{q^2} v(p_2) \quad (7.5)$$

$p_i^2 = k^2 = 0$ so

$$g_S^2 \text{Tr}[T^a T^a] \bar{u}(p_1) \frac{2p_1^\mu}{2p_1 \cdot q} \frac{2p_2^\nu}{2k \cdot p_2} \frac{\eta_{\mu\nu}}{q^2} v(p_2) \quad (7.6)$$

$p_1^\mu = |p_1|n_1^\mu, p_2^\nu = |p_2|n_2^\nu$

$$g_S^2 \text{Tr}[T^a T^a] \bar{u}(p_1) \frac{1}{q^2} \frac{n_1 \cdot n_2}{(n_1 \cdot q)(n_2 \cdot q)} v(p_2) = \mathcal{B} \mathcal{X} \quad (7.7)$$

We defined:

$$\mathcal{X}_V = g_S^2 \text{Tr}[T^a T^a] \int \frac{dq}{q^2} \frac{n_1 \cdot n_2}{(n_1 \cdot q)(n_2 \cdot q)} \quad (7.8)$$

which is a diagram of two eikonal lines which exchange a gluon. The result for the matrix element is the Born diagram \mathcal{B} times \mathcal{X}_V .

7.1.2 Two virtual gluons

The next step is to look at two exchanged gluons. In this case there are two possible diagrams, that have to be added, see figure 7.3. Let us start with the left diagram. Its matrix element is:

$$\int \frac{d^n q_1}{(2\pi)^n} \frac{d^n q_2}{(2\pi)^n} \bar{u}(p_1) \gamma^\mu \frac{\not{p}_1 + \not{q}_2}{(p_1 + q_2)^2} \frac{\eta_{\mu\nu}}{q_2^2} \gamma^\alpha \frac{\not{p}_1 + \not{q}_1 + \not{q}_2}{(p_1 + q_1 + q_2)^2} \frac{\not{p}_2 - \not{q}_1 - \not{q}_2}{(p_2 - q_1 - q_2)^2} \gamma^\beta \frac{\eta_{\alpha\beta}}{q_1^2} \frac{\not{p}_2 - \not{q}_2}{(p_2 - q_2)^2} \gamma^\nu v(p_2) \quad (7.9)$$

Again making the eikonal approximation (in this case $q_i \ll p_j$) and using the Dirac equation:

$$\int \frac{d^n q_1}{(2\pi)^n} \frac{d^n q_2}{(2\pi)^n} \bar{u}(p_1) \frac{2p_1^\mu}{2p_1 \cdot q_2} \frac{\eta_{\mu\nu}}{q_2^2} \frac{2p_1^\alpha}{2p_1 \cdot (q_1 + q_2)} \frac{2p_2^\beta}{2p_2 \cdot (q_1 + q_2)} \frac{\eta_{\alpha\beta}}{q_1^2} \frac{2p_2^\nu}{2p_2 \cdot q_2} v(p_2) \quad (7.10)$$

The result for the second possible diagram is the same with respect to the second part of the integral, but in the first part q_1 and q_2 are interchanged. Therefore, the sum of these two diagrams is:

$$\int \frac{d^n q_1}{(2\pi)^n} \frac{d^n q_2}{(2\pi)^n} \bar{u}(p_1) \frac{\eta_{\mu\nu}}{q_2^2} \left[\frac{p_1^\mu}{p_1 \cdot q_2} \frac{p_1^\alpha}{p_1 \cdot (q_1 + q_2)} + \frac{p_1^\mu}{p_1 \cdot q_1} \frac{p_1^\alpha}{p_1 \cdot (q_1 + q_2)} \right] \times \frac{p_2^\beta}{p_2 \cdot (q_1 + q_2)} \frac{\eta_{\alpha\beta}}{q_1^2} \frac{p_2^\nu}{p_2 \cdot q_2} v(p_2) \quad (7.11)$$

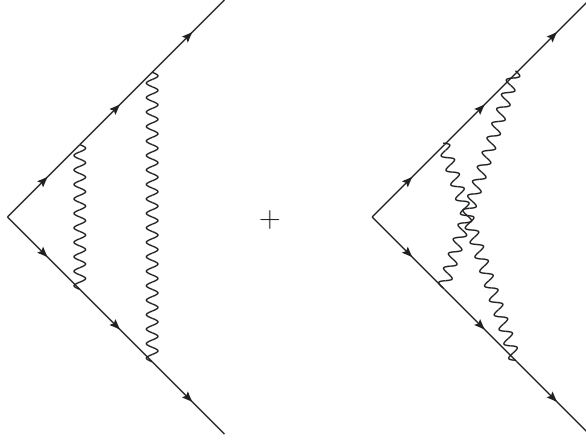


Figure 7.3: The two possible diagrams for exchanging two virtual gluons.

But since we can rewrite

$$\frac{1}{p_1 \cdot q_2} \frac{1}{p_1 \cdot (q_1 + q_2)} + \frac{1}{p_1 \cdot q_1} \frac{1}{p_1 \cdot (q_1 + q_2)} = \frac{1}{p_1 \cdot q_1} \frac{1}{p_1 \cdot q_2} \quad (7.12)$$

we end up with:

$$\int \frac{d^n q_1}{(2\pi)^n} \frac{d^n q_2}{(2\pi)^n} \bar{u}(p_1) \frac{\eta_{\mu\nu}}{q_2^2} \frac{p_1^\mu p_1^\alpha}{(p_1 \cdot q_1)(p_1 \cdot q_2)} \frac{p_2^\beta}{p_2 \cdot (q_1 + q_2)} \frac{\eta_{\alpha\beta}}{q_1^2} \frac{p_2^\nu}{p_2 \cdot q_2} v(p_2) \quad (7.13)$$

Now in the lower part of the diagram we can interchange q_1 and q_2 , so $p_2 \cdot q_1 \leftrightarrow p_2 \cdot q_2$ and add those to the original, while compensating with a factor $\frac{1}{2!}$ to avoid overcounting. This symmetrises the second part of the integral and we get:

$$\begin{aligned} & \frac{1}{2!} \int \frac{d^n q_1}{(2\pi)^n} \frac{d^n q_2}{(2\pi)^n} \bar{u}(p_1) \frac{1}{q_2^2} \frac{(p_1 \cdot p_2)^2}{(p_1 \cdot q_1)(p_1 \cdot q_2)} \frac{1}{p_2 \cdot (q_1 + q_2)} \frac{1}{q_1^2} \left(\frac{1}{p_2 \cdot q_2} + \frac{1}{p_2 \cdot q_1} \right) v(p_2) \\ &= \frac{1}{2!} \bar{u}(p_1) v(p_2) \int \frac{d^n q_1}{(2\pi)^n} \frac{d^n q_2}{(2\pi)^n} \frac{1}{q_1^2} \frac{1}{q_2^2} \frac{(p_1 \cdot p_2)^2}{(p_1 \cdot q_1)(p_1 \cdot q_2)(p_2 \cdot q_1)(p_2 \cdot q_2)} \end{aligned} \quad (7.14)$$

We can disentangle this to get:

$$\bar{u}(p_1) \frac{1}{2!} \left(\int \frac{d^n q}{(2\pi)^n} \frac{1}{k^2} \frac{n_1 \cdot n_2}{(n_1 \cdot q)(n_2 \cdot q)} \right)^2 v(p_2) = \mathcal{B} \times \frac{1}{2!} \mathcal{X}_V^2 \quad (7.15)$$

Generalizing to m gluons and taking the limit $m \rightarrow \infty$, we conclude that the sum of all possible gluon exchanges is the exponent of X_V times the Born diagram.

$$\sum_{\text{all possible gluon exchanges}} = \lim_{m \rightarrow \infty} \sum_m \bar{u}(p_1) \frac{1}{m!} \left(\int \frac{d^n q}{(2\pi)^n} \frac{1}{q^2} \frac{n_1 \cdot n_2}{(n_1 \cdot q)(n_2 \cdot q)} \right)^m v(p_2) = \mathcal{B} e^{\mathcal{X}_V} \quad (7.16)$$

7.2 Real emissions

To first order there are two possible diagrams, the two from the chapter 2. Using that notation, the approximation $k \ll p_i$ means that $p_{a,b}$ is to be replaced by $p_{2,1}$ in the numerator and $p_{a,b}^2$ by $2p_{2,1} \cdot p_3$ in the denominator. Hence we see that:

$$|A|^2 \propto (p_a \cdot p_2)(p_a \cdot p_1) - p_a^2(p_2 \cdot p_1) + (p_a \cdot p_1)(p_2 \cdot p_a) \rightarrow p_2^2(p_2 \cdot p_1) - p_2^2(p_2 \cdot p_1) + (p_2 \cdot p_1)p_2^2 = 0 \quad (7.17)$$

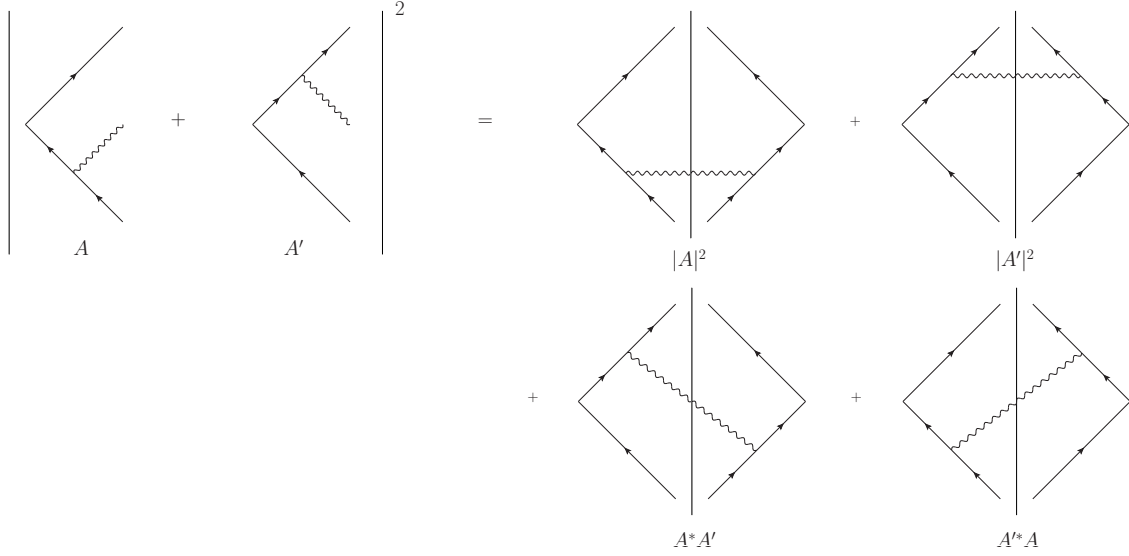


Figure 7.4: The four parts of the calculation of chapter 2.

Similarly $|A'|^2 = 0$. Hence we are only left with the cross terms:

$$A^*A' = A'^*A = 16g_S^2 \text{Tr}[T^a T^a] \frac{(p_a \cdot p_b)(p_1 \cdot p_2)}{p_a^2 p_b^2} \rightarrow 4g_S^2 \text{Tr}[T^a T^a] \frac{(p_1 \cdot p_2)^2}{(p_1 \cdot k)(p_2 \cdot k)} \quad (7.18)$$

(where we left out the factor $2e^2 e_q^2$, associated with the QED vertex). So the squared sum becomes in total:

$$|A + A'|^2 = 2A^*A' = 8g_S^2 \text{Tr}[T^a T^a] p_1 \cdot p_2 \frac{n_1 \cdot n_2}{(n_1 \cdot k)(n_2 \cdot k)} \quad (7.19)$$

We define \mathcal{X}_R as

$$\mathcal{X}_R = g_S^2 \text{Tr}[T^a T^a] \frac{n_1 \cdot n_2}{(n_1 \cdot k)(n_2 \cdot k)} \quad (7.20)$$

Since we calculated in chapter 2 that the amplitude squared of the non-emission process (\mathcal{B}) is $4p_1 \cdot p_2$ (again leaving out $2e^2 e_q^2$) this equals:

$$|\bar{\mathcal{M}}|^2 = |\mathcal{B}|^2 \mathcal{X}_R \quad (7.21)$$

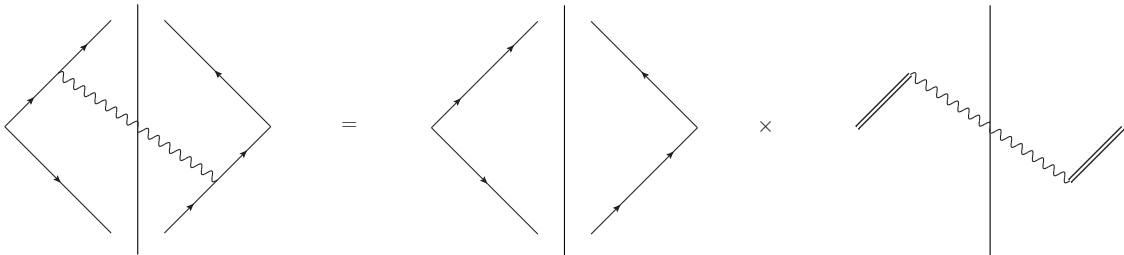


Figure 7.5: The only non-zero diagram of figure 7.4 can be written as the product of the Born diagram and the eikonal diagram.

7.2.1 Two emissions

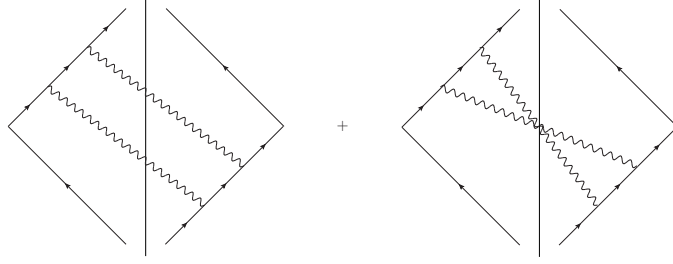


Figure 7.6: The two non-zero diagrams for two real emissions..

$$A = \bar{u}(p_1) \frac{\not{p}_2 + \not{k}_1 + \not{k}_2}{(p_2 + k_1 + k_2)^2} (-ig_S \gamma^{\mu'} T_{ij}^a) \frac{\not{p}_2 + \not{k}_1}{(p_2 + k_1)^2} (-ig_S \gamma^{\nu'} T_{ij}^a) v(p_2) \epsilon_\mu(k_2) \epsilon_\nu(k_1) \quad (7.22)$$

$$A' = \bar{u}(p_1) (-ig_S \gamma^\mu T_{ij}^a) \frac{\not{p}_1 + \not{k}_2}{(p_1 + k_2)^2} (-ig_S \gamma^\nu T_{ij}^a) \frac{\not{p}_1 + \not{k}_1 + \not{k}_2}{(p_1 + k_1 + k_2)^2} v(p_2) \epsilon_\mu(k_2) \epsilon_\nu(k_1) \quad (7.23)$$

So the cross product becomes in the eikonal approximation:

$$A^* A' = g_S^2 Tr[T^a T^a]^2 \frac{1}{2p_2 \cdot (k_1 + k_2)} \frac{1}{2p_2 \cdot k_1} \frac{1}{2p_1 \cdot k_2} \frac{1}{2p_1 \cdot (k_1 + k_2)} Tr[\not{p}_2 \gamma_\nu \not{p}_2 \gamma_\mu \not{p}_2 \not{p}_1 \gamma^\mu \not{p}_1 \gamma^\nu \not{p}_1] \quad (7.24)$$

Using the relation

$$\gamma^\alpha \gamma^\mu \gamma^\nu \gamma_\alpha = 4g^{\mu\nu} \quad (7.25)$$

this becomes

$$A^* A' = 4g_S^2 Tr[T^a T^a]^2 (p_1 \cdot p_2)^3 \frac{1}{p_2 \cdot (k_1 + k_2)} \frac{1}{p_2 \cdot k_1} \frac{1}{p_1 \cdot k_2} \frac{1}{p_1 \cdot (k_1 + k_2)} \quad (7.26)$$

Again, the second diagram (where the gluons cross) has for the p_2 part $k_1 \leftrightarrow k_2$, which becomes using a similar identity as (7.12)

$$\begin{aligned} A^* A' &= g_S^2 Tr[T^a T^a]^2 4(p_1 \cdot p_2)^3 \frac{1}{p_1 \cdot k_2} \frac{1}{p_1 \cdot (k_1 + k_2)} \frac{1}{p_2 \cdot (k_1 + k_2)} \left(\frac{1}{p_2 \cdot k_1} + \frac{1}{p_2 \cdot k_2} \right) \\ &= g_S^4 Tr[T^a T^a]^2 (p_1 \cdot p_2)^3 \frac{1}{p_1 \cdot k_2} \frac{1}{p_1 \cdot (k_1 + k_2)} \frac{1}{p_2 \cdot k_1} \frac{1}{p_2 \cdot k_2} \end{aligned} \quad (7.27)$$

We can again add the relabeled $k_1 \leftrightarrow k_2$ version and compensate by a factor $\frac{1}{2!}$ to get:

$$\begin{aligned} A^* A' &= \frac{1}{2!} 4g_S^4 Tr[T^a T^a]^2 (p_1 \cdot p_2)^3 \frac{1}{p_1 \cdot (k_1 + k_2)} \frac{1}{p_2 \cdot k_1} \frac{1}{p_2 \cdot k_2} \left(\frac{1}{p_1 \cdot k_2} + \frac{1}{p_1 \cdot k_1} \right) \\ &= \frac{1}{2!} 4(p_1 \cdot p_2)^3 g_S^4 Tr[T^a T^a]^2 \frac{1}{p_1 \cdot k_1} \frac{1}{p_1 \cdot k_2} \frac{1}{p_2 \cdot k_1} \frac{1}{p_2 \cdot k_2} \\ &= 4(p_1 \cdot p_2) \frac{1}{2!} \left(g_S^2 Tr[T^a T^a] \frac{n_1 \cdot n_2}{(n_1 \cdot k)(n_2 \cdot k)} \right)^2 \\ &= \mathcal{B} \frac{1}{2!} \mathcal{X}_R^2 \end{aligned} \quad (7.28)$$

Again generalising to m emissions and taking the sum in the limit $m \rightarrow \infty$ we get:

$$|\bar{\mathcal{M}}|^2 = |\mathcal{B}|^2 \sum_{m=0}^{\infty} \frac{1}{m!} \mathcal{X}_R^m = |\mathcal{B}|^2 e^{\mathcal{X}_R} \quad (7.29)$$

7.3 Virtual and real emissions

Next we consider diagrams which have both virtual and real emissions.

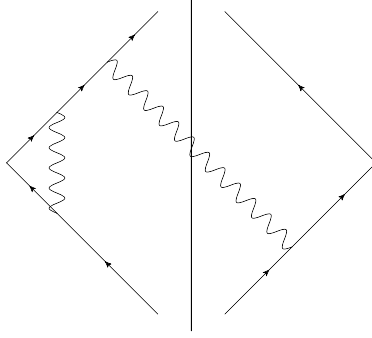


Figure 7.7: A diagram with one virtual and one real emission.

7.3.1 One virtual gluon, one real gluon

Let us start with one virtual gluon and one real gluon. Denote in figure 7.7 the two diagrams A($\gamma \rightarrow q(\bar{q} \rightarrow qg)$) and B(one virtual gluon and then $q \rightarrow qg$) Then we get for the diagram in figure 7.7 A^*B :

$$A^*B = g_S^4 Tr[T^a T^a] \sum_s \int \frac{d^d q}{(2\pi)^d} \frac{1}{q^2} \bar{v}(p_2) \gamma_\mu \frac{\not{p}_2 - \not{k}}{(p_2 - k)^2} u(p_1) \times \\ \times \bar{u}(p_1) \gamma^\mu \frac{\not{p}_1 + \not{k}}{(p_1 + k)^2} \gamma^\alpha \frac{\not{p}_1 + \not{k} + \not{q}}{(p_1 + k + q)^2} \frac{\not{p}_2 - \not{q}}{(p_2 - q)^2} \gamma_\alpha v(p_2) \quad (7.30)$$

In the eikonal approximation we get:

$$A^*B = g_S^4 Tr[T^a T^a]^4 \int \frac{d^d q}{(2\pi)^d} \frac{1}{q^2} \frac{1}{2^4} \frac{1}{p_2 \cdot k} \frac{1}{p_2 \cdot k} \frac{1}{p_1 \cdot (k+q)} \frac{1}{p_2 \cdot q} Tr[\not{p}_2 \gamma_\mu \not{p}_2 \not{p}_1 \gamma^\mu \not{p}_1 \gamma^\alpha \not{p}_1 \not{p}_2 \gamma_\alpha] \\ = g_S^4 Tr[T^a T^a]^4 \int \frac{d^d q}{(2\pi)^d} \frac{1}{q^2} \frac{1}{2^4} \frac{1}{p_2 \cdot k} \frac{1}{p_2 \cdot k} \frac{1}{p_1 \cdot (k+q)} \frac{1}{p_2 \cdot q} 4^3 (p_1 \cdot p_2)^3 \quad (7.31)$$

Add the diagram in which the real gluon is emitted before the virtual one, i.e. replace $p_1 \cdot k \rightarrow p_1 \cdot q$ and use the same trick as before to get:

$$A^*B = 4(p_1 \cdot p_2) g_S^4 Tr[T^a T^a]^4 \frac{n_1 \cdot n_2}{(n_1 \cdot k)(n_2 \cdot k)} \int \frac{d^d q}{(2\pi)^d} \frac{1}{q^2} \frac{n_1 \cdot n_2}{(n_1 \cdot q)(n_2 \cdot q)} \quad (7.32)$$

which is as expected $|\mathcal{B}|^2 \mathcal{X}_V \mathcal{X}_R$

7.3.2 Two virtual gluons, one real gluon

Now consider one extra virtual gluon. There are six possible permutations of the gluons, one of which has the real gluon being emitted after the loops, as in figure (7.8). This gives the

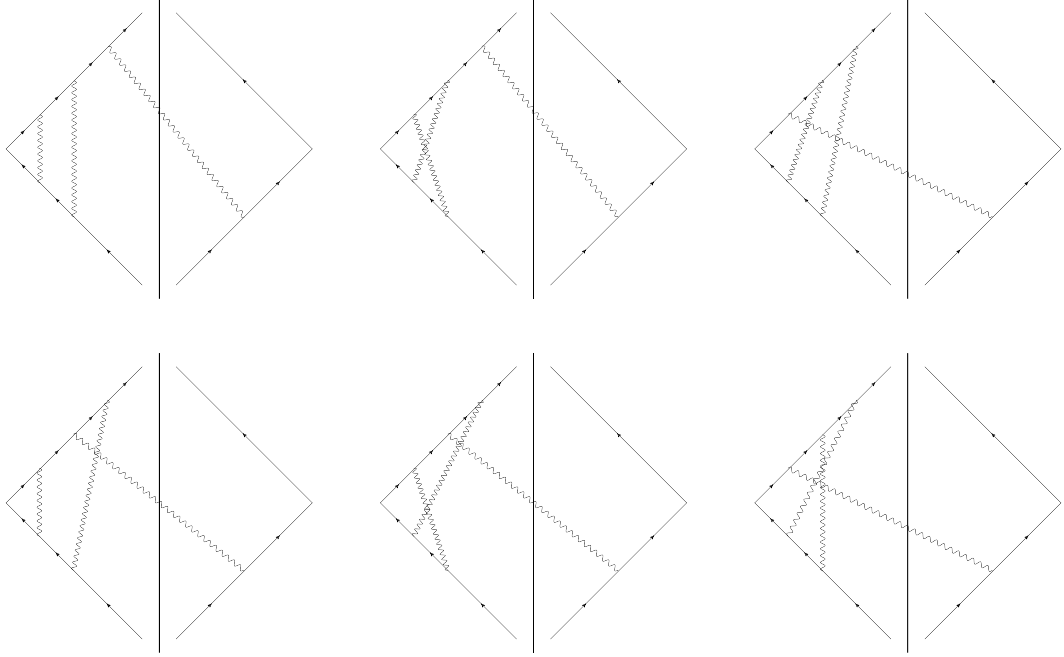


Figure 7.8: The six possible diagrams with one virtual and two real emissions. Left column are D_1 and D_2 in the text, second D_3 and D_4 , third D_5 and D_6 .

contribution to the matrix element:

$$\begin{aligned}
& g_S^4 \text{Tr}[T^a T^a]^4 \sum_s \int \frac{d^d q}{(2\pi)^d} \frac{1}{q^2} \bar{v}(p_2) \gamma_\mu \frac{\not{p}_2 - \not{k}}{(p_2 - k)^2} u(p_1) \times \\
& \times \bar{u}(p_1) \gamma^\mu \frac{\not{p}_1 + \not{k}}{(p_1 + k)^2} \gamma^\alpha \frac{\not{p}_1 + \not{k} + \not{q}_1}{(p_1 + k + q_1)^2} \gamma^\beta \frac{\not{p}_1 + \not{k} + \not{q}_1 + \not{q}_2}{(p_1 + k + q_1 + q_2)^2} \frac{\not{p}_2 - \not{q}_1 - \not{q}_2}{(p_2 - q_1 - q_2)^2} \gamma^\beta \frac{\not{p}_2 - \not{q}_1}{(p_2 - q_1)^2} \gamma_\alpha v(p_2)
\end{aligned} \tag{7.33}$$

which becomes in eikonal approximation:

$$\begin{aligned}
& g_S^4 \text{Tr}[T^a T^a]^4 \int \frac{d^d q_1}{(2\pi)^d} \frac{1}{q_1^2} \int \frac{d^d q_2}{(2\pi)^d} \frac{1}{q_2^2} \frac{1}{2^6} \frac{1}{p_2 \cdot k} \frac{1}{p_1 \cdot k} \frac{1}{p_1 \cdot (k + q_1)} \frac{1}{p_1 \cdot (k + q_1 + q_2)} \frac{1}{p_2 \cdot (q_1 + q_2)} \frac{1}{p_2 \cdot q_1} \times \\
& \times \text{Tr}[\not{p}_2 \gamma_\mu \not{p}_2 \not{p}_1 \gamma^\mu \not{p}_1 \gamma^\alpha \not{p}_1 \gamma^\beta \not{p}_1 \not{p}_2 \gamma_\beta \not{p}_2 \gamma_\alpha] \\
& = g_S^4 \text{Tr}[T^a T^a]^4 \int \frac{d^d q_1}{(2\pi)^d} \frac{1}{q_1^2} \int \frac{d^d q_2}{(2\pi)^d} \frac{1}{q_2^2} \frac{4^4}{2^6} (p_1 \cdot p_2)^4 \frac{1}{2^6} \times \\
& \times \frac{1}{p_2 \cdot k} \frac{1}{p_1 \cdot k} \frac{1}{p_1 \cdot (k + q_1)} \frac{1}{p_1 \cdot (k + q_1 + q_2)} \frac{1}{p_2 \cdot (q_1 + q_2)} \frac{1}{p_2 \cdot q_1}
\end{aligned} \tag{7.34}$$

Given the six diagrams (D_1 to D_6) in figure (7.8), we can add them in the following way, only looking at the fractions:

$$\begin{aligned}
D_1 + D_2 & \propto \frac{1}{p_1 \cdot (k + q_1)} \frac{1}{p_1 \cdot k} + \frac{1}{p_1 \cdot (k + q_1)} \frac{1}{p_1 \cdot q_1} = \frac{1}{p_1 \cdot k} \frac{1}{p_1 \cdot q_1} \\
D_3 + D_4 & \propto \frac{1}{p_1 \cdot (k + q_2)} \frac{1}{p_1 \cdot k} + \frac{1}{p_1 \cdot (k + q_2)} \frac{1}{p_1 \cdot q_2} = \frac{1}{p_1 \cdot k} \frac{1}{p_1 \cdot q_2} \\
D_5 + D_6 & \propto \frac{1}{p_1 \cdot (q_1 + q_2)} \frac{1}{p_1 \cdot q_1} + \frac{1}{p_1 \cdot (q_1 + q_2)} \frac{1}{p_1 \cdot q_2} = \frac{1}{p_1 \cdot q_1} \frac{1}{p_1 \cdot q_2}
\end{aligned} \tag{7.35}$$

Now we have for all six diagrams added, now including the common factor associated with the innermost propagator:

$$\sum_i D_i \propto \frac{1}{p_1 \cdot (k + q_1 + q_2)} \left(\frac{1}{p_1 \cdot k} \frac{1}{p_1 \cdot q_1} + \frac{1}{p_1 \cdot k} \frac{1}{p_1 \cdot q_2} + \frac{1}{p_1 \cdot q_1} \frac{1}{p_1 \cdot q_2} \right) = \frac{1}{p_1 \cdot k} \frac{1}{p_1 \cdot q_1} \frac{1}{p_1 \cdot q_2} \quad (7.36)$$

Now for the p_2 part we interchange q_1 and q_2 , add and compensate by a factor $\frac{1}{2!}$ like we have done before. We end up with for the whole expression:

$$\begin{aligned} |\mathcal{M}|^2 &= \frac{1}{2!} \frac{4^4}{2^6} (p_1 \cdot p_2)^4 g_S^4 \text{Tr}[T^a T^a]^4 \int \frac{d^d q_1}{(2\pi)^d} \frac{1}{q_1^2} \int \frac{d^d q_2}{(2\pi)^d} \frac{1}{q_2^2} \frac{1}{p_1 \cdot k} \frac{1}{p_1 \cdot q_1} \frac{1}{p_1 \cdot q_2} \frac{1}{p_2 \cdot k} \frac{1}{p_2 \cdot q_1} \frac{1}{p_2 \cdot q_2} \\ &= 4(p_1 \cdot p_2) g_S^4 \text{Tr}[T^a T^a]^4 \frac{n_1 \cdot n_2}{(n_1 \cdot k)(n_2 \cdot k)} \frac{1}{2!} \left(\int \frac{d^d q}{(2\pi)^d} \frac{1}{q^2} \frac{n_1 \cdot n_2}{(n_1 \cdot q)(n_2 \cdot q)} \right)^2 \end{aligned} \quad (7.37)$$

7.4 Generalisation

Now without further prove, we state that for n virtual gluons with momentum q_n and m real gluons with momentum k_m we get for the matrix element squared:

$$|\mathcal{M}_{n,m}|^2 = g_S^{n+m} \text{Tr}[T^a T^a]^4 \frac{4^{1+n+m}}{2^{2(n+m)}} (p_1 \cdot p_2) \frac{1}{n!} \left(\int \frac{d^d q}{(2\pi)^d} \frac{1}{q^2} \frac{n_1 \cdot n_2}{(n_1 \cdot q)(n_2 \cdot q)} \right)^n \frac{1}{m!} \left(\frac{n_1 \cdot n_2}{(n_1 \cdot k)(n_2 \cdot k)} \right)^m \quad (7.38)$$

where the numerical fraction (which is of course always four) is written in this way to indicate that each new line gives one extra factor four because of the extra contraction, but also gives two new propagators, which each give another factor two in the denominator. This yields the final result for the sum of all possible diagrams:

$$|M|^2 = \sum_{n,m} |\mathcal{M}_{n,m}|^2 = |\mathcal{B}|^2 e^{\chi_V + \chi_R} \quad (7.39)$$

Chapter 8

Comparison CDM and Webs

The two models we have discussed work in the same soft limit. It is therefore interesting to compare the two.

8.1 Comparison including phase space

We have so far ignored the phase space in the real emission part of the Webs model. This was no problem, because it exponentiates as well, since the phase space factorises. For simplicity we will for now ignore all numerical prefactors. The phase space for each emission i is:

$$\prod_i d^4 k^i \delta^{(4)}((k^i)^2) \quad (8.1)$$

For the total expression, the conservation of momentum is added:

$$\delta^{(4)}(Q - \sum_i^n k^i) \quad (8.2)$$

When comparing with the result of the CDM, we are interested in the differential cross section to the transverse momentum squared. To this end, insert a carefully chosen unity into the phase space:

$$1 = \int d^2 p_\perp \delta^{(2)}(p_\perp - \sum_i^n k_\perp^i) \quad (8.3)$$

Now the differential cross section to k_\perp^2 strips of the integral. This part does not factorise, therefore we will take the Fourier transform to impact parameter space:

$$F\left(\frac{d\sigma}{dp_\perp^2}\right) = \int d^2 p_\perp e^{-ib \cdot p_\perp} \frac{d\sigma}{dp_\perp^2} \quad (8.4)$$

The delta function from equation (8.3) now makes that the phase space per emission becomes:

$$d^4 k \delta(k^2) e^{-ib \cdot k_\perp} \quad (8.5)$$

and this does factorise. Now remember that the matrix element for one real emission in the Webs model is given by:

$$\mathcal{M}_R^{(1)} = \frac{p_1 \cdot p_3}{(p_1 \cdot k)(p_3 \cdot k)} \quad (8.6)$$

We write p_1 and p_3 back to back and use lightcone variables:

$$\begin{aligned} p_1 &= (p, 0, 0, p) \\ p_3 &= (p, 0, 0, -p) \\ k^\pm &= \frac{1}{\sqrt{s}}(k^0 \pm k^3) \\ \vec{k}_\perp &= (k^1, k^2) \end{aligned} \quad (8.7)$$

This means that $p_1 \cdot p_3 = 2p^2$ and $(p_1 \cdot k)(p_3 \cdot k) = p^2(k^0 + k^3)(k^0 - k^3) = 2k^+k^-$. Now using the fact that $k^2 = 0$ according to the delta function and therefore $2k^+k^- = k_\perp^2$ we conclude that

$$\mathcal{M}_R^{(1)} \propto \frac{1}{k_\perp^2} \quad (8.8)$$

And in combination with the phase space we now have:

$$\mathcal{M}_R^{(1)} d\Phi \propto \frac{1}{k_\perp^2} d^4k \delta(k^2) e^{-ib \cdot k_\perp} \quad (8.9)$$

Rewrite the whole expression to lightcone variables:

$$\mathcal{M}_R^{(1)} d\Phi \propto \frac{1}{k_\perp^2} dk^+ dk^- d^2k_\perp \delta(2k^+k^- - k_\perp^2) e^{-ib \cdot k_\perp} \quad (8.10)$$

We now introduce two Heaviside step functions to ensure soft emission:

$$\theta\left(\frac{Q}{\sqrt{s}} - k^+\right) \theta\left(\frac{Q}{\sqrt{s}} - k^-\right) \quad (8.11)$$

Let us now carry out the azimuthal part of the integration d^2k_\perp . We have to consider the inner product in the exponent to do this, which has a ϕ dependence. Remember now the definition of the first Bessel function J_0

$$J_0(z) = \frac{1}{2\pi} \int_0^{2\pi} d\phi e^{iz \cos \phi} \quad (8.12)$$

We therefore get:

$$\mathcal{M}_R^{(1)} \propto \frac{1}{k_\perp^2} dk^+ dk^- dk_\perp^2 J_0(bk_\perp) \delta(2k^+k^- - k_\perp^2) \theta\left(\frac{Q}{\sqrt{s}} - k^+\right) \theta\left(\frac{Q}{\sqrt{s}} - k^-\right) \quad (8.13)$$

Up until now we have only considered the real emissions. When including the virtual corrections, we need to deal with the fact that their q_\perp is zero. In chapter 3 we already solved this problem by introducing the plus prescription. We do that here as well, so:

$$\left(\frac{e^{ibq_\perp}}{q_\perp^2}\right)_+ = \frac{e^{ibq_\perp} - e^0}{q_\perp^2} = \frac{1}{q_\perp^2} (e^{ibk_\perp} - 1) \quad (8.14)$$

So the whole expression becomes:

$$\mathcal{M}_R^{(1)} \propto \frac{1}{k_\perp^2} dk^+ dk^- dk_\perp^2 (J_0(bk_\perp)) \delta(2k^+k^- - k_\perp^2) \theta\left(\frac{Q}{\sqrt{s}} - k^+\right) \theta\left(\frac{Q}{\sqrt{s}} - k^-\right) \quad (8.15)$$

We have not explicitly written down the integration limits so far, but the heaviside functions give limits for k^+ and k^- . To avoid UV divergence, we demand that $k_\perp < Q^2$. We then have:

$$\mathcal{M}_R^{(1)} \propto \int_0^{Q/\sqrt{2}} dk^+ \int_0^{Q/\sqrt{2}} dk^- \int_0^{Q^2} dk_\perp^2 \frac{1}{k_\perp^2} (J_0(bk_\perp)) \delta(2k^+k^- - k_\perp^2) \quad (8.16)$$

We now use the delta function to do the integration over k^- . To this end, rewrite the delta function as:

$$\delta(2k^+k^- - k_\perp^2) = \delta\left(2k^+ \left[k^- - \frac{k_\perp^2}{2k^+}\right]\right) = \frac{1}{2k^+} \delta\left(k^- - \frac{k_\perp^2}{2k^+}\right) \quad (8.17)$$

We want to have k_\perp^2 as the outermost integral, so we also need to consider the limits of the integrals: The limits of k^- become those of $k_\perp^2/2k^+$, so

$$0 < k^- < \frac{Q}{\sqrt{2}} \Rightarrow 0 < \frac{k_\perp^2}{2k^+} < \frac{Q}{\sqrt{2}} \Rightarrow \frac{k_\perp^2}{\sqrt{s}Q} < k^+ < \frac{Q}{\sqrt{2}} \quad (8.18)$$

The phase space integrals therefore become:

$$\int_0^{Q^2} \frac{dk_{\perp}^2}{k_{\perp}^2} \int_{k_{\perp}^2/\sqrt{2}Q}^{Q/\sqrt{2}} \frac{dk^+}{2k^+} = \frac{1}{2} \int_0^{Q^2} \frac{dk_{\perp}^2}{k_{\perp}^2} \ln \left(\frac{Q^2}{k_{\perp}^2} \right) \quad (8.19)$$

Now we can make the following approximation:

$$J_0(bk_{\perp}) - 1 \approx -\theta \left(k_{\perp} - \frac{1}{b} \right) \quad (8.20)$$

Let us show this by integrating both sides over:

$$\int_0^{Q^2} \frac{dk_{\perp}}{k_{\perp}} \ln^n k_{\perp} \quad (8.21)$$

For $n = 0$, the right hand side becomes:

$$-\int_0^{Q^2} \frac{dk_{\perp}}{k_{\perp}} \theta \left(k_{\perp} - \frac{1}{b} \right) = -\int_{1/b}^{Q^2} \frac{dk_{\perp}}{k_{\perp}} = -\ln b \quad (8.22)$$

The left hand side becomes:

$$\int_0^{2\pi} d\phi \int \frac{dk_{\perp}}{k_{\perp}} \left(e^{ibk_{\perp}} - \frac{1}{2\pi} \right) \quad (8.23)$$

Using a Laplace transformation one can show that this also becomes $-\ln b$. We now have as total result, using the heaviside function from equation (8.20) to change the lower integration limit:

$$\exp \left[\int_{1/b^2}^{Q^2} \frac{dk_{\perp}^2}{k_{\perp}^2} \ln \left(\frac{Q^2}{k_{\perp}^2} \right) \right] \quad (8.24)$$

The integration can be solved by a variable substitution $k_{\perp}^2 = Q^2 x$

$$\begin{aligned} \int \frac{dk_{\perp}^2}{k_{\perp}^2} \ln \left(\frac{Q^2}{k_{\perp}^2} \right) &= -\int \frac{dx}{x} \ln x = -\ln^2 x \Big|_{x_{\min}}^{x_{\max}} + \int \frac{dx}{x} \ln x \\ &= -\frac{1}{2} \ln^2 x \Big|_{x_{\min}}^{x_{\max}} = \frac{1}{2} \ln^2 \left(\frac{k_{\perp}^2}{Q^2} \right) \Big|_{1/b^2}^{Q^2} \\ &= \frac{1}{2} \ln^2 (Q^2 b^2) \end{aligned} \quad (8.25)$$

Let us call this expression $S(Q, b)$. We now need to undo the Fourier transformation:

$$\frac{1}{\sigma_0} \frac{d\sigma}{dp_{\perp}^2} \propto \int_0^{\infty} d^2 b e^{ibp_{\perp}} e^{S(Q, b)} = \int_0^{\infty} b J_0(bp_{\perp}) e^{S(Q, b)} \quad (8.26)$$

Let us now rescale $b = \hat{b}/p_{\perp}$. We now have:

$$\frac{1}{\sigma_0} \frac{d\sigma}{dp_{\perp}^2} \propto \frac{1}{p_{\perp}^2} \int_0^{\infty} d\hat{b} \hat{b} J_0(\hat{b}) \exp S(Q, \hat{b}/p_{\perp}) \quad (8.27)$$

Now use one property of the Bessel functions:

$$x J_0(x) = \frac{d}{dx} (x J_1(x)) \quad (8.28)$$

We then get:

$$\begin{aligned} \frac{1}{\sigma_0} \frac{d\sigma}{dp_{\perp}^2} &\propto \frac{1}{p_{\perp}^2} \int_0^{\infty} d\hat{b} \frac{d}{d\hat{b}} \left(\hat{b} J_1(\hat{b}) \right) \exp S(Q, \hat{b}/p_{\perp}) \\ &= \hat{b} J_1(\hat{b}) \exp S(Q, \hat{b}/p_{\perp}) \Big|_0^{\infty} - \frac{1}{p_{\perp}^2} \int_0^{\infty} \hat{b} J_1(\hat{b}) \left(\frac{d}{d\hat{b}} S(Q, \hat{b}/p_{\perp}) \right) \exp S(Q, \hat{b}/p_{\perp}) \end{aligned} \quad (8.29)$$

The first term is zero because $J_0(\hat{b})$ goes to zero as $\hat{b} \rightarrow \infty$. Now the derivative to \hat{b} of $S(Q, \hat{b}/p_\perp)$ is:

$$\frac{d}{d\hat{b}} S(Q, \hat{b}/p_\perp) = \frac{d}{d\hat{b}} \left[\frac{1}{2} \ln^2 \left(\frac{Q^2 \hat{b}^2}{p_\perp^2} \right) \right] = \frac{2}{\hat{b}} \ln \left(\frac{Q^2 \hat{b}^2}{p_\perp^2} \right) \quad (8.30)$$

So the differential cross section becomes:

$$\frac{1}{\sigma_0} \frac{d\sigma}{dp_\perp} \propto 2 \frac{1}{p_\perp^2} \int_0^\infty J_1(\hat{b}) \ln \left(\frac{Q^2 \hat{b}^2}{p_\perp^2} \right) \exp S(Q, \hat{b}/p_\perp) \quad (8.31)$$

This is can be rewritten to:

$$\frac{1}{\sigma_0} \frac{d\sigma}{dp_\perp^2} \propto \frac{d}{dp_\perp^2} \int_0^\infty J_1(\hat{b}) \exp S(Q, \hat{b}/p_\perp) \quad (8.32)$$

because the derivative to p_\perp^2 of $S(Q, \hat{b}/p_\perp)$ is:

$$\frac{d}{dp_\perp^2} S(Q, \hat{b}/p_\perp) = \frac{d}{dp_\perp^2} \left[\frac{1}{2} \ln^2 \left(\frac{Q^2 \hat{b}^2}{p_\perp^2} \right) \right] = -\frac{1}{p_\perp^2} \ln \left(\frac{Q^2 \hat{b}^2}{p_\perp^2} \right) \quad (8.33)$$

Notice this minus sign cancels the one obtained by partial integration in equation (8.29). Let us now define:

$$\begin{aligned} S &= aL^2 \\ a &\propto \frac{\alpha_S C_F}{2\pi} \\ L &= \ln \left(\frac{Q^2 \hat{b}^2}{p_\perp^2} \right) = l + l_b \\ l &= \ln \left(\frac{Q^2}{p_\perp^2} \right) \\ l_b &= \ln(\hat{b}) \end{aligned} \quad (8.34)$$

In these variables, we have:

$$S = -al^2 - 2all_b - al_b^2 \quad (8.35)$$

The result for the differential cross section is then:

$$\frac{1}{\sigma_0} \frac{d\sigma}{dp_\perp^2} \propto \left[\frac{d}{dp_\perp^2} e^{-al^2} \right] \int_0^\infty d\hat{b} J_1(\hat{b}) e^{-2all_b - l_b^2} \quad (8.36)$$

The \hat{b} integral is a complicated integral that we will not perform here, but the result is:

$$\frac{1}{\sigma_0} \frac{d\sigma}{dp_\perp^2} \propto \left[\frac{d}{dp_\perp^2} e^{-al^2} \right] \frac{\Gamma(1 - 2al)}{1 + 2al} 2^{-4al} \quad (8.37)$$

This last factor does not influence the showering behaviour, so we just expand it to lowest order in a (which is proportional to α_S). This lowest order term is simply 1. This means that, to leading logarithmic accuracy, our final result for the differential cross section is:

$$\frac{1}{\sigma_0} \frac{d\sigma}{dp_\perp^2} \propto \left[\frac{d}{dp_\perp^2} \exp \left\{ -a \ln^2 \left(\frac{Q^2}{p_\perp^2} \right) \right\} \right] = \frac{1}{p_\perp^2} \ln \left(\frac{Q^2}{p_\perp^2} \right) \exp \left[-a \ln^2 \left(\frac{Q^2}{p_\perp^2} \right) \right] \quad (8.38)$$

This is remarkable similar to the differential distribution for the first branching possibility in the CDM, see equation (6.23).

8.2 Short discussion

This comparison has been somewhat heuristic. We have left out a lot of numerical factors. Furthermore, we have left out the branching possibilities $P_{ij}(z)$ in our model for the CDM and have treated the eikonal approach in its Abelian simplification. More elaborate analysis is needed to conclude whether the two models are truly the same. But certain is that they have striking similarities.

Chapter 9

Conclusions, Discussion and Outlook

In this thesis we have discussed the Colour Dipole Model (CDM) and the Webs model for parton showering. To leading logarithmic accuracy, the Webs model turns out to give the same differential cross section as the CDM gives for the first emission. However, a more detailed evaluation is needed to conclude whether these two models are truly the same or only very similar.

The program I wrote to test the sampling of distributions gives the results it should give and can be expanded easily to test other distributions. The toy model program I wrote to test the showering of partons in the CDM gives reasonable results, but there is still room for improvement: So far, it only considers general 'partons' and does not discriminate between the different splitting possibilities.

Most textbooks and papers on the CDM do not go beyond the first emission and then give the general idea of how to proceed further. Only when I looked at steps further down the shower, it turned out that the parton shower in the CDM is non Markovian. This is a remarkable result, since parton showers in general are Markovian: One does not expect that a parton should 'memorise' a step in the shower before the last one. This might be an intrinsic property of looking at dipoles instead of partons or it might have to do with the way we 'go through the phase space triangle'. In our case, with angular ordering imposing strongly ordered transverse momentum, we go 'from top to bottom'. But there are other ways of going through the triangle, like from the left and right to the center (ordering in rapidity), or even along the diagonals towards zero rapidity and transverse momentum. In any case, the non Markovian nature of our shower is not so bad as being completely non Markovian: Not all steps in between are needed, only the last one and the last time the choice between left and right changed.

Chapter 10

Populaire samenvatting

Een vereist deel van scripties aan de Universiteit van Amsterdam is de populaire samenvatting. Het idee is dat dit de scriptie samenvat voor een geïnteresseerd, maar niet gespecialiseerd publiek, zoals de lezers van de wetenschapsbijlages in kranten.

10.1 Modellen voor LHC fysica

De *Large Hadron Collider* (LHC) is de nieuwste en krachtigste deeltjesversneller ter wereld. De versneller bevindt zich in een tunnel van 27 kilometer onder de grond, op de grens van Zwitserland en Frankrijk, net buiten Geneve, bij het deeltjesfysica instituut Cern. Hadronen zijn kerndeeltjes, zoals protonen. In de versneller worden protonen versnelt tot bijna de lichtsnelheid om ze vervolgens op elkaar te laten botsen. Door wat er daarvan overblijft te bestuderen, hopen wetenschappers nieuwe deeltjes te ontdekken en meer te weten te komen over het achterliggende model, het Standaard Model, dat alle krachten en deeltjes beschrijft, en wellicht zelfs een uitbreiding daarvan te ontdekken. De LHC is niet de eerste versneller die zich in deze kilometers lange buis bevindt. De voorganger van de LHC heette de *Large Electron Positron Collider* (LEP) en versnelde elektronen en positronen (de anti-deeltjes van elektronen) om die op elkaar te botsen. Als een elektron en een positron bij elkaar komen, annihilieren ze elkaar: de deeltjes worden via $E = mc^2$ omgezet in energie. Uit die energie ontstaan vervolgens weer nieuwe deeltjes. Dit is dus een zeer 'schoon' proces: Alles wat gemeten wordt in de detectoren komt vrijwel zeker uit de interactie tussen een elektron en een positron. Dit maakt alle berekeningen relatief eenvoudig, vandaar dat versnellers jarenlang dus ook elektron-positron versnellers waren. Er is echter een probleem met ronde versnellers zoals de LEP en LHC en dat is dat elektrisch geladen deeltjes die worden afgebogen straling gaan afgeven. De hoeveelheid energie die daarmee verloren gaat neemt toe naarmate ze harder versneld worden. Er is dus een limiet aan hoe ver men kan gaan met het versnellen van elektronen en positronen: Op een gegeven moment gaat alle extra energie die erin gestopt wordt meteen weer verloren als straling. Toch willen deeltjesfysici steeds hogere energieën, omdat daarbij zwaardere deeltjes gevormd kunnen worden (wederom $E = mc^2$) en daarmee dus nieuwe deeltjes kunnen worden ontdekt. Een manier om toch naar hogere energieën te komen is door protonen te gebruiken. Protonen zijn meer dan duizend keer zwaarder dan elektronen of positronen en bij zwaardere deeltjes is het stralingseffect minder. Maar deze winst heeft een duidelijke keerzijde: Protonen zijn bepaald niet 'schoon'. Men kan zich protonen beter voorstellen als een grote zak met daarin een wisselende samenstelling van deeltjes. Er zitten standaard drie quarks (nog kleinere deeltjes) in, maar die maken minder dan de helft uit van de massa. De rest wordt gevormd door tijdelijke quark anti-quark paren, die gevormd worden uit energie en elkaar weer annihilieren, en uit gluonen, zogehete lijmdeeltjes, die het proton bij elkaar houden. Quarks en gluonen worden partonen genoemd: 'deeldeeltjes'. Deze onoverzichtelijke zak van deeltjes wordt in de LHC op een andere zak van deeltjes gebotst. In dit proces reageren hopelijk twee quarks of gluonen met elkaar, maar er zijn ook resten van de protonen die een hoop achtergrond deeltjes

geven en misschien zelfs onderling reageren. Vervolgens kunnen de deeltjes uit zowel het primaire proces (de zogeheten harde interactie) als uit de achtergrond verder gaan stralen: ze splitsen en vormen meer deeltjes. Uiteindelijk vormt zich een *parton shower*, een douche van partonen. Partonen kunnen niet los leven, dus uiteindelijk, als de energie per deeltje laag genoeg is, zullen ze nieuwe hadronen gaan vormen. Deze hadronen hoeven niet per se stabiel te zijn, dus ze kunnen ook nog vervallen tot nog meer deeltjes. Pas tijdens of na dit stadium staat de detector: Dit zijn de deeltjes die men kan meten. Het moge duidelijk zijn dat de berekeningen die met deze processen samenhangen niet meer snel met de hand kunnen worden gedaan. Hier is veel rekenkracht voor nodig en daarnaast is het nodig dat we het proces van *parton showers* goed kunnen modeleren.

10.1.1 Kleurdipoolmodel

Hiervoor zijn meerdere modellen. Een ervan kijkt naar de zogehete 'kleur' van de partonen. Dit is geen echte kleur, maar een eigenschap van deeltjes die eenzelfde symmetrie vertoont als echte kleuren. In kleur kan men (in licht) rood, groen en blauw mengen om wit te krijgen. Als alleen blauw en groen worden gemengd verkrijgt men cyaan. Men zou cyaan dus anti-rood kunnen noemen, omdat het rood annihileert tot wit. Als men nu in een proces een 'rood' quark en een 'anti-rood' anti-quark vormt, dan kan het geheel beschreven worden als een kleurdipool. Een normale, elektrische dipool is aan de ene kant positief geladen en aan de andere kant negatief. Een kleurdipool kan rood en anti-rood als uiterste hebben, maar ook blauw en anti-blauw of groen en anti-groen. Gluonen dragen zowel een kleur als een antikleur, bijvoorbeeld rood/anti-groen of anti-blauw/groen. Als van dat quark anti-quark paar dat het kleurdipool vormt er nu een gluon afstraalt, dan vormen zich twee nieuwe kleurdipolen. Als bijvoorbeeld een rood quark een rood anti-groen gluon afstraalt, dan wordt het quark zelf groen. Er zijn nu dus twee kleurdipolen: een tussen het groene quark en de anti-groene kant van het gluon en een tussen de rode kant van het gluon en het anti-rode anti-quark. In totaal is er dus een kleurdipool gesplitst in twee kleurdipolen. Op die manier kan een *parton shower* worden beschreven door een opeenvolging van splitsingen van kleurdipolen.

10.1.2 Webbenmodel

Een ander model dat wordt behandeld in deze scriptie is het Webben model. In dit model wordt de berekening van de kans op het afstralen van een deeltje vergeleken met dat van twee deeltjes, van drie deeltjes enzovoorts tot oneindig veel deeltjes. Het blijkt dat al deze mogelijkheden bij elkaar opgeteld voor bepaalde gevallen eigenlijk kunnen worden uitgedrukt in een gedeelte van de eerste berekening, die van de kans op het afstralen van een deeltje. Het heet het Webben model omdat bij het tekenen van het afstralen van heel veel deeltjes er een hoop lijnen kruisen in bepaalde patronen en dus enige gelijkenis met een spinnenweb beginnen te vertonen. Op deze manier kan dus een ingewikkelde *parton shower* worden uitgedrukt in de veel simpelere berekening van slechts een splitsing.

10.1.3 Vergelijking Kleurdipoolmodel en Webbenmodel

Het eindresultaat van deze twee, in eerste instantie totaal verschillende modellen, blijkt behoorlijk op elkaar te lijken. In deze scriptie wordt nog niet zover gegaan om te zeggen dat de twee modellen identiek zijn, maar het gedrag dat ze vertonen lijkt zeer sterk op elkaar.

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Appendix A

Sampling Distributions program

A.1 script.sh

```
g++ -o main *.c;
main | root -q -b -l rootmacro.cxx;
root DrawRoot.cxx;
```

A.2 main.c

```
#include <iostream>
#include <ctime>
#include <cstdlib>
#include <cmath>
#include "functions.h"
using namespace std;

void noveto(int Bart, int n, double a);
void veto(int Bart, int n, double a);

int main(){
/*-----SPECIFY YOUR PARAMETERS-----*/
    int n = 1000000;    //number of events
    double a = 1.;     //coefficient for Bart, should be >0.25
    int Bart = 2;      //Bart=0: f(t)=ln(t), Bart=1: f(t)=t. Bart=2: CDM first emission
    bool VetoBool = 1; //Use Veto-algorithm or don't. Do not set to 0 when Bart=0
/*-----*/

    if(VetoBool==0){
        noveto(Bart,n,a);
    }
    if(VetoBool==1){
        veto(Bart,n,a);
    }

    FILE *Bartfile = fopen("Bart.txt","w");
    fprintf(Bartfile,"%i",Bart);
    fclose(Bartfile);

    return 0;
}
```

A.3 functions.h

```
#ifndef FUNCTIONS_H
#define FUNCTIONS_H

double f(int Bart, double t);
double F(int Bart, double t);
double Finv(int Bart, double x);
double g(int Bart, double t, double a);
double G(int Bart, double t, double a);
double Ginv(int Bart, double x, double a);

#endif
```

A.4 functions.c

```
#include "functions.h"
#include <cmath>

double f(int Bart, double t){
    if(Bart==0){return log(t+1);}
    if(Bart==1){return t;}
    if(Bart==2){return 1/t;}
}

double F(int Bart, double t){
    if(Bart==0){return (t+1)*log(t+1)-t;}
    if(Bart==1){return 0.5*t*t;}
    if(Bart==2){return 0.5*log(1/t)*log(1/t);}
}

double Finv(int Bart, double x){
    if(Bart==0){throw Bart; return 0;}
    if(Bart==1){return sqrt(2*x);}
    if(Bart==2){return exp(-sqrt(2*x));}
}

double g(int Bart, double t, double a){
    if(Bart==0){return t+a;}
    if(Bart==1){return t*t+a;}
}

double G(int Bart, double t, double a){
    if(Bart==0){return 0.5*t*t+a*t;}
    if(Bart==1){return t*t*t/3.+a*t;}
}

double Ginv(int Bart, double x, double a){
    if(Bart==0){return sqrt(a*a+2*x)-a;}
    if(Bart==1){return (-pow(2*a*a*a/(3*x+sqrt(4*a*a*a+9*x*x)),1./3.)
        +pow((3*x+sqrt(4*a*a*a+9*x*x))/2,1./3.));}
}
```

A.5 veto.c

```
#include <iostream>
#include <ctime>
```

```

#include <cstdlib>
#include <cmath>
#include "functions.h"

using namespace std;

void veto(int Bart=0, int n=1000000, double a=1.){
    srand(time(0));
    cout << n << " " << Bart << endl;
    double told;
    double tnext;
    double R1,R2;
    bool b1,b2;
    for(int i=0;i<n;i++){
        told=0.;
        b1 = true;
        while(b1){
            b2 = true;
            while(b2){
                R1 = rand();R1 = R1/RAND_MAX;
                tnext = Ginv(Bart,G(Bart, told,a)-log(R1), a);
                if(tnext > told){b2 = false;}
            }
            R2 = rand();R2 = R2/RAND_MAX;
            if((f(Bart, tnext)/g(Bart, tnext,a)) > R2){b1 = false;}
            told=tnext;
        }
        cout << tnext << endl;
    }
    return;
}

```

A.6 noveto.c

```

#include <iostream>
#include <ctime>
#include <cstdlib>
#include <cmath>
#include "functions.h"

using namespace std;

void noveto(int Bart=0, int n=1000000, double a=1.){
    srand(time(0));
    cout << n << " " << Bart << endl;
    double t;
    double R;
    for(int i=0;i<n;i++){
        R=rand();R=R/RAND_MAX;
        t=Finv(Bart, F(Bart, 0.)-log(R));
        if(Bart==2){
            t=Finv(Bart, F(Bart, 1.)-log(R));
        }
        cout << t << endl;
    }
    return;
}

```

A.7 rootmacro.cxx

```
#include "Riostream.h"
#include <string>
#include <cmath>
using namespace std;

void rootmacro(){
    TFile *f = new TFile("file.root","RECREATE");
    TNtuple *ntuple = new TNtuple("ntuple","title","x");
    int nbins = 80;
    double xmin = 0.;
    double xmax = 4.;
    Float_t in;
    int n;
    int Bart=0;
    int i=0;
    cin >> n >> Bart;
    if(Bart==2){xmax = 1.;}
    double binwidth = (xmax-xmin)/nbins;
    TH1F *h1 = new TH1F("h1","title",nbins,xmin,xmax);
    while(true){
        cin >> in;
        if(!cin.good()){break;}
        h1->Fill(in);
        ntuple->Fill(in, 1./(n*binwidth));
        if((i \% (n/20)) == 0){cout << "i=" << i << endl;}
        i++;
    }
    h1->Scale(1./(n*binwidth));
    f->Write();
}
```

A.8 DrawRoot.cxx

```
#include "Riostream.h"
#include "TFile.h"
#include "TTree.h"
#include "TNtuple.h"
#include <cstring>

void DrawRoot(){
    int Bart;
    int *ptr;
    FILE *Bartfile = fopen("Bart.txt","r");
    fscanf(Bartfile,"%i",&Bart);
    cout << Bart << endl;
    fclose(Bartfile);
    gROOT->SetStyle("Plain");
    TFile *f = new TFile("file.root");
    TCanvas *c1 = new TCanvas("c1","c1",0,0,1500,1200);
    c1->SetLeftMargin(0.1);
    c1->SetBottomMargin(0.1);
    TF1 *P0 = new TF1("P0","log(x+1)*exp(-((x+1)*log(x+1)-x))",0,4);
    TF1 *P1 = new TF1("P1","x*exp(-(0.5*x*x))",0,4);
    TF1 *P2 = new TF1("P2","-log(x)*exp(-0.5*log(1/x)*log(1/x))/x",0,1);
    // c1->Divide(2,2);
    if(Bart==1){
```

```

        h1->SetTitle(0);
//        h1->SetTitle(f(t)=t);
    }
    if(Bart==0){
        h1->SetTitle(0);
//        h1->SetTitle("f(t)=ln(t)");
    }
    if(Bart==2){
        h1->SetTitle(0);
//        h1->SetTitle("Not Veto Algorithm CDM");
    }
    h1->GetXaxis()->SetTitle("t");
    h1->GetXaxis()->SetTitleSize(0.03);
    h1->GetXaxis()->SetLabelSize(0.02);
    h1->GetYaxis()->SetTitle("Relative occurence");
    h1->GetYaxis()->SetTitleSize(0.03);
    h1->GetYaxis()->SetLabelSize(0.02);
    h1->SetFillColor(kCyan-10);
    h1->SetLineWidth(1.);
    h1->SetStats(0);
//    c1->cd(1);
    h1->Draw();
    if(Bart==0){
        P0->SetLineWidth(2);
        P0->SetLineColor(2);
        P0->Draw("same");
    }
    if(Bart==1){
        P1->SetLineWidth(2);
        P1->SetLineColor(2);
        P1->Draw("same");
    }
    if(Bart==2){
        P2->SetLineWidth(2);
        P2->SetLineColor(2);
        P2->Draw("same");
    }
}
//    c1->cd(3);
//    P0->Draw();
//    c1->cd(4);
//    P1->Draw();
//    c1->cd(2);
//    P2->Draw();
}

```

Appendix B

Shower Toy Model program

B.1 script.sh

```
#!/usr/bin/tcsh
#AUTHOR:    Hans van Deurzen
#DATE:      June 2011

#Compile main
g++ -o main main.c XS.c ConeVariables.c Shower2.c;

#MAIN:
#options are:
#      -c for cut off value                default: 0.1
#      -f for filename of the ntuple      default: "defaultfile.root"
#      -n for number of events            default: 1000
#      -p for choice of variables, 0 is (x1,x3), 1 is (pt,y) default: 1
#      -S for CM energy                   default: 1.0
#      -s for parton shower: 0 is off, 1 is on default: 1
#      -t for title of the histogram      default: "histogramtitle"

#pipe directly to rootmacro.cxx or via an eventlog:

main -p 1 -t title -f output.root | root -q -b -l rootmacro.cxx
#main -p 1 -n 50000 -t title -f output.root > eventlog.txt;

#ROOTMACRO:
# -qbl to process quietly without splash screen
#root -q -b -l rootmacro.cxx < eventlog.txt;

#VERYSMALLROOTSCRIPT
#This is just to plot the rootntuple
root verysmallrootscript.cxx;
```

B.2 main.c

```
#include <iostream>
#include <cstdlib>
#include <ctime>
#include "string.h"
#include "XS.h"
#include "ConeVariables.h"
#include "Shower2.h"
```

```

using namespace std;

int main(int argc, char **argv){

    //defaults
    int parse      = 1;
    int nevents    = 1000000;
    double S = 1.;
    double cutvalue = 0.1;
    char* histogramtitle = "histogramtitle";
    char* filename = "defaultfile.root";
    bool ShowerOn = true;

    //parsing arguments
    for(int i=0; i<argc; i++){
        //CM energy
        if(strcmp(argv[i], "-S")==0){
            S=atof(argv[i+1]);
        }
        //parse=0: dXSqq, parse=1: dXScone
        if(strcmp(argv[i], "-p")==0){
            parse=atoi(argv[i+1]);
        }
        //number of events
        if(strcmp(argv[i], "-n")==0){
            nevents=atoi(argv[i+1]);
        }
        //cut value
        if(strcmp(argv[i], "-c")==0){
            cutvalue=atof(argv[i+1]);
        }
        //histogram title
        if(strcmp(argv[i], "-t")==0){
            histogramtitle=argv[i+1];
        }
        //filename
        if(strcmp(argv[i], "-f")==0){
            filename=argv[i+1];
        }
        //shower
        if(strcmp(argv[i], "-s")==0){
            ShowerOn = true;
        }
    }
    // cutvalue = S*cutvalue;
    cout << filename << " " << histogramtitle << " "
    << nevents << " " << S << " " << cutvalue << endl;

    srand(time(0));
    //random loop
    int showercalls=0;
    for (int i=0; i<nevents; i++){

        //The part below is an older, more general version.
        //It did not contribute to any of the figures in the main text.
        //This part can be used to calculate cross sections using the variables x1 and x3,
        //put the shower off etc.
        /*          double r1, r2, x1, x3;

```

```

while(true){
    r1 = rand();
    r2 = rand();
    x1 = r1/RAND_MAX;
    x3 = r2/RAND_MAX;

    //          cout << "x1=" << x1 << endl;
    //          cout << "x3=" << x3 << endl;

    //cuts
    if (x1 < 0.99 && x3 < 0.99){
        if(parse==0){
            if (x1+x3 > cutvalue) break;
        }
        if(parse==1){
            //cout << "pt2=" << pt2(S,x1,x3) << endl;
            if (pt2(S,x1,x3) > cutvalue) break;
        }
    }
}

if(ShowerOn==false){
    //XSqq
    if (parse==0){
        double _pt = pt(S,x1,x3);
        double sigma = XSqq(x1,x3);
        cout << _pt << " " << sigma << endl;
    }

    //XScone
    if (parse==1){
        double _pt2 = pt2(S,x1,x3);
        double sigmacone = XScone(S,x1,x3);
        cout << _pt2 << " " << sigmacone << endl;
    }
}

*/

if(ShowerOn==true){

    //          double _pt2 = pt2(S,x1,x3);
    //          showercalls++;
    //          Shower(_pt2, cutvalue);
    Shower2(S,cutvalue);
}

}

cout << "SHOWERCALLS=" << showercalls << endl;
return 0;
}

```

B.3 XS.h

```

#ifndef XS_H
#define XS_H

```

```

double XSqq(double x1, double x3);
double XSgq(double x1, double x3);
double XSgg(double x1, double x3);
double XScone(double E, double x1, double x3);

#endif

```

B.4 XS.c

```

#include "XS.h"
#include "ConeVariables.h"

double XSqq(double x1, double x3){
    return (x1*x1+x3*x3) / (1-x1) / (1-x3);
}

double XSgq(double x1, double x3){
    return (x1*x1*x1+x3*x3) / (1-x1) / (1-x3);
}

double XSgg(double x1, double x3){
    return (x1*x1*x1+x3*x3*x3) / (1-x1) / (1-x3);
}

double XScone(double S, double x1, double x3){
    double _pt2 = pt2(S, x1, x3);
    double y = rapidity(x1, x3);
    return 1./_pt2;
}

```

B.5 ConeVariables.h

```

#ifndef CONEVARIABLES_H
#define CONEVARIABLES_H

double pt2(double S, double x1, double x3);
double pt(double S, double x1, double x3);
double rapidity(double x1, double x3);

#endif

```

B.6 ConeVariables.c

```

#include "ConeVariables.h"
#include <math.h>

double pt2(double S, double x1, double x3){
    return S*(1-x1)*(1-x3);
}

double pt(double S, double x1, double x3){
    return sqrt(S*(1-x1)*(1-x3));
}

```

```
double rapidity(double x1, double x3){
    return .5*(log(1-x1)-log(1-x3));
}
```

B.7 Shower2.h

```
#ifndef SHOWER2_H
#define SHOWER2_H

#include <iostream>
#include <stdlib.h>
#include <time.h>
#include "string.h"
using namespace std;

void f(signed int H, double kappatop, double ytop, signed int Htop,
    double kappaold, double yold, signed int Hold, double s0, double cutvaluekappa);
void Shower2(double pt2, double cutpt2);

#endif
```

B.8 Shower2.c

```
#include <iostream>
#include <cstdlib>
#include <ctime>
#include "string.h"
#include "math.h"
#include "SplittingFunction.h"
using namespace std;

void f(signed int H, double kappatop, double ytop, signed int Htop,
    double kappaold, double yold, signed int Hold, double s0, double cutvaluekappa);

void Shower2(double pt2, double cutpt2){
    //s0 is the scale factor of the logarithm
    //level and totalL are bug finders and can be ignored

    double s0 = 0.1;
    double L = log(pt2/s0);
    int level = 0;
    double cutkappa = log(cutpt2/s0);
    f(0, L, 0, 0, L, 0, 0, s0, cutkappa);
    cout << endl;
    return;
}

void f(signed int H, double kappatop, double ytop, signed int Htop,
    double kappaold, double yold, signed int Hold, double s0, double cutvaluekappa){
    double r1, kappa, A;
    r1 = rand();
    r1 = r1/RAND_MAX;
    A = 0.5*(kappatop-kappaold)+H*yold+Htop*ytop;
    kappa = kappaold + 0.5*A - 0.5*sqrt(A*A-4*log(r1));

    //When the partons are below the cut off value in pt^2,
```

```

//the pt^2 and y before the cut off value are printed.
if(kappa<cutvaluekappa){
    cout << exp(kappaold)*s0 << " " << yold << " ";
    return;
}
else{
    double r2 = rand();
    r2 = r2/RAND_MAX;
    double y;
    if(H==1){
        y = yold-A-0.5*(kappaold-kappa)+r2*(kappaold-kappa+A);
    }
    else{
        y = yold-0.5*(kappaold-kappa)+r2*(kappaold-kappa+A);
    }

    //Call the function again for both the left and right triangle
    if(H!=0){
        f(H, kappatop, ytop, Htop, kappa, y, H, s0, cutvaluekappa);
        f(-H, kappaold, yold, Hold, kappa, y, H, s0, cutvaluekappa);
    }
    else{
        f(1, kappatop, ytop, Htop, kappa, y, H, s0, cutvaluekappa);
        f(-1, kappatop, ytop, Htop, kappa, y, H, s0, cutvaluekappa);
    }
}
}
}

```

B.9 rootmacro.cxx

```

#include "Riostream.h"
#include <string>

void rootmacro(){

    cout << "BEGIN ROOTMACRO" << endl;

    Float_t x;
    Float_t w;
    Int_t nlines = 0;

    string stringfilename;
    string stringtitle;
    int nevents;
    double S;
    double cutvalue;
    cin >> stringfilename >> stringtitle >> nevents >> S >> cutvalue;
    const char * title = stringtitle.c_str();
    //    const char * filename = stringfilename.c_str();
    const char * filename = "output.root";

    TFile *f = new TFile(filename,"RECREATE");
    TNtuple *ntuple = new TNtuple("ntuple","title","x");

    int nbins = 200;
    double ymin = -log(10.)/2;
    double ymax = log(10.)/2;

```

```

nlines++;
TH1F *h1 = new TH1F("h1",title,nbins,cutvalue*0.9,S*1.01);
// TH1F *h1 = new TH1F("h1",title,200,-1,2);
TH1F *h2 = new TH1F("h2",title,nbins,-log(10.)/2,log(10.)/2);
int realnevents=0;
int cinnotgood=0;
int cineol=0;
while (true){
    cin >> x >> w;
    if (!cin.good()){
        cinnotgood++;
        break;
    }
    h1->Fill(x,1);
    ntuple->Fill(x,1);
    h2->Fill(w,1);
    ntuple->Fill(w,1);
    nlines++;
    realnevents++;
}

cout << "NEVENTS=" << nevents << endl;
cout << "REALEVENTS=" << realnevents << endl;
cout << "CINNOTGOOD=" << cinnotgood << endl;
cout << "CUTVALUE=" << cutvalue << endl;
double ybinwidth = (ymax-ymin)/nbins;

//Normalising the integrated histograms to unity.
h1->Scale(1./realnevents);
h2->Scale(1./(ybinwidth*realnevents));

f->Write();
cout << filename << endl;
cout << "END OF ROOTMACRO.CXX" << endl;
}

```

B.10 verysmallrootscript.cxx

```

#include "Riostream.h"
#include "TFile.h"
#include "TTree.h"
#include "TNtuple.h"
#include "string.h"

void verysmallrootscript(){
//    string s;
//    while (true){
//        cin >> s;
//        if(cin.eof()){break;}
//    }
//    const char * filename = s.c_str();
const char * filename = "output.root";
TFile *f = new TFile(filename);
gROOT->SetStyle("Plain");
TCanvas *c1 = new TCanvas("c1","c1",0,0,1500,1200);
TF1 *P2 = new TF1("P2","0.35*(log(10.)-2*abs(x))",-log(10.)/2,log(10.)/2);

```

```

//      c1->Divide(2,2);
c1->SetLeftMargin(0.1);
c1->SetBottomMargin(0.1);

//      c1->cd(1);
h1->SetTitle(0);
h1->SetStats(0);
h1->GetXaxis()->SetTitle("pt2 (GeV)");
h1->GetXaxis()->SetTitleSize(0.03);
h1->GetXaxis()->SetLabelSize(0.02);
h1->GetYaxis()->SetTitle("Relative occurrence");
h1->GetYaxis()->SetTitleSize(0.03);
h1->GetYaxis()->SetLabelSize(0.02);
h1->SetFillColor(kBlue-10);
//      h1->Draw();

//      c1->cd(2);
h2->SetTitle(0);
h2->SetStats(0);
h2->GetXaxis()->SetTitle("y (GeV)");
h2->GetXaxis()->SetTitleSize(0.03);
h2->GetXaxis()->SetLabelSize(0.02);
h2->GetYaxis()->SetTitle("Relative occurrence");
h2->GetYaxis()->SetTitleSize(0.03);
h2->GetYaxis()->SetLabelSize(0.02);
h2->SetFillColor(kBlue-10);
h2->Draw();

//      P2->Draw("same");
//      P0->SetLineWidth(2);
//      P0->SetLineColor(2);
//      P0->Draw("same");
//      c1->cd(3);
//      P2->Draw();
}

```