Master Thesis

Dressed states and the infrared problem

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Abstract

Infrared divergences appear in massless gauge theories as a consequence of the false assumption that the interactions vanish in the asymptotic region. We claim that these divergences cancel when S matrix elements are taken between appropriate asymptotic dressed states. Dressed states are defined and a calculation to support this claim is shown. Properties of dressed states are used for a derivation of the well known result of exponentiation of soft divergences in amplitudes. In the final part a different type of dressing is introduced. The asymptotic fields are dressed in such a way that they become locally gauge invariant and they create physical charged particles.

Contents

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1 Introduction

Particle physics is nowadays on its way to cracking the Standard Model. To achieve that goal a lot of effort is invested in setting up scattering experiments as well as in making good theoretical predictions for such experiments. A central role in the theoretical approach is played by the S matrix which is used to compute probability amplitudes for different processes. Assume that the asymptotic states of a scattering experiment represent physical particles with well defined momentum, spin, charge,mass, etc. We define the S matrix element for a given process as the overlap of an asymptotic state defined in the far past with another asymptotic state from the far future. A prediction for a certain scattering process is then obtained by integrating the modulus squared of the amplitude over the phase-space of the final state particles weighted by a certain function that defines the observable we want to predict.

Such a prediction exists if indeed the asymptotic states are physical and if the S matrix elements are well defined (so finite). This is not always the case. Most of the scattering experiments we perform involve either abelian or non-abelian massless gauge bosons. In such situations the S matrix elements are plagued with divergences associated with soft or collinear massless particles or with very energetic ones.

Gauge field theories like QED or QCD are classically conformal invariant. This implies that our calculations describe phenomena that happen at any length scale. For example, in a pertubative series, in addition to the leading order, we have all kind of higher orders loop integrals. In such a loop at least one virtual quanta is exchanged. If the gauge boson is massless then such a loop correction might be divergent because we allow the virtual quanta to have any energy which leads to interactions at any length scale. The only problem is that we always include approximation in our theory and it might be the case that some of these approximations do not hold at certain length scales. This is the main cause of divergent S matrix elements. Let us explain this with more details.

A virtual particle of very high energy (compared to the energy scale of the scattered particles) is the signature of an interaction that happen at a very short distance. At this scale we do not know for sure which degrees of freedom are relevant so assuming the theory is fully applicable we end up with a UV divergence. This type of divergences can be cured by renormalization. It has been proved that the high-energy contributions decouple from the low energy ones and they can be absorbed somehow in the definition of parameters like coupling constants or masses.

On the other side of the spectrum the gauge bosons mediate long ranged interactions which cannot be neglected. In this case building a perturbative expansion in terms of free states is wrong because the true asymptotic fields are not free. Unfortunately we use the LSZ reduction formula to make the connection between Green functions and S matrix elements even though this applies only for theories in which the interaction vanishes in the asymptotic region. As we shall see in one of the chapters of this thesis the S matrix has well defined elements when defined between the true asymptotic states, eigenstates of a Hamiltonian that includes the non-vanishing asymptotic interaction.

The IR problems can be cured in QED using the KLN theorem. Even though the S matrix elements are ill defined the physically measurable probabilities can be obtained if we sum over all the degenerate final states. In QCD this does not always work. When it works we have to sum over both final and initial degenerate states which turns out to be very complicated because of non-perturbative effects like confinment. The solution comes from factorization. This is based on the fact that high energy inclusive cross-sections can be written as convolutions of short-distance finite partonic cross-sections with long-distance factors such as parton distributions or fragmentation functions which are non-perturbative but can be associated with hadronic wave functions. Proving factorization to all orders in perturbation is difficult but rewarding because a factorization theorem is a sufficient condition to perform a resummation of perturbation theory. Resummation is vital for probing the all-order structure of perturbation theory. This area has developed a lot in the past decades leading to a large amount of literature.

In the present thesis we tackle the problem of infrared divergences from a different angle. We try to understand the cause of these divergences and find a way to contruct infrared finite amplitudes in perturbation theory. We are concerned as well with constructing well defined asymptotic charges. We show in the second chapter that, in the asymptotic region, the charges that emerge from the ordinary field theory of QED have no singl-particle interpretation. In fact there is no such thing as a lonely charged particle in the asymptotic region. It turns out that in that regio charged particles are surrounded by a cloud of soft and collinear gauge bosons. Such a state is known as a coherent state in the Fock space of the gauge bosons. From this point of view we can remark that the S matrix is not well defined since the number of gauge bosons in the final state is unknown.

The structure of the thesis is as follows. We begin with a semi-classical example of a classical conserved current interacting with a quantized electromagnetic field. If the interaction takes place in a limited amount of time the final state is a coherent state. We also show that the number of soft photons emitted in the process tends to infinity if the amount of radiated energy is finite. This is the so-called infrared catastrophe. Then we turn to quantum theory and show that the asymptotic Dirac field in the Heisenberg picture creates fermions that are surrounded by a cloud of soft and collinear photons.

One big part of the thesis is dedicated to building finite S matrix elements. We start with a review of scattering theory and the way the S matrix is defined. We identify the root of the infrared divergences and find a way to overcome them. Later on we define dressed states in such a way that the S matrix elements between them are finite. One entire chapter is dedicated to an application for this theory. We consider the process $\gamma \to 2jets$ and show that up to NLO its S matrix elements between dressed states are finite. Once this is done we use the properties of dressed states and the finite S matrix elements to derive an important result, the exponentiation of soft divergences in amplitudes. The leading order exponent of QED for a certain process is found. The same approach holds in QCD as well and the first two webs and the corresponding color factors are found for a given process.

The last part of the thesis presents a method of dresing the asymptotic fields in such a way that they become gauge invariant and the particles they create / annihilate have the desired properties of physical particles. The resulting asymptotic charged particles are free of any kind of IR singularities and they display the electric field of moving particles. We also show that the asymptotic interaction Hamiltonian involving such dressed field vanishes and a LSZ reduction formalism can be implemented.

2 A few examples

2.1 Introduction

Quantum Field Theory is full of approximations which many times are pushed beyond their limits leading to all kind of problems. One of these approximations states that the fields and particles do not interact in the asymptotic region. The corresponding problem asociated to this approximation in massless gauge theories is known to be the infrared divergences problem. We shall show in this chapter that, in QED, the asymptotic states (or initial and final states if you want) are not free states but rather coherent states. Such states contain fermions and a number gauge bosons which can diverge in certain situations if the bosons are soft. Similar examples can be studied for QCD but the problem is more complicated there given the non-abelian structure of the theory and the confinement property of particles carying a color charge.

In what follows we show two examples. One treats the intreraction of a classical current with a quantized electromagnetic field. It is shown that, in the approximation of free asymptotic fields, the final state is a coherent state that contains a divergent number of soft photons. The other example shows that,in the Heisenberg picture, the quantized fermionic field is not a free field in the asymptotic region because the interactions do not vanish. The particles created by this field have no single-particle interpretation given the cloud of photons that surrounds the asymptotic fermions. The number of soft photons in the cloud diverges and the so-called infrared catastrophe occurs once again.

2.2 A semi-classical example

This first section is dedicated to a simple QED application which considers the interaction of a classical conserved current with a quantized electromagnetic field. The whole calculation is subject to the approximation that the interaction is 'switched on' only a finite amount of time in such a way that the incoming and outgoing electromagnetic field has the features of a free field.

We work in the Fock space of the incoming photons and the main goal is to determine and study the final state given that the initial state is the electromagnetic vacuum i.e. no field quanta are present.

In order to find the final state we need to determine the so called S-operator which satisfies

$$|out>= S^{\dagger}|in>= S^{-1}|in> \quad \text{and} \quad A^{\mu}_{out}(x) = S^{-1}A^{\mu}_{in}(x)\, S, \tag{2.1}$$

where $|out>$ and $|in>$ are the states describing the system in the distant future / past.

The starting point is the wave equation of the electromagnetic field. The conserved current j^{μ} acts as a source for the field and, in the Feynman gauge, the equation of motion reads

$$\Box A^{\mu} = j^{\mu}. \tag{2.2}$$

This equation can be solved using the Greens functions method (for a deeper insight of this method see [5]). A general solution has the following form

$$A^{\mu}(x) = A^{\mu}_{0}(x) + \int d^4y\, G(x-y)\, j^{\mu}(y), \tag{2.3}$$

where A^{μ}_{0} is a free field, the solution of the source-less wave equation, and $G(x-y)$ is a Green function, a solution of the wave equation

$$\Box G(x) = \delta^{(4)}(x). \tag{2.4}$$

Such a solution can be also written in the 'Feynman form'

$$G(x) = \frac{-1}{(2\pi)^4} \int d^4k \frac{\exp(-ikx)}{k^2 + i\epsilon}. \tag{2.5}$$

Other solutions of the same wave equation (2.4) are the so called 'retarded' and 'advanced' Green functions that have the special property

$$G_{ret}(x) = 0 \quad \text{for} \quad x_0 < 0,$$

while

$$G_{adv}(x) = 0 \quad \text{for} \quad x_0 > 0. \tag{2.6}$$

Thus they can be written in a form similar to Eq. (2.5)

$$G_{ret/adv}(x) = \frac{-1}{(2\pi)^4} \int d^4k \frac{\exp(-ikx)}{k^2 + i\epsilon} \theta(\pm x_0). \tag{2.7}$$

The general solution of Eq. (2.3) can now be written in a particular way in terms of these Green functions. That is

$$A^\mu(x) = A^\mu_{in}(x) + \int d^4y\, G_{ret}(x-y)\, j^\mu(y) = A^\mu_{out}(x) + \int d^4y\, G_{adv}(x-y)\, j^\mu(y). \tag{2.8}$$

where the general free field was replaced by the incoming and outgoing electromagnetic fields ($t \equiv x^0$)

$$A^\mu_{in}(x) = A^\mu(x)|_{t\to-\infty} \qquad \text{and} \qquad A^\mu_{out}(x) = A^\mu(x)|_{t\to\infty}. \tag{2.9}$$

Eq. (2.8) takes us one step closer to determining the S operator since

$$\begin{aligned} A^\mu_{out}(x) &= A^\mu_{in}(x) + \int d^4y\, [G_{ret}(x-y) - G_{adv}(x-y)]\, j^\mu(y) \\ &\equiv A^\mu_{in}(x) + \int d^4y\, G^{(-)}(x-y)\, j^\mu(y). \end{aligned} \tag{2.10}$$

All the calculations we've done so far were classical and no quantization of any kind was imposed. Let us take a step further and quantize the electromagnetic field. We do this in the canonical way by imposing certain commutation relations between the field components and the corresponding conjugated fields, which coincide with the time derivative of the field. These relations are given by

$$[A^\mu(x), \dot{A}^\nu(y)]|_{x_0=y_0} = -i\, g^{\mu\nu} \delta^{(3)}(\vec{x} - \vec{y}). \tag{2.11}$$

Following the usual approach we can write the quantized field in terms of the particle creation and annihilation operators. For later use we can formally decompose it into two terms that contain either creation or annihilation operators : $A^\mu(x) = A^{\mu-}(x) + A^{\mu+}(x)$. Given the commutation relation for the creation and annihilation operators we have that

$$[A^{\mu-}(x), A^{\nu+}(y)] = g^{\mu\nu} \int d^4k \frac{1}{(2\pi)^3} e^{-ik(x-y)} \theta(k^0)\delta(x^2). \tag{2.12}$$

where θ is the Heaviside step function and $g^{\mu\nu}$ is the Minkowsky metric.

Also based on the commutation relations of field operators another important relation can be derived, a direct connection between the commutator of two gauge fields and the Green's functions.

$$[A^\mu_{in}(x), A^\nu_{in}(y)] = ig^{\mu\nu} \left(G_{adv}(x-y) - G_{ret}(x-y)\right). \tag{2.13}$$

Remark that Eq. (2.13) fits into Eq. (2.10) to arrive at

$$A^\mu_{out}(x) = A^\mu_{in}(x) + i \int d^4y \; [A_{in}(y)\, j(y), A^\mu_{in}(x)]. \tag{2.14}$$

At this point we can derive the S operator easily using the Hadamard lemma

$$e^A B e^{-A} = B + [A, B] + \frac{1}{2!}[A, [A, B]] + ... \tag{2.15}$$

and so from Eq. (2.1) and Eq. (2.14) and Eq. (2.15) we find

$$S = \exp\left[-i \int d^4x A_{in}(x)\, j(x)\right] = \exp\left[-i \int d^4x A_{out}(x)\, j(x)\right]. \tag{2.16}$$

The last part of Eq. (2.16) is obtained easily if we take Eq. (2.1), (2.10) and (2.14), write the incoming field in terms of the outgoing field and use Hadamard lemma once again.

Our initial goal has been reached now and we can proceed and study the final state emerging from a given initial state.

Suppose the initial state has no photons. The final state follows

$$|0, out>= \exp\left[i \int d^4x A_{in}(x)\, j^*(x)\right] \; |0, in>. \tag{2.17}$$

In the notation for the final state 0 is there to point out that this state corresponds to the initial state with no photons. It does not mean the final state has no photons. The exact number of photons will be computed later.

Note that this final state is a coherent state as the exponential in Eq. (2.17) can be brought into a form similar to Eq. (3.13). This is easily done using the BCH formula and the decomposition of A^μ into positive and negative frequency parts

$$S = \exp\left[-i \int d^4x A^+_{in}(x)\, j(x)\right] \; \exp\left[-i \int d^4x A^-_{in}(x)\, j(x)\right] \times$$
$$\times \exp\left\{\frac{1}{2} \int \int d^4x d^4y \left[A^+_{in}(x)\, j(x), A^-_{in}(y)\, j(y)\right]\right\}. \tag{2.18}$$

The last term of Eq. (2.18) is just a function of the source given the commutation relation of Eq. (2.12)

$$\exp\left\{\frac{1}{2} \int \int d^4x d^4y \left[A^+_{in}(x)\, j(x), A^-_{in}(y)\, j(y)\right]\right\} = \exp\left[\frac{1}{2} \int d\tilde{k} J^*(k) J(k)\right], \tag{2.19}$$

where $J(k)$ is the Fourier transform of the current. This current can be now decomposed into one longitudinal part and two transversal parts along some defined polarisation vectors (remember that a physical photon has two transversal polarizations) :

$$J^\mu(k) = k^\mu J_{long}(k) + J^\mu_{tr}(k). \tag{2.20}$$

Using the fact that we work on the mass shell and that the photon is massless we arrive at

$$J^*(k)J(k) = J^*_{tr}(k)J_{tr}(k) = |J_1(k)|^2 + |J_2(k)|^2. \tag{2.21}$$

This leads to

$$S = \exp\left[-i\int d^4x A^+_{in}(x)\, j(x)\right]\ exp\left[-i\int d^4x A^-_{in}(x)\, j(x)\right]\ \exp\left\{\frac{1}{2}\int d\widetilde{k}\left[|J_1(k)|^2 + |J_2(k)|^2\right]\right\}, \tag{2.22}$$

which has the form of a coherent state operator as defined in the next chapter in Eq. (3.13)

It is straight forward to find the average number of photons that are emitted in this process. It is shown in the chapter on coherent states the number of particles in a coherent state is finite and given (see Eq. 3.8) by

$$n_{out} = \int d\widetilde{k}\left[|J_1(k)|^2 + |J_2(k)|^2\right] = \int d\widetilde{k}|J(k)|^2. \tag{2.23}$$

In the next paragraphs we use these results to compute the probability of photon emissions and also the average energy that is emitted by the source. In the end a connection between the average number of emitted particles and the average emitted energy will be made in order to arrive at the so called 'infrared catastrophe'.

Let us begin with the probability that n photons are emitted. This is given by

$$p_n =< 0, out|P_n|0, in >, \tag{2.24}$$

with

$$P_n = \frac{1}{n!}\int d\tilde{k}_1 d\tilde{k}_2 ... d\tilde{k}_n \sum_{\lambda_i=1,2} |k_1\lambda_1, ..., k_n\lambda_n >< k_1\lambda_1, ..., k_n\lambda_n| \tag{2.25}$$

the projector on n photon states. These states are usual Fock states containing photons. Each photon is described by its momentum and polatization in such a way that

$$|k_1\lambda_1, ..., k_n\lambda_n >= a^{\dagger(\lambda_1)}(k_1)a^{\dagger(\lambda_2)}(k_2)...a^{\dagger(\lambda_n)}(k_n)|0> . \tag{2.26}$$

To find the desired probabilities we need to compute the matrix element

$$< k_1\lambda_1, ..., k_n\lambda_n, out|0, in >=< k_1\lambda_1, ..., k_n\lambda_n, in|S|0, in >=$$

$$= \exp\left[-\frac{1}{2}\int d\widetilde{k}(|J_1(k)|^2 + |J_2(k)|^2)\right] \quad < k_1\lambda_1, ..., k_n\lambda_n, in|e^{-i\int d^4x A^+_{in}(x)j(x)}|0, in > . \tag{2.27}$$

If we perform a Fourier transform for the current j

$$e^{-i\int d^4x A_{in}^{+}(x)j(x)} = e^{-i\int d\tilde{k}\sum_{\lambda=1,2} a^{\dagger(\lambda)}(k)J_\lambda(k)} \tag{2.28}$$

and then making use of normalization relation for the Fock states we find

$$< k_1\lambda_1, ..., k_n\lambda_n, in|e^{-i\int d^4x A_{in}^{+}(x)j(x)}|0, in >=$$

$$= \frac{(-i)^n}{n!} < k_1\lambda_1, ..., k_n\lambda_n| \int d\tilde{q}_1...d\tilde{q}_n a^{\dagger(\sigma_1)}(q_1)J_{\sigma_1}(q_1)...a^{\dagger(\sigma_n)}(q_n)J_{\sigma_n}(q_n)|0 >$$

$$= (-i)^n J_{\lambda_1}(k_1)....J_{\lambda_n}(k_n). \tag{2.29}$$

The probability turns out to be

$$p_n = \frac{1}{n!}\int d\tilde{k_1}...d\tilde{k_n} \sum_{\lambda_i=1,2} \left[J_{\lambda_1}^*(k_1)....J_{\lambda_n}^*(k_n)\; J_{\lambda_1}(k_1)....J_{\lambda_n}(k_n)\right] \times$$

$$\times \exp\left\{-\int d\tilde{k}\left[|J_1(k)|^2 + |J_2(k)|^2\right]\right\}$$

$$= \frac{1}{n!}\left\{\int d\tilde{k}\left[|J_1(k)|^2 + |J_2(k)|^2\right]\right\}^n \exp\left\{-\int d\tilde{k}\left[|J_1(k)|^2 + |J_2(k)|^2\right]\right\}. \tag{2.30}$$

This has the form of a Poisson distribution

$$p_n = \frac{1}{n!}\overline{n}^n\, e^{-\overline{n}}, \tag{2.31}$$

if we denote

$$\overline{n} = \int d\tilde{k}(|J_1(k)|^2 + |J_2(k)|^2). \tag{2.32}$$

Eq. (2.31) reveals the statistical independence of successive emissions.

From the distribution in Eq. (2.31) we find the average number of emitted photons to be

$$\sum_0^\infty np_n = \sum_0^\infty n\frac{1}{n!}\overline{n}^n\, e^{-\overline{n}} = \overline{n}\, e^{-\overline{n}}\sum_1^\infty \frac{1}{(n-1)!}\overline{n}^{n-1} = \overline{n}\, e^{-\overline{n}}e^{\overline{n}} = \overline{n}. \tag{2.33}$$

In Eq. (2.23) we computed the number of particles in the coherent state $|0, out>$. As expected this number coincides with the average number of photons emitted by the source.

Let us now compute the emitted energy of the interacting electromagnetic field. This is given by

$$\overline{\varepsilon} =< 0, in|H(A_{out})|0, in >=< 0, in|S^{-1}H(A_{in})\, S|0, in >, \tag{2.34}$$

where

$$H(A_{in}) = \int d\widetilde{k}\, k^0 \sum_{\lambda=1,2} a_{in}^{(\lambda)+}(k)\, a_{in}^{(\lambda)}(k) \tag{2.35}$$

is the Hamiltonian of the incoming field and so the average emitted energy is

$$\overline{\varepsilon} =< 0, in|S^{-1} \int d\widetilde{k}\, k^0 \sum_{\lambda=1,2} a_{in}^{(\lambda)+}(k)\, a_{in}^{(\lambda)}(k)\, S|0, in >=$$

$$=< 0, in| \int d\widetilde{k}\, k^0 \sum_{\lambda=1,2} S^{-1} a_{in}^{(\lambda)+}(k)\, S\, S^{-1} a_{in}^{(\lambda)}(k)\, S|0, in >=$$

$$=< 0, in| \int d\widetilde{k}\, k^0 \sum_{\lambda=1,2} \left[a_{in}^{(\lambda)+}(k) - iJ_\lambda(k)\right] \left[a_{in}^{(\lambda)}(k) + iJ_\lambda^*(k)\right] |0, in >=$$

$$= \int d\widetilde{k}\, k_0 \left[|J_1(k)|^2 + |J_2(k)|^2\right]. \tag{2.36}$$

This result is the same to that obtained in the classical theory of the electromagnetic field.

The relation between the average emitted energy and the average number of emitted photons in an element of phase-space $d\widetilde{k}$ comes out as

$$d\overline{n} = \frac{1}{k_0} d\overline{\varepsilon}. \tag{2.37}$$

A problem arises in the case of low frequency photons because k^0 is small and for a finite emitted energy the number of photons will tend to infinity. This is the so called infrared catastrophy.

Indeed there's a problem when $\overline{n} \to \infty$because all the emission probabilities will go to zero. For example

$$p_0 = | < 0, out|0, in| > |^2 = e^{-\overline{n}} \to 0. \tag{2.38}$$

In conclusion the final state, which is a coherent state, will include an unknown number of soft photons. In this case the S operator will be ill-defined.

2.3 A quantum example

In order to make the connection between Green's functions of a certain process and the corresponding S matrix element a so called reduction formula can be used. Most common formula of this type is the one introduced by Lehmann, Symanzik and Zimmermann (LSZ). In this formalism the asymptotic fields are taken as free fields and the S matrix elements are the residues of the poles that arise in the Fourier transform of the correlation functions when four-moments of the external particles are put on-shell. However, in QED the asymptotic fields are not free and the interaction is carried by very soft photons. We shall show in what follows that, in the Heisenberg picture, the asymptotic fields would create charged 'objects' that have no single particle interpretation.

It is shown in chapter 6 that in QED the asymptotic interaction Hamiltonian in the interaction picture has the following form

$$H_I^{as}(t) = -e \int d\mathbf{x} A_\mu(t, \mathbf{x}) J_{as}^\mu(t, \mathbf{x}), \tag{2.39}$$

where $A_\mu(t,\mathbf{x})$ is the free electromagnetic field and $J^\mu_{as}(t,\mathbf{x})$ is the asymptotic matter current defined as

$$J^\mu_{as}(t,\mathbf{x}) = \int d\mathbf{p}\left(b^\dagger(\mathbf{p})b(\mathbf{p}) - d^\dagger(\mathbf{p})d(\mathbf{p})\right)\frac{p^\mu}{\omega(\mathbf{p})}\delta^{(3)}\left(\mathbf{x} - \mathbf{p}t/\omega(\mathbf{p})\right). \quad (2.40)$$

Remember that the free fermionic field and the free photon field that enter in the expression of the asymptotic current and the Hamiltonian in Eq. (2.39) are actually the expressions of the fields in the interaction picture. If we want to get the fields in the asymptotic Heisenberg picture then we invert Eq (4.32) and write explicitly the evolution operators in the interaction picture to find

$$O^{as}_H(t) = \tilde{T}\exp\left(i\int_{t_0}^t d\tau H^{as}_I(\tau)\right)O^{as}_I(t,t_0)T\exp\left(i\int_{t_0}^t d\tau H^{as}_I(\tau)\right). \quad (2.41)$$

We also prove later that, in QED, the time ordered exponential of the interaction Hamiltonian can be written in a simple manner given that the commutator of two of these Hamiltonians is a c-number as the asymptotic currents commute with eachother. Then it is straight forward to get expressions for the Heisenberg picture fields.

A final step before recovering the Heisenberg picture' expression of the fields is the use of the Hadamard lemma

$$e^A Be^{-A} = B + [A,B] + \frac{1}{2!}[A,[A,B]] + ... \quad (2.42)$$

In this way the expression obtained for the electromagnetic field is similar to the one we got in the last paragraph when we considered the electromagnetic field interacting with a classical current (remember in that case the interaction was available a limited time). We find here

$$A^{as}_\mu(x) = A_\mu(x) + e\int_{-\infty}^t d\tau d^3y G^{(-)}(\tau - t, \mathbf{y} - \mathbf{x})J^{as}_\mu(\tau,\mathbf{y}). \quad (2.43)$$

Where $G^{(-)}(x)$ is defined in Eq. (2.10).

In order to get the similar expression for the fermionic field we need the commutator of the interaction Hamiltonian with the free fermionic field which basically involves commutators of the charge density operator (Eq. 6.10) coming from the asymptotic current with the particle creation and annihilation operators. These commutators are of the form : $[\rho(p), b(q)] = -b(q)\delta^{(3)}(\mathbf{p}-\mathbf{q})$ and $[\rho(p), b^\dagger(q)] = b^\dagger(q)\delta^{(3)}(\mathbf{p}-\mathbf{q})$.

In the end, by applying Hadamard lemma and using the commutation relation of the charge density with the fermionic field operators, we find the field in the Heisenberg picture to be

$$\Psi^{as}(x) = \int d^3p \frac{1}{(2\pi)^3\sqrt{2\omega(\mathbf{p})}}D(p,t)\left[b(p)u(p)e^{-ipx} + d^\dagger(p)\bar{v}(p)e^{ipx}\right], \quad (2.44)$$

where $D(p,t)$ is an operator that involves photonic creation and annihilation operators.

Remark from Eq. (2.41) that in the case when the asymptotic Hamiltonian vanishes the asymptotic fields will simply be free fields.

This is not the case in massless gauge theories where the interaction is long-ranged and does not vanish in time. Thus the asymptotic fields in Heisenberg picture will not be free and the LSZ reduction formula cannot be used. The price we pay if we use this formula is the appearance of infrared divergences.

Let us return to Eq. (2.44) and add some comments. The Hadamard lemma (Eq. 2.42) has an infinite number of terms but given that each commutator of the asymptotic Hamiltonian with the creation and anihilation operators gives back a term that has only photonic operators multiplied by the initial fermionic operators, the n'th commutator in Eq. (2.42) will be simply the n'th power of one commutator multiplied by the creation or annihilation operator. This is how we get the expression in Eq. (2.44) where the operator $D(p,t)$ contains only photonic operators and it is the one responsible for the infrared divergencies. It has two parts : a phase coming from terms that result from writing the time-ordered exponential as a usual exponential and the soft photons part coming from the commutator of the interaction Hamiltonian with the fermionic field. The latter is given by

$$D_{soft}(p,t) = \exp\left[-e\int_{soft} d^3k \frac{1}{(2\pi)^3 2\omega(\mathbf{k})}\left(\frac{p\cdot a(k)}{p\cdot k}e^{-ikpt/\omega(\mathbf{p})} - \frac{p\cdot a^\dagger(k)}{p\cdot k}e^{ikpt/\omega(\mathbf{p})}\right)\right]. \quad (2.45)$$

As a conclusion of this section we remark that the asymptotic fields in the Heisenberg picture are not free fields and they create some sort of composite fermions. The 'distortion' operator of Eq. (2.45) creates a cloud of soft photons around the original fermion. Thus, just like in the semi-classical example, the final state of a QED scattering experiment is a coherent state. Since the 'distortion' operator is divergent for very soft photons the average number of photons in the coherent state is divergent. Moreover any attempt to compute the amplitude of a scattering process involving these fields would lead to divergent results.

Summary

In both examples studied in this chapter we found that the final state of a generic interaction process is a coherent state of photons. The asymptotic fields are not free but rather interact exchanging soft photons. The number of these photons is ill-defined and any scattering amplitude computed for a process that involves interactions is infrared divergent. A charged particle in the asymptotic region appears as being surrounded by a cloud of photons which can be soft and causes the single particle description to fail.

3 Coherent states

This chapter is an introduction to coherent states. The word 'coherent' in their name comes from Quantum Optics where this kind of states are used to describe different phenomena involving electromagnetic radiation emitted by a laser which is coherent. We shall discuss first about photonic coherent states and give some of their peoperties. Then generalized coherent states are introduced for a general Lie group. The connection to photonic coherent states is made by chosing the Weyl-Heisenberg group as a starting point. It is shown that the results of the latter calculation coincide with the ones we obtain in the beginning of the chapter.

3.1 Coherent states in Hilbert space

The simplest definition for a coherent state is based on the properties of the Fock space. This space is mostly used to describe systems with many degrees of freedom such as systems of harmonic oscillators or many-particles systems.

Any vector of this space is characterized by the so called 'occupation numbers' each of these counting the number of particles or harmonic oscillators characterized by a certain quantum number. If we talk about particles these numbers will be positive integers (including zero) for bosons or, in case of fermions, their value is either zero or one due to Pauli principle. These occupation numbers can be changed by acting with a so called 'ladder' operator in the harmonic oscillator case or, equivalently, with a creation or annihilation operator for a system of particles. Each of these operators will modify one occupation number by unity (if possible). Alternatively a Fock state for a system of n particles can be parametrized by a n-dimensional vector that has as components the quantum numbers of the n particles. That is the parametrization we will mostly use in this part.

The creation and annihilation operators come with different properties depending on the type of particles they create / annihilate. Bosonic operators are represented by complex-valued matrices while fermionic operators are Grassmann numbers. This is reflected in their commutation (for bosons) or anticommutation (for fermions) relations

$$[a(k_1), a^\dagger(k_2)]_\mp \sim \delta^{(3)}(\mathbf{k_1} - \mathbf{k_2}). \tag{3.1}$$

Let us consider the Fock vector associated to a system of n particles with given momenta $k_1, ..., k_n$

$$|k_1, ..., k_n >\equiv C(k_1, ..., k_n)a^+(k_1)a^+(k_2).....a^+(k_n)|0>, \tag{3.2}$$

where $C(k_1, ..., k_n)$ is a combinatorial factor which has to be taken in account if two or more momenta are equal.

Acting with $a(q)$ on $|k_1, ..., k_n >$ we simply anihilate one particle of momentum q, if any, or get zero otherwise. This is formally equivalent to acting with a derivative with respect to $a^+(q)$ on $|k_1, ..., k_n >$

$$a(q)|k_1, ..., k_n >= \frac{\partial}{\partial a^+(q)}|k_1, ..., k_n > . \tag{3.3}$$

The natural question that arise now is : 'Can we find any eigenstates of the annihilation operators ?' The answer is positive. Based on the Eq. (3.3) we expect that such an eigenstate has some kind of an exponential form just because the derivative of an exponential remains an exponential.

Consider a normalised state of the form

$$|\alpha >= e^{-\frac{1}{2}\int d\widetilde{k}|\alpha(k)|^2}\, e^{\int d\widetilde{k}\alpha(k)a^\dagger(k)}\, |0>, \tag{3.4}$$

where, for particles with mass m, the integration measure is defined as

$$d\tilde{k} = \frac{1}{(2\pi)^3 2(\mathbf{k^2} + m^2)^{1/2}} d^3\mathbf{k}.$$

The definition in Eq. (3.4) makes sense only if $\alpha(k)$ is a normalizable function that commutes with any other function

$$\int d\tilde{k}|\alpha(k)|^2 < \infty. \tag{3.5}$$

Then we can easily check that such a state is an eigenstate of a

$$a(q)|\alpha >= \frac{\partial}{\partial a^+(q)} \left[e^{-\frac{1}{2}\int d\tilde{k}|\alpha(k)|^2} e^{\int d\tilde{k}\alpha(k)a^\dagger(k)}|0>\right] = \alpha(q)\,|\alpha > . \tag{3.6}$$

In general these eigenstates are called coherent states. Since these are eigenstates of the annihilation operator they cannot be brought to the vacuum state by any function of annihilation operators, not even by an exponential because

$$e^{\int d\tilde{k}\beta(k)a(k)} e^{\int d\tilde{k}\alpha(k)a^\dagger(k)}\,|0>=$$

$$= e^{\int d\tilde{k}\alpha(k)a^\dagger(k)} e^{\int d\tilde{k}\beta(k)a(k)} e^{\int d\tilde{k}\beta(k)\alpha(k)[a(k),a^\dagger(k)]}|0>=$$

$$= e^{\int d\tilde{k}\beta(k)\alpha(k)} e^{\int d\tilde{k}\alpha(k)a^\dagger(k)}|0> . \tag{3.7}$$

Does this last statement imply that the number of particles in a coherent state is infinite ? Not at all because, according to their definition, the total number of particles in such a state is

$$< \alpha|\hat{N}|\alpha >\equiv \int d\tilde{k} < \alpha|a^\dagger(\mathbf{k})a(\mathbf{k})|\alpha >= \int d\tilde{k}|\alpha(k)|^2. \tag{3.8}$$

This number is finite if the state is normalized as in Eq. (3.5).

On the other hand we know that the Fock states are orthogonal in the sense that, for two states that differ by at least one particle of any momentum, the inner product is zero. This is not true for the coherent states because

$$< \alpha_1|\alpha_2 >= e^{-\frac{1}{2}\int d\tilde{k}(|\alpha_1(k)|^2 + \alpha_2(k)|^2)} < 0|e^{\int d\tilde{q}\alpha_1^*(q)a(q)} e^{\int d\tilde{k}\alpha_2(k)a^\dagger(k)}|0>$$

$$= e^{\int d\tilde{k}(i\,Im(\alpha_1^*(k)\alpha_2(k)) - \frac{1}{2}|\alpha_1(k)-\alpha_2(k)|^2)}, \tag{3.9}$$

since

$$< 0|e^{\int d\tilde{q}\alpha_1^*(q)a(q)} e^{\int d\tilde{k}\alpha_2(k)a^\dagger(k)}|0>= e^{\int d\tilde{k}\alpha_1^*(k)\alpha_2(k)}, \tag{3.10}$$

and also

$$|\alpha_1 - \alpha_2|^2 = |\alpha_1|^2 + |\alpha_2|^2 - 2Re(\alpha_1)Re(\alpha_2) - 2Im(\alpha_1)Im(\alpha_2),$$

$$i\, Im(\alpha_1^* \alpha_2) = i\,(Re(\alpha_1) Im(\alpha_2) - Im(\alpha_1) Re(\alpha_2))\,. \tag{3.11}$$

In conclusion any two coherent states defined as above are not orthogonal because for any two normalizable functions α_1 and α_2 the modulus square of Eq. (3.9) is nonzero

$$| < \alpha_1 | \alpha_2 > |^2 = \exp\left(- \int d\widetilde{k} |\alpha_1(k) - \alpha_2(k)|^2 \right). \tag{3.12}$$

The exponential form of a coherent state defined as in Eq. (3.4) is not based on a guess but it can be proved by an alternative definition of coherent states. These kind of states can be created acting on the vacuum with an unitary operator

$$D(\eta) = \exp\left(- \int d\widetilde{k} \left[\eta^*(k) a(k) - \eta(k) a^\dagger(k) \right] \right). \tag{3.13}$$

This operator is often called a translation operator because in the space of coherent states its action is equivalent to a translation

$$D(\eta)|\alpha >= \exp\left(\frac{1}{2} \int d\tilde{k} \left[\eta(k)\alpha^*(k) - \alpha(k)\eta^*(k) \right] \right) |\alpha + \eta > . \tag{3.14}$$

This property is easily proved using different variations of the Campbell-Baker-Hausdorff formula for the special case $[A,[A,B]] = [B,[A,B]] = 0$

$$e^A e^B = e^B e^A e^{[A,B]} \qquad \text{and} \qquad e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]}. \tag{3.15}$$

First

$$D(\eta)|0 >= \exp\left(- \int d\widetilde{k} [\left[\eta^*(k) a(k) - \eta(k) a^\dagger(k) \right] \right) |0 >$$

$$= e^{-\frac{1}{2}\int d\widetilde{k} |\eta(k)|^2} e^{\int d\widetilde{k} \eta(k) a^\dagger(k)} |0 >= |\eta >, \tag{3.16}$$

and the translation property is proved by

$$D(\eta)|\alpha >= e^{-\int d\widetilde{k} [\eta^*(k)a(k) - \eta(k)a^\dagger(k)]} e^{-\int d\widetilde{k} [\alpha^*(k)a(k) - \alpha(k)a^\dagger(k)]} |0 >$$

$$= e^{\frac{1}{2}\int d\widetilde{k} [\eta^*(k)a(k) - \eta(k)a^\dagger(k), \alpha^*(k)a(k) - \alpha(k)a^\dagger(k)]} e^{-\int d\{[\eta^*(k)+\alpha^*(k)]a(k) - [\eta(k)+\alpha(k)]a^\dagger(k)\}} |0 >$$

$$= e^{\frac{1}{2}\int d\tilde{k} (\eta(k)\alpha^*(k) - \alpha(k)\eta^*(k))} D(\eta + \alpha)|0 >, \tag{3.17}$$

because of the commutator

$$\left[\eta^*(k)a(k) - \eta(k)a^\dagger(k)\, ,\, \alpha^*(k)a(k) - \alpha(k)a^\dagger(k) \right] = -\eta^*(k)\alpha(k) + \eta(k)\alpha^*(k). \tag{3.18}$$

The last definition holds for bosons alone since all the time we used commutation relations. In fact if we consider the operators as creating or annihilating photons Eq. (3.13) is the starting point in defining photonic coherent states.

The physical interpretation of this special type of states was first given by Dirac and it can be easily understood through a simple example.

Consider one quantum harmonic oscillator. It is well known that its energy is quantized in levels given by $E_n = (n+\frac{1}{2})\hbar\omega$. Then the Fock state corresponding to this oscillator can be parametrized by n, the number of the 'occupied' level. The ladder operators bring the oscillator either on the next or previous energy level (if possible). Moreover we can express the momentum and position of the oscillator in terms of these operators as $\hat{p} = \frac{1}{i\sqrt{2}}(a - a^\dagger)$ and $\hat{q} = \frac{1}{\sqrt{2}}(a + a^\dagger)$. Where a is the operators that decreases the energy level by a unity while $a^\dagger$ takes the oscillator to the next energy level. Given the commutation relations of the momentum and position operators ladder operators obey $[a, a^\dagger] = 1$.

We can now define a normalized coherent state by

$$|\alpha >= e^{-\frac{|\alpha|^2}{2}} e^{\alpha a^\dagger} |0> . \tag{3.19}$$

Although this is not an eigenstate of the Hamiltonian it can be shown that it is the state of minimum uncertainty for the oscillator because the momentum and position uncertainties obey

$$(\triangle p)^2 (\triangle q)^2 = \left(\frac{1}{2}\right)^2 \quad \text{and } \triangle p = \triangle q, \tag{3.20}$$

with

$$(\triangle p)^2 =< \alpha|(\hat{p}- < \hat{p} >)^2|\alpha > . \tag{3.21}$$

In a similar manner $(\triangle q)^2$ is defined and $< \hat{p} >=< \alpha|\hat{p}|\alpha >$.

For the states $|\alpha >$ defined in Eq. (3.19) the momentum and position averages are

$$< \hat{p} >= \frac{1}{i\sqrt{2}}(\alpha - \alpha^*) = \sqrt{2} Im\alpha\,,$$

$$< \hat{q} >= \frac{1}{\sqrt{2}}(\alpha + \alpha^*) = \sqrt{2} Re\alpha\,. \tag{3.22}$$

Then the averages of the operators sqared read

$$< \hat{p}^2 >= \frac{1}{2} + 2(Im\alpha)^2 \quad \text{and} \quad < \hat{q}^2 >= \frac{1}{2} + 2(Re\alpha)^2. \tag{3.23}$$

To find in the end

$$(\triangle p)^2 = (\triangle q)^2 = \frac{1}{2}. \tag{3.24}$$

This confirms the interpretation of states of minimum uncertainty.

3.2 Generalized coherent states

In mathematical physics the concept of coherent states presented above had been generalized and coherent states for arbitrary Lie groups were defined. In what follows we will present the definition of a generalized coherent state of a Lie group and some of its properties as they are given by Perelomov [8].

Let G be a Lie group and T an unitary and irreducible representation of this group acting in the Hilbert space $\mathcal{H}$. Let $|\psi_0>$ be a fixed vector of this Hilbert space. The system of states $\{|\psi_g>\}$ with $|\psi_g>= T(g)|\psi_0>$,where g are elements of G, is called the coherent states system associated to the representation T. If H is the isotropy subgroup for the state $|\psi_0>$ then a coherent state $|\psi_g>$ is determined by a point $x = x(g)$ in the coset space G/H corresponding to the element g. In other words $|\psi_g>= e^{i\alpha(g)}|x(g)>$.

The scalar product of two coherent states will then be given by

$$< \psi_{g_1}|\psi_{g_2} >=< \psi_0|T(g_1^{-1}g_2)|\psi_0 >= e^{i(\alpha(g_2)-\alpha(g_1))} < x(g_1)|x(g_2) > . \tag{3.25}$$

This leads to $| < x(g_1)|x(g_2) > | < 1$ for $g_1 \neq g_2$ because the representation is irreducible.

After the generalized coherent states are defined we can discuss few of their properties. Let us start with the proof of existance of a resolution of unity in terms of coherent state. If such a resolution exist then any other state can be decomposed in terms of these coherent states.

Suppose a measure $dy(g)$ that is invariant to left and right shifts exists in G. This induces an invariant measure dx in G/H.

Consider the operator

$$\hat{A} = \int dx|x >< x|, \tag{3.26}$$

where x are points in the coset G/H.

According to the definition of coherent states

$$T(g_1)|x(g) >= e^{-i\alpha(g)}T(g_1)T(g)|\psi_0 >= e^{i(\alpha(g_1 g)-\alpha(g))}|g_1 x >\equiv e^{i\beta(g_1,g)}|g_1 x > \tag{3.27}$$

This leads to

$$T(g_1)\hat{A}[T(g_1)]^{-1} = e^{i\beta(g_1,g)}e^{-i\beta(g_1,g)}\hat{A} = \hat{A} \tag{3.28}$$

Because the measure dx is invariant.

Eq. (3.28) shows that $\hat{A}$ commutes with all the operators $T(g)$. Thus according to Schur's lemma it must be such that $\hat{A} = a\hat{I}$ with a one constant which can be absorbed in the definition of the measure in such a way that $d\mu(x) = \frac{1}{a}dx$. Then

$$\int d\mu(x)|x >< x| = \hat{I}. \tag{3.29}$$

Moreover if we take a starting vector that is normalised, $< \psi_0|\psi_0 >= 1$, we find

$$< \psi_0|\hat{I}|\psi_0 >= 1 = \int d\mu(x)| < \psi_0|x > |^2. \tag{3.30}$$

This shows that a resolution of unity in terms of coherent states exists for any group that has a square-integrable coherent states system. Eq. (3.30) is in fact the convergence condition that assures the existence of the operator $\hat{A}$ in the first place.

Since a resolution of unity in terms of coherent states is possible any state can be expanded in terms of coherent states as well.

$$|\psi>= \int d\mu(x) < x|\psi> \ |x> . \tag{3.31}$$

This means any coherent state can be also expanded in terms of the other coherent states

$$|x>= \int d\mu(y) < y|x> \ |y> . \tag{3.32}$$

This proves that the coherent states are non-orthogonal because they are linear dependent since the expansion' coefficients $K(y,x) \equiv < y|x>$ are non-vanishing. Moreover $K(x,y)$ which is called 'the kernel' is reproducing in the sense that

$$K(x,z) = \int d\mu(xy) K(x,y) K(y,z) \tag{3.33}$$

In conclusion the set of coherent states defined for one group is overcomplete because they are linear dependent but the also hold a resolution of unity.

Another remark has to be made about the starting point in the definition of a coherent state system, the fixed vector $|\psi_0>$ of the Hilbert space the representation acts in. There is a freedom of choice for it but we expect to find coherent states with different properties for different fixed vectors.

We have seen before that for a quantum harmonic oscillator the coherent states are states of minimum uncertainty thus states that are the closest to the classical case. We sill show in the next paragraphs that the generalized coherent states for the Heisenberg-Weyl group coincide with the coherent states defined in the beginninig of this chapter for a certain choice of $|\psi_0>$. That particular choice leads then to states of minimum uncertainty.

Let us apply now the theory presented above to a well known group and prove that the mathematical formalism presented above is in agreement with the physical picture we introduced in the beginning of the chapter.

Suppose G is the Heisenberg-Weyl group W_N. This group is associated to a quantum system with N degrees of freedom described by the coordinate operators $\hat{q}_i$ and momentum operators $\hat{p}_j$with $i,j = 1, ..., N$. The Lie algebra of the group contains then 2N+1 elements: the coordinate and momentum operators plus the identity operator. Ifwe consider $\hbar = 1$ these oprators satisfy the Heisenberg commutation relations:

$$[\hat{q}_i, \hat{p}_j] = i\delta_{ij}\hat{I}\,, \qquad [\hat{q}_i, \hat{q}_j] = 0\,, \qquad [\hat{p}_i, \hat{p}_j] = 0. \tag{3.34}$$

We can change the operator basis to a more familiar one: the 'ladder' operators basis. Define

$$a_j = \frac{1}{\sqrt{2}}(\hat{q}_j + i\hat{p}_j)\,, \qquad a_j^\dagger = \frac{1}{\sqrt{2}}(\hat{q}_j - i\hat{p}_j). \tag{3.35}$$

Then the commutation relations of Eq. (3.34) become

$$[a_i, a_j^\dagger] = \delta_{ij}\hat{I}\,, \qquad [a_i, a_j] = [a_i, \hat{I}] = 0\,, \qquad [a_i^\dagger, a_j^\dagger] = [a_i^\dagger, \hat{I}] = 0. \tag{3.36}$$

The system described here is equivalent to a system of N identical quantum harmonic oscillators. The ladder operators a_j and $a_j^\dagger$ decrease / increase the energy of oscillator j by one quanta thus they modify the energy quantum number by unity. A general normalized vector in this Hilbert space will be given by $|n_1, n_2, ..., n_N >$where each number n_i is a positive integer related to the energy of the respective oscillator $E_i = (n_i + \frac{1}{2})\hbar\omega$.

The vacuum is the vector corresponding to the lowest energy state i.e. the state of the system when all the oscillators are in the ground state. By definition the vacuum is a normalized vector with the property

$$a_i|0>=0 \qquad \forall i = 1, ..., N\,. \tag{3.37}$$

Then a general normalized vector in the Hilbert space can be written in terms of creation operators acting on the vacuum as

$$|n_1, ..., n_N >= \frac{1}{\sqrt{n_1!...n_N!}}(a_1^\dagger)^{n_1}...(a_N^\dagger)^{n_N}|0>\,. \tag{3.38}$$

And the action of a creation or annihilation operator on such a vector will be

$$a_i|n_1, ..., n_i, ..., n_N >= \sqrt{n_i}|n_1, ..., n_i - 1, ..., n_N >;$$

$$a_i^\dagger|n_1, ..., n_i, ..., n_N >= \sqrt{n_i + 1}|n_1, ..., n_i + 1, ..., n_N >\,. \tag{3.39}$$

In the theory of Lie groups any element of such a group can be expanded in terms of the generators of the Lie algebra. If x is an element of the group then $x = e^{i\hat{X}}$ where $\hat{X}$ is an element of the Lie algebra which can be written in terms of the generators. In our case the generators are the ladder operators plus the identity. Thus

$$\hat{X} = s\hat{I} - i(\alpha a^\dagger - \alpha^* a) \tag{3.40}$$

Where a and $a^\dagger$ are N dimensional vectors whose components are the ladder operators and α and α^* are vectors formed by the coefficients of the expansion. In the end the finite group element is given by

$$x = e^{i\hat{X}} = e^{is\hat{I}}D(\alpha), \tag{3.41}$$

where

$$D(\alpha) = e^{(\alpha a^\dagger - \alpha^* a)}. \tag{3.42}$$

Thus a unitary irreducible representation of W_N is

$$T(s, \alpha) = e^{is}D(\alpha). \tag{3.43}$$

According to the definition of this Lie group the coherent states are given by

$$|\psi_{s,\alpha} >= T(s, \alpha)|0 >= e^{is}D(\alpha)|0>\,. \tag{3.44}$$

Or, if we consider that states that differ only by a phase are the same, coherent states are defined by

$$|\alpha>= D(\alpha)|0>, \tag{3.45}$$

which coincides with the definition of photon coherent states in Eq. (3.16).

The fact that we chose $|\psi_0>=|0>$ made the system of coherent states here defined to have the property of minimal uncertainty. The proof goes in the same way as for a single quantum harmonic oscillator Eq. (3.22)-Eq. (3.24) and the result is identical for each of the oscillators

$$(\triangle p_j)^2 = (\triangle q_j)^2 = \frac{1}{2}. \tag{3.46}$$

In conclusion the link between a mathematical description of coherent states and their physical meaning was made. However, generalized coherent states can be built for different Lie groups that have an application in phsyics. Different applications of this theory are available in a variety of domains, especially in Quantum Optics and the Theory of Quantum Information.

3.3 Quantum optics

This last part of the chapter aims to show how coherent states proved to be useful in the late 60's. Although coherent states are widely used in physics these days by far the most famous use is in Quantum Optics. We shall see in what follows why is it so.

Quantum optics offers a description of phenomena related to light and its interactions using the tools of quantum mechanics. The relation between light and quantum mechanics seem natural given the fact that experiments involving electromagnetic waves led to the idea of quantization. It was Max Planck the first to prove the quantization of the blackbody radiation. Few years later Einstein completed the image explaining the photoelectric effect.

The electromagnetic waves satisfy Maxwell's equation and are described by a pair of fields: either $\mathbf{E}$ and $\mathbf{B}$ -the electric and magnetic field, or $\mathbf{A}$ and φ -the vector and scalar potential. The relation between the two sets of fields in vacuum is simply

$$\mathbf{B} = \nabla \times \mathbf{A} \qquad \mathbf{E} = -\frac{\partial}{\partial t}\mathbf{A} - \nabla\varphi. \tag{3.47}$$

In this paragraph we work in a gauge in which $\varphi = 0$;

If a quantization procedure is imposed the vector potential can be written in terms of creation and annihilation operators as

$$\mathbf{A}(x) = \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega(\mathbf{k})}} \sum_{k,\lambda} \left(a^\lambda(k)\varepsilon^\lambda(k)e^{-ikx} + a^{\lambda\dagger}(k)\varepsilon^{\lambda *}(k)e^{ikx}\right), \tag{3.48}$$

with $[a^\lambda(k), a^{\dagger\rho}(k')] = g^{\lambda\rho}\delta^{(3)}(\mathbf{k}-\mathbf{k}')$ and λ standing for the helicity.

We use Eq. (3.48) to find the electric field and observe that this has a form which allows us to write it as a sum of terms that contain only creation or annihilation operators.

According to Glauber [10] in Quantum Optics the coherence of light is described in terms of a correlation function of the electromagnetic field. The n'th order correlation function expresses the correlation of values of the fields in 2n different points in space-time. This is given by

$$G^{(n)}(x_1, ...x_{2n}) = Tr\left(\rho E^-(x_1)...E^-(x_n)E^+(x_{n+1})...E^+(x_{2n})\right), \tag{3.49}$$

where $E^{\pm}$are the negative and positive frequency components of the field operator and ρ is the density.

In his study of these correlations functions Glauber proves that a condition for coherence is a factorization of the correlation functions

$$G^{(n)}(x_1, ...x_{2n}) = \mathcal{E}^*(x_1)...\mathcal{E}^*(x_n)\mathcal{E}(x_{n+1})...\mathcal{E}(x_{2n}), \tag{3.50}$$

where $\mathcal{E}(x)$ is a function independent of n.

This coherence condition is fulfilled if we can find eigenstates of $E^{\pm}(x)$ which could form a complete set of states in such a way that the identity operator and the density operator can be expanded in terms of them. Then $\mathcal{E}(x)$ will be just the eigenvalues of the field operators corresponding to those eigenstates.

Eigenstates of E^+ exist and they are called photon coherent state. They are defined as

$$|\alpha >= e^{-\frac{1}{2}\int d^3k \sum_\lambda |\alpha^\lambda(k)|^2} e^{\int d^3k \sum_\lambda \alpha^\lambda(k) a^{\lambda\dagger}(k)}|0> . \tag{3.51}$$

Then it turns out that

$$\mathcal{E}(x) = i\int \frac{d^3k}{(2\pi)^{3/2}}\sqrt{\frac{\omega(k)}{2}}\sum_\lambda \alpha^\lambda(k)\varepsilon^\lambda(k)e^{-ikx}. \tag{3.52}$$

We can show that these states are not orthogonal and they form an overcomplete set. However, the identity operator can be expanded in terms of them as

$$I = \int d\alpha d\alpha^* \frac{1}{\pi}|\alpha >< \alpha|. \tag{3.53}$$

The density operator' expansion is

$$\rho = \int d\alpha d\alpha^* |\alpha >< \alpha| P(\alpha), \tag{3.54}$$

where $P(\alpha)$ is a weight function defined in such a way that

$$Tr\rho = \int d\alpha d\alpha^* P(\alpha) = 1 \tag{3.55}$$

Based on the above relations Glauber and Sudarsham showed that the laser beams can be completely described in the quantum formalism in terms of coherent states. This plays an essential role in modern quantum optics.

Summary

The definition and a few properties of photonic coherent states were discussed in this chapter. A general definition for coherent states associated to a Lie group was introduced and some of their properties were given. The general definition can be applied to the Weyl-Heisenberg group and the known definition and properties of photonic coherent states are found. In the end we made a short discussion about how coherent states were first used in Quantum Optics.

4 Field theory

4.1 Introduction

The S matrix plays a very important role in the theory of scattering since it is the tool we usually use to make theoretical predictions. In this chapter we define the S matrix in the usual way and discuss about the problems that might obstruct our attempt of making good predictions. These problems usually occur in massless theories such as QED or QCD and are associated with particles that have very low energy or with sets of collinear particles. We understand what is the cause of the divergences that appear and define different sets of states in order to avoid such problems. The keywords of this chapter are : the S matrix , the Moller operators, the asymptotic states, the asymptotic Hamiltonian and the dressed states.

4.2 The S matrix

Let us suppose a scattering experiment is performed. We call the 'in' state or the initial state that state in which the system can be found long before any scattering took place. In other words this is the state of the system at $t \to -\infty$. In principle this is prepared to be $|\alpha, + >$. It depends on the quantum numbers generically denoted by α which characterize the eigenvalues of operators that commute with the Hamiltonian. The '+' sign in the notation labels the 'in' character of the state in a convention similar to the advanced / retarded Green's functions notation.

The probability that the system ends up in a state $|\beta, - >$, long after the scattering took place (that is at $t \to \infty$), is given by the S matrix element corresponding to this process. By definition

$$S_{\alpha\beta} = < \beta, -|\alpha, + > . \tag{4.1}$$

The 'out' state, denoted $|\beta, - >$, is beyond our control and it is a time-evolved state determined by the full Hamiltonian of the system. That is the reason of talking about probabilities.

Both the 'in' and 'out' states are eigenstates of the full Hamiltonian H. If they coincide, i.e. the S matrix element takes the value 1, then no interaction took place in the time interval $-\infty < t < \infty$ and so the Hamiltonian of the system must coincide with the free Hamiltonian H_0. If the S matrix element is different from one then interactions must be part of the scenery and the Hamiltonian does not coincide with the free Hamiltonian in all points of the space-time. As a consequence we don't necessarily expect the 'in' and 'out' states to be free states. From now on we consider only the nontrivial case when the Hamiltonian has also an interacting part.

In the trivial case that lacks any interaction the Hamiltonian can be diagonalized and its eigenstates are fully determined. However, the non-trivial case is mathematically more complicated so we cannot always diagonalize the full Hamiltonian. Then we need to find a way to relate its eigenstates with the eigenstates of another Hamiltonian. For instance we can find a connection between 'in' and 'out' states and asymptotic free states. To do so we need to make the assumption that the interactions die off quick enough in such a way that in the asymptotic area (the far past and future) their effects can be neglected and the wave packets corresponding to the 'in' or 'out' states resemble wave packets of some free states.

We talk about wave packets due to restrictions imposed by the time-energy uncertainty relation which implies that energy eigenstates cannot be localized in time. Thus we cannot relate directly the eigenstates of the full Hamiltonian to free states in the asymptotic region a.k.a the boundary of the time interval in which the system is under observation. Instead we introduce the wave packets corresponding to these states. We define these wavepackets as superpositions of eigenstates of the Hamiltonian corresponding to eigenvalues that are in a finite range of energies $\triangle E$. For a rigurous definition we introduce a distribution function $g(\alpha)$ that is smoothly varying and not vanishing everywhere over the finite range of energies $\triangle E$. We do not give any specific form for this distribution as it is not important at the moment. By this definition one of these wave packets looks like

$$\int d\alpha g(\alpha)|\alpha, + > . \tag{4.2}$$

The resemblance between free states and full theory eigenstates does not come as a strong convergeance of their wave packets, i.e. a measure of how well the two packets would overlap, but rather through a weak limit of these packets' time evolution.

First let us define such a time evolution. Consider a state in the Schrodinger picture (a discussion about other pictures and the corresponding relations between them will be done later in this chapter). In this picture any state is time-dependent and obey the Schrodinger equation, which leads to a time evolution generated by the total Hamiltonian of the system

$$i\frac{\partial}{\partial t}|\Psi_S(t) >= H|\Psi_S(t) >,$$

$$\Rightarrow \qquad |\Psi_S(t) >= e^{-iH(t-t_0)}|\Psi_S(t_0) >\equiv U_H(t,t_0)|\Psi_S(t_0) > . \tag{4.3}$$

With this consideration we define the 'in' and 'out' states as eigenstates of the full Hamiltonian in such a way that the weak limit of the time evolution of their wave packet corresponds to the time evolution of the free Hamiltonian eigenstates wave packet in the limit of long times. That is, for $t - t_0 \to \infty$,

$$e^{-iH(t-t_0)} \int d\alpha g(\alpha)|\alpha, \pm >\to e^{-iH_0(t-t_0)} \int d\alpha g(\alpha)|\phi(\alpha) > . \tag{4.4}$$

We denoted by $|\phi(\alpha) >$ an eigenstate of the free Hamiltonian characterized by the quantum number(s) α.

In this way the relation between eigenstates is given by

$$|\alpha, \pm >= \Omega^{(\pm)}_{H,H_0}|\phi(\alpha) >, \tag{4.5}$$

where we introduced the Moller operators defined as

$$\Omega^{(\pm)}_{H,H_0} = \lim_{t\to\mp\infty}\Omega_{H,H_0}(t) \qquad \text{with} \qquad \Omega_{H,H_0}(t) = e^{iHt}e^{-iH_0t}. \tag{4.6}$$

Using these definitions the S matrix elements can be related to elements of the Moller operators between free states

$$S_{\alpha\beta} =< \beta, -|\alpha, + >=< \phi(\beta)|\Omega^{(-)\dagger}_{H,H_0}\Omega^{(+)}_{H,H_0}|\phi(\alpha) > . \tag{4.7}$$

Since in Eq. (4.7) S matrix elements are equal to matrix elements of a given operator we can call this operator S and use it in further calculations instead of using its matrix elements. We define

$$S = \Omega^{(-)\dagger}_{H,H_0}\Omega^{(+)}_{H,H_0}. \tag{4.8}$$

Let us get back to the Moller operator of Eq. (4.6) and study it in more detail. According to Eq. (4.5), this operator maps a free state to a 'in' or 'out' state. The inverse mapping is also possible and we can write in general

$$|\phi(\alpha)>= \Omega^{(\pm)\dagger}_{H,H_0}|\alpha,\pm> . \tag{4.9}$$

It follows that

$$|\phi(\alpha)>= \Omega^{(\pm)\dagger}_{H,H_0}\Omega^{(\pm)}_{H,H_0}|\phi(\alpha)> . \tag{4.10}$$

Since the free states $|\phi(\alpha)>$ span the entire Hilbert space we conclude that $\Omega^{(\pm)\dagger}_{H,H_0}\Omega^{(\pm)}_{H,H_0} = 1$. Operators with this property are called isometric. If we perform the same type of manipulations but now for the 'in' and 'out' states instead of the free states we could conclude that the Moller operators are unitary because also $\Omega^{(\pm)}_{H,H_0}\Omega^{(\pm)\dagger}_{H,H_0} = 1$. This last statement does not hold in general because the 'in' or 'out' states could contain bound states. Then an equation identical to Eq. (4.10) cannot be written for them since they do not span the entire Hilbert space of the problem. From now on we shall consider only the case when no bound states are involved as initial or final states so that the Moller operators shall be unitary.

Everything we have done so far is based on the assumption that the weak limit of Eq. (4.4) holds and the wave packets that evolve according to the full Hamiltonian go through the interaction region in a finite amount of time. This is true, in general, for massive theories but the convergeance in Eq. (4.4) cease to hold if massless particles are part of the theory.

Massless particles are the signature of infinite-range interactions. For example, the photon, which turns out to be massless, carries the electromagnetic interaction which is known to act at the distance. Moreover there is no good reason why this interaction should 'switch off' after some amount of time. In conclusion our assumptions fail in this situation.

On the other hand, the presence of massless particles is associated to states that are degenerate in energy. In the spectrum of the free Hamiltonian one state containing one electron of energy E has the same energy with a state that includes the same electron and any number of photons of very low energy. Another example of a degeneracy that occurs involves massless colinear particles. In QCD for example the gluons, which are the gauge bosons, carry color charge themselves and they can interact with other gluons. It is possible for a gluon to split into a pair of two different gluons. Given their masslessness the two resulting particles can be collinear giving rise to another degeneracy in energy. Together the two particles have the same energy as the initial gluon and they cannot be detected as two different particles given their collinearity. These degeneracies are the source of the so called infrared divergencies.

The weak limit of Eq. (4.4) is violated whenever such degeneracies occur because in such a situation the Moller operators would be divergent. This will be proved later, when a more explicit form for these operator is introduced.

The usual way to avoid degeneracies among energy eigenstates is to include extra quantum numbers in the set describing the states (in our case this is denoted by α or β). This is not possible in the massless theories because the degeneracies arise due to the lack of mass not because of some other variables that were neglected. We conclude that in such cases the wave packets that evolve according to the full Hamiltonian will never leave the interaction region so they will never fully overlap with free wave packets.

This problem can be solved in a certain way. One method to get rid of divergences would be to introduce a small mass for the massless particles. This is called mass regularization. It is efficient in eliminating divergences but it can lead to other problems. For example, if we give a mass to a fermion that was considered massless, this might break the chiral symmetry of the system.

Another approach is the so called dimensional regularization in which the integrals are evaluated in a sligthly modified number of dimensions. The integrals will then be finite but dependent on the regulator. When the regulator is eliminated at some point further complications may occur.

Anyhow, in the end the claim is that when calculating infrared safe observables such as cross-sections the potential singularity in the regulator cancels and we are left with finite results (more or less) such that the regulator can be removed.

Another way to approach the problem is to replace the free Hamiltonian in all the relations above with a different asymptotic Hamiltonian, one that satisfies the weak limit of Eq. (4.4) and has no degenerate states in its spectrum of eigenstates.

In what follows we will introduce the new asymptotic Hamiltonian but the general approach towards finite S matrix elements include also what we discussed up to now. That is why from now on we will consider a regulated theory in which all the operators that are regulated will carry an extra index ε to show this fact. We will keep in mind that the operators and the S matrix are well defined in the regulated theory as long as we do not put the regulator to vanish.

4.3 Another asymptotic Hamiltonian

In order to have well defined Moller operators we need to introduce a Hamiltonian which is in the same convergeance class as the full Hamiltonian such that an equation similar to Eq. (4.4) holds. That implies that the new asymptotic Hamiltonian should also contain interaction terms in such a way that all the soft and collinear singularities of the full Hamiltonian are included. We call it simply 'the asymptotic Hamiltonian' and denote it by H_A. Then the time evolution of the 'in' and 'out' wave packets overlap well over the wave packets of the new asymptotic states

$$e^{-iHt}\int d\alpha g(\alpha)|\alpha,\pm>\to e^{-iH_At}\int d\alpha g(\alpha)|\psi^{\pm}(\alpha)>^A . \tag{4.11}$$

Here $|\psi^{\pm}(\alpha)>^A$ are eigenstates of the asymptotic Hamiltonian which we will call asymptotic states and $t\to\mp\infty$.

At his point we define again Moller operators similar to those of Eq. (4.6) but involve different Hamiltonians

$$\Omega^{(\pm)}_{H,H_A}=lim_{t\to\mp\infty}\Omega_{H,H_A}(t) \qquad \text{with} \qquad \Omega_{H,H_A}(t)=e^{iHt}e^{-iH_At}. \tag{4.12}$$

In the end the 'in' and 'out' states are related to the asymptotic states by

$$|\alpha,\pm>=\Omega^{(\pm)}_{H,H_A}|\psi^{\pm}(\alpha)>^A . \tag{4.13}$$

We could also define an asymptotic S operator which now has to be evaluated between asymptotic states to give the elements of the original S matrix

$$S_{\alpha\beta}={}^A<\psi^+(\beta)|S_A|\psi^-(\alpha)>^A . \tag{4.14}$$

Using the definition of the asymptotic states we conclude that

$$S_A=\Omega^{(-)\dagger}_{H,H_A}\Omega^{(+)}_{H,H_A}. \tag{4.15}$$

We could have applied the formalism of the asymptotic Hamiltonian from the very beginning and later on identify which is the right asymptotic Hamiltonian. For massive theories it is the free Hamiltonian. How to define the right asymptotic Hamiltonian will be discussed in the following paragraphs.

The right choice for the asymptotic Hamiltonian is motivated by the need of convergence of Moller operators alone. As a matter of fact is no specific prescription for such a choice. The only rule is that this Hamiltonian should contain all the mass and collinear singularities present in the full Hamiltonian such that the convergence is fullfilled. A resonable choice is to take it as the sum of the free Hamiltonian and the soft part of the interaction Hamiltonian (by the soft part we mean the part that includes all infrared singularities). This choice has a certain amount of arbitrariness in it because it depends on the way the infrared region is defined. Usually one introduces a set of infrared and collinear parameters that are associated with the definition of the collinear and infrared region. We will denote this set of parameters $\triangle$. This set of parameters has as equivalent in experiments the set of resolutions that characterize the experimental facilities. No matter if we talk about energy or angular resolutions, they both describe the phase-space of degenerate states present in the experiment.

When such parameters are introduced and a distinction between 'soft' and 'hard' particles (gauge bosons in our case of interest) can be made the interaction Hamiltonian can be split in two parts. Using such a technique the full Hamiltonian can be witten as $H = H_0 + H_{int} = H_0 + H_S(\triangle) + H_H(\triangle) \equiv H_A(\triangle) + H_H(\triangle)$. Then according to the choice presented above the asymptotic Hamiltonian can be identified as the sum of the free Hamiltonian and the soft part of the interaction Hamiltonian. Note that such a splitting is frame-dependent and the theory will lose its Lorentz invariance. However, this does not imply that the observables we operate with are not Lorentz-invariant themselves. In particular the matrix elements of S_A between asymptotic states conserve energy because, as we shall see later, S_A commutes with H_A .

Another discussion can be made on how to choose the infrared parameter $\triangle$ in order to obtain results that are in agreement with experiments.

Suppose that we have a QED scattering experiment and that the detectors we use have a resolution $\triangle_e$. If the resolution is nonvanishing the experiment will not be able to distinguish between, say, an electron alone and an electron accompanied by a number of soft and/or collinear photons. Theoretically, if the infrared parameters are taken to be equal to the experimental resolutions, the measured cross-section should be equal with the cross-section computed from S_A. This is a consequence of the fact that the matrix elements of the asymptotic S matrix vanish when taken between states that contain any soft or collinear photons because the corresponding S operator contains no operators that creates or annihilates soft / collinear photons. Thus

$$d\sigma_{exp}(\triangle^e) = |<l^{'}|S_A[\triangle^e]|l>|^2$$

.

Suppose that we choose a parameter $\triangle$ smaller than the experimental resolutions. In this case we have to add to the ellastic cross-section of S_A the innelastic terms that the experiment cannot distinguish. On the other hand if the experimental resolution $\triangle_e$ is smaller than $\triangle$ we have to substract from the elastic cross section the contributions that the experiment detects but we consider undetectable.

The last phrase can be formally written as

$$d\sigma_{obs}(\triangle_e) = |<l^{'}|S_A[\triangle^e]|l>|^2 \pm \sum_{i \neq f} |<f|S_A(\triangle)|i>|^2, \tag{4.16}$$

where the first term is the ellastic contribution and the second term is the sum of indistinguishable contributions which we assumed as distinguishable (or distinguishable contributions considered undetectable).

To complete this discussion we also take into account the posibility of having different resolutions for the initial and final states in the experiment. In this case, according to its definition, we have to modify S_A to

$$S_{A',A}[\triangle_f, \triangle_i] = \Omega^{(-)\dagger}_{H,H_{A'}} \Omega^{(+)}_{H,H_A} = \Omega^{(-)}_{H_{A'},H_A} S_A[\triangle_i], \tag{4.17}$$

where different Hamiltonians correspond to different resolutions

$$H_{A'} \equiv H_A(\triangle_f) \quad and \quad H_A \equiv H_A(\triangle_i). \tag{4.18}$$

A more evolved discussion about the connection to the experiment and calculations that support these statements can be found in [4].

4.4 Dressed states

Our main goal of doing scattering theory is to compute the cros-sections of different processes. Since the scattering amplitudes are the main ingredient of this calculation we have to make sure the amplitudes we obtain are finite. In the usual treatment these are well defined if a regularization procedure is involved. However, each amplitude alone might be infinite in the limit of vanishing regulator. It is well known that the regulator disappears in the cross section if a summation of squared scattering amplitudes over degenerate final and initial states is performed. In this way we need not worry about infinities when the regulator vanishes.

It is somehow odd to work with matrix elements that diverge when the regulator is removed even if they are well defined as long as the regulator is there and not vanishing. A way to cure this is to work with the matrix elements of the asymptotic S matrix between the newly introduced asymptotic states. We know those elements are free of infrared singularities. However it might be imposible to diagonalize the Hamiltonian in order to find its eigenstates to compute the corresponding scattering amplitudes.

Another approach would be to find a different basis of states in which the ordinary S operator elements are finite. In this way we need not introduce a new S operator. Still the calculations are not simpler. The only advantage is that, at the end of the day, we have matrix elements that do not depend on the regulator even though the S operator and the new states depend on it.

We call these new states asymptotic dressed states because in their definition the asymptotic Hamiltonian is involved and also they are defined by dressing the usual Fock states as follows

$$|\{\psi;\pm\}>=\Omega^{(\pm)\dagger}_{H_A^{(\epsilon)},H_0^{(\epsilon)}}|\psi> . \tag{4.19}$$

In the literature these states are sometimes called 'generalised coherent states'. We will avoid using this terminology for the simple reason that in some cases, e.g. QCD, they don't have the usual properties of coherent states. On the other hand, for QED they have the remarkable property of being eigenstates of the annihilation operator so we can call them coherent states.

The Moller operators used in this definition $\Omega^{(\pm)}_{H_A^{(\epsilon)},H_0^{(\epsilon)}}$ map eigenstates of the free Hamiltonian to eigenstates of the asymptotic Hamiltonian that have the same eigenvalue. If the theory was not regulated these operators would be ill-defined due to the infrared singularities found in the asymptotic Hamiltonian.

Using this definition we remark that the matrix elements of S between this kind of states are equal to the elements of the asymptotic S operator between free states

$$< \phi|S_A^{(\epsilon)}|\psi>=<\phi|\Omega^{(-)\dagger}_{H^{(\epsilon)},H_A^{(\epsilon)}}\Omega^{(+)}_{H^{(\epsilon)},H_A^{(\epsilon)}}|\psi>=$$

$$=<\phi|\Omega^{(-)}_{H_A^{(\epsilon)},H_0^{(\epsilon)}}\Omega^{(-)\dagger}_{H^{(\epsilon)},H_0^{(\epsilon)}}\Omega^{(+)}_{H^{(\epsilon)},H_0^{(\epsilon)}}\Omega^{(+)\dagger}_{H_A^{(\epsilon)},H_0^{(\epsilon)}}|\psi>=<\{\phi;-\}|S[H^{(\epsilon)}]|\{\psi;+\}> . \tag{4.20}$$

If we are able to show that the matrix elements of the asymptotic S operator are finite then the S matrix elements between dressed states are finite as well. Since the Moller operators $\Omega^{(\pm)}_{H^{(\epsilon)},H_A^{(\epsilon)}}$

are well defined the corresponding S matrix should be finite as well. There is still one question remaining. We are used with evaluating S_A between proper asymptotic states and find a finite result. Can we really evaluate it between free states and would it give the same cross-section in the end ?

At this point we still work in the regulated theory and it can be shown that, in this case, the set of free states and the set of asymptotic states are both basis of the Fock space related by a unitary transformation. Thus there is no problem with evaluating asymptotic S operator elements between free states. Moreover, in the end, when a summation over the possible final and initial states is performed, the cross-section computed this way would coincide with the usual result. Formally this is proved by

$$\sum_{in,out}\int_{ph.sp}|\mathcal{A}|^2\times J=\sum_{in,out}\int_{ph.sp}<\phi_{out}|\Omega^{(-)}_{H_A^{(\epsilon)},H_0^{(\epsilon)}}\,S\,\Omega^{(+)\dagger}_{H_A^{(\epsilon)},H_0^{(\epsilon)}}|\psi_{in}>\times$$

$$\times<\psi_{in}|\Omega^{(+)}_{H_A^{(\epsilon)},H_0^{(\epsilon)}}\,S^\dagger\,\Omega^{(-)\dagger}_{H_A^{(\epsilon)},H_0^{(\epsilon)}}|\phi_{out}>\times J=$$

$$=\sum_{in,out}\int_{ph.sp}Tr\left(\Omega^{(-)}_{H_A^{(\epsilon)},H_0^{(\epsilon)}}\,S\,\Omega^{(+)\dagger}_{H_A^{(\epsilon)},H_0^{(\epsilon)}}\Omega^{(+)}_{H_A^{(\epsilon)},H_0^{(\epsilon)}}\,S^\dagger\,\Omega^{(-)\dagger}_{H_A^{(\epsilon)},H_0^{(\epsilon)}}\right)\times J=$$

$$=\sum_{in,out}\int_{ph.sp}Tr(SS^\dagger)\times J,$$

where J is a certain weight function that depends on the momenta of the external particles. For the total cross-section we can take $J=1$.

The reason why the S operator elements between dressed states is finite is simple. The Moller operators 'dress' the usual non-interacting final (or initial) states in a cloud of soft and collinear gauge bosons and this dressing generates infrared and collinear singularities that cancel those generated by the full scattering operator S.

In the next chapter we will show in one application that indeed the S matrix elements between this dressed states are finite.

4.5 Properties of the Moller operators

We have defined the Moller operators for different pairs of Hamiltonians but we omit to discuss what the properties of these operators are. All we mentioned was the isometry property which, if absent, signals the presence of false assumptions. Leaving out the posibility of having bound states we conclude that the Moller operators are unitary if the asumptions are correct or if the theory was regulated and the operators are well defined for non-vanishing regulator.

In order to show another property of these operators recall that the states involved, free states, asymptotic states or eigenstates of the full Hamiltonian, they all obey Schrodinger equations of the usual type. For example, the free states satisfy

$$i\frac{\partial}{\partial t}|\phi(\alpha)>=H_0|\phi(\alpha)>\,. \tag{4.21}$$

The dependence on time is not specified in the state's notation for simplicity but states do have a time-dependence associated with the picture we work in.

Applying a time derivative to Eq. (4.5) and using the fact that the Moller operators are time-independent here (we are in the Schrodinger picture) we find that

$$i\frac{\partial}{\partial t}|\phi(\alpha)>= i\frac{\partial}{\partial t}\left(\Omega^{(\pm)\dagger}_{H^{(\varepsilon)},H_0^{(\varepsilon)}}|\alpha,\pm>\right) = \Omega^{(\pm)\dagger}_{H^{(\varepsilon)},H_0^{(\varepsilon)}} i\frac{\partial}{\partial t}|\alpha,\pm>$$

$$H_0\Omega^{(\pm)\dagger}_{H^{(\varepsilon)},H_0^{(\varepsilon)}}|\alpha,\pm>= \Omega^{(\pm)\dagger}_{H^{(\varepsilon)},H_0^{(\varepsilon)}} H\,|\alpha,\pm> . \tag{4.22}$$

Thus $H_0\Omega^{(\pm)\dagger}_{H^{(\varepsilon)},H_0^{(\varepsilon)}} = \Omega^{(\pm)\dagger}_{H^{(\varepsilon)},H_0^{(\varepsilon)}} H$ and by a hermitian conjugation operation $H\Omega^{(\pm)}_{H^{(\varepsilon)},H_0^{(\varepsilon)}} = \Omega^{(\pm)}_{H^{(\varepsilon)},H_0^{(\varepsilon)}} H_0$.

Remark that the basis of this property is the fact that Moller operators are isometric and they map eigenstates of one Hamiltonian to the eigenstates of the other one with the condition that the two energy eigenvalues are equal. Similar calculations can be done for other Moller operators so we will take this result as a generic one.

On the other hand, the claim that dressed states are not eigenvalues of any of these Hamiltonians becomes clear. For example, the dressed state obtained from $|\phi(\alpha)>^{(\epsilon)}$ is not an eigenstate of the asymptotic Hamiltonian because

$$H_A^{(\epsilon)}|\{\psi(\alpha),\pm\}>= H_A^{(\epsilon)}\Omega^{(\pm)\dagger}_{H_A^{(\epsilon)},H_0^{(\epsilon)}}|\phi(\alpha)>^{(\epsilon)}$$

$$\neq \Omega^{(\pm)\dagger}_{H_A^{(\epsilon)},H_0^{(\epsilon)}} H_0^{(\epsilon)}|\phi(\alpha)>^{(\epsilon)}= E_\alpha\Omega^{(\pm)\dagger}_{H_A^{(\epsilon)},H_0^{(\epsilon)}}|\phi(\alpha)>^{(\epsilon)} . \tag{4.23}$$

The 'commutation' property obtained above can be used to prove that the S operator commutes with the Hamiltonian and so it conserves energy. Take for example the ordinary S operator of Eq. (4.9). It commutes with the free Hamiltonian because

$$H_0^{(\epsilon)}S = H_0^{(\epsilon)}\Omega^{(-)\dagger}_{H^{(\epsilon)},H_0^{(\epsilon)}}\Omega^{(+)}_{H^{(\epsilon)},H_0^{(\epsilon)}} = \Omega^{(-)\dagger}_{H^{(\epsilon)},H_0^{(\epsilon)}}H^{(\epsilon)}\Omega^{(+)}_{H^{(\epsilon)},H_0^{(\epsilon)}}$$

$$= \Omega^{(-)\dagger}_{H^{(\epsilon)},H_0^{(\epsilon)}}\Omega^{(+)}_{H^{(\epsilon)},H_0^{(\epsilon)}}H_0^{(\epsilon)} = SH_0^{(\epsilon)}. \tag{4.24}$$

Since $[S,H_0]=0$ the energy is conserved at the level of matrix elements. In the same manner S_A commutes with H_A and again energy is conserved even though we suspect H_A not to be Lorentz invariant. In the spirit of this discussion it can be shown that $[S_A,H_0]\neq 0$ and that energy conservation is lost when computing matrix elements of S_A between free states.

4.6 The asymptotic states

Suppose we work in the frame of a regulated theory such that all the Moller operators defined above exist and are well defined. In this case we can build up the space of asymptotic states and the space of dressed states starting from the Fock space of the free states.

Let us first make the connection between the asymptotic states and the free states. Using Eq. (4.5) and Eq. (4.14) we conclude that this connection is given by a Moller operator that involve the free and the asymptotic Hamiltonian

$$|\psi^\pm(\alpha)>^{A,(\epsilon)}= \Omega^{(\pm)}_{H_A^{(\epsilon)},H_0^{(\epsilon)}}|\phi(\alpha)>^{(\epsilon)} . \tag{4.25}$$

Given the properties of Moller operators shown above we remark that the states $|\psi^\pm(\alpha)>^{A,(\epsilon)}$ are eigenstates of H_A with the same eigenvalue which $|\phi(\alpha)>^{(\epsilon)}$, the eigenstate of H_0, has.

If we denote the space of free states $\mathcal{H}_f$ then, according to Eq. (4.25), the space of asymptotic states is shown to be $\mathcal{H}_A = \Omega^{(\pm)}_{H_A^{(\epsilon)},H_0^{(\epsilon)}}\mathcal{H}_f$. Note that, if we take the asymptotic Hamiltonian to be the free Hamiltonian, the Moller operators used here would be simply identity operators and the asymptotic states would be free states.

This is the way to define the space of dressed states. According to their definition in Eq. (4.20) the dressed states span the space $\mathcal{H}_D = \Omega^{(\pm)\dagger}_{H_A^{(\epsilon)},H_0^{(\epsilon)}}\mathcal{H}_f$.

In the regulated theory all these Moller operators are well defined and unitary so the space of asymptotic states and the one of dressed states are Fock spaces as the space of free states is. In this limit Eq. (4.25) is simply a change of basis in the Fock space since the free states span this space and are linearly independent (thus form a basis). That means the three sets of states $\{|\psi^+(\alpha)>^{A,(\epsilon)}\}, \{|\psi^-(\alpha)>^{A,(\epsilon)}\}, \{|\phi(\alpha)>^{(\epsilon)}\}$ form three equivalent basis for the Fock space for a finite ϵ.

Given the definition of dressed states in Eq. (4.20) one can think of them as being another basis of the Hilbert space since their definition looks like a change-of-basis in Fock space relation when Moller operators are unitary. But these kind of states cannot be defined in the limit of vanishing regulator being no eigenstates of one of the Hamiltonians and so it makes no sense to take them together with the other states in the discussion to come.

When the regulator is removed (in other words put to zero) things change dramatically because many operators used above become ill-defined. All the Moller operators that include the free Hamiltonian will be singular again and all the relation between different states and the free states cease to exist. The set of dressed states cease to exist as well since they are defined using such a singular operator. The other operators are supposed to approach smoothly their non-regularized form. For example

$$\Omega^{(\pm)}_{H^{(\epsilon)},H_A^{(\epsilon)}}|\psi> \longrightarrow \Omega^{(\pm)}_{H,H_A}|\psi> . \tag{4.26}$$

The space of asymptotic states still exist even for a vanishing regulator but it cannot be unitarily related to the Fock space of free states any more. Thus we can say the space of physical states (the space of asymptotic states as well as the space of 'in' and 'out' states) will be non-Fock. This is related to the property that any two states of a Fock space are related by one unitary operator which is not the case anymore. The question you might ask now is: 'what kind of space are the physical states in this case ?'. It can be snowed that they become part of the more general van Neumann space in which the Fock space can be embedded. In this space operators of the form e^{iA} , with A a hermitian operator that can be singular, are defined as null operators considering $e^{i\infty} = 0$. This implies that the sets of states of Eq. (4.26) that once formed equivalent basis of the Fock space are now orthogonal to each other. In other words the space of physical states will fe orthogonal of the Fock space of free states.

This discussion can be made from a different perspective. One can consider the physical 'in' and 'out' states together with the asymptotic states as being in the Fock space and then for a vanishing regulator the free states will be non-Fock states. This is the case in [4] and a larger discussion on this problem is made in one of their appendices.

The above discussion was made for completness and will not need it in the rest of this thesis. As long as the theory is regulated all these states belong to the Fock space since all the Moller operators that connect them are unitary we do not need to worry about the intricaces of van Neumann space and other difficulties that arise from that discussion.

4.7 More about Moller operators

After this short discussion about the asymptotic states let us get back to the Moller operators and study some more properties. Up to now we specified their time-independent form and identified some of their properties. In this part we will search for a time-dependent form for these operators. We will proceed with a brief discussion on the so called 'pictures' in which we can describe the evolution of our states and operators. We will also see what is the connection between the Moller operators and the evolution operators of different pictures.

When describing the time-evolution of states and operators in quantum mechanics different points of view can be employed. We can think as if the states are time dependent while observables and their associated operators do not change in time. On the other hand, we can take time-depenent operators and stationary states or both states and operators as time-dependent. All these points of view are called 'pictures' and their theory is built in such a way that they are equivalent in the sense that the inner product of states and the expectation values of operators are the same in any picture. We will discuss about three of the possible pictures and suppose that at some reference time t_0 the states and operators in all these pictures coincide.

Let us start with the Schrodinger picture in which operators are time-independent and states evolve in time according to the full Hamiltonian

$$|\Psi_S(t)>= U_H(t,t_0)|\Psi_S(t_0)> \quad \text{and} \quad \frac{\partial}{\partial t}O_S = 0, \tag{4.27}$$

where we define the evolution operator as

$$U_H(t,t_0) = \exp\left[-iH(t-t_0)\right]. \tag{4.28}$$

Note that the index of the evolution operator refers to the Hamiltonian that produces the evolution and not to the picture because they are not related to any picture.

In the Heisemberg picture the states are time-independent but operators are not

$$|\Psi_H(t)>= |\Psi_H(t_0)>= |\Psi_S(t_0)> \quad \text{and} \quad i\frac{\partial}{\partial t}O_H(t) = [O_H(t), H]$$

$$\Rightarrow \qquad O_H(t) = e^{iH(t-t_0)}O_H(t_0)e^{-iH(t-t_0)} = U_H^\dagger(t,t_0)O_S(t)U_H(t,t_0). \tag{4.29}$$

The claim we made that inner products and expectation values are picture-independent can be checked now. Because the evolution operators are unitary the inner product is

$$< \Psi_S(t)|\Psi_S(t) >=< \Psi_S(t_0)|U_H^\dagger(t,t_0)U_H(t,t_0)|\Psi_S(t_0) >=< \Psi_S(t_0)|\Psi_S(t_0) >=< \Psi_H(t)|\Psi_H(t) > . \tag{4.30}$$

The last picture we introduce is the asymptotic interaction picture in which both states and operators have a given time-evolution generated by the asymptotic Hamiltonian. If the asymptotic Hamiltonian is taken to be the free Hamiltonian we have the 'usual' interaction picture but we prefer to work in a more general frame and consider an asymptotic Hamiltonian in general.

The evolution in this picture is then

$$|\Psi_{AI}(t) >= U_{H_A}^\dagger(t,t_0)|\Psi_S(t) >\equiv U_{AI}(t,t_0)|\Psi_S(t_0) >= U_{AI}(t,t_0)|\Psi_H(t) > . \tag{4.31}$$

And for operators

$$O_{AI}(t) = U_{H_A}^\dagger(t,t_0)O_S(t_0)U_{H_A}(t,t_0)$$

$$= U^{\dagger}_{H_A}(t,t_0)U_H(t,t_0)O_H(t)U^{\dagger}_H(t,t_0)U_{H_A}(t,t_0)$$

$$\equiv U_{AI}(t,t_0)O_H(t)U^{\dagger}_{AI}(t,t_0), \tag{4.32}$$

where we defined a different evolution operator by

$$U_{AI}(t,t_0) = U^{\dagger}_{H_A}(t,t_0)U_H(t,t_0) = e^{iH_A(t-t_0)}e^{-iH(t-t_0)}. \tag{4.33}$$

This is the point where a link to the Moller operators can be made. Remark that

$$\Omega^{(\pm)}_{H,H_A} = lim_{t\to\mp\infty}U^{\dagger}_{AI}(t,0). \tag{4.34}$$

Thus, if we are able to get a more specific form for the evolution operator of the interaction picture, the Moller operators will be determind taking the limit of Eq. (4.35). In this equation we took already the reference time $t_0 = 0$. Basically any other finite choice for t_0 would be good and we will obtain an integral form for this operator with the reference time as one of the intregration limits. The choice $t_0 = 0$ was made to simplify further calculations.

The evolution operator of Eq. (4.34) satishies a Schrodinger equation which comes from the fact that the states that evolve in time according to it obey such an equation. In analogy with this we expect the time-evolution operator of the asymptotic picture to obey a Scrodinger-like equation with a certain Hamiltonian that is to be determined. That is

$$i\frac{\partial}{\partial t}U_{AI}(t,t_0) = H_{AI}(t)U_{AI}(t,t_0). \tag{4.35}$$

The left hand side of this equation leads to the following relation

$$i\frac{\partial}{\partial t}U_{AI}(t,t_0) = e^{iH_A(t-t_0)}(H-H_A)e^{-iH(t-t_0)} =$$

$$= e^{iH_A(t-t_0)}(H-H_A)e^{-iH_A(t-t_0)}e^{iH_A(t-t_0)}e^{-iH(t-t_0)} \equiv H_{AI}(t)U_{AI}(t,t_0). \tag{4.36}$$

The newly defined Hamiltonian $H_{AI}(t)$ is called the asymptotic interaction Hamiltonian in the asymptotic interaction picture.

Note that Eq. (4.37) has the same solution as the integral equation

$$U_{AI}(t,t_0) = 1 - i\int_{t_0}^{t} dt_1 H_{AI}(t_1)U_{AI}(t_1,t_0), \tag{4.37}$$

with the initial condition $U(t_0,t_0) = 1$. This equation enables us to write the evolution operator as an integral form as

$$U_{AI}(t,t_0) = 1 - i\int_{t_0}^{t} dt_1 H_{AI}(t_1) + (-i)^2\int_{t_0}^{t} dt_1\int_{t_0}^{t_1} dt_2 H_{AI}(t_1)H_{AI}(t_2) + \dots . \tag{4.38}$$

Or in other words the evolution operator is a time-ordered exponential of the interaction Hamiltonian time integral

$$U_{AI}(t,t_0) = T \exp\left[-i\int_{t_0}^{t} dt_1 H_{AI}(t_1)\right]. \tag{4.39}$$

According to Eq. (4.35), if the reference time is taken to be zero, the Moller are determined via

$$\Omega^{(\pm)\dagger}_{H,H_A} = lim_{t\to\mp\infty} U_{AI}(t,0) = T \exp\left[-i\int_{0}^{\mp\infty} dt_1 H_{AI}(t_1)\right]. \tag{4.40}$$

In a similar manner other Moller operators can be found. In the case of $\Omega^{(\pm)}_{H,H_0}$ the asymptotic Hamiltonian is to be taken equal to the free Hamiltonian in all expresions above and we arrive at

$$\Omega^{(\pm)}_{H,H_0} = lim_{t\to\mp\infty} U^{\dagger}_I(t,0) = lim_{t\to\mp\infty} U_I(0,t) = T \exp\left[i\int_{0}^{\mp\infty} dt_1 H_I(t_1)\right], \tag{4.41}$$

where $H_I(t) = e^{iH_0t}(H - H_0)e^{-iH_0t}$ is the interaction Hamiltonian in the usual interaction picture. Remark also that, because the free Hamiltonian commutes with the full Hamiltonian, we have $U^{\dagger}_I(t,0) = U_I(0,t)$.

In the same spirit we can show that $\Omega^{(\pm)}_{H_A,H_0} = T \exp\left[i\int_0^{\mp\infty} dt_1 H_{IS}(t_1)\right]$, where the role of the interaction Hamiltonian is played by the soft part of the full interaction Hamiltonian in the interaction picture.

Using these relations we find that the S operator is given by

$$S = \Omega^{(-)\dagger}_{H,H_0}\Omega^{(+)}_{H,H_0} = U_I(\infty,0)U^{\dagger}_I(-\infty,0) = U_I(\infty,-\infty) \tag{4.42}$$

The theoretical aspects discussed thus far should be enough in order to show that finite matrix elements of the ordinary S operator exist. This is the subject of the next chapter.

Summary

The S matrix elements can be related to the matrix elements of a certain operator between free states. In massless gauge theories this is not possible anymore unless a regularization procedure is introduced. In this case the free states are not suitable candidates for asymptotic states. The true asymptotic states are introduced and the S matrix elements are shown to be the matrix elements of a well defined operator (here called the asymptotic S operator) between the asymptotic states. In the regulated theory the matrix elements of the asymptotic S operator between free states are finite and they are equal to the matrix elements of the usual S operator between dressed states.

5 An application

In order prove the validity of the theory developed in Chapter 4 we consider a well known example. For a given process, using a perturbative approach, we find the regulated S matrix elements between dressed states to be finite and independent on the regulator.

Starting from the original idea of Chung [4] many papers applied this method both in QED [1,11,16] and QCD [17-22]. The variety of studies range from proving the finiteness of the S matrix in this basis of dressed states to computing different form factors up to the renormalization of the theory that includes dressed states. We shall not study most of these problems but focus on showing the finiteness of the S matrix in the basis of dressed states.

The outline of this chapter is a follows. First we introduce the dressed states specific for this example, define the Hamiltonian which enters the definition of dressed states and give the Feynman rules that apply. Afterwards we evaluate a series of diagrams corresponding to different matrix elements of S to prove that divergences cancel. In the end we remark that the cross-section computed from these amplitudes coincides [13] with the well known cross-section of the studied process computed in the 'usual' way.

5.1 Introduction

The whole discussion of this chapter goes around the process $e^+ + e^- \to \gamma \to 2\,jets$. For us the interesting part of this is the strong interaction present in the pair creation $\gamma \to 2\,jets$. The electron-positron annihilation process is plagued with infrared divergencies as well and we could cure them using either Kinoshita-Lee-Nauemberg method, which works very well for QED, or this method of dressed states. We will not spend time doing this but instead focus on the more difficult strong interaction part. Details regarding the hadronization processes that involve the 2 jets in the final state are beyond the scope of this thesis so they are not included in any way. We shall restrict our calculations to showing the finiteness of scattering amplitudes up to NLO.

Besides the non-abelian nature of the strong interaction which introduces extra difficulties in our calculations the difference between this and QED is the presence of extra singularities associated with collinear particles that are created or annihilated. In QCD the gauge boson, the gluon, carries color charge itself and it can split into a pair of gluons which can be collinear with each other. Moreover, if we consider the quarks massless, as it is done many times in the literature, the emission of a collinear gluon by a quark or antiquark is singular. It is clear that the full Hamiltonian for this example should include both the strong and electromagnetic interaction since quarks carry both types of charges. Moreover the two parts of the interaction Hamiltonian would commute with each other and so the Moller operators built from the full interaction Hamiltonian can be factorised in a product of electromagnetic one strong interactions operators which commute with each other. Thus the singularities coming from two different interactions can be trated separately. As a matter of fact we will not even include the electromagnetic interaction into our Hamiltonian but focus only on the strong interactions.

We emphasis again that the dressed states' definition is valid only in a regulated theory which is also the framework we use in this chapter. For simplicity of notations we will ommit the ϵ index from the regularized operators.

Remember the aim of this example is to show that S operator matrix elements between dressed states are finite. We could show this by proving that matrix elements of S_A between free states are finite but we also want to perform calculations that involve dressed states. According to Eq. (4.21) these matrix elements are defined as

$$< \{f\}|S|\{i\} > = < f\,|\Omega^{(-)}_{H_A,H_0} S \Omega^{(+)\dagger}_{H_A,H_0}|i > . \tag{5.1}$$

In our case the final state contains the two jets which at the lowest order in perturbation correspond to a quark-antiquark pair. The initial state is the photon which obviously does not interact strongly thus, from that point of view, this state is nothing but the vacuum of the strong interaction. This motivates us to take from now on $|i>\equiv|0>$. Of course the photon-quark-antiquark vertex will be present in the calculation via its corresponding Feynman rule even.

In this approach we can either compute S matrix elements between dressed states straight away by brute force or relate these to ordinary matrix elements between free states. The latter can be done rather easy by inserting complete sets of free states between the three operators of Eq. (5.1). That leads to

$$< f|\Omega^{(-)}_{H_A,H_0} S \Omega^{(+)\dagger}_{H_A,H_0}|i>= \int df^{'} \int di^{'} < f\,|\Omega^{(-)}_{H_A,H_0}|f^{'}> \otimes < f^{'}|S|i^{'}> \otimes < i^{'}|\Omega^{(+)\dagger}_{H_A,H_0}|i> . \tag{5.2}$$

The integration over the new states $f^{'}$ and $i^{'}$ is the formal way of saying that we sum over the two complete set of states. In this approach the difficulty of evaluating unusual matrix elements between dressed states was exchanged for the calculation of a double number of matrix elements between free states. These will be evaluated by means of perturbation theory and we will see in the end that even though the matrix elements taken alone are not finite (in the sense that they still depend on the regulator that is vanishing) when the summation over the complete sets of states is performed the dependence on the regulator disappears and the finiteness is proved. We wish to remind the reader that these calculation are concerned only with infrared divergences and any ultra violet divergences that might occur are to be treated with usual tools such as renormalization.

One more remark regards the evaluation of the matrix element $< i^{'}|\Omega^{(+)\dagger}_{H_A,H_0}|i>$. We argued above that the choice $|i>\equiv|0>$ is legitimate. Given that the Hamiltonian that enters the Moller operator is normal ordered it is straight forward to show that at any order in perturbation theory this matrix element vanishes unless $< i^{'}| =< 0|$ as well. In other words $< i^{'}|\Omega^{(+)\dagger}_{H_A,H_0}|i>\sim \delta_{|i'>,|0>}$. In this way we can give up the summation over the states $i^{'}$ since they include only the vacuum of the strong interaction and also take the last matrix element of Eq. (5.2) to be $< 0|\Omega^{(+)\dagger}_{H_A,H_0}|0>= 1$.

In this way Eq. (5.2) has the simplified form

$$< f|\Omega^{(-)}_{H_A,H_0} S \Omega^{(+)\dagger}_{H_A,H_0}|i>= \int df^{'} < f\,|\Omega^{(-)}_{H_A,H_0}|f^{'}> \otimes < f^{'}|S|0> \tag{5.3}$$

Next step is to introduce the Hamiltonian that is involved in the Moller operator and find the diagrammatic rules to be used for the evaluation of these matrix elements.

5.2 The Hamiltonian and Feynman rules

According to the definition of Moller operators in Eq. (4.41) we need to define the asymptotic interaction Hamiltonian in the interaction picture. This should include all the soft and collinear interactions present in the full interaction Hamiltonian in this picture.

If we neglect any other possible interaction between the two jets, besides the strong interaction, the Hamiltonian we will use is the usual QCD interaction Hamiltonian

$$H_I = g \int d\mathbf{x} : \bar{\Psi} T^a \gamma^\mu \Psi : A^a_\mu, \tag{5.4}$$

where g is the coupling constant, T^a are the color group generators and γ^μ the Dirac matrices.

Since we work in the interaction picture the field operators are defined as free fields according to

$$\Psi(x) = \int d\tilde{k} \left(u(r,\mathbf{k}) b(r,\mathbf{k}) e^{-ikx} + v(r,\mathbf{k}) d^{\dagger}(r,\mathbf{k}) e^{ikx} \right),$$

$$\bar{\Psi}(x) = \int d\tilde{k} \left(\bar{u}(r,\mathbf{k}) b^{\dagger}(r,\mathbf{k}) e^{ikx} + \bar{v}(r,\mathbf{k}) d(r,\mathbf{k}) e^{-ikx} \right),$$

$$A_{\mu}(x) = \int d\tilde{k} \left(\varepsilon_{\mu}(\lambda,\mathbf{k}) a(\lambda,\mathbf{k}) e^{-ikx} + \varepsilon^{*}_{\mu}(\lambda,\mathbf{k}) a^{\dagger}(\lambda,\mathbf{k}) e^{ikx} \right). \tag{5.5}$$

The integration measure used in these definitions will be kept the same throughout this chapter and includes a sum over the gluon's helicities

$$d\tilde{k} = \frac{d^{D-1}\mathbf{k}}{(2\pi)^{D-1} 2\omega(\mathbf{k})} \sum_{1,2}. \tag{5.6}$$

The operators $b^{\dagger}(r,\mathbf{k})$ and $d^{\dagger}(r,\mathbf{k})$ (or $b(r,\mathbf{k})$ and $d(r,\mathbf{k})$) creates (or anihilates) one fermion or anti-fermion of momentum $\mathbf{k}$ and helicity r while u and v are D-dimensional spinors. On the other hand $a^{\dagger}(\lambda,\mathbf{k})$ (or $a(\lambda,\mathbf{k})$) creates (or anihilates) one gluon of momentum $\mathbf{k}$ and helicity λ while ε denotes the polarization of the gluon. Creation and annihilation operators come with the usual commutation relation for the gauge bosons and anticommutation relations for fermions.

$$[a(\lambda_1,\mathbf{k_1}), a^{\dagger}(\lambda_2,\mathbf{k_2})]_{-} = (2\pi)^3 2\omega(\mathbf{k_1}) g_{\lambda_1\lambda_2} \delta(\mathbf{k_1} - \mathbf{k_2})$$

$$[b(\lambda_1,\mathbf{k_1}), b^{\dagger}(\lambda_2,\mathbf{k_2})]_{+} = (2\pi)^3 2\omega(\mathbf{k_1}) g_{\lambda_1\lambda_2} \delta(\mathbf{k_1} - \mathbf{k_2}) \tag{5.7}$$

When the corresponding expressions for the field operators are plugged in Eq. (5.4) the Hamiltonian can be written as

$$H_I = gT^a \int d\mathbf{x} \int d\tilde{k}_1 d\tilde{k}_2 d\tilde{k}_3 \sum_{i=1}^{8} V_i(\mathbf{k_1},\mathbf{k_2},\mathbf{k_3}) e^{i\sum_{j=1}^{3}\sigma_{ij}\mathbf{k_j}\mathbf{x}} e^{-it\sum_{j=1}^{3}\sigma_{ij}\omega(\mathbf{k_j})}, \tag{5.8}$$

where σ is a matrix that stores information about incoming and outgoing particles. Its elements take values $+1$ for each incoming particle and -1 for any outgoing one. We have also denoted by $V_i(\mathbf{k_1},\mathbf{k_2},\mathbf{k_3})$ all possible 3-particle vertices. In terms of particle creation and annihilation opeartors all possible verices are included in the following list :

$$V_1 = b^{\dagger}(\mathbf{k_1}) b(\mathbf{k_2}) a(\mathbf{k_3}) \bar{u}(\mathbf{k_1}) \varepsilon_{\mu}(\mathbf{k_3}) \gamma^{\mu} u(\mathbf{k_2}),$$

$$V_2 = b^{\dagger}(\mathbf{k_1}) d^{\dagger}(\mathbf{k_2}) a(\mathbf{k_3}) \bar{u}(\mathbf{k_1}) \varepsilon_{\mu}(\mathbf{k_3}) \gamma^{\mu} v(\mathbf{k_2}),$$

$$V_3 = d(\mathbf{k_1}) b(\mathbf{k_2}) a(\mathbf{k_3}) \bar{v}(\mathbf{k_1}) \varepsilon_{\mu}(\mathbf{k_3}) \gamma^{\mu} u(\mathbf{k_2}),$$

$$V_4 = -d^{\dagger}(\mathbf{k_1}) d(\mathbf{k_2}) a(\mathbf{k_3}) \bar{v}(\mathbf{k_2}) \varepsilon_{\mu}(\mathbf{k_3}) \gamma^{\mu} v(\mathbf{k_1}),$$

$$V_5 = b^{\dagger}(\mathbf{k_1}) b(\mathbf{k_2}) a^{\dagger}(\mathbf{k_3}) \bar{u}(\mathbf{k_1}) \varepsilon^{*}_{\mu}(\mathbf{k_3}) \gamma^{\mu} u(\mathbf{k_2}),$$

$$V_6 = d(\mathbf{k_1})b(\mathbf{k_2})a^\dagger(\mathbf{k_3})\bar{v}(\mathbf{k_1})\varepsilon^*_\mu(\mathbf{k_3})\gamma^\mu u(\mathbf{k_2}),$$

$$V_7 = b^\dagger(\mathbf{k_1})d^\dagger(\mathbf{k_2})a^\dagger(\mathbf{k_3})\bar{u}(\mathbf{k_1})\varepsilon^*_\mu(\mathbf{k_3})\gamma^\mu v(\mathbf{k_2}),$$

$$V_8 = -d^\dagger(\mathbf{k_1})d(\mathbf{k_2})a^\dagger(\mathbf{k_3})\bar{v}(\mathbf{k_2})\varepsilon^*_\mu(\mathbf{k_3})\gamma^\mu v(\mathbf{k_1}). \tag{5.9}$$

Not all these vertices are part of our calculations. Since we are interested in the infrared problems we will take only those vertices which are divergent whenever a soft and/or collinear gluon is emitted or absorbed. In such a case only vertices that have vanishing energy' exponents in Eq. (5.8) should be taken into account. The reason why vanishing energy at the vertices would give rise to divergences is simple. According to Eq. (4.40) the Moller operators are defined in terms of time integrals of the Hamiltonian. As long as the time interval is limited to $0 \leq t < \infty$ the integral will bring down the energy coefficient of the exponent as a denominator leading to a divergent result.

Vertices V_3 and V_7 are to be excluded from the beginning since they would create / annihilate three particles out of nothing, process which is absent up to NLO in our example. On the other hand vertices V_2 and V_6 enter the category of interest only when both the fermions and the glon are collinear. These two contribute at higher order terms in perturbation theory but not at the order we shall work. As a conclusion vertices 1,4,5 and 8 are the ones to be included in the effective Hamiltonian.

As a matter of fact the Hamiltonian of Eq. (5.4) is not exactly the one we need in the Moller operators for the dressed states. Besides selecting the vertices that may be divergent we also need to introduce the distinction between the hard and soft parts of the Hamiltonian (as well as collinear / non-collinear parts) via theparameter $\triangle$. This will be introduced using a Heaviside step function remembering that the vertices give divergent contributions only when the energy coefficient in the time exponential of Eq. (5.8) vanishes. The form of $\triangle$ does not matter too much here as long as it very small compared to the energy scale of the fermions. In this way we shall find the 'soft' Hamiltonian to be

$$H_I^S(\triangle) = gT^a \int d\mathbf{x} \int d\tilde{k}_1 d\tilde{k}_2 d\tilde{k}_3 \sum_{i=1,4,5,8}^{8} V_i(\mathbf{k_1}, \mathbf{k_2}, \mathbf{k_3}) e^{i\sum_{j=1}^{3}\sigma_{ij}\mathbf{k_j}\mathbf{x}} e^{-it\sum_{j=1}^{3}\sigma_{ij}\omega(\mathbf{k_j})} \times$$

$$\times \Theta(\triangle - |\sum_j \sigma_{ij}\omega(\mathbf{k_j})|). \tag{5.10}$$

Note that the integration over the space variable can be performed in Eq. (5.10) and leads to a delta function that ensure momentum conservation at any vertex since

$$\int d\mathbf{x} e^{-i\sum_{j=1}^{3}\sigma_{ij}\mathbf{k_j}\mathbf{x}} = \delta^{(D-1)}\left(\sum_{j=1}^{3}\sigma_{ij}\mathbf{k_j}\right). \tag{5.11}$$

Remark that energy is not conserved at any of the vertices because this would automatically lead to vanishing energy coefficients of the time exponential of Eq. (5.10) and give rise to divergences. Thus Lorentz invariance is lost once again.

At this moment we have all the tools we need to start the calculation. Since our approach is perturbative, in analogy to usual perturbation theory, we shall introduce a set of diagrams together with the corresponding diagrammatic rules. The situation at hand is different from the usual applications because we need to evaluate two different types of amplitudes according to Eq. (5.3). One amplitude involves the ordinary S operator and all Feynman rules and calculations would be of the usual type. The other matrix element involves a time-ordered exponential which makes us use the 'old fashioned' time-oredered perturbation theory. In this theory amplitudes are built up from (virtual) unperturbed states through which the system passes while making its time-evolution. The vertices are time-ordered and different diagrammatic rules are defined. Moreover in this theory all particles are on-shell. We shall not deduce all the diagrammatic rules here but oly show an example of how to get the rules for vertices. More information about time-ordered perturbation theory can be found in section 9.5 of [14]

Consider one of the vertices that enters the Hamiltonian of Eq. (5.10). In the usual theory the Feynman rule for this would include a momentum-conserving delta function, a Dirac matrix together a color matrix. In this case things are different because according to the definition

$$\Omega^{(-)}_{H_A,H_0} = Te^{i\int_0^\infty dt\, H_I^S(t)}. \tag{5.12}$$

And

$$\int_0^\infty dt\, H_I^S(t) = -lim_{\eta\to 0} gT^a \int d\tilde{k}_1 d\tilde{k}_2 d\tilde{k}_3 \sum_{i=1,4,5,8}^{8} V_i(\mathbf{k_1},\mathbf{k_2},\mathbf{k_3})\times$$

$$\times\frac{1}{\sum_{j=1}^{3}\sigma_{ij}\omega(\mathbf{k_j})-i\eta}\delta^{(D-1)}\left(\sum_{j=1}^{3}\sigma_{ij}\mathbf{k_j}\right)\Theta(\triangle-|\sum_j\sigma_{ij}\omega(\mathbf{k_j})|). \tag{5.13}$$

Note that the vertex function in Eq. (5.13) includes one Dirac matrix and one color matrix. These together with the coefficient of the vertex function give the diagrammatic rules for each vertex. These are listed below:

the gluon absorption vertex:

$$igT^a\gamma^\mu\frac{\delta^{(3)}(\mathbf{p_3}+\mathbf{p_2}-\mathbf{p_1})}{\omega(\mathbf{p_3})+\omega(\mathbf{p_2})-\omega(\mathbf{p_1})-i\epsilon}\Theta(\triangle-|\omega(\mathbf{p_3})+\omega(\mathbf{p_2})-\omega(\mathbf{p_1})|); \tag{5.14}$$

the gluon emission vertex:

$$-igT^a\gamma^\mu\frac{\delta^{(3)}(\mathbf{p_1}+\mathbf{p_2}-\mathbf{p_3})}{\omega(\mathbf{p_1})+\omega(\mathbf{p_2})-\omega(\mathbf{p_3})-i\epsilon}\Theta(\triangle-|\omega(\mathbf{p_1})+\omega(\mathbf{p_2})-\omega(\mathbf{p_3})|); \tag{5.15}$$

fermion 'propagator':

$$\frac{1}{2\omega(\mathbf{p})}ip_\mu\gamma^\mu; \tag{5.16}$$

gluon 'propagator':

$$\frac{\delta^{ab}}{2\omega(\mathbf{p})}\left(-g_{\mu\nu}+\frac{p_\mu\bar{p}_\nu+p_\nu\bar{p}_\mu}{(p\bar{p})}\right); \tag{5.17}$$

The last two rules give the propagator-like functions. Such a function represents the transition between two vertices but it is not the inverse of a Green function as a usual propagator would be.

5.3 The amplitudes

In this section we compute different amplitudes for the studied process in a pertubative manner up to next-to-leading order. Up to this order we have to include two types of amplitudes. One includes diagrams with purely virtual corrections while the diagrams with free gluons in the final state are part of a second category. In this example we will be deal with the virtual diagrams and the ones that have one gluon in the final state. All these amplitudes will be evaluated at most up to second order in the coupling constant.

Before actually starting the discussion let us introduce a series of notations for various matrix elements.

The Fock states to be used in this example contain three types of particles : quarks denoted q, anti-quarks $\bar{q}$ and gluons denoted g (attention is required as the coupling constant has the same notation). The helicity and momentum of a particle are denoted by a single label e.g. $q(r_1, \mathbf{p_1}) \equiv q_{p1}$.

Regarding the matrix elements we shall denote the order $O(g^n)$ amplitudes as follows:

the Moller operator matrix elements

$$g^n D^{(n)}(q_{p1}, \overline{q}_{p2}, g_{p3}, ..; q_{q1}, \overline{q}_{q2}, ..) \equiv$$

$$\equiv < q_{p1}, \bar{q}_{p2}, g_{p3}...|\Omega^{(-)}_{\triangle}|q_{q1}, \bar{q}_{q2}, g_{q3}, ... > |_{g^n}; \tag{5.18}$$

the usual S matrix elements

$$g^n A^{(n)}(q_{q1}, \bar{q}_{q2}, g_{q3}, ..; \gamma) \equiv < q_{q1}, \bar{q}_{q2}, g_{q3}...|S|0 > |_{g^n}; \tag{5.19}$$

the S matrix elements between coherent states

$$g^n A^{(n)}(\{q_{q1}, \bar{q}_{q2}, g_{q3}, ..\}; \gamma) \equiv < \{q_{p1}, \bar{q}_{p2}, g_{p3}...\}|S|0 > |_{g^n}. \tag{5.20}$$

If we want to evaluate the S matrix element between dressed states up to a order n in perturbation theory, according to the factorization of Eq. (5.3), we need to take the n'th order factor of the product of the perturbation series for the two matrix elements. As mentioned we shall use a diagrammatic representation of these amplitudes. We put together the two matrix elements on a single diagram and separate the two by a dashed line. Eveything located on the left of the dashed line refers to the Moller operator matrix element while the right hand side stands for the ordinary S operators matrix element.

In all the diagrams the final state contains external fermions so the amplitudes are 'sandwiched' between spinors. The spinor for an incoming fermion of momentum p is denoted $|p>$. Not to be confused with a Fock state !

5.3.1 The amplitude $A(\{q(p_1), \bar{q}(p_2)\}; \gamma)$

We will start the actual calculation by evaluating the purely virtual term up to NLO. The actual process to be studied is $\gamma \to q + \bar{q}$ and we shall denote its total amplitude by $A(\{q(p_1), \bar{q}(p_2)\}; \gamma)$

This is the simplest case possible for a 2 jets final state. According to the factorization in Eq.(5.3), up to second order in the coupling constant, the amplitude has the form

$$A(\{q_{p1},\bar{q}_{p2}\};\gamma) \equiv \int d\tilde{q}_1 d\tilde{q}_2 D^{(0)}\left(q_{p1},\bar{q}_{p2};q_{q1},\bar{q}_{q2}\right) \times A^{(0)}\left(q_{q1},\bar{q}_{q2};\gamma(P)\right)$$

$$+g^2 \int d\tilde{q}_1 d\tilde{q}_2 D^{(0)}\left(q_{p1},\bar{q}_{p2};q_{q1},\bar{q}_{q2}\right) \times A^{(2)}\left(q_{q1},\bar{q}_{q2};\gamma(P)\right)$$

$$+g^2 \int d\tilde{q}_1 d\tilde{q}_2 D^{(2)}\left(q_{p1},\bar{q}_{p2};q_{q1},\bar{q}_{q2}\right) \times A^{(0)}\left(q_{q1}\bar{q}_{q2};\gamma(P)\right)$$

$$+g^2 \int d\tilde{q}_1 d\tilde{q}_2 d\tilde{q}_3 D^{(1)}\left(q_{p1},\bar{q}_{p2};q_{q1},\bar{q}_{q2},g_{q3}\right) \times A^{(1)}\left(q_{q1},\bar{q}_{q2},g_{q3};\gamma(P)\right) + O(g^4). \quad (5.21)$$

Terms having odd powers of g do not appear because they represent diagrams in which at least one gluon is emitted (or absorbed) without being absorbed (or emitted) later (or earlier) or in other words they suggest that one or more gluons are present in the final (or initial) state which is not the case here.

The leading order in perturbation is the first term in Eq. (5.21). This can be evaluated straight away as it is given by the photon-quark-antiquark vertex.

$$A^{(0)}\left(\{q_{p1},\bar{q}_{p2}\};\gamma\right) = A^{(0)}\left(q_{p1},\bar{q}_{p2};\gamma(P)\right), \quad (5.22)$$

because

$$D^{(0)}\left(q_{p1},\bar{q}_{p2};q_{q1},\bar{q}_{q2}\right) =$$

$$= (2\pi)^3 2\omega(\mathbf{p_1})\delta_{r1,s1}\delta(\mathbf{p_1}-\mathbf{q_1})(2\pi)^3 2\omega(\mathbf{p_2})\delta_{r2,s2}\delta(\mathbf{p_2}-\mathbf{q_2}). \quad (5.23)$$

At next to leading order we have to take into account all the one loop amplitudes: both the corrections to the quark-photon-antiquark vertex and the corrections to the asymptotic (time-ordered) part of the diagrams. In the evaluation of these amplitudes we have to pay extra attention to the different diagrammatic rules that apply for different parts of the diagrams.

The second term in Eq. (5.21) is a first correction to the photon-quark-antiquark vertex. Some of the diagrams that contribute to this correction are shown in Fig. 1. There are four such diagrams but some of them give the same contribution.

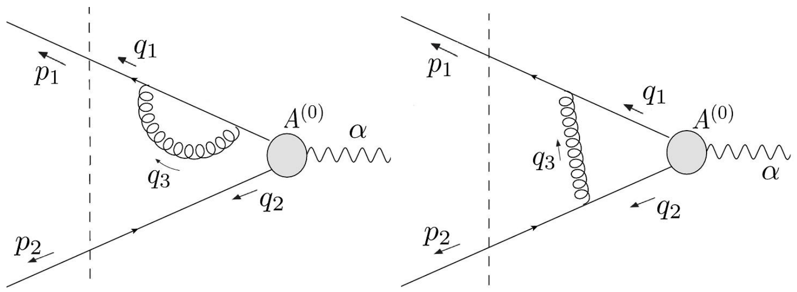

Fig.1 First order corrections to the $q\bar{q}\gamma$ vertex

Usual Feynman rules are used to get the amplitude

$$A^{(2)}\left(q_{q1},\bar{q}_{q2};\gamma(P)\right)=(-ie)g^2\int\frac{d^{D-1}q_3}{\omega(\mathbf{q_3})}\frac{1}{2(2\pi)^{D-1}}T^a_{ik}T^a_{kj}(-g^{\mu\nu}+\frac{q_3^\mu q_3^\nu}{q_3^2})\times$$

$$\times<p_1|\left\{\gamma_\mu\frac{i(p_1-q_3)^\lambda\gamma_\lambda}{(p_1-q_3)^2}\gamma_\nu\frac{ip_1^\rho\gamma_\rho}{p_1^2-i\eta}\gamma^\alpha-\gamma^\alpha\frac{ip_2^\lambda\gamma_\lambda}{p_2^2-i\eta}\gamma_\mu\frac{i(p_2-q_3)^\rho\gamma_\rho}{(p_2-q_3)^2}\gamma_\nu+\right.$$

$$\left.+\gamma_\mu\frac{i(p_1+q_3)^\lambda\gamma_\lambda}{(p_1+q_3)^2}\gamma^\alpha\frac{i(p_2-q_3)^\rho\gamma_\rho}{(p_2-q_3)^2}\gamma_\nu-\gamma_\mu\frac{i(p_1-q_3)^\lambda\gamma_\lambda}{(p_1-q_3)^2}\gamma^\alpha\frac{i(p_2+q_3)^\rho\gamma_\rho}{(p_2+q_3)^2}\gamma_\nu\right\}|p_2>. \quad (5.24)$$

This amplitude is shown to be singular in the limit $\epsilon\to 0$ and has the form

$$A^{(2)}\left(q_{q1},\bar{q}_{q2};\gamma(P)\right)=C_F\frac{\alpha_s}{2\pi}\left(\frac{\mu^2}{s}\right)^\epsilon\left(-\frac{1}{\epsilon^2}-\frac{3}{2\epsilon}-4+\frac{\pi^2}{12}\right)\times A^{(0)}\left(q_{q1},\bar{q}_{q2};\gamma(P)\right). \quad (5.25)$$

Dimensional regularization is used always in this chapter so the dimension of space-time is $D=4-2\epsilon$.

Notice that in the usual treatment of this process, up to the second order in perturbation theory, the amplitude would be given by the first two terms of Eq. (5.21). It is obvious that their sum is not finite while the complete sum in Eq. (5.21) turns out to be finite.

The third term in Eq. 5.(21) is the sum of four one-loop amplitudes : two self-interaction diagrams and two one-gluon exchange diagrams. For all these diagrams the special diagrammatic rules introduced in Eq. (5.14-5.17) need to be used.

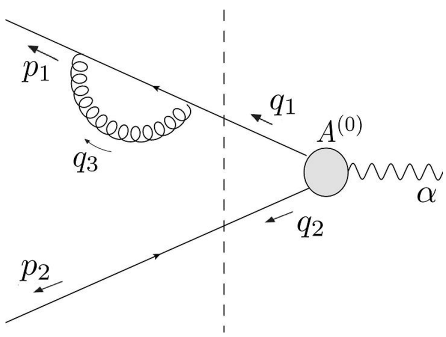

Fig.2 Self-energy of the quark in the asymptotic region

Consider the two self interaction diagrams. One is a correction to the quark propagator and includes the vertices V_1 and V_5. The other one is a one loop correction for the anti-quark propagator and contains the vertices V_4 and V_8. Since their contribution is equal only one of these amplitudes need to be computed. Fig.2 shows the fermion self-energy diagram. Using the diagrammatic rules defined above this amplitude is given by

$$a^{(2,0)}_{15}\equiv g^2\int d\tilde{q}_1 d\tilde{q}_2 D^{(2)}_{15}\left(q_{p1},\bar{q}_{p2};q_{q1},\bar{q}_{p2}\right)\times A^{(0)}\left(q_{q1}\bar{q}_{p2};\gamma(P)\right)$$

$$=-\frac{(-ie)g^2}{2}\int d^{D-1}q_1 d^{D-1}q_2 d^{D-1}q_3\frac{1}{(2\pi)^{D-1}}T^a_{ik}T^b_{kj}\times$$

$$\frac{\delta^{ab}}{2\omega(\mathbf{q_3})}\left(-g^{\mu\nu}+\frac{q_3^\mu\bar{q}_3^\nu+q_3^\nu\bar{q}_3^\mu}{(q_3\bar{q}_3)}\right)<p_1|\gamma_\mu q_2^\lambda\gamma_\lambda\gamma_\nu q_1^\beta\gamma_\beta\gamma^\alpha|p_2>\times$$

$$\frac{\Theta(\triangle - |\omega(\mathbf{q_3}) + \omega(\mathbf{q_2}) - \omega(\mathbf{p_1})|)\Theta(\triangle - |\omega(\mathbf{q_3}) + \omega(\mathbf{q_2}) - \omega(\mathbf{q_1})|)}{2\omega(\mathbf{q_1})(\omega(\mathbf{q_3}) + \omega(\mathbf{q_2}) - \omega(\mathbf{p_1}))2\omega(\mathbf{q_2})(\omega(\mathbf{q_3}) + \omega(\mathbf{q_2}) - \omega(\mathbf{q_1}))} \times$$

$$\delta^{(D-1)}(\mathbf{q_2} + \mathbf{q_3} - \mathbf{p_1})\delta^{(D-1)}(\mathbf{q_2} + \mathbf{q_3} - \mathbf{q_1})(2\pi)^D \delta^{(D)}(P - q_1 - p_2). \tag{5.26}$$

The two D-1 dimensional delta functions assure momentum conservation at the vertices and they can be used to evaluate the integrals over q_1 and q_2. Due to lack of Lorentz invariance in the time ordered perturbation theory the energy is not conserved at the vertices. Still the two momenta q_1 and q_2 are on shell and so their first components are well determined. For this amplitude we use the notation $\{p_1 - q_3\} \equiv q_2 = (\omega(\mathrm{p_1}-\mathbf{q_3}), \mathrm{p_1}-\mathbf{q_3})$. It is also useful to introduce a shorter notation for the energy loss at every vertex, energy that appear for example in the denominator of Eq. (5.26). Denote $\rho(\mathbf{k_1}, \mathbf{k_2}) \equiv \omega(\mathbf{k_1}) + \omega(\mathbf{k_2}) - \omega(\mathbf{k_1} + \mathbf{k_2})$.

After performing the integrals over q_1 and q_2 in Eq. (5.26) and using the new notations the amplitude reads

$$a_{15}^{(2,0)} = -\frac{(-ie)g^2}{2} T^a_{ik} T^a_{kj} (2\pi)^D \delta^{(D)}(P - p_1 - p_2) \times$$

$$\int d\tilde{q}_3 \left(-g^{\mu\nu} + \frac{q_3^\mu \bar{q}_3^\nu + q_3^\nu \bar{q}_3^\mu}{(q_3 \bar{q}_3)} \right) \Theta(\triangle - |\rho(\mathbf{q_3}, \mathbf{p_1} - \mathbf{q_3})|) \times$$

$$\frac{< p_1 | \gamma_\mu \{p_1 - q_3\}^\lambda \gamma_\lambda \gamma_\nu p_1^\beta \gamma_\beta \gamma^\alpha | p_2 >}{2\omega(\mathbf{p_1}) 2\omega(\mathbf{p_1} - \mathbf{q_3})(\rho(\mathbf{q_3}, \mathbf{p_1} - \mathbf{q_3}))^2}. \tag{5.27}$$

Using the properties of the γ matrices in D dimensions, their anti-commutation relations and the Dirac equation and remarking that $\{p_1 - q_3\} = p_1 - q_3 + r$ with $r = (\rho(\mathbf{q_3}, \mathbf{p_1} - \mathbf{q_3}), \mathbf{0}) \equiv (r_0, \mathbf{0})$ we can further simplify the above expression to

$$a_{15}^{(2,0)} = -\frac{(-ie)g^2}{2} T^a_{ik} T^a_{kj} (2\pi)^D \delta^{(D)}(P - p_1 - p_2) \times$$

$$\int d\tilde{q}_3 \Theta(\triangle - |\rho(\mathbf{q_3}, \mathbf{p_1} - \mathbf{q_3})|) \frac{< p_1 | \gamma^\alpha | p_2 >}{2\omega(\mathbf{p_1}) 2\omega(\mathbf{p_1} - \mathbf{q_3})(\rho(\mathbf{q_3}, \mathbf{p_1} - \mathbf{q_3}))^2} \times$$

$$\times \left[(D-2)((p_1 q_3) - (p_1 r)) - \frac{4(p_1 q_3)(p_1 \bar{q}_3)}{(q_3 \bar{q}_3)} \right]. \tag{5.28}$$

It is clear that Eq. (5.28) is divergent for soft q_3 or for $\mathbf{q_3} || \mathbf{p}_1$ because the denominator vanishes. We leave the explicit evaluation of this amplitude for the appendix. Still we compute it partially there and then combine it with another amplitude in order to get a simpler result. This amplitude alone was computed by D.Forde [13]

The one gluon-exchange term, shown in Fig.3, contains two diagrams which are symmetric under exchange of all momenta so they give the same contribution.

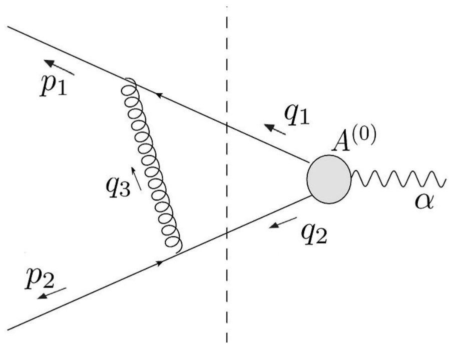

Fig.3 Gluon exchange in the asymptotic region

One of the two diagrams includes the vertices $V_1$and V_8 and the other one contains V_4 and V_5. Again we compute only one of the two amplitudes. The diagrammatic rules lead to

$$a_{18}^{(2,0)} \equiv g^2 \int d\tilde{q}_1 d\tilde{q}_2 D_{18}^{(2)} \left(q_{p1}, \bar{q}_{p2}; q_{q1}, \bar{q}_{q2}\right) \times A^{(0)} \left(q_{q1} \bar{q}_{q2}; \gamma(P)\right)$$

$$= \frac{(-ie)g^2}{2} \int d^{D-1}q_1 d^{D-1}q_2 d^{D-1}q_3 \frac{1}{(2\pi)^{D-1}} T_{ik}^a T_{kj}^b \times$$

$$\frac{\delta^{ab}}{2\omega(\mathbf{q_3})} \left(-g^{\mu\nu} + \frac{q_3^\mu \bar{q}_3^\nu + q_3^\nu \bar{q}_3^\mu}{(q_3 \bar{q}_3)}\right) < p_1|\gamma_\mu q_1^\lambda \gamma_\lambda \gamma_\nu q_2^\beta \gamma_\beta \gamma^\alpha |p_2 > \times$$

$$\frac{\Theta(\triangle - |\omega(\mathbf{q_3}) + \omega(\mathbf{q_1}) - \omega(\mathbf{p_1})|)\Theta(\triangle - |\omega(\mathbf{q_3}) + \omega(\mathbf{p_2}) - \omega(\mathbf{q_2})|)}{2\omega(\mathbf{q_1})(\omega(\mathbf{q_3}) + \omega(\mathbf{q_1}) - \omega(\mathbf{p_1}))2\omega(\mathbf{q_2})(\omega(\mathbf{q_3}) + \omega(\mathbf{p_2}) - \omega(\mathbf{q_2}))} \times$$

$$\delta^{(D-1)}(\mathbf{q_1} + \mathbf{q_3} - \mathbf{p_1})\delta^{(D-1)}(\mathbf{p_2} + \mathbf{q_3} - \mathbf{q_2})(2\pi)^D \delta^{(D)}(P - q_1 - q_2). \tag{5.29}$$

The three particles in the asymptotic region are all on shell and the momentum is conserved at the vertices. We define $\{p_1 - q_3\} \equiv q_1 = (\omega(\mathbf{p_1} - \mathbf{q_3}), \mathrm{p}_1 - \mathbf{q_3})$ and $\{p_2 + q_3\} \equiv q_2 = (\omega(\mathrm{p}_2 + \mathbf{q_3}), \mathrm{p}_2 + \mathbf{q_3})$. Thus the expression in Eq. (5.29) is reduced to

$$a_{18}^{(2,0)} = \frac{(-ie)g^2}{2} T_{ik}^a T_{kj}^a \int \frac{d^{D-1}q_3}{(2\pi)^{D-1}} (2\pi)^D \delta^{(D)}(P - \{p_1 - q_3\} - \{p_2 + q_3\}) \times$$

$$\frac{1}{2\omega(\mathbf{q_3})} \left(-g^{\mu\nu} + \frac{q_3^\mu \bar{q}_3^\nu + q_3^\nu \bar{q}_3^\mu}{(q_3 \bar{q}_3)}\right) < p_1|\gamma_\mu \{p_1 - q_3\}^\lambda \gamma_\lambda \gamma_\nu \{p_2 + q_3\}^\beta \gamma_\beta \gamma^\alpha |p_2 > \times$$

$$\frac{\Theta(\triangle - |\rho(\mathbf{q_3}, \mathbf{p_1} - \mathbf{q_3})|)\Theta(\triangle - |\rho(\mathbf{q_3}, \mathbf{p_2})|)}{2\omega(\mathbf{p_1} - \mathbf{q_3})\rho(\mathbf{q_3}, \mathbf{p_1} - \mathbf{q_3})2\omega(\mathbf{p_2} + \mathbf{q_3})\rho(\mathbf{q_3}, \mathbf{p_2})}. \tag{5.30}$$

The divergences in this case come from the soft q_3 but the collinear singularities vanish because, for $\mathbf{q_3}||\mathbf{p_2}$ or $\mathbf{q_3}||\mathbf{p_1}$, the numerator vanishes.

Suppose q_3 is collinear to p_1 or in other words $\mathbf{q_3} = z\,\mathbf{p_1}$,where z is a real constant. Then $\{p_1 - q_3\} = (1 - z)p_1$. Thus the numerator of Eq. (5.30) vanishes because of the Dirac equation $< p|(p^\mu \gamma_\mu) = 0$.

The rest of the calculation is schetched in the appendix and the final result reads

$$2a_{18}^{(2,0)} = \frac{C_F}{2}\frac{\alpha_s}{2\pi}\left(\frac{\mu^2}{s}\right)^{\epsilon}\left(\frac{1}{\epsilon} + g_2(\triangle) + F_2\right) \times A^{(0)}\left(q_{q1}, \bar{q}_{q2}; \gamma(P)\right), \tag{5.31}$$

where

$$g_2(\triangle) = 2log2 - 2log\left(\frac{\triangle}{2}\right), \tag{5.32}$$

and the finite function

$$F_2 = (2\pi)^{D-1}\delta^{(D-1)}(\vec{P} - \vec{p_1} - \vec{p_2}) < p_1|\gamma^{\alpha}|p_2 > \times$$

$$\times \int d\tilde{q}_3 \Theta(\triangle - |\rho(\mathbf{q_3}, \mathbf{p_1} - \mathbf{q_3})|)\Theta(\triangle - |\rho(\mathbf{q_3}, \mathbf{p_2})|)\, f_2(p_1, p_2, q_3). \tag{5.33}$$

In the last line the function $f_2(p_1, p_2, q_3)$ has a finite contribution and the total factor F_2 is shown to vanish if we put $\triangle \to 0$.

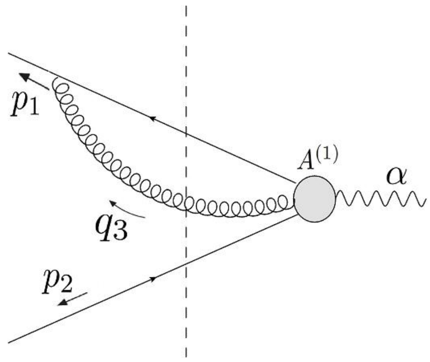

Fig.4 Cross term correction

The forth term in Eq. (5.21) is represented in Fig.4. There are two diagrams corresponding to the absorption of the gluon on the fermion line (vertex V_1) or the antifermion line (vertex V_4). The two contributions are equal and we need to compute only one of them. Besides each of these contributions contain two different terms as the gluon can be initially emitted by the quark or by the anti-quark. This is shown in Fig.5

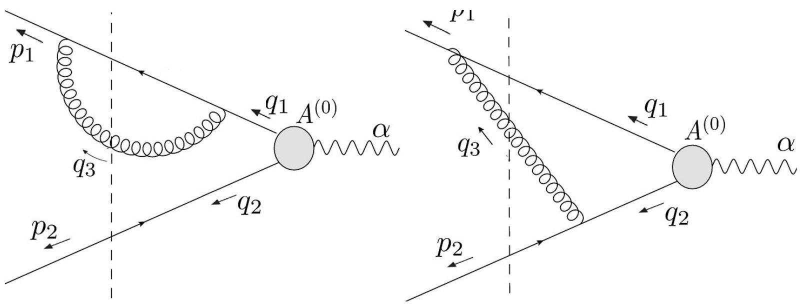

Fig.5 Cross-terms corrections in detail

The amplitude reads

$$a_1^{(1,1)} \equiv g^2 \int d\tilde{q}_1 d\tilde{q}_2 d\tilde{q}_3 D^{(1)}\left(q_{p1}, \bar{q}_{p2}; q_{q1}, \bar{q}_{q2}, g_{q3}\right) \times A^{(1)}\left(q_{q1}, \bar{q}_{q2}, g_{q3}; \gamma(P)\right)$$

$$= (-ie)g^2 \int d^{D-1}q_1 d^{D-1}q_2 \frac{d^{D-1}q_3}{(2\pi)^{D-1}} \frac{\delta^{ab}}{2\omega(\mathbf{q_3})} T^a_{ik} T^b_{kj} \times$$

$$\left(-g^{\mu\nu} + \frac{q_3^\mu \bar{q}_3^\nu + q_3^\nu \bar{q}_3^\mu}{(q_3\bar{q}_3)}\right) \frac{\Theta(\triangle - |\omega(\mathbf{q_3}) + \omega(\mathbf{q_1}) - \omega(\mathbf{p_1})|)}{2\omega(\mathbf{q_1})(\omega(\mathbf{q_3}) + \omega(\mathbf{q_1}) - \omega(\mathbf{p_1}))}$$

$$< p_1|\gamma_\mu q_1^\lambda \gamma_\lambda \left(\frac{\gamma^\nu (q_1+q_3)^\beta \gamma_\beta \gamma^\alpha}{2(q_1q_3)} - \frac{\gamma^\alpha (q_2+q_3)^\beta \gamma_\beta \gamma^\nu}{2(q_2q_3)}\right)|p_2>$$

$$\delta^{(D-1)}(\mathbf{q_1} + \mathbf{q_3} - \mathbf{p_1})\delta^{(D-1)}(\mathbf{p_2} - \mathbf{q_2})(2\pi)^D \delta^{(D)}(P - q_1 - q_2). \tag{5.34}$$

The singularities that are present in Eq. (5.34) come again from the soft gluon limit and also from the collinear region $\mathbf{q_3}||\mathbf{p_1}$

This amplitude is computed in the Appendix and in the end it is added to $a_{15}^{(2,0)}$ to simplify the calculation as the two amplitudes contain integrals over the same phase-space. The result reads

$$2a_{15}^{(2,0)} + 2a_1^{(1,1)} = \frac{C_F}{2}\frac{\alpha_s}{2\pi}\left(\frac{\mu^2}{s}\right)^\epsilon \left(\frac{1}{\epsilon^2} + \frac{1}{2\epsilon} + g_1(\triangle) + F_1\right) \times A^{(0)}\left(q_{q1}, \bar{q}_{q2}; \gamma(P)\right), \tag{5.35}$$

where

$$g_1(\triangle) = \frac{7}{2} - \frac{3}{2}\left(\frac{\triangle}{2}\right)^2 + \left[\frac{1}{2} + 3\frac{\triangle}{2} - \frac{1}{2}\left(\frac{\triangle}{2}\right)^2\right] log\left(\frac{\triangle}{2}\right) - log^2\left(\frac{\triangle}{2}\right) -$$

$$-2log^2\left(1 + \frac{\triangle}{2}\right) - 4Li_2\left(\frac{2}{2+\triangle}\right). \tag{5.36}$$

The finite factor

$$F_1 = (2\pi)^{D-1}\delta^{(D-1)}(\vec{P} - \vec{p_1} - \vec{p_2}) < p_1|\gamma^\alpha|p_2> \times$$

$$\times \int d\tilde{q}_3 \Theta(\triangle - |\rho(\mathbf{q_3}, \mathbf{p_1} - \mathbf{q_3})|)\Theta(\triangle - |\rho(\mathbf{q_3}, \mathbf{p_2})|)\ (f_1(p_1, p_2, q_3) + f_3(p_1, p_2, q_3)), \tag{5.37}$$

vanishes as well if we put the infrared resolution to zero.

If we add up all the contributions for the purely virtual diagram we find that, up to NLO, the result reads

$$A(\{q_{p1}, \bar{q}_{p2}\}; \gamma) = \left\{1 + \frac{C_F}{2}\frac{\alpha_s}{2\pi}\left[-\frac{1}{2} - \frac{3}{2}log\left(\frac{\triangle}{2}\right) - log^2\left(\frac{\triangle}{2}\right)\right]\right\} A^{(0)}\left(q_{q1}, \bar{q}_{q2}; \gamma(P)\right). \tag{5.38}$$

This is a finite result as all the terms that contain the regulator ϵ vanish. We are still left with terms that depend on $\triangle$. The finite terms $F_{1,2}$ were neglected since they would vanish anyway for a vanishing resolution. The terms that depend on $\triangle$ look like potentially divergent terms if $\triangle \to 0$ but they will disappear in the total cross-section being canceled by similar terms from another amplitude.

5.3.2 The amplitude $A(\{q(p_1), \bar{q}(p_2), g(p_3)\}; \gamma)$

In this section we will compute the amplitude of the case when a gluon is present in the final state. Up to the desired order in the coupling constant this amplitude is given by

$$A(\{q_{p1}, \bar{q}_{p2}, g_{p3}\}; \gamma) \equiv \int d\tilde{q}_1 d\tilde{q}_2 g D^{(1)}\left(q_{p1}, \bar{q}_{p2}, g_{p3}; q_{q1}, \bar{q}_{q2}\right) \times A^{(0)}\left(q_{q1}, \bar{q}_{q2}; \gamma(P)\right)$$

$$+ \int d\tilde{q}_1 d\tilde{q}_2 g D^{(0)}\left(q_{p1}, \bar{q}_{p2}, g_{p3}; q_{q1}, \bar{q}_{q2}, g_{p3}\right) \times A^{(1)}\left(q_{q1}, \bar{q}_{q2}, g_{p3}; \gamma(P)\right) + O(g^3). \tag{5.39}$$

The appearance of only odd powers of g in the expansion is due to the presence of only one gluon in the final state without any gluons in the initial state. One gluon emission gives one power of g and any other higher order corrections give even powers of g.

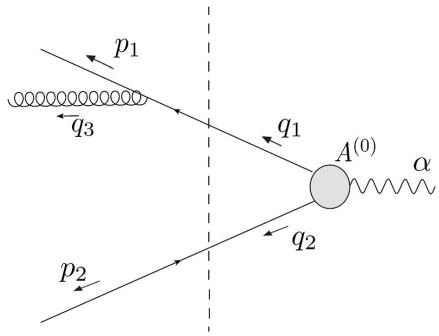

Fig.6 Gluon emission in the asymptotic region

There are two diagrams corresponding to the first term in Eq. (5.39) one for the quark emitting the gluon (vertex V_5) and one for the anti-quark (vertex V_8). Summed up together the two amplitudes read

$$b^{(1,0)} \equiv g \int d\tilde{q}_1 d\tilde{q}_2 D^{(1)}\left(q_{p1}, \bar{q}_{p2}, g_{p3}; q_{q1}, \bar{q}_{q2}\right) \times A^{(0)}\left(q_{q1}, \bar{q}_{q2}; \gamma(P)\right)$$

$$= (-ie) g T^a_{ij} (2\pi)^D \delta^{(D-1)}(\mathbf{P} - \mathbf{p_1} - \mathbf{p_2} - \mathbf{p_3}) < p_1| \times \{$$

$$\frac{(-\varepsilon^{\lambda}_{p3}\gamma_\lambda)\{p_1 + p_3\}^\rho \gamma_\rho \gamma^\alpha}{2\omega(\mathbf{p_1} + \mathbf{p_3})\rho(\mathbf{p_1}, \mathbf{p_3})} \Theta(\triangle - |\rho(\mathbf{p_1}, \mathbf{p_3})|) \delta(\sqrt{s} - \omega(\mathbf{p_1} + \mathbf{p_3}) - \omega(\mathbf{p_2})) +$$

$$\frac{\gamma^\alpha \{p_2 + p_3\}^\rho \gamma_\rho \varepsilon^{\lambda}_{p3}\gamma_\lambda}{2\omega(\mathbf{p_2} + \mathbf{p_3})\rho(\mathbf{p_2}, \mathbf{p_3})} \Theta(\triangle - |\rho(\mathbf{p_2}, \mathbf{p_3})|) \delta(\sqrt{s} - \omega(\mathbf{p_2} + \mathbf{p_3}) - \omega(\mathbf{p_1}))\} \times |p_2 > . \tag{5.40}$$

In the delta function $\sqrt{s}$ is the center of mass energy. The two integrals over q_1 and q_2 were both canceled by the delta functions that assure momentum conservation at vertices.

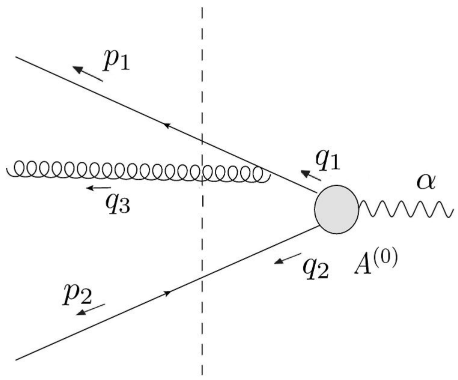

Fig.7 Cut diagram with three particles in the final state

The second term in Eq. (5.39) gives a cut diagram as in Fig.7 . The three particles in the asymptotic region in the final state do not interact such that the amplitude is simply

$$\int d\tilde{q}_1 d\tilde{q}_2 g D^{(0)}\left(q_{p1}, \bar{q}_{p2}, g_{p3}; q_{q1}, \bar{q}_{q2}, g_{p3}\right) \times A^{(1)}(q_{q1}, \bar{q}_{q2}, g_{p3}; \gamma(P))$$

$$= (-ie) g T^a_{ij} (2\pi)^D \delta^{(D)}(P - p_1 - p_2 - p_3) \times$$

$$< p_1| \left(\frac{\varepsilon^\lambda_{p3}\gamma_\lambda (p_1+p_3)^\beta \gamma_\beta \gamma^\alpha}{2(p_1 p_3)} - \frac{\gamma^\alpha (p_2+p_3)^\beta \gamma_\beta \varepsilon^\lambda_{p3}\gamma_\lambda}{2(p_2 p_3)} \right) |p_2> . \tag{5.41}$$

Here we have made use of usual Feynman rules for fermions and the fact that the final gluon can be emitted either by the fermion or by the anti-fermion.

In order to show that the total amplitude up to this order is finite in the infrared limit we split the amplitude in Eq. (5.41) in two pieces by introducing two Heaviside functions which added up give unity. To make the expression simpler denote $E = \sqrt{s} - \omega(\mathbf{p_1}) - \omega(\mathbf{p_2}) - \omega(\mathbf{p_3})$. The total amplitude reads:

$$A\left(\{q_{p1}, \bar{q}_{p2}, g_{p3}\}; \gamma\right) = (-ie) g T^a_{ij} (2\pi)^{(D)} \delta^{(D-1)}(\mathbf{P} - \mathbf{p_1} - \mathbf{p_2} - \mathbf{p_3}) < p_1| \times \{$$

$$\left[-\frac{\varepsilon^\lambda_{p3}\gamma_\lambda \{p_1+p_3\}^\rho \gamma_\rho \gamma^\alpha}{2\omega(\mathbf{p_1}+\mathbf{p_3})\rho(\mathbf{p_1},\mathbf{p_3})} \delta(E + \rho(\mathbf{p_1},\mathbf{p_3})) + \frac{\varepsilon^\lambda_{p3}\gamma_\lambda (p_1+p_3)^\beta \gamma_\beta \gamma^\alpha}{2(p_1 p_3)} \delta(E) \right] \times$$

$$\Theta(\triangle - |\rho(\mathbf{p_1},\mathbf{p_3})|) + \frac{\varepsilon^\lambda_{p3}\gamma_\lambda (p_1+p_3)^\beta \gamma_\beta \gamma^\alpha}{2(p_1 p_3)} \delta(E) \Theta(|\rho(\mathbf{p_1},\mathbf{p_3})| - \triangle) +$$

$$+ \left[\frac{\gamma^\alpha \{p_2+p_3\}^\beta \gamma_\beta \varepsilon^\lambda_{p3}\gamma_\lambda}{2\omega(\mathbf{p_2}+\mathbf{p_3})\rho(\mathbf{p_2},\mathbf{p_3})} \delta(E + \rho(\mathbf{p_2},\mathbf{p_3})) - \frac{\gamma^\alpha (p_2+p_3)^\beta \gamma_\beta \varepsilon^\lambda_{p3}\gamma_\lambda}{2(p_2 p_3)} \delta(E) \right] \times$$

$$\Theta(\triangle - |\rho(\mathbf{p_2},\mathbf{p_3})|) - \frac{\gamma^\alpha (p_2+p_3)^\beta \gamma_\beta \varepsilon^\lambda_{p3}\gamma_\lambda}{2(p_2 p_3)} \delta(E) \Theta(|\rho(\mathbf{p_2},\mathbf{p_3})| - \triangle)\} \times |p_2> . \tag{5.42}$$

Suppose now that the gluon is soft and/or collinear to the quark or anti-quark. In this case only the terms multiplied by $\Theta(\triangle - |\rho(\mathbf{p_1},\mathbf{p_3})|)$ or $\Theta(\triangle - |\rho(\mathbf{p_2},\mathbf{p_3})|)$ survive. These terms could be divergent in the limit $\triangle \to 0$ but they are not.

On the other hand if $\rho(\mathbf{p_1},\mathbf{p_3}) < \triangle$ or $\rho(\mathbf{p_2},\mathbf{p_3}) < \triangle$ in the limit $\triangle \to 0$ we have that

$$\{p_1 + p_3\} \equiv p_1 + p_3 - \rho(\mathbf{p_1},\mathbf{p_3}) \to (p_1 + p_3), \tag{5.43}$$

or for the other factor

$$\text{or } \{p_2 + p_3\} \equiv p_2 + p_3 - \rho(\mathbf{p_2},\mathbf{p_3}) \to (p_2 + p_3). \tag{5.44}$$

In a similar manner we find that in this limit

$$\delta(\sqrt{s} - \omega(\mathbf{p_1}) - \omega(\mathbf{p_2}) - \omega(\mathbf{p_3}) + \rho(\mathbf{p_1},\mathbf{p_3})) \to \delta(\sqrt{s} - \omega(\mathbf{p_1}) - \omega(\mathbf{p_2}) - \omega(\mathbf{p_3}))$$

$$\delta(\sqrt{s} - \omega(\mathbf{p_1}) - \omega(\mathbf{p_2}) - \omega(\mathbf{p_3}) + \rho(\mathbf{p_2},\mathbf{p_3})) \to \delta(\sqrt{s} - \omega(\mathbf{p_1}) - \omega(\mathbf{p_2}) - \omega(\mathbf{p_3}))$$

$$\omega(\mathbf{p_1} + \mathbf{p_3})\rho(\mathbf{p_1},\mathbf{p_3}) \to (p_1 p_3) \text{ or } \omega(\mathbf{p_2} + \mathbf{p_3})\rho(\mathbf{p_2},\mathbf{p_3}) \to (p_2 p_3). \tag{5.45}$$

Once we insert these relations in the Eq. (5.42) all the soft and collinear divergences cancel.

If the gluon is neither soft nor collinear (that is $\rho(\mathbf{p_1},\mathbf{p_3}) > \triangle$and $\rho(\mathbf{p_2},\mathbf{p_3}) > \triangle$) the part that is left in the total amplitude Eq. (5.42) is finite as the denominators are non-vanishing. This is simply

$$A_{hard}\left(\{q_{p1}, \bar{q}_{p2}, g_{p3}\};\gamma\right) = (-ie)gT^a_{ij}(2\pi)^{(D)}\delta^{(D-1)}(\mathbf{P} - \mathbf{p_1} - \mathbf{p_2} - \mathbf{p_3}) < p_1| \times \{$$

$$\frac{\varepsilon^{\lambda}_{p3}\gamma_\lambda(p_1+p_3)^\beta\gamma_\beta\gamma^\alpha}{2(p_1p_3)}\delta(E)\Theta(|\rho(\mathbf{p_1},\mathbf{p_3})| - \triangle) -$$

$$-\frac{\gamma^\alpha(p_2+p_3)^\beta\gamma_\beta\varepsilon^{\lambda}_{p3}\gamma_\lambda}{2(p_2p_3)}\delta(E)\Theta(|\rho(\mathbf{p_2},\mathbf{p_3})| - \triangle)\} \times |p_2> \tag{5.46}$$

5.4 The cross-section

In the previous sections we managed to prove the finiteness of S operator matrix elements between dressed states up to NLO. This section is ment to present the result of the cross-section calculation for the amplitudes computed above. The calculation will not be shown here in detail, only the final result as obtained by Forde & Signer will be displayed in order to make a comparison with the well known result obtained with usual methods.

The total cross-section, up to second order in perturbation theory, is the sum of the cross-sections of the two precesses presented above : the one loop corection without any gluons in the final state and the process with one free gluon in the final state.

To obtain the cross-section from the one-loop correction amplitude we need to integrate over the two-particles phase space the modulus sqare of the amplitude obtained in Eq. (5.38). Formally that is

$$\sigma_{q\bar{q}} = \int d\Phi_2 |A(\{q(p_1), \bar{q}(p_2)\};\gamma)|^2$$

$$= \left[1 + C_F \frac{\alpha_s}{2\pi}\left(-\frac{1}{2} + 2log2 - \frac{3}{2}log\left(\frac{\triangle}{2}\right) - log^2\left(\frac{\triangle}{2}\right)\right)\right]^2 \sigma_0. \quad (5.47)$$

Where $\sigma_0 = \left|A^{(0)}(\{q(p_1), \bar{q}(p_2)\}; \gamma)\right|^2$ is the cross-section of the process at tree-level.

The cross-section of the process with one gluon in the final state is more difficult to compute as it involves an integration over three-particles phase space and according to Signer & Forde its value is

$$\sigma_{q\bar{q}g} = \int d\Phi_3 |A(\{q(p_1), \bar{q}(p_2), g(p_3)\}; \gamma)|^2$$

$$= C_F \frac{\alpha_S}{\pi}\left(\frac{5}{4} - 2log2 + \frac{3}{2}log\left(\frac{\triangle}{2}\right) + log^2\left(\frac{\triangle}{2}\right)\right)\sigma_0. \quad (5.48)$$

If we add up the two cross-sections it is remarkable that up to first order in α_S any dependence on $\triangle$ vanishes and the result is finite

$$\sigma = \left(1 + \frac{3\alpha_S}{4\pi} C_F + O(\alpha_S^2)\right)\sigma_0$$

What is also remarkable is that this result coincides with the cross-section of the photon-quark-antiquark process computed in the usual way at NLO.

Summary

For a well known process in QCD, the S operator matrix elements between dressed states are shown to be finite. Using a perturbative approach and a certain factorization of the matrix elements we show that up to NLO the amplitudes do not depend on the vanishing regulator so they are free of infrared singularities. The total cross-section of the process, calculated up to NLO, gives the well known result.

6 Exponentiation

6.1 Introduction

Infrared divergences may occur in both real and virtual corrections in any scattering process. The higher the correction order are, the more complicated the divergences. Nevertheless it has been shown that, when adding up the corrections to all orders in perturbation theory, these divergences exponentiate [17,28]. For the virtual corrections such an exponential comes out at the level of scattering amplitudes while divergences coming from soft real corrections exponentiate when the cross-section is computed.

The aim of this chapter is to present a different approach to exponentiation starting from the theory of dressd states introduced in Chapter 4. We use the fact that S matrix elements between dressed states are finite to derive exponential of infrared divergences as it would show up in the S matrix element between usual Fock states. Once more the dressed states for QED and QCD are presented in detail, the mechanism by which the divergences cancel is discussed and eikonal exponentiation for two simple processes is derived. The approach seem rather simple since it reduces to manipulations of operators in the exponent of the Moller operators and involves no expansion of the exponential expressions.

The structure of this chapter is as follows. We shall begin with some mathematical tools which are used throughout the chapter. We then continue with a discussion about the QED dressed states and a derivation of QED exponentiation of virtual corrections. We end with a similar discussion for QCD.

6.2 Mathematical tools

Though the mathematics of this chapter are not extremely complicated, it is good to introduce a few tools and tricks that will turn out to be very useful in the calculations.

Since the discussion is about dressed states the starting point is their definition, which according to Eq. (4.20) is given by

$$|\{\psi;\pm\}>=\Omega_{H_I}^{(\pm)\dagger}|\psi>, \tag{6.1}$$

with the time-independent Moller operators defined in terms of the asymptotic interaction Hamiltonian

$$\Omega_{H_I}^{(\pm)}=T\,\exp\left(-i\int_{\mp\infty}^{0}dt_1H_I^{as}(t_1)\right), \tag{6.2}$$

where T stands for time-ordering. In the calculations to come it is handy to work with usual exponentials rather than time-ordered ones. The relation between the two is provided by a theorem due to Magnus[24]. This states that

$$T\,\exp(-i\int_{t_0}^{t}dt_1H_I^{as}(t_1)) = \exp\left(A\right), \tag{6.3}$$

where

$$A=-i\int_{t_0}^{t}dt_1H_I^{as}(t_1)+\frac{1}{2}(-i)^2\int_{t_0}^{t}dt_1\int_{t_0}^{t_1}dt_2\left[H_I^{as}(t_1),H_I^{as}(t_2)\right]+$$

$$+\frac{1}{4}(-i)^3\int_{t_0}^{t}dt_1\int_{t_0}^{t_1}dt_2\int_{t_0}^{t_2}dt_3\,[H_I^{as}(t_1),[H_I^{as}(t_2),H_I^{as}(t_3)]]$$

$$+\frac{1}{12}(-i)^3\int_{t_0}^{t}dt_1\int_{t_0}^{t_1}dt_2\int_{t_0}^{t_1}dt_3\,[[H_I^{as}(t_1),H_I^{as}(t_2)]\,,H_I^{as}(t_3)]+... \tag{6.4}$$

Note that the coefficients of each commutator belong to the set of Bernoulli numbers.

In the same context of dressed states it will be shown that for QED these states have the same form as the coherent states of Chapter 3 so they share the same properties. However, this is not the case for QCD. Due to the non-abelian nature of this theory the dressed states are no longer coherent states. Though the structure of the Hamiltonian is more complicated than the next example we can prove this statement as follows.

For simplicity we consider a gluon dressed state of the form $\exp\left(\int d\tilde{k}T_i a_i^\dagger(k)\right)|0>$. If this would be an eigenstate of $a_j(q)$ then

$$a_j(q)\exp\left(\int d\tilde{k}T_ia_i^\dagger(k)\right)|0>=T_j\exp\left(\int d\tilde{k}T_ia_i^\dagger(k)\right)|0> \tag{6.5}$$

We know that any two annihilation operators commute and also commute with the colour matrices T_i. Then

$$a_j(q_1)a_k(q_2)\ \exp\left(\int d\tilde{k}T_ia_i^\dagger(k)\right)|0>=a_k(q_2)a_j(q_1)\ \exp\left(\int d\tilde{k}T_ia_i^\dagger(k)\right)|0>. \tag{6.6}$$

Then Eq. (6.5) leads to

$$T_kT_j\exp\left(\int d\tilde{k}T_ia_i^\dagger(k)\right)|0>=T_jT_k\exp\left(\int d\tilde{k}T_ia_i^\dagger(k)\right)|0>, \tag{6.7}$$

which is not true because the colour matrices do not commute : $[T_i,T_j]=f_{ij}^kT_k$, where the coeficients f_{ij}^k are the structure constants of the color group.

To see this another way, let us expand the coherent state in a power series of creation operators. Once again we find that it is not an eigenstate of the annihilation operator since

$$a_j(q)\exp\left(\int d\tilde{k}T_ia_i^\dagger(k)\right)|0>=\frac{\partial}{\partial a_j^\dagger(q)}\exp\left(\int d\tilde{k}T_ia_i^\dagger(k)\right)|0>$$

$$=\frac{\partial}{\partial a_j^\dagger(q)}\left(1+\int d\tilde{k}t_ia_i^\dagger(k)+\frac{1}{2}\int d\tilde{k_1}d\tilde{k_2}T_ia_i^\dagger(k_1)T_la_l^\dagger(k_2)+...\right)|0>$$

$$=(T_j+\frac{1}{2}\int d\tilde{k}(T_jT_i+T_iT_j)a_i^\dagger(k)+...)|0>\neq T_j\exp\left(\int d\tilde{k}T_ia_i^\dagger(k)\right)|0>. \tag{6.8}$$

The last inequality holds provided that $\frac{1}{2}(T_jT_i+T_iT_j)=T_jT_i+\frac{1}{2}f_{ij}^kT_k\neq T_jT_i$.

The full QCD dressed state shows a more complex structure than this simple case. However, the same conclusion holds and they lose the property of being states of minimum uncertainty or in other words closest to classical states. As a matter it makes no sense to talk about states that are closest to classical since there is no classical limit for QCD.

Another tool we shall use is the Baker-Campbell-Hausdorf formula which states that for non-commuting A_2 and A_1

$$e^{A_2}e^{A_1} = \exp\left(A_1 + A_2 + \frac{1}{2}[A_2, A_1] + \frac{1}{12}[A_2, [A_2, A_1]] + \frac{1}{12}[A_1, [A_1, A_2]] - \frac{1}{24}[A_1, [A_2, [A_2, A_1]]] + ...\right). \tag{6.9}$$

Besides this formula we will also use Schur's Lemma stating that

$$e^X Y e^{-X} = Y + [X, Y] + \frac{1}{2!}[X, [X, Y]] + \frac{1}{3!}[X, [X, [X, Y]]] + ... \equiv Z. \tag{6.10}$$

This can be extended to

$$e^X e^Y e^{-X} = e^X(1 + Y + \frac{1}{2}Ye^{-X}e^X Y + \frac{1}{3!}Ye^{-X}e^X Ye^{-X}e^X Y + ...)e^{-X} = e^Z. \tag{6.11}$$

Most calculations of this chapter involve operators and their commutators and in order to make the equations simpler we shall use different tricks. Take for example the following commutator which involves both creation and annihilation operators and color matrices.

$$\left[T_i a_i, T_j a_j^\dagger\right] = [T_i, T_j]\, a_j^\dagger a_i + T_i T_j \left[a_i, a_j^\dagger\right] \tag{6.12}$$

The first part of the rhs is a normal ordered expression and the second part is a c-number. In the same way it can be proved that the commutator of two normal ordered expressions will be normal ordered.

Regarding normal-ordered expressions it can be shown that the exponential of a sum of normal ordered terms that might contain color matrices evaluated between states that contain no gauge bosons (here we refer mainly at bosonic operators) will act as identity operator. For example

$$< 0|\exp\left(T_i T_j a_i^\dagger a_j + T_i a_i + [T_i, T_j]\, T_k a_i^\dagger a_k^\dagger a_j\right)|0>=$$

$$=< 0|\left(1 + T_i T_j a_i^\dagger a_j + T_i a_i + [T_i, T_j]\, T_k a_i^\dagger a_k^\dagger a_j + ...\right)|0>=< 0|0> . \tag{6.13}$$

This is due to the normal-ordered character of the exponent and based on the assumption that color matrices commute with the field operators. In any higher order term of the exponential series there is at least one annihilation operator acting on the right state or one creation operator acting on the left state so that such terms do not contribute.

One of the aims of this chapter is to approach the exponentiation of infrared divergences starting from the formalism introduced in the past chapters. To that end it turns out that we need to use the connection between the commutation relation of creation and annihilation operators and the gauge bosons' propagator. We shall derive this connection for QCD.

Suppose we work in the Feynman gauge and we expand the gauge field in terms of its Fourier modes

$$A_\mu^a(x) = \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2\omega(\mathbf{k})}} \left(a_\mu^a(k)e^{-ikx} + a_\mu^{a\dagger}(k)e^{ikx}\right), \tag{6.14}$$

and take the commutation relation between the creation and annihilation operators to be

$$\left[a_\mu^a(k), a_\nu^{b\dagger}(k')\right] = -g_{\mu\nu}(2\pi)^3\delta^{ab}\delta^{(3)}(\mathbf{k} - \mathbf{k}')$$

$$\left[a_\mu^a(k), a_\nu^b(k^{'})\right] = \left[a_\mu^{a\dagger}(k), a_\nu^{b\dagger}(k^{'})\right] = 0. \tag{6.15}$$

Remark that the polarization vectors of the gauge bosons are included in the definition of the field operators.

Then the gluon propagator for $x^0 > y^0$ is defined as

$$D_{\mu\nu}^{ab}(x-y) =< 0|TA_\mu^a(x)A_\nu^b(y)|0 >= ...$$

$$= \int d\tilde{k}_1 d\tilde{k}_2 [a_\mu^a(k_1), a_\nu^{b\dagger}(k_2)] e^{-ik_1x+ik_2y} = -g_{\mu\nu}\delta^{ab} \int \frac{d^3k}{(2\pi)^3}\frac{1}{2\omega(\mathbf{k})} e^{-ik(x-y)} =$$

$$= \int d^4k(-ig_{\mu\nu})\delta^{ab}\frac{1}{k^2+i\epsilon}e^{-ik(x-y)} \equiv \int d^4k(-g_{\mu\nu})\delta^{ab}\tilde{D}(k)e^{-ik(x-y)}. \tag{6.16}$$

For QED the same derivation holds, except that there are no color indices a, b,etc.

6.3 QED dressed states

We are now ready to start a discussion about the QED dressed states. Their definition is given in Eq. (4.20) and they are generated by the QED interaction Hamiltonian in the interaction picture. This is defined as

$$H_I = (-e)\int d\mathbf{x} : \bar{\Psi}(x)\gamma^\mu\Psi(x) : A_\mu(x). \tag{6.17}$$

We ignore for the moment the fact that only the soft part of this interaction Hamiltonian contributes. Whenever this aspect becomes important we shall make use of it.

In this picture the both the fermionic and the gauge fields are taken as free fields defined in the usual way

$$\Psi(x) = \int d^3k \frac{1}{(2\pi)^{\frac{3}{2}}\left(2\omega(\mathbf{k})\right)^{\frac{1}{2}}}\left(u(\mathbf{k})b(\mathbf{k})e^{-ikx} + v(\mathbf{k})d^\dagger(\mathbf{k})e^{ikx}\right),$$

$$\bar{\Psi}(x) = \int d^3k \frac{1}{(2\pi)^{\frac{3}{2}}\left(2\omega(\mathbf{k})\right)^{\frac{1}{2}}}\left(\bar{u}(\mathbf{k})b^\dagger(\mathbf{k})e^{ikx} + \bar{v}(r,\mathbf{k})d(\mathbf{k})e^{-ikx}\right),$$

$$A_\mu(x) = \int d^3k \frac{1}{(2\pi)^{\frac{3}{2}}\left(2\omega(\mathbf{k})\right)^{\frac{1}{2}}}\left(a_\mu(\mathbf{k})e^{-ikx} + a_\mu^\dagger(\mathbf{k})e^{ikx}\right). \tag{6.18}$$

A sum over helicities in the matter field decomposition is understood and the polarization vectors of the photon are included in the definition of the field operators via

$a_\mu(\mathbf{k}) \equiv \varepsilon_\mu(\mathbf{k})a(\mathbf{k})$ and $a_\mu^\dagger(\mathbf{k}) \equiv \varepsilon_\mu^*(\mathbf{k})a^\dagger(\mathbf{k})$.

The fermionic creation and anihilation operators $b^\dagger, b, d^\dagger, d$ come with the usual anti-commutation relations while the bosons satisfy Eq. (5.7).

Let us return for a moment to Eq. (6.17). Once the Fourier expansions of Eq. (6.18) are plugged in eight different terms appear. Each term has a time dependance $e^{i\omega t}$. In the asymptotic region $t \to \pm\infty$ and so only terms that have $\omega \to 0$ contribute. In this way the eight terms we obtain can be divided in two groups. One group contains the products of both electron and positron creation/anihilation operators. These terms are not singular for vanishing photon momentum because ω in the exponential is never vanishing. Suppose the electron has momentum $\mathbf{p}$ and the incoming/outgoing photon has momentum $\pm\mathbf{k}$. Then the positron has a momentum $\mathbf{p}\pm\mathbf{k}$ due to the momentum conservation at the vertex. In this way the energy in the time exponential will be $\omega = (\omega(\mathbf{p}) + \omega(\mathbf{p}\pm\mathbf{k}) \pm \omega(\mathbf{k}))$ which is never vanishing for vanishing $\mathbf{k}$. Still there is a possibility for ω to vanish. Since those vertices represent pair-creation or pair-annihilation processes, it might be the case that the photon is collinear to one of the fermions leading to a vanishing ω. Anyway we are not concerned about collinear singularities in this chapter so we shall simply neglect them. In conclusion between the singular terms we shall not have any particle-antiparticle 'mixing' terms.

It can be shown that in the asymptotic limit the interaction Hamiltonian has the same limit as

$$H_I^{as}(t) = -e\int d\mathbf{x} A_\mu(t,\mathbf{x}) J^\mu_{as}(t,\mathbf{x}), \tag{6.19}$$

with

$$J^\mu_{as}(t,\mathbf{x}) = \int d\mathbf{p}\rho(\mathbf{p})\frac{p^\mu}{\omega(\mathbf{p})}\delta^{(3)}(\mathbf{x}-\mathbf{p}t/\omega(\mathbf{p})), \tag{6.20}$$

and

$$\rho(\mathbf{p}) \equiv b^\dagger(\mathbf{p})b(\mathbf{p}) - d^\dagger(\mathbf{p})d(\mathbf{p}). \tag{6.21}$$

the charge density.

Eq. (6.19) holds provided that we work in the eikonal approximation in which the fermions move on straight lines with a well defined momentum and any photon that is emitted or absorbed is soft. This is the origin of the delta function in the asymptotic current. The absence of spinors in the current is explaned by the following relation $\sum_{spins}\bar{u}(p)\gamma^\mu u(p) = p^\mu$ if the fermions are massless and on-shell.

A simpler form of the asymptotic Hamiltonian is given by

$$H_I^{as}(t) = -e\int d\tilde{k}d\mathbf{p}\frac{p^\mu}{\omega(p)}\rho(p)\left(a_\mu(k)e^{-ipkt/\omega(p)} + a^\dagger_\mu(k)e^{ipkt/\omega(p)}\right) \tag{6.22}$$

Let us return to the dressed states. We know that the Moller operators involving the asymptotic interaction Hamiltonian are time-ordered exponentials which according to a theorem by Magnus [24] can be written as ordinary exponentials just like in Eq. (6.3). The important point in QED is that the series of commutators entering that exponential stops after the first commutator. According to the definition in Eq. (6.19) we find that

$$[H_I(t_1), H_I(t_2)] = -2ie^2\int d\mathbf{p}d\mathbf{q}d\mathbf{k}\frac{1}{(2\pi)^3\,\omega(\mathbf{k})}\frac{p^\mu q^\nu}{\omega(\mathbf{p})\omega(\mathbf{q})}\rho(\mathbf{p})\rho(\mathbf{q})g_{\mu\nu}\sin\left(\frac{kqt_2}{\omega(\mathbf{q})} - \frac{kpt_1}{\omega(\mathbf{p})}\right), \tag{6.23}$$

which is a c-number since the charge-density operators commute. Thus for any 3 time moments $t_{1,2,3}$ we would have

$$[H_I(t_1), [H_I(t_2), H_I(t_3)]] = 0. \tag{6.24}$$

In conclusion, for QED, Eq. (6.3) becomes

$$T\exp(i\int_{-\infty}^{0} dt_1 H_I(t_1)) = \exp\left(i\int_{-\infty}^{0} dt_1 H_I(t_1) + \frac{1}{2}(i)^2\int_{-\infty}^{0} dt_1 \int_{-\infty}^{t_1} dt_2\, [H_I(t_1), H_I(t_2)]\right). \tag{6.25}$$

Since the second term in the exponential commutes with the first term we can write all this as

$$T\exp\left(i\int_{-\infty}^{0} dt_1 H_I(t_1)\right) = \exp(R)\exp(i\Phi), \tag{6.26}$$

where

$$R = i\int_{-\infty}^{0} dt_1 H_I(t_1),$$

$$\Phi = \frac{1}{2}i\int_{-\infty}^{0} dt_1\int_{-\infty}^{t_1} dt_2 [H_I(t_1), H_I(t_2)] \tag{6.27}$$

The Moller operator for the dressed state in the asymptotic future is to be treated in the same manner.

Let us discuss about this dressing a bit further. According to Eq. (6.26) the dressing consists of a cloud of soft photons that will cancel the divergences in the S matrix and a phase which comes from the interaction of the cloud with the particles to dress. The photonic cloud forms a coherent state since the operator R has a form equivalent to the coherent state operator of Eq. (3.13). This will cancel all the divergences of the S matrix, which itself is made of two coherent state operators (in QED).

On the other hand according to Eq. (6.23) the phase is given by

$$\Phi = e^2\int_{-\infty}^{0} dt_1\int_{-\infty}^{t_1} dt_2\int d\mathbf{p}d\mathbf{q}d\mathbf{k}\frac{1}{(2\pi)^3\,\omega(\mathbf{k})}\frac{p^\mu q^\nu}{\omega(\mathbf{p})\omega(\mathbf{q})}\rho(\mathbf{p})\rho(\mathbf{q})g_{\mu\nu}\sin\left(\frac{kqt_2}{\omega(\mathbf{q})} - \frac{kpt_1}{\omega(\mathbf{p})}\right) \tag{6.28}$$

This is the so called 'Coulomb' phase. In the infinite time limit this is not finite. According to [11] if the evolution in the asymptotic region takes place in the time interval $[t_0, t]$ and the fermions have mass m then this phase is equal to

$$\Phi = e^2\frac{1}{8\pi}\int d\mathbf{p}d\mathbf{q}\ :\rho(\mathbf{p})\rho(\mathbf{q}):\ \frac{pq}{\sqrt{(pq)^2 - m^4}}\,\text{sign}\ t\log\frac{|t|}{t_0}. \tag{6.29}$$

In our application $t_0 = 0$ and $t \to \pm\infty$ so the phase blows up. As a matter of fact similar phases appear in the S matrix and they cancel against these phases. In the usual theory these phases make the S matrix ill-defined but they cancel at the level of cross-section the cross-section.

6.4 QED exponentiation

Let us consider the simplest application possible for the above discussion and show that exponentiation of infrared divergences can be obtained starting from the finite scattering amplitudes defined in chapter 4. Consider an electron coupled to an electromagnetic field. Suppose the electron has in the beginning a momentum $\mathbf{p_1}$ and emits one hard photon at the time moment $t = 0$ in such a way that it changes its momentum to $\mathbf{p_2}$. Suppose no other photons are present in the initial or final state of this process so only virtual correction to the electron-photon vertex should be taken into account.

We know that the S matrix element for this process is not finite since any virtual photon that is exchanged can be soft giving rise to divergences. The electron is on its mass shell and thus any soft emission (or absorption) adds a divergence as for every emission (or absorption) one more electron propagator has to be included in the amplitude. This propagator is divergent if the electron is on its mass shell and the photon is soft.

According to Chapter 4 a finite S matrix amplitude can be defined and for the present example this should be

$$< \{p_2\}|S|\{p_1\} > \equiv < p_2|\Omega_{H_I}^{(-)} S \Omega_{H_I}^{(+)\dagger}|p_1 > . \tag{6.30}$$

Based on the discussion made in the last section and on the fact that the electron is on shell in the asymptotic region we have that

$$< p_2|\Omega_{H_I}^{(-)} = < p_2|e^{R_2} \qquad \text{and} \qquad \Omega_{H_I}^{(+)\dagger}|p_1 > = e^{R_1}|p_1 >, \tag{6.31}$$

where

$$R_2 = -ie \int_0^\infty dt_2 \frac{p_2^\mu}{\omega(\mathbf{p}_2)} A_\mu \left(\frac{p_2 t_2}{\omega(\mathbf{p}_2)} \right),$$

$$R_1 = -ie \int_{-\infty}^0 dt_1 \frac{p_1^\mu}{\omega(\mathbf{p}_1)} A_\mu \left(\frac{p_1 t_1}{\omega(\mathbf{p}_1)} \right). \tag{6.32}$$

The corresponding Coulomb phases vanish because they include the following expression $pq\ :\rho(p)\rho(q):$. Since the incoming and outgoing states are eigenstates of the charge operator $\rho$and the fermions are on the mass-shell these expressions have a vanishing contribution.

If we write the vector field as in Eq. (6.18) we see that the two dressing operators have the form similar to the coherent state operators of Eq. (3.4)

Then Eq. (6.30) becomes

$$< \{p_2\}|S|\{p_1\} > \equiv < p_2|e^{R_2} S e^{R_1}|p_1 > . \tag{6.33}$$

Since this matrix element is finite the would-be infrared divergences in S have to be canceled by the action of the dressing on the two states. As we have seen the dressing has the form of a coherent state. On the other hand there exist a hard-soft factorization for the Moller operators entering the S matrix since the interaction Hamiltonian there contain both hard and soft interactions and the two commute. This is a consequence of the fact that any two photonic operators with different momenta commute. Thus the S matrix can be written as a product of exponentials which have the form of coherent states. The soft part will then cancel the two coherent states coming from the dressing. We know that the only way to annihilate a coherent state created by the operator $D(\eta)$ is by acting with $D(-\eta)$ on it. This is exactly what happens since the coherent states in the S matrix have a different sign in the exponent compared to the coherent states of the dressing. The claim which comes as a conclusion to all these remarks is that the infrared structure of a usual matrix element $< p_2|S|p_1 >$ coincides with the infrared structure of an overlap of dressed states $< \{p_2\}|\{p_1\} >$. In what follows we will compute the latter and show that it gives the expected result that we would find by adding up all the virtual corrections to the electron-photon vertex.

Let us proceed and compute $<\{p_2\}|\{p_1\}>$. According to the discussion above that reduces to calculating $<p_2|e^{R_2}e^{R_1}|p_1>$. For more precise calculations we shall include also a hard interaction that commutes with both of the coherent states such and stands generically for the hard photon emission vertex. Let us call that V.

Since the electron is on shell so $p_1^2 = p_2^2 = 0$ we can further simplify the expression of the two coherent states operators. Take as an example $e^{R_1}|p_1>$. We find that

$$e^{R_1}|p_1>= \exp\left(-ie\int_{-\infty}^{0} dt_1 \int d\tilde{k} \frac{p_1^\mu}{\omega(\mathbf{p}_1)}\left(a_\mu(k)e^{-ik\frac{p_1 t}{\omega(p_1)}} + a_\mu^\dagger(k)e^{ik\frac{p_1 t}{\omega(p_1)}}\right)\right)|p_1>$$

$$= \exp\left(-ie\int_{-\infty}^{0} dt_1 \int d\tilde{k} \frac{p_1^\mu}{\omega(\mathbf{p}_1)} a_\mu^\dagger(k)e^{ik\frac{p_1 t}{\omega(p_1)}}\right)|p_1>\equiv e^{A_1}|p_1> . \tag{6.34}$$

That makes sence because the commutator of a_μ and $a_\nu^\dagger$ would give a $g_{\mu\nu}$ which would contract the coefficient in front of it which is $p_1^\mu p_1^\nu$. Then we are left to evaluate $<p_2|e^{A_2}Ve^{A_1}|p_1>$. Using BCH formula, the fact that $e^{A_2}|p_1>=|p_1>$ and also $<p_2|e^{A_1}=<p_2|$ and the remark that the commutator of A_1 and A_2 is a c-number we find that

$$<p_2|e^{A_2}Ve^{A_1}|p_1>= e^{[A_2,A_1]}<p_2|V|p_1> . \tag{6.35}$$

Finally

$$[A_2, A_1] = -\int_{-\infty}^{0} dt_1 \int_{0}^{\infty} dt_2 \frac{p_1^\mu}{\omega(\mathbf{p}_1)}\frac{p_2^\nu}{\omega(\mathbf{p}_2)}\int d\tilde{k_1}\int d\tilde{k_2}\times$$

$$\times\left[a_\nu(k_2), a_\mu^\dagger(k_1)\right] e^{-ik_2\frac{p_2 t_2}{\omega(p_2)}+ik_1\frac{p_1 t_1}{\omega(p_1)}}$$

$$= -\int_{-\infty}^{0} dt_1 \int_{0}^{\infty} dt_2 \frac{p_1^\mu}{\omega(\mathbf{p}_1)}\frac{p_2^\nu}{\omega(\mathbf{p}_2)}\int d^4k(-ig_{\mu\nu})\frac{1}{k^2+i\epsilon}e^{-ik(\frac{p_2 t_2}{\omega(p_2)}-\frac{p_1 t_1}{\omega(p_1)})}$$

$$= \int d^4k(-ig_{\mu\nu})\frac{1}{k^2+i\epsilon}\frac{1}{p_1 k - i\epsilon}\frac{1}{p_2 k - i\epsilon} \tag{6.36}$$

If we are to represent the result of Eq. (6.36) in terms of a Feynman diagram that would look like Picture 6.1 and it corresponds to the leading order contribution to QED exponentiation.

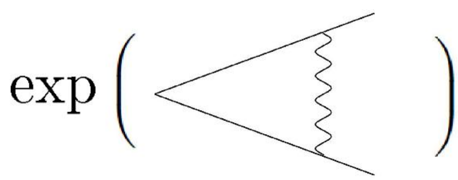

Fig.6.1 Leading order in QED exponentiation

6.5 QCD dressed states

The dressed states of QCD have a much more complicated structure than those of QED. This is due to, firstly, the non-abelian nature of the theory and secondly the non-trivial interactions between the gauge bosons which carry a color charge themselves. In the literature many atempts had been made to describe these states. The first attempts were straight forward generalizations of QED calculations [12,13]. These attempts ignore the gluon correlations providing results at lower orders in perturbation theory. Later on [22,23] gluon correlations were included in the formalism and the structure of the dressing operators was studied order by order in perturbation theory. The advanced techniques introduced by Catani and its collaborators allowed them to study different aspects of the infrared divergences present in the S matrix. Instead of using the finite matrix elements between dressed states they write the usual S matrix in terms of the asymptotic S matrix, which is finite, and the soft Moller operators, which contain all the soft and collinear infrared divergences. This procedure allows one to reproduce easier certain important physical predictions in QCD. For example, it can be proved that the Bloch-Nordsieck theorem does not hold when subleading singularities are included. The QCD version of this theorem states that the infrared divergences cancel in the cross-section if a summation upon both final and initial degenerate states is done. This is rather hard to accomplish in practice.

The asymptotic dressed states are defined as usual (see Eq. (6.1)) and the asymptotic interaction Hamiltonian is a sum of three different terms: the quark (or anti-quark) -gluon vertex, the three-gluon vertex and the four-gluon vertex. The last two terms constitute the major difference between this Hamiltonian and the one in the abelian theory. Thus $H_I^{as} = H_{qg} + H_{3g} + H_{4g}$. Also gauge fixing terms and interactions for ghosts could be added but they are not of much relevance for what follows.The explicit expressions of these Hamiltonians are given by

$$H_{qg} = -g \int d\tilde{k} d\mathbf{p} \frac{p^\mu}{\omega(\mathbf{p})} \rho_a(p) A_\mu^a \left(\frac{pt}{\omega(\mathbf{p})} \right), \tag{6.37}$$

where $A_\mu^a(x)$ is defined in Eq. (6.14) and the color charge density is

$$\rho_a(p) = b^\dagger(p) T_a b(p) - d^\dagger(p) T_a^* d(p). \tag{6.38}$$

The three-gluon interaction is of the form

$$H_{3g}(t) = -ig \frac{f_{abc}}{2} (2\pi)^3 \int d\tilde{k}_1 d\tilde{k}_2 d\tilde{k}_3 \times$$

$$\times \Big\{ a_\mu^{a\dagger}(k_1) a_\nu^b(k_2) a_\lambda^c(k_3) \delta^{(3)}\left(k_1 - k_2 - k_3\right) e^{-i(\omega(\mathbf{k_1}) - \omega(\mathbf{k_2}) - \omega(\mathbf{k_3}))t} V_{\mu\nu\lambda}(k_1, -k_2, -k_3) -$$

$$-\frac{1}{3} a_\mu^a(k_1) a_\nu^b(k_2) a_\lambda^c(k_3) \delta^{(3)}\left(k_1 + k_2 + k_3\right) e^{-i(\omega(\mathbf{k_1}) + \omega(\mathbf{k_2}) + \omega(\mathbf{k_3}))t} V_{\mu\nu\lambda}(k_1, k_2, k_3) + h.c. \Big\}, \tag{6.39}$$

where the 3-gluon vertex function is

$$V_{\mu\nu\lambda}(k_1, k_2, k_3) = g_{\mu\nu}(k_1 - k_2)_\lambda + g_{\nu\lambda}(k_2 - k_3)_\mu + g_{\lambda\mu}(k_3 - k_1)_\nu. \tag{6.40}$$

Finally the 4 gluon interaction

$$H_{4g}(t) = -g^2 \frac{1}{4!} (2\pi)^3 V_{\mu\nu\lambda\eta}^{abcd} \int d\tilde{k}_1 d\tilde{k}_2 d\tilde{k}_3 d\tilde{k}_4 \times$$

$$\{6\, a_\mu^{a\dagger}(k_1) a_\nu^{b\dagger}(k_2) a_\lambda^c(k_3) a_\eta^d(k_4) \delta^{(3)}\left(k_1+k_2-k_3-k_4\right) e^{-i(\omega(\mathbf{k_1})+\omega(\mathbf{k_2})-\omega(\mathbf{k_3})-\omega(\mathbf{k_4}))t} +$$

$$+4\, a_\mu^{a\dagger}(k_1) a_\nu^{b}(k_2) a_\lambda^c(k_3) a_\eta^d(k_4) \delta^{(3)}\left(k_1-k_2-k_3-k_4\right) e^{-i(\omega(\mathbf{k_1})-\omega(\mathbf{k_2})-\omega(\mathbf{k_3})-\omega(\mathbf{k_4}))t} +$$

$$+a_\mu^{a}(k_1) a_\nu^{b}(k_2) a_\lambda^c(k_3) a_\eta^d(k_4) \delta^{(3)}\left(k_1+k_2+k_3+k_4\right) e^{-i(\omega(\mathbf{k_1})+\omega(\mathbf{k_2})-+\omega(\mathbf{k_3})+\omega(\mathbf{k_4}))t} + h.c.\}, \quad (6.41)$$

with the 4 gluon vertex

$$V_{\mu\nu\lambda\eta}^{abcd} = f_{abe} f_{cde}\left(g_{\mu\lambda} g_{\nu\eta} - g_{\mu\eta} g_{\nu\lambda}\right) + f_{ace} f_{bde}\left(g_{\mu\nu} g_{\lambda\eta} - g_{\mu\eta} g_{\nu\lambda}\right) + f_{ade} f_{cbe}\left(g_{\mu\lambda} g_{\nu\eta} - g_{\mu\nu} g_{\lambda\eta}\right). \quad (6.42)$$

Given the Hamiltonian we could proceed as in Eq. (6.3) and compute different terms that enter the exponential. Then all the complex terms would contribute to a phase which is the non-abelian equivalent of the Coulomb phase. Given that the exponent as given in Eq. (6.4) is not a closed expression this phase would have a very complicated structure and it is expected to be infinite. Since the S matrix element between dressed states is finite this phase should cancel the similar phase coming from the S matrix. However we shall not enter the intricacies of this calculation.

6.6 QCD exponentiation

It can be shown [25] that the infrared divergences exponentiate in QCD at the level of the amplitudes as well as well as in cross-sections if a summation is done on the right set of diagrams. The structure of the exponent is much more complicated than in QED and the diagrams it contains belong to the class of diagrams called webs. They have the special property that from a higher-order web no lower order web can be obtained by cutting the eikonal lines exactly once.

In this section we shall approach exponentiation in a similar way we did for QED. Starting from a very simple example of a quark emitting a hard photon and changing momenta from the initial p_1 to a final p_2 we find the leading and next-to-leading order webs that enter the exponential. To that end we need not include in the interaction Hamiltonian the three and four gluons vertices but keep only the quark-gluon part of the Hamiltonian given in Eq. (6.37). Moreover only virtual corrections will be taken into account since we consider the initial and final state to include the quark alone.

Consider then the scattering amplitude $< \{p_2\}|S|\{p_1\} >$ which is expected to be finite according to chapters 4 and 5. The S matrix includes as a hard part the quark-photon vertex which we shall denote simply V. The claim is that such an amplitude is finite because the potential infrared divergences in the S matrix are cancelled by divergences of the same type from the dressing factors. Similar to QED we expect the overlap of initial and final dressed states to lead to an exponentiation of infrared divergences of the same form the S matrix element between usual Fock states would display.

In what follows we shall derive a result for exponentiation starting from these dressed states. If we evaluate $< \{p_2\}|V|\{p_1\} >$ we find an exponential result which has no closed form but up to the desired order in the coupling constant it contains the wanted webs.

The starting point is again the definition of dressed states which for the application at hand read

$$< \{p_2\}| = < p_2| T \exp\left[-ig\int_0^\infty dt\, \frac{p_2^\mu}{\omega(p_2)} T_a A_\mu^a\left(\frac{p_2 t}{\omega(p_2)}\right)\right]$$

$$|\{p_1\}>= T\exp\left[-ig\int_{-\infty}^{0} dt\,\frac{p_1^\mu}{\omega(p_1)}T_a A_\mu^a\left(\frac{p_1 t}{\omega(p_1)}\right)\right]|p_1> \tag{6.43}$$

The two time-ordered exponential can be transformed into ordinary exponentials using Magnus theorem displayed in Eq. (6.3) with the Hamiltonian replaced by $H_I(p,t) = -g\,\frac{p^\mu}{\omega(p)}T_a A_\mu^a\left(\frac{pt}{\omega(p)}\right)$. Note that the series of commutators in Eq. (6.4) would never end in the sense of reaching a term which is a c-number since neither the color matrices nor the bosonic operators do not commute (color matrices commute with bosonic operators however).

Up to the order in g we are interested in we can consider only the first two terms of the series in Eq. (6.4). Moreover the commutator of two such Hamiltonians reduce to the commutator of color matrices since the momenta in the final and initial states are on-shell. That leads to

$$T\exp\left[i\int_{-\infty}^{0} dt\,H_I(p,t)\right] = \exp\left[-ig\int_{-\infty}^{0} dt\,\frac{p^\mu}{\omega(p)}T_a\int d\tilde{k}\left(a_\mu^a(k)e^{-ikpt/\omega(p)} + a_\mu^{a\dagger}(k)e^{ikpt/\omega(p)}\right)\right.$$

$$-\frac{1}{2}g^2\int_{-\infty}^{0} dt_1\int_{-\infty}^{t_1} dt_2\,\frac{p^\mu p^\nu}{(\omega(p))^2}[T_a,T_b]\int d\tilde{k}_1 d\tilde{k}_2\times$$

$$\left.\left(a_\mu^a(k_1)e^{-ik_1pt_1/\omega(p)} + a_\mu^{a\dagger}(k_1)e^{ik_1pt_1/\omega(p)}\right)\left(a_\nu^b(k_2)e^{-ik_2pt_2/\omega(p)} + a_\nu^{b\dagger}(k_2)e^{ik_2pt_2/\omega(p)}\right)\right]. \tag{6.44}$$

Note that any product of bosonic operators in the second term of Eq. (6.43) can be written in a normal-ordered way since any commutator of two operators would either vanish or give a $g_{\mu\nu}$ which would contract the on-shell momenta coefficients giving a vanishing contribution. In this way the whole exponent of Eq. (6.44) can be normal-ordered.

The claim is that if some expression E is normal ordered than its exponential can be written as $e^E = e^B e^C e^A$ where A contains only creation (or only annihilation) operators and B has the rest of the normal ordered expression of the exponent E. According to BCH formula C would contain all kind of commutators between A and B. As explained above any commutator of creation and annihilation operators in this exponent would have vanishing coefficients. Thus C will contain only commutators of color matrices and the operators can be again normal ordered. Moreover any normal-ordered exponential acting on the vacuum leaves it unchanged. Then the exponentials of the dressing factors will take a simpler form such that $<\{p_2\}| = <p_2|e^{A_2}$ and $|\{p_1\}> = e^{A_1}|p_1>$. Here

$$A_1 = -ig\int_{-\infty}^{0} ds\,\frac{p_1^\nu}{\omega(p_1)}T_b\int d\tilde{q}\,a_\nu^{b\dagger}(q)e^{iqp_1s/\omega(p_1)} -$$

$$-\frac{1}{2}g^2\int_{-\infty}^{0} ds_1\int_{-\infty}^{s_1} ds_2\,\frac{p_1^{\nu_1}p_1^{\nu_2}}{(\omega(p_1))^2}[T_{b_1},T_{b_2}]\int d\tilde{q}_1 d\tilde{q}_2 a_{\nu_1}^{b_1\dagger}(q_1)a_{\nu_2}^{b_2\dagger}(q_2)e^{iq_1p_1s_1/\omega(p_1)+iq_2p_1s_2/\omega(p_1)} + ... \tag{6.45}$$

and

$$A_2 = -ig\int_{0}^{\infty} dt\,\frac{p_2^\mu}{\omega(p_2)}T_a\int d\tilde{k}\,a_\mu^a(k)e^{-ikp_2t/\omega(p_2)} -$$

$$-\frac{1}{2}g^2\int_0^\infty dt_1\int_{t_1}^\infty dt_2\frac{p_2^{\mu_1}p_2^{\mu_2}}{(\omega(p_2))^2}\left[T_{a_2},T_{a_1}\right]\int d\tilde{k}_1 d\tilde{k}_2 a_{\mu_1}^{a_1}(k_1)a_{\mu_2}^{a_2}(k_2)e^{-ik_1p_2t_1/\omega(p_2)-ik_2p_2t_2/\omega(p_2)}+... \tag{6.46}$$

Including these last relations in the overlap of dressed states we use the BCH formula and Eq. (6.11) to arrive at

$$<\{p_2\}|V|\{p_1\}>=<p_2|e^{A_2}Ve^{A_1}|p_1>=<p_2|Ve^{A_2}e^{A_1}e^{-A_2}e^{A_2}|p_1>$$

$$=<p_2|Ve^{A_2}e^{A_1}e^{-A_2}|p_1>=<p_2|Ve^{A_1}e^{A_3}|p_1>=<p_2|Ve^{A_3}|p_1> \tag{6.47}$$

where

$$A_3=[A_2,A_1]+\frac{1}{2}[A_2,[A_2,A_1]]+\frac{1}{2}[A_1,[A_1,A_2]]+\frac{1}{4}[A_2,[A_1,[A_1,A_2]]]+other\,terms \tag{6.48}$$

The terms that are useful for the present calculation are written explicitly in Eq. (6.48). All the others are grouped under the generic name 'other terms'. This is motivated by the following. Since we don't go further than order g^4 and we moreover take into account only the virtual corrections the explicit commutators in Eq. (6.48) include A_1 or A_2 at most twice. On the other hand we shall keep into the expandion of A_1 and A_2 only the terms up to second order in g.

In the appendix all these commutators are computed and in the end we find that up to desired order

$$A_3|_{g^4}=-g^2C_F\int d^4k\frac{i}{k^2+i\epsilon}\frac{p_1p_2}{p_1k+i\epsilon}\frac{1}{p_2k+i\epsilon}+g^4(-\frac{C_AC_F}{2})\int\frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4}\times$$

$$\times\frac{i}{k_1^2+i\epsilon}\frac{i}{k_2^2+i\epsilon}(p_1p_2)^2\frac{1}{(k_1p_1-i\epsilon)(k_2p_2-i\epsilon)\,(p_1(k_1+k_2)-i\epsilon)\,(p_2(k_1+k_2)-i\epsilon)} \tag{6.49}$$

In this way exponentiation up to NLO is proved since the amplitudes in Eq. (6.49) correspond to the lower order webs.

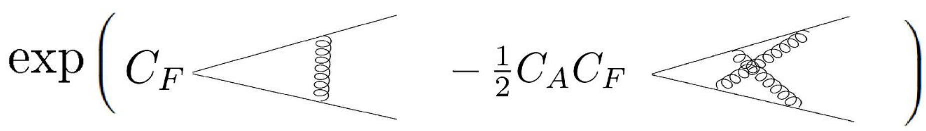

Fig.6.2 First two webs in QCD exponentiation

Summary

The finitness of S operator elements between dressed states implies that the infrared properties of usual S operator elements between free states can be derived directly studying the infrared behaviour of the dressed states. We showed that the infrared divergences present in the dressed states exponentiate in the overlap of two such states. The leading factors in the exponent coincide with the factors obtained in the usual proofs of exponentiation both for QED and QCD.

7 Dressing the fields

7.1 Introduction

In the previous chapters we presented few aspects related to the asymptotic dynamics of gauge theories. We also introduced the concept of dressed states as an alternative to the usual method of dealing with infrared divergences. The aim of that discussion was to show that in a certain basis of states the S matrix is well defined and finite and cross-section can be computed directly leading to correct results, results that coincide with the ones we get by usual not-so-correct methods. On the other hand the motivation of having dressed fields was also given by the argument that in the laboratory we do not detect lonely electrons but rather electrons that are surrounded by a cloud of soft and collinear photons which are not detected by our limited devices that have a non vanishing resolution. However, these ideas are not enough to support the idea that the dressed states used there are a good theoretical approximation of the true asymptotic states .

The purpose of this chapter is to define the photon 'cloud' in a more rigurous way and to arrive at physical states, states that can be detected in experiments and are free of infrared singularities. The formalism described in this thesis refers only to QED but a similar approach can be done for the QCD Taking into account the non-abelian nature of the theory together with the properties of confinement and asymptotic freedom the dressed fermions are identified there with the constituent quarks [30].

We shall begin by discussing about QED physical. Then we will introduce Dirac's proposal for a locally gauge-invariant dressed field, that is historically the first attempt of this kind. Later on we shall define a new dressing in such a way that the dressed fermion field is locally gauge invariant and the particles it creates display the usual electromagnetic field. In the end we will prove that the infrared divergences cancel if using such a dressing and also we arrive at scattering amplitudes that do not suffer from infrared problems.

7.2 Physical states

The discussion of this paragraph is focused on some basic aspects of QED. We seek for the principle which makes the distinction between physical and un-physical states and which separates the physical operators from the un-physical ones. Since we are dealing with a gauge theory we will see that a basic ingredient in this quest is gauge invariance.

Let us start from the lagrangian density of the theory which has the general form

$$L = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\bar{\Psi}\gamma^{\mu}D_{\mu}\Psi - m\bar{\Psi}\Psi + B\partial_{\mu}A^{\mu} + \frac{1}{2}\alpha B^2, \tag{7.1}$$

where B is a gauge-fixing term $D_\mu \equiv \partial_\mu - ieA_\mu$ is the covariant derivative and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic tensor. The gauge parameter α vanishes in the Landau gauge and it's equal to one in the Feynman gauge.

If there were no gauge fixing terms in Eq. (7.1) the Lagrangian density would be invariant under a change of the gauge field $A_\mu(x) \to A_\mu(x) + \partial_\mu\theta(x)$ only if the fermionic field would have changed as well according to $\Psi(x) \to e^{ie\theta(x)}\Psi(x)$. The function $\theta(x)$ should be a c-number function that satisfies $\Box\theta(x) = 0$.

By usual variational method we can derive the equations of motion for each field in the Lagrangian. The EOM for the electromagnetic field, also known as the quantum Maxwell equations, are given by

$$\partial_\mu F^{\mu\nu} - \partial^\nu B = -ej^\nu \quad \text{and} \quad \partial^\mu A_\mu + \alpha B = 0, \tag{7.2}$$

where the conserved current is $j^\mu = \bar{\Psi}\gamma^\mu\Psi$. On the other hand the EOM for B lead to $\Box B = 0$,which is the equation of a free massles field.

The Dirac field and the photon field can be quantized imposing certain (anti)commutation relations between the fields and their canonical conjugate. For the fermionic field we take Ψ and $\Psi^\dagger$ as canonical conjugates while for A_μ the conjugate is given by $\pi_\mu = \frac{\partial}{\partial(\partial_0 A^\mu)}L = F_{0\mu} + \eta_{0\mu}B$. In this way the equal time commutations are given by $[A_\mu(x), A_\nu(y)]_0 = 0$, $[\pi_\mu(x), \pi_\nu(y)]_0 = 0$ and $[\pi^\mu(x), A_\nu(y)]_0 = -i\delta^\mu_\nu\delta^{(3)}(\mathbf{x}-\mathbf{y})$. The anti-commutation relations for fermions read $[\Psi(x), \Psi(y)]_0^+ = \left[\Psi^\dagger(x), \Psi^\dagger(y)\right]_0^+ = 0$ and $\left[\Psi^\dagger(x), \Psi(y)\right]_0^+ = i\delta^{(3)}(\mathbf{x}-\mathbf{y})$.

Since B is a free field it can be expressed in terms of its Fourier modes in the usual way

$$B(x) = \int \frac{d^3k}{(2\pi)^3 2\omega(k)} \left(c(k)e^{-ikx} + c^\dagger(k)e^{ikx}\right). \tag{7.3}$$

We denoted its Fourier modes with $c(k)$ not to be confused with the fermionic modes usualy denoted $b(k)$. Then we can impose the so called 'Gupta subsidiary condition' to define the physical states as those states which are annihilated by the positive Fourier modes of B

$$c(k)|phys >= 0. \tag{7.4}$$

It can be shown [27] that the set of physical states form a subspace of the Hilbert space that is Poincare invariant.

Given the definition of B and Eq. (7.4) we conclude that the matrix elements of B between physical states vanish $< phys|B(x)|phys >= 0$.

Suppose $\hat{O}(x)$ is the operator associated to a physical observable. Then its eigenstates should be physical states. In other words $\hat{O}(x)|phys >= o(x)|phys >$. Then $c(k)\hat{O}(x)|phys >= c(k)o(x)|phys >= 0$ and $\hat{O}(x)c(k)|phys >= 0$. Thus a physical observable satisfies $\left[\hat{O}(x), c(k)\right] = 0$.

Having said these let us return to the fields defined above. Using a little bit of algebra, the canonical commutation relations and the EOM it can be shown [27] that the electromagnetic field and the Dirac field are not physical observables since

$$[A_\mu(x), B(y)] = -i\partial_\mu D(x-y),$$

$$[\Psi(x), B(y)] = e\Psi(x)D(x-y), \tag{7.5}$$

where $D(x-y)$ is given by

$$D(x-y) = \frac{1}{(2\pi)^4}\int d^4k \left[\frac{exp(-ik(x-y))}{k^2+i\epsilon}\theta\left((x-y)_0\right) - \frac{exp(-ik(x-y))}{k^2+i\epsilon}\theta\left(-(x-y)_0\right)\right]. \tag{7.6}$$

The question that arises now is : how come the Dirac and the electromagnetic field that enter the Lagrangian are not physical according to Eq. (7.5) ? First of all we know that not all the vector potential components are physical since the photons are polarized and the polarization vector lye in a plane normal to the direction of motion i.e. the photon has two degrees of freedom instead of four. On the other hand the Dirac field is not physical because what the fermionic field in the asymptotic region is not the free Dirac field that enters the Lagrangian. The asymptotic field is the physical one. If somehow the interaction would vanish in time, i.e. the coupling constant would go to zero for large times, the second commutator in Eq. (7.5) would vanish in the asymptotic region and the fermionic field would be physical.

In conclusion the physical states are those annihilated by the positive freqency modes of B and the physical observables are those whose operators commute with these modes in any point of the space-time.

7.3 Dirac's proposal

We have seen in section 2.3 that the particles created by the asymptotic Dirac field in the Heisenberg have no single-particle interpretation. Moreover the surrounding cloud of photons is not well defined in the infrared region. The point is that such fields are very different from the free fields entering the QED Lagrangian. Same situation occurs in other theorie where infinite ranged interactions do not vanish in the asymptotic limit. The conclusion is that we cannot contruct well defined physical charged particles out of these theories by the usual treatment.

The first idea of how to construct a charged field in QED belonged to Dirac who introduced a method to build locally gauge invariant matter fields that display a certain electromagnetic field.

To overcome the problem of fields that are not gauge-invariant Dirac proposed a new field built on the basis of the usual fermionic one. This field is locally gauge invariant and has the general form

$$\Psi_f(x) = \exp\left(-ie\int d^4z f^\mu(x-z)A_\mu(z)\right)\Psi(x), \tag{7.7}$$

with f^μ in such a way that $\partial_\mu f^\mu(x) = \delta^4(x)$.

The additional condition that the fermions created by this field should have the right electric field would give us f . What comes out in the end reads

$$\Psi_D(x) = \exp\left(-ie\frac{\partial_i A_i}{\nabla^2}\right)\Psi(x). \tag{7.8}$$

Let us explain the expression above. $\frac{1}{\nabla^2}$ stands for the inverse of ∇^2. Its action on a function of coordinates would then be

$$\frac{1}{\nabla^2}f(x) = -\frac{1}{4\pi}\int d^3y\frac{f(y)}{|\mathbf{x}-\mathbf{y}|}. \tag{7.9}$$

Having this form of the dressed field we can check that indeed it gives rise to a certain electric field. This field is defined as the time-derivative of the vector potential $\mathbf{E}(x) = \partial_0\mathbf{A}(x)$. Note that its components do not commute with the vector field' components. Their commutator is (see Eq. 2.11) $[E_i(x), A_j(y)] = i\delta_{ij}\delta^{(3)}(\mathbf{x}-\mathbf{y})$.

Given this and the fact that the usual fermionic field operator acting on the vacuum gives a state without an electric field (i.e. $E_i(x)\Psi(y)|0>= 0$ since $[E_i(x), \Psi(y)] = 0$ and $E_i(x)|0>= 0$) we find that the electric field created by the composite operator of Eq. (7.8) is

$$E_i(x)\Psi_D(y)|0>= \frac{e}{4\pi}\frac{x_i - y_i}{|\mathbf{x}-\mathbf{y}|^3}\Psi_D(y)|0> \tag{7.10}$$

In Dirac's interpretation the dressed field in Eq. (7.8) creates a fermion together with its electric field. Note that the respective Coulomb field is the one of a static charge located at the point y in space-time.

7.4 The dressing equation

The proposal of Dirac seems fine but its disadvantage is that it describes only fields that create static charged particles. This conclusion follows from the fact that those dressed particles create the field of a static charge. What we are looking for is a gauge invariant dressed field that creates particles which would have the right electric field when their four momentum is on the mass shell.

Throughout these calculation we consider that fermions have a non-vanishing mass. The mass becomes important here as it provides the dominant energy-scale. This is motivated by the fact that in the asymptotic region the interaction is mediated by long wavelength photons a.k.a soft photons.

Consider a free massive Dirac field. Since there are no interaction a massive fermion created by this field would move on a straight line in space-time. The variation of the field along that line vanishes. In other words

$$u \cdot \partial\psi(x) = 0, \tag{7.11}$$

where the four velocity is defined as $u^\mu = \gamma(\eta + v)^\mu$, with the following notations $\gamma = (1 - \mathbf{v}^2)^{-1/2}$, $v = (0, \mathbf{v})$ and $\eta = (1, \mathbf{0})$.

Let us consider the fermionic field to be minimally coupled to a gauge field. In the limit in which the mass of the fermion is much larger than the energy of the emitted (or absorbed) photons this field satisfies an equation equivalent to Eq. (7.11). The only difference come from the fact that the Dirac field is coupled to the electromagnetic field and the usual differential operator should be replaced by the covariant derivative. $D_\mu = \partial_\mu - ieA_\mu$.

The limit of large mass holds in our study because we work in the asymptotic region where all photons are supposed to be soft. As we have seen in Chapter 2 a good description for a charged particle in the asymptotic region does not exist. That is the reason of searching for a suitably dressed field which is locally gauge invariant but also behaves like a free field. The last demand comes from the fact that usual in the ordinary perturbative calculation we need to use reduction formulas as the one of Lehmann, Symanzik and Zimmermann (LSZ). This formalism works only for fields that are free in the asymptotic region. In all other cases such an approach leads to infrared divergences.

Consider a dressed field defined as $\Psi(x) = h^{-1}(x)\psi(x)$. If the initial fermionic field transforms under a gauge change as $\psi(x) \to e^{ie\theta(x)}\psi(x)$ we expect the dressing to transform as $h^{-1}(x) \to h^{-1}(x)e^{-ie\theta(x)}$ in order to have a locally gauge invariant dressed field. If we impose the condition that this field acts as a free field we have again

$$u \cdot \partial\Psi(x) = 0. \tag{7.12}$$

Since the origial Dirac field is coupled to an electromagnetic field and satisfies Eq. (7.12) it is straight-forward to find an equation for the dressing operator. This reads

$$u \cdot \partial h^{-1}(x) = -ieh^{-1}(x)u \cdot A(x). \tag{7.13}$$

This is called the dressing equation. Because the 4 velocity is involved we expect the dressing to depend on it. In the next sections we will determine the dressing for the cases when the fermion is moving with a given velocity.

7.5 Static charge

The gauge dependece of the dressing together with the dressing equation are the necessary ingredients to find the form of the dressing. We shall determine it first for a static charge and later on extend the calculations for a moving charge.

Consider a charge that in certain frame has a vanishing spacial velocity i.e. $u = (1, \mathbf{0})$. We can solve the dressing equation and try to find a dressing that would make the asymptotic dressed particle to create the electric field of a static particle.

If we replace the expression of the four velocity in Eq. (7.13) we get the dressing equation for this particular case which reads

$$\partial_0 h^{-1}(x) = -ieh^{-1}(x)A_0(x). \tag{7.14}$$

A general solution is a time ordered exponential

$$h^{-1}(x, t_0) = C\, T\text{exp}\left(-ie\int_{t_0}^{t} ds\, A_0(s, \mathbf{x})\right), \tag{7.15}$$

where C is a term constant in time which we will determine later and t_0 is a reference time.

A solution of the same type with Eq. (7.15) is suitable for our purpose if it has the desired change under gauge transformation (i.e. $h^{-1}(x) \to h^{-1}(x)e^{-ie\theta(x)}$) . The time ordered exponential transforms as

$$T\text{exp}\left(-ie\int_{t_0}^{t} ds\, A_0(s, \mathbf{x})\right) \to T\text{exp}\left(-ie\int_{t_0}^{t} ds\, A_0(s, \mathbf{x})\right) \exp\left(ie\theta(\mathbf{x}, t_0) - ie\theta(x)\right). \tag{7.16}$$

This is not exactly the wanted transformation. However the extra term $e^{ie\theta(x,t_0)}$ disappears if the time-independent term C changes under a gauge transformation as $C \to Ce^{-ie\theta(x,t_0)}$.

Remark that the Dirac proposal for the dressing of a fermion does not have the form of our solution but it has a transformation under a gauge change exactly as the one we need for C because, for a any time moment t_0

$$e^{-ie\frac{\partial_i A_i}{\nabla^2}(t_0,\mathbf{x})} \to e^{-ie\frac{\partial_i A_i}{\nabla^2}(t_0,\mathbf{x}) - ie\theta(t_0,\mathbf{x})}. \tag{7.17}$$

Since this does not depent explicitely on the time and has the wanted behaviour under gauge transformationswe can set

$$C = e^{-ie\frac{\partial_i A_i}{\nabla^2}(t_0,\mathbf{x})}. \tag{7.18}$$

As this is time-independent and has the wanted behaviour under gauge transformations. In this way the particular solution to the dressed equation has the form

$$h^{-1}(x, t_0) = e^{-ie\frac{\partial_i A_i}{\nabla^2}(t_0,\mathbf{x})} T\text{exp}\left(-ie\int_{t_0}^{t} ds\, A_0(s, \mathbf{x})\right) \tag{7.19}$$

The time independent term can be introduced in the time ordered exponential and put as a total time derivative under the integral. Using this and neglecting possible commutators between the photonic field components (that would result in some phases) we write Eq. (7.19) as

$$h^{-1}(x, t_0) = T\text{exp}\left(-ie\int_{t_0}^{t} ds\, \left(A_0(s, \mathbf{x}) - \frac{\partial_0 \partial_i A_i}{\nabla^2}(s, x)\right)\right) \mathrm{e}^{-ie\frac{\partial_i A_i}{\nabla^2}(x)}. \tag{7.20}$$

Now the Dirac-like term is time dependent while the first term can be written in a simpler way using the definition and properties of $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. In this way we get the final form of the dressing

$$h^{-1}(x,t_0) = T\exp\Big(ie\int_{t_0}^{t} ds\,\frac{\partial^\mu F_{\mu 0}}{\nabla^2}(s,x)\Big)\,\mathrm{e}^{-ie\frac{\partial_i A_i}{\nabla^2}(x)} \tag{7.21}$$

What is amazing about this solution is that it resembles the electric field of a stationary charge. Using the commutation relation introduced in section 7.2 we see that the first term of Eq. (7.21) commutes with the electromagnetic potential in the asymptotic region. Then the electric field acting on a dressed state is equal to the electric field acting on a state dressed only in Dirac way which we have seen it is the correct value for a static charge.

Since the fields are dressed in the asymptotic region the fields that enter Eq. (7.21) are nothing but the asymptotic fields. Moreover, if we use Magnus theorem to write the time-ordered exponential as a ordinary exponential and also use the commutation properties of $F_{\mu\nu}$ we finally get

$$h^{-1}(x,t_0) = \exp\Big(ie\int_{t_0}^{t} ds\,\frac{\partial^\mu F^{as}_{\mu 0}}{\nabla^2}(s,x)\Big)\,\mathrm{e}^{-ie\frac{\partial_i A^{as}_i}{\nabla^2}(x)}. \tag{7.22}$$

Note that due to anti-symmetry $\partial^\mu F^{as}_{\mu 0} = -\partial^i F^{as}_{i0}$. In addition the Gauss' law in electrodynamics states that $\partial^i F^{as}_{i0} = -eJ^{as}_0$ where for the present case the current J^{as} is the conserved asymptotic current defined in section 7.2. Thus the first term in Eq. (7.22) is gauge invariant and will only contribute as a phase factor. Also according to Eq. (2.43) the asymptotic electromagnetic potential is the sum of a free potential and a part that has no photonic operators. That second part will contribute to the phase as well. Thus Eq. (7.22) can be written as

$$h^{-1}(x,t_0) = e^{i\phi(x,t_0)}\mathrm{e}^{-ie\frac{\partial_i A_i}{\nabla^2}(x)}, \tag{7.23}$$

where the electromagnetic field is free and the phase is gauge-invariant. We shall not specify the form of ϕ as it is not useful here. All we know is that it does not contain any photonic operators. The other part, call it minimal dressing, coincides with the Dirac's proposal and it generates the wanted electric field and the wanted gauge dependence.

We could discuss here how the infrared divergences present in the cloud that surrounds a static fermion in the Heisenberg picture cancel when the fermion is dressed in the way described here. The discussion is made later for the more general case of a moving charged particle.

7.6 Moving charged particle

In this section we shall find the form of the dressing for a charged particle moving with 4-velocity $u = \gamma(\eta + v)$. The approach is similar to the static charge case and includes solving the dressing equation and using arguments of gauge invariance and different commutation relations to determine the dressing that would make the asymptotic fields create well-defined physical particles.

To make our life easier we begin with an ansatz regarding the form of the dressing factor $h^{-1}(x)$ Based on the result we have in the static case we expect the dressing to be the product of two terms. That is

$$h^{-1}(x) = e^{-ieK(x)}e^{-ie\chi(x)}, \tag{7.24}$$

with $K(x)$ a gauge invariant function while $\chi(x) \to \chi(x) + \theta(x)$ under a gauge change.

Note that there is no time ordering imposed on these exponentials. The only reason for this is that the time-ordered exponential of an expression that is linear in $A^\mu(x)$ can be written as a product of two ordinary exponentials. One of these two is a phase and the other one is linear in $A^\mu(x)$. In this way we may assume that all the commutators that would contribute to that phase are included in the gauge invariant part of Eq. (7.24)

Let us discuss first about the gauge dependent term $\chi(x)$. Given its transformation law we would expect it to have a form similar to the dressing of Eq. (7.7). That reads

$$\chi(x) = \int d^4z \; f^\mu(x-z)A_\mu(z), \tag{7.25}$$

with $\partial_\mu^{(z)} f^\mu(x-z) = \delta^4(x-z)$. Similar to the previous case $f^\mu(x)$ is not just a complex function but can contain differential operators that commute with eachother. One difficulty we may encounter is the appearence of unwanted surface terms that arise when an integration by parts is done in Eq. (7.25) after the gauge transformation $A_\mu(x) \to A_\mu(x) + \partial_\mu\theta(x)$ is performed. This problem can be solved if we restrict our gauge choice to gauges that vanish at spatial infinity and at infinite times. While vanishing gauges at spatial infinity are usual because of the $\frac{1}{r}$ fall of the potential energy, vanishing gauges at infinite times do not have a physical support. In that case we have to restrict the choice of $f^0(x)$ to functions that vanish outside of a finite time interval.

Suppose that f^μ is the product of a complex function with a first order differential operator $f^\mu(x-z) = G(x-z)\mathcal{G}^\mu$. The condition $\partial_\mu^{(z)} f^\mu(x-z) = \delta^4(x-z)$ is translated to $\mathcal{G}\cdot\partial\, G(x-z) = \delta^{(4)}(x-z)$ and the restriction to gauges vanishing at large times leads to the condition that $\mathcal{G}\cdot\partial$ contains no time derivatives. We can choose $G(x-z) = \delta(x^0 - z^0)G(\mathbf{x}-\mathbf{z})$ which leads in the end to $\mathcal{G}\cdot\partial\, G(x-z) = \delta^{(3)}(\mathbf{x}-\mathbf{z})$. The final form of $\chi(x)$ is

$$\chi(x) = \int d^3z G(\mathbf{x}-\mathbf{z})\mathcal{G}^\mu_{(z)}A_\mu(t,\mathbf{z}) \equiv \frac{\mathcal{G}\cdot A}{\mathcal{G}\cdot\partial}(x). \tag{7.26}$$

The exact form of the operator $\mathcal{G}^\mu$ can be determined once the dressing equation Eq. (7.13) is solved with the ansatz in Eq. (7.24). In order to solve the equation easier we can expand the dressing operator in a power series in the coupling constant $K(x) = K_1(x) + eK_2(x) + O(e^2)$. Having done this we solve the equation order by order in the coupling constat to determine $K(x)$ and $\chi(x)$.

Let us solve the dressing equation $u\cdot\partial h^{-1}(x) = -ieh^{-1}(x)u\cdot A(x)$. If we perform the expansion in e and use the fact that χ involves no powers of this coupling constant we find the lowest order of the dressing equation to be

$$(\eta+v)^\mu\partial_\mu(K_1+\chi) = (\eta+v)^\mu A_\mu. \tag{7.27}$$

The general solution of this is

$$K_1(x) + \chi(x) = \int_{t_0}^t (\eta+v)^\mu A_\mu(x(s))ds + C \tag{7.28}$$

Here C is constant in time and it turns out that the special choice $C = \chi(x(t_0))$ would make K_1 gauge invariant. Thus the gauge invariant part of lowest order in the coupling constant is

$$K_1(x) = \int_{t_0}^t ds\; (\eta+v)^\mu \left(A_\mu(x(s)) - \partial_\mu\chi(x(s))\right) \tag{7.29}$$

To obtain this result we made use of $(\eta+v)^\mu\partial_\mu\chi(x(s)) = \frac{\partial x^\mu}{\partial s}\frac{\partial\chi(x(s))}{\partial x^\mu} = \frac{\partial\chi(x(s))}{\partial s}$.
Given the form of $\chi(x)$ in Eq. (7.26) we find $K_1(x)$ to be

$$K_1(x) = \int_{t_0}^t (\eta+v)^\mu\frac{\mathcal{G}^\nu F_{\nu\mu}}{\mathcal{G}\cdot\partial}(x(s))ds. \tag{7.30}$$

In Eq. (7.30) the lower limit of the integral is a reference time t_0. Since we are in the asymptotic region trying to define physical fields we must make sure the outcome of the present calculation is physical as well. Thus the dependence of $K(x)$ over this reference time should be physical. In other words $\frac{\partial K}{\partial t_0}$ must vanish whenever put between physical states so it must be some function of the field B.

Suppose

$$\frac{\partial K}{\partial t_0} = (\eta + v)^\mu \frac{\partial_\mu B}{\mathcal{G} \cdot \partial}(x(t_0)). \tag{7.31}$$

In the Feynman gauge $B = -\partial^\mu A_\mu$. Given the definition of $F_{\mu\nu}$ and of the equation of motion in this gauge, $-eJ_\mu = \partial^\nu \partial_\nu A_\mu$, Eq. (7.31) becomes

$$\frac{\partial K(x)}{\partial t_0} = (\eta + v)^\mu \left(\frac{\partial^\nu F_{\nu\mu}}{\mathcal{G} \cdot \partial}(x(t_0)) + e\frac{J_\mu}{\mathcal{G} \cdot \partial}(x(t_0)) \right). \tag{7.32}$$

Expanding K in a power series of the coupling constant, like above, we find that

$$\frac{\partial K_1(x)}{\partial t_0} = (\eta + v)^\mu \frac{\partial^\nu F_{\nu\mu}}{\mathcal{G} \cdot \partial}(x(t_0)) \tag{7.33}$$

and also

$$\frac{\partial K_2(x)}{\partial t_0} = (\eta + v)^\mu \frac{J_\mu}{\mathcal{G} \cdot \partial}(x(t_0)). \tag{7.34}$$

On the other hand taking this derivative in Eq. (7.30) we find

$$\frac{\partial K_1(x)}{\partial t_0} = -(\eta + v)^\mu \frac{\mathcal{G}^\nu F_{\nu\mu}}{\mathcal{G} \cdot \partial}(x(t_0)). \tag{7.35}$$

If we compare Eq. (7.33) to Eq. (7.35) we can determine the form of $\mathcal{G}^\mu$. The initial assumptions were that this is a first order differential operator and that $\mathcal{G} \cdot \partial$ contains no time derivatives. Suppose $\mathcal{G}^\mu$ is constructed out of ∂^μ, η^μ and v^μin such a way that

$$\mathcal{G}^\mu = -\partial^\mu + (\eta + v)^\mu(\alpha\eta \cdot \partial + \beta v \cdot \partial). \tag{7.36}$$

The first part of $\mathcal{G}^\mu$ is directly inspired by Eq. (7.33) to Eq. (7.35) while the second part should vanish when multiplied by $(\eta + v)^\nu$ and applied on $F_{\mu\nu}$. The coefficients α and β are determined by the condition that no time derivatives are present in $\mathcal{G} \cdot \partial$. From Eq. (7.36) we find

$$\mathcal{G} \cdot \partial = (\alpha - 1)(\eta \cdot \partial)^2 + (\alpha + \beta)(\eta \cdot \partial)(v \cdot \partial) + \beta(v \cdot \partial)^2 + \nabla^2. \tag{7.37}$$

Thus $\alpha = 1$ and $\beta = -1$ and so $\mathcal{G}^\mu = -\partial^\mu + (\eta + v)^\mu(\eta \cdot \partial - v \cdot \partial) = -\partial^\mu + (\eta + v)^\mu(\eta - v) \cdot \partial$.

With these values Eq. (7.37) becomes

$$\mathcal{G} \cdot \partial = -(\mathbf{v} \cdot \nabla)^2 + \nabla^2. \tag{7.38}$$

In the end, due to the antisymmetry of $F_{\mu\nu}$, Eq. (7.30) leads to

$$K_1(x) = \int_{t_0}^{t} (\eta + v)^\mu \frac{\partial^\nu F_{\nu\mu}}{\mathcal{G}\cdot\partial}(x(s))ds. \tag{7.39}$$

In a similar manner K_2 can be found. Eq. (7.34) implies that

$$K_2(x) = \int_{-\infty}^{t_0} (\eta + v)^\mu \frac{J_\mu}{\mathcal{G}\cdot\partial}(x(s))ds + K_2^{'}(x) \tag{7.40}$$

Where $K_2^{'}(x)$ is independent of t_0. The exact form of K_2 can be found if we return to the dressing equation and solve it for the second order in the coupling constant. Remember that we took $K(x) = K_1(x) + eK_2(x)$. The equation at the second order reads

$$(\eta + v)^\mu \partial_\mu (K_1\chi + K_1^2 + \chi^2 + iK_2) = (K_1 + \chi)(\eta + v)^\mu A_\mu, \tag{7.41}$$

or written in neat form

$$i(\eta + v)^\mu \partial_\mu K_2 = \frac{1}{2}(\eta + v)^\mu \left(\partial_\mu [\chi, K_1] - [A_\mu, K_1 + \chi]\right). \tag{7.42}$$

Using the form of K_1 and χ and the commutation relations for A_μ the two commutators can be computed. It can be shown [15] that they give rise to

$$i(\eta + v)^\mu \partial_\mu K_2 = -\frac{1}{2} i\gamma^{-2} \int d^3k \frac{1}{(2\pi)^3} \frac{1}{(\omega(\mathbf{k}))^2 - (\mathbf{k}\cdot\mathbf{v})^2}, \tag{7.43}$$

where $\gamma = (1 - \mathbf{v^2})^{-1}$.

It is easy now to identify the t_0-independent term K_2' from Eq. (7.43) and we find

$$K_2(x) = \int_{-\infty}^{t_0} (\eta + v)^\mu \frac{J_\mu}{\mathcal{G}\cdot\partial}(x(s))ds - \frac{1}{2} u\cdot x\, \gamma^{-1} \int d^3k \frac{1}{(2\pi)^3} \frac{1}{(\omega(\mathbf{k}))^2 - (\mathbf{k}\cdot\mathbf{v})^2}. \tag{7.44}$$

Higher order contributions for $K(x)$ and be found in a similar manner but they are less interesting for us at this moment.

The dressing we defined is fully acceptable if the dressed fields create charged particles that are free of infrared problems and create the known electromagnetic field corresponding to the charge they carry. We shall check the first condition in the next section and compute the electric field of a dressed charged particle here.

We compute the electric field created by such a particle in the same manner as for the Dirac's proposal. Thus all we need is the commutator between the dressing operator and the time derivative of the vector potential. Note that the dressing consists of two parts out of which one is gauge-invariant. Given the symmetry properties of $F_{\mu\nu}$ and the commutation relations of the gauge field we have that $[\partial^\mu F_{\mu\nu}(x), F_{\lambda\rho}(y)] = 0$. Thus K will give no cotribution to the electric field since it contains the gauge field only via the tensor $F_{\mu\nu}$ which commutes with the electric and magnetic field. That is why, for the moment, we will put apart this component of the dressing and keep only the minimal part $e^{-ie\chi}$.

First we need to clarify some aspects of this operator χ. According to Eq. (7.26) it is

$$\chi(x) = \frac{\mathcal{G}\cdot A}{\mathcal{G}\cdot\partial}(x), \tag{7.45}$$

with $\mathcal{G}$ given below Eq. (7.37).

The two differential operators acting on A commute. Thus we can first compute $\frac{1}{\mathcal{G}\cdot\partial}A(x)$ and then act with $\mathcal{G}$ on it. To this end let us use Eq. (7.26) and determine $G(\mathbf{x}-\mathbf{z})$. So

$$\frac{1}{\mathcal{G}\cdot\partial}A(x) = \int d^3z G(\mathbf{x}-\mathbf{z})A(z), \tag{7.46}$$

in such a way that $\mathcal{G}\cdot\partial G(\mathbf{x}-\mathbf{z}) = \delta^{(3)}(\mathbf{x}-\mathbf{z})$. By a Fourier transform of this last equation a form for $G(\mathbf{x}-\mathbf{z})$ is found provided $\mathcal{G}\cdot\partial = \nabla^2 - (v\cdot\partial)^2$. That is

$$\left(\nabla^2 - (v\cdot\partial)^2\right)\int \frac{d^3k}{(2\pi)^3}\tilde{G}(\mathbf{k})e^{i\mathbf{k}(\mathbf{x}-\mathbf{z})} = \int \frac{d^3k}{(2\pi)^3}e^{i\mathbf{k}(\mathbf{x}-\mathbf{z})}. \tag{7.47}$$

Thus

$$\tilde{G}(\mathbf{k}) = -\frac{1}{\mathbf{k}^2 - (\mathbf{v}\cdot\mathbf{k})^2}. \tag{7.48}$$

And by inverse Fourier transform

$$G(\mathbf{x}-\mathbf{z}) = -\frac{1}{4\pi}\frac{\gamma}{\sqrt{|\mathbf{x}-\mathbf{z}|^2 + \gamma^2((\mathbf{x}-\mathbf{z})\cdot\mathbf{v})^2}}. \tag{7.49}$$

Let us assume that the bare fermionic field does not create any electric field, that is $E_i(x)\Psi(y)|0>= 0$, because $[E_i(x),\Psi(y)] = 0$. If we dress this field in the manner described above i.e. $\Psi_v(x) = e^{-ie\chi(x)}\Psi(x)$ then we find the electric field of the dressed fermion to be

$$E_i(x)\Psi_v(y)|0>= -ie[E_i(x),\chi(y)]\Psi_v(y)|0> . \tag{7.50}$$

And its eigenvalue for the direction i corresponding to a charge that is located in the point y is

$$\varepsilon_i(x) = \frac{e}{4\pi}\frac{\gamma(\mathbf{x}-\mathbf{y})_i}{\left(|\mathbf{x}-\mathbf{y}|^2 + \gamma^2((\mathbf{x}-\mathbf{y})\cdot\mathbf{v})^2\right)^{3/2}}. \tag{7.51}$$

Note that this is indeed the electric field of a positron moving with velocity $\mathbf{v}$.

7.7 Cancellation of divergences

The last test the dressing has to pass is the cancelation of infrared divergences that cause the charged particle to be ill-defined in the asymptotic region. In this section we will use only the 'minimal' part of the dressing, that is $e^{-ie\chi(x)}$, because the gauge invariant part will contribute only with a phase which can be neglected in the following calculations.

Let us begin by writing this minimal dressing in the momentum space. Given the expression of the differential operator $\mathcal{G}$ below Eq. (7.37), its corresponding form in the momentum space is $V^\mu = -k^\mu + (\eta+v)^\mu(\eta-v)\cdot k$. This form, together with the Fourier expansion of the free gauge field, lead to

$$\chi(x) = \frac{\mathcal{G}\cdot A}{\mathcal{G}\cdot\partial}(x) = i\int \frac{d^3k}{(2\pi)^3}\frac{1}{\omega(k)}\left(\frac{V\cdot a(k)}{V\cdot k}e^{-ikx} - \frac{V\cdot a^\dagger(k)}{V\cdot k}e^{ikx}\right). \tag{7.52}$$

And so the dressing operator becomes

$$h^{-1}(x) \equiv \exp(-ie\chi(x)) = \exp\left(e\int \frac{d^3k}{(2\pi)^3}\frac{1}{\omega(k)}\left(\frac{V\cdot a(k)}{V\cdot k}e^{-ikx} - \frac{V\cdot a^\dagger(k)}{V\cdot k}e^{ikx}\right)\right). \quad (7.53)$$

Note that this expression holds also in the limit of a static charge if we put $v = 0$.

What we have to show is that the dressed field $\Psi_v(x) = h^{-1}(x)\Psi(x)$ acting on the vacuum will create something that has a single-particle interpretation and it is free of infrared divergences.

Starting from the Heisenberg field $\Psi(x)$ we can create an electron of momentum $p^\mu = \gamma m(\eta + v)^\mu$ by acting on the vacuum with the creation operator

$$b^\dagger(p) = \int d^3x \frac{1}{2\omega(p)}\Psi^\dagger(x)u(p)e^{-ipx}. \quad (7.54)$$

This would have a single particle interpretation if the field $\Psi(x)$ was a free field in the asymptotic limit. This is not the case in QED and the asymptotic particle is 'distorted' by a cloud of soft photons as shown in Eq.(2.44).

Let us consider now the dressed Dirac field $\Psi_v^\dagger(x) = \Psi^\dagger(x)h(x)$. It should create a dressed electron via

$$b_v^\dagger(p) = \int d^3x \frac{1}{2\omega(p)}\Psi^\dagger(x)h(x)u(p)e^{-ipx}. \quad (7.55)$$

Remember that in the asymptotic region we work in the eikonal approximation and so the fermions move on a line in the direction of their momentum p. That means their coordinate on the world line is $x = \frac{p}{\omega(p)}t$.

To make a link between this operator and the electronic creation operator we have to compare the dressing in Eq. (7.53) with the distortion operator in Eq. (2.45). We see that they both have the form of a coherent state operator and they commute. And so

$$D^\dagger(p,t)h_{soft}(p,t,v) =$$

$$= \exp\left[-e\int \frac{d^3k}{(2\pi)^3}\frac{1}{\omega(k)}\left(\left(\frac{V\cdot a(k)}{V\cdot k} - \frac{p\cdot a(k)}{p\cdot k}\right)e^{-i\frac{kp}{\omega(p)}t} - \left(\frac{V\cdot a^\dagger(k)}{V\cdot k} - \frac{p\cdot a^\dagger(k)}{p\cdot k}\right)e^{i\frac{kp}{\omega(p)}t}\right)\right]. \quad (7.56)$$

At the first sight the dressed creation operator for one electron is equal to the creation operator for a free electron distorted by this new operator in Eq. (7.56). The important remark is that, in certain conditions, the operator includes the creation and annihilation operators that enter the definition of the B field in Eq. (7.3). In this case they will give no contribution when put between physical states.

Note that the coeficients of the creation and annihilation operators in the exponent of Eq. (7.59) have a trivial form when the momentum of the particle is on-shell $p^\mu = m\gamma(\eta + v)^\mu$ because

$$\frac{V^\mu}{V\cdot k} - \frac{p^\mu}{p\cdot k} = \frac{(\eta+v)^\mu(\eta+v)\cdot k}{(\eta+v)\cdot k\,(\eta+v)\cdot k} - \frac{(\eta+v)^\mu}{(\eta+v)\cdot k} = \frac{k^\mu}{V\cdot k}. \quad (7.57)$$

In this case we find that

$$D^{\dagger}(p,t)h_{soft}(p,t,v) = \exp\left[-e\int \frac{d^3k}{(2\pi)^3}\frac{1}{\omega(k)}\frac{1}{V\cdot k}\left(c(k)e^{-i\frac{kp}{\omega(p)}t} - c^{\dagger}(k)e^{i\frac{kp}{\omega(p)}t}\right)\right]$$

When this operator is put between physical states it gives no contribution because of the definition of the physical states and because the real photons that make up the cloud are on-shell.

In conclusion a fermion dressed in this way will have a single-particle interpretation and it will be free of infrared divergences.

Let us end this chapter with a discussion concerning the interaction hamiltonian for dressed fields. In the beginning, when we derived the dressing equation, we stated that the dressed fields whould act like free fields in the asymptotic region. In this case the interaction hamiltonian has to vanish. Let us prove this in the following paragraphs.

The starting point is the QED Lagrangian density. This contains both kinetic and interaction terms. We shall work only with those parts that might change when the Dirac field is dressed. Let us denote the ordinary fermionic field by $\Psi(x)$ and the dressed field by $\Psi_h(x) = h^{-1}(x)\Psi(x)$. Then we can write the parts of the Lagrangian we are interested in terms of these dressed fields as

$$i\bar{\Psi}\gamma^{\mu}\left(\partial_{\mu} - ieA_{\mu}\right)\Psi = i\bar{\Psi}hh^{-1}\gamma^{\mu}\left(\partial_{\mu} - ieA_{\mu}\right)hh^{-1}\Psi = i\bar{\Psi}_h h^{-1}\gamma^{\mu}\left(\partial_{\mu} - ieA_{\mu}\right)h\Psi_h$$

$$= i\bar{\Psi}_h\gamma^{\mu}\left(\partial_{\mu} - ieh^{-1}A_{\mu}h - (\partial_{\mu}h^{-1})h\right)\Psi_h \equiv i\bar{\Psi}_h\gamma^{\mu}\left(\partial_{\mu} - ieA^h_{\mu}\right)\Psi_h, \tag{7.58}$$

where we have denoted $A^h_{\mu}(x) = h^{-1}(x)A_{\mu}(x)h(x) + \frac{1}{ie}(\partial_{\mu}h^{-1}(x))h(x)$. The asymptotic interaction Hamiltonian derived from this lagrangean density has the same form with the one in Eq. (6.19) but the fermionic field is to be replaced by the dressed field and the vector potential by $A^h_{\mu}(x)$. That is

$$H^h_{int}(t) = -e\int d^3\mathbf{x}A^h_{\mu}(t,\mathbf{x})J^{\mu}_{as}(t,\mathbf{x})$$

$$= -e\int d^3\mathbf{x}\int \frac{d^3p}{(2\pi)^3}\frac{1}{\omega(p)}p^{\mu}A^h_{\mu}(t,\mathbf{x})\rho(p)\delta^{(3)}\left(\mathbf{x} - \frac{\mathbf{p}t}{\omega(p)}\right) \tag{7.59}$$

The crucial observation is that the dressing satisfies Eq. (7.13) which states that if the dressed fermion is on the mass shell and has the four velocity u then $u\cdot\partial h^{-1}(x) = -ieh^{-1}(x)u\cdot A(x)$. This is equivalent to saying that for a dressed on-shell fermion $uA^h(t,\mathbf{x}) = 0$ given the definition of $A^h(t,\mathbf{x})$. The Hamiltonian of Eq. (7.59) usually acts on an eigenstate of the charge density operator $\rho(p)$ which in the asymptotic region is simply a charged particle moving with a given on-shell momentum $p = mu$. Given this we conclude that $p^{\mu}A^h_{\mu}(t,\mathbf{x}) = 0$ and the asymptotic interaction Hamiltonian vanishes.

Since the asymptotic interaction vanishes the usual theoretical methods (like LSZ reduction,etc) can be applied and a perturbation theory for the dressed fields can be set up. Even though we did not mention anything about the UV divergences it is possible to cure them as well using renormalization. The approach is the same as in the ordinary case but the diagrams involved and the corresponding Feynman rules are different. For a discussion on renormalization see [28].

Summary

In the usual theory a single-particle interpretation for the charged fermion in the asymptotic region does not exist in the Heisenberg picture. A dressing for the fields is defined in such a way that the asymptotic fields are locally gauge-invariant and free and the particles they create are physical.

8 Conclusion

The theme of this thesis was the infrared problem that we encounter in massless gauge theories in the usual perturbative approach. We did not study the ways to deal with the infrared divergences that occur but instead tried to find a different approach to scattering theory, one that avoids these divergences. Part of this scenery was the attempt to 'build' physical asymptotic charged particles that are free of any infrared divergences. The new techniques presented here did not necessarily make our life easier but rather explore the theory of scattering from a different point of view. At different stages we argued that the usual theory is not applied in a correct manner and that our approach is closer to the physical reality of the problem. At the end of the day the result for any physical observable should coincide no matter approach we use.

In the second chapter we used two examples to show that the final state of a system of fermions interacting with an electromagnetic field is not a free state. It turns out that such a final state contains an indefinite number of photons so it is a coherent state. For example the asymptotic Dirac field in the Heisenberg picture creates a fermion that is surrounded by a cloud of soft photons. This is a consequence of the fact that the asymptotic interaction hamiltonian of QED does not vanish. Moreover, in the asymptotic region, the interaction is carried only by soft photons which make the cloud to be ill-defined because the number of these photons diverges when their energy is very small.

Since coherent states are always involved in our story in QED we added a separate chapter to talk about them. In the third chapter the definition of photonic coherent states was given and a few properties were discussed. Later on, generalized coherent states were defined for any Lie group and few properties were discussed and the results obtained for photonic coherent states were recovered.

The forth chapter was the starting point of our discussion on scattering theory. We defined the S matrix elements and made the connection between these and the Moller operators. We found that infrared divergences are rooted in the false assumption that, for massless gauge theories, the asymptotic hamiltonian coincides with the free hamiltonian. If we make the right choice for the asymptotic hamiltonian the whole theory is well defined but difficulties arise if we try to diagonalize this hamiltonian which includes non-trivial interactions. Otherwise the Moller operators are ill-defined. A connection to free states has to be made, connection that is possible only if the massless theory is regularized. In that case the Moller operators involving the free hamiltonian are well defined again, if the regulator does not vanish, and the S operator can be used to compute S matrix elements. We argued that the asymptotic S operator matrix elements between free states are finite and defined the so called 'dressed states'. In this new basis of states the matrix elements of the usual S operator are finite giving rise to new scattering amplitudes that are also finite.

The fifth chapter was dedicated to an application for the new method of dressed states. The process $\gamma \to 2\,jets$ is studied in perturbation theory up to NLO and the amplitudes are shown to be finite. The calculations are slightly different from the usual scattering amplitudes due to the time ordering that enters the definition of the dressed states. A certain factorization of the S operator matrix elements between dressed states was possible. In this way amplitudes were constructed from different pieces which taken alone were not necessarily finite but when added up they lead to finite amplitudes. The calculations were carried out up to NLO and the amplitudes that were obtained did not depend on the regulator anymore but displayed a certain dependence on a set of parameters we used to define the infrared region. However this dependence disappeared when the total cross-section was computed. Up to NLO the total cross-section was the same with the one obtained in the usual approach.

Using the fact that the S operator matrix elements between dressed states are finite we found a new way to derive the well known exponentiation of infrared divergences in the eikonal approximation. Given the finiteness of those matrix elements we argued that the infrared properties of the S operator matrix elements between free states are reproduced by a certain overlap of dressed states. In this way we found the lowest order of the exponent for a QED example and the first two webs of the exponent for a QCD example. The advantage of this method is that all calculation are done in the exponent taking advantage of the exponential form of the dressed states. In the usual approach a large number of diagrams with increasing number of loops have to be added to derive the same result. The results here obtained can be extended to obtain higher order corrections in the exponent if higher order terms are taken into account in the dressing factors.

The last chapter was dedicated to a study of dressed fields. Starting from the observation that the asymptotic fermionic field is not gauge-invariant in QED and, as shown in the second chapter, the charged particles it creates are ill-defined due to the infrared divergent cloud that surround them, we looked for a suitable dressing to overcome all these problems. A certain dressing was defined and we showed that the asymptotic dressed fields are free and the particles they create are physical and free of any kind of infrared divergencies.

This thesis focused on the infrared problems alone and no concern was shown for the ultraviolet divergences. Because the two problems decouple from eachother we could think as if the theory was renormalized in such a way that we did not have to worry about the short-distance interaction phenomena.

9 Appendix

9.1 Appendix of chapter 5

We shall compute here some of the integrals in chapter 5.

For simplicity we work in the center of mass frame and parametrize the momentum of the on-shell particles as follows

$$P = \sqrt{s}(1, \mathbf{0}, 0), \qquad p_1 = \frac{1}{2}\xi_{p1}(1, \mathbf{0}, 1), \qquad p_2 = \frac{1}{2}\xi_{p1}(1, \mathbf{0}, -1),$$

$$q_3 = \frac{1}{2}\xi_{p1} z \left(1, \sqrt{1-y^2}\mathbf{e}_T, y\right). \tag{9.1}$$

We have introduced the following notations: $\mathbf{e}_T$ the unit vector in the $(2-2\epsilon)$ dimensional transverse momentul space , $\mathbf{0}$ the null vector in the same space and $0 \le z \le \infty$, $-1 \le y \le 1$. These two parameters describe how soft and/or how collinear is the gluon. When $z \to 0$ we deal with a soft singularity and for $y \to 1$ (or $y \to -1$) we have a collinear singularity corresponding to $q_3||p_1$ (or $q_3||p_2$).

The change of variables to z and y also changes the integration measure. Since the particles are on-shell the initial integration was done over the momentum $q \equiv q_3$in $3-2\epsilon$ dimensions ($D = 4-2\epsilon$). Only for the following calculation q_3 stands for the longitudinal component of the momentum q , q_T is the transversal component of q and $\Omega_{2-2\epsilon}$ is the volume of the unit sphere in $2-2\epsilon$ dimensions . Due to the energy conserving delta function we have that $\sqrt{s} = \xi_{p1}$ .The transformation of the integral to the new coordinates leads to

$$\int d\widetilde{q} = \int \frac{d^{D-1}q}{(2\pi)^{D-1}2\omega(\mathbf{q})} = \int \frac{d^{D-2}q\; dq_3}{(2\pi)^{D-1}2\omega(\mathbf{q})} = \Omega_{2-2\epsilon}\int \frac{dq_3 dq_T q_T^{1-2\epsilon}}{(2\pi)^{D-1}2\omega(\mathbf{q})} = \frac{\Omega_{2-2\epsilon}}{2}\int \frac{dq_3 d(q_T^2) q_T^{-2\epsilon}}{(2\pi)^{D-1}2\omega(\mathbf{q})}$$

$$= \frac{\Omega_{(2-2\epsilon)}}{2(2\pi)^{3-2\epsilon}}\frac{s}{4}\left(\frac{\mu^2}{s}\right)^{\epsilon}\int dy\, dz\; z^{1-2\epsilon}\left(1-y^2\right)^{-\epsilon}. \tag{9.2}$$

It is easy to show that the volume of the unit sphere in $2-2\epsilon$ dimensions is

$$\Omega_{(2-2\epsilon)} = \frac{2\pi^{1-\epsilon}}{\Gamma(1-\epsilon)}, \tag{9.3}$$

where $\Gamma(x) = \int_0^\infty dt\, e^{-t}t^{x-1}$ is the well known gamma function.

After the change in variables the phase-space limits are supposed to be$-1 \le y \le 1$ and $0 \le z \le \infty$. However, in different amplitudes they are restricted by the presence of different Heaviside functions .

Let us start with the amplitude $a_{15}^{(2,0)}$. First we show how the phase space look like in this new parametrization. Due to the presence of the Heaviside function $\Theta\left(\triangle - |\rho(\mathbf{q_3}, \mathbf{p_1}-\mathbf{q_3})|\right)$ we get the following constraints

$$\triangle \geqslant |\rho(\mathbf{q_3}, \mathbf{p_1}-\mathbf{q_3})| = \sqrt{s}\left(z-1+\sqrt{1-2zy+z^2}\right). \tag{9.4}$$

After we solve this for z and y and rescale $\triangle$ to $\frac{\triangle}{\sqrt{s}}$ we find the phase-space limits to be

$$0 \leq z \leq \frac{\triangle}{2} \frac{2+\triangle}{1-y+\triangle} \text{ with } -1 \leq y \leq 1. \tag{9.5}$$

Having the phase space limits we can get back to the amplitude calculation. As mentioned before the singularities occur in for $z \to 0$ and also for $y \to 1$. The finite parts of the integrand will give a finite result after integration and in the end if crossections are computed this finite parts will give vanishing contribution. That is why we focus only on the singular parts and compute them.

Denote the integrand as $F(y,z)$. Then

$$F(y,z) = \frac{(D-2)\left(1-zy-\sqrt{1-2zy+z^2}\right) - 2(1-y^2)}{\sqrt{1-2zy+z^2}\left(z-1+\sqrt{1-2zy+z^2}\right)}. \tag{9.6}$$

In order to isolate the divergent pieces we can factorize this function as

$$F(y,z) = F(y,0) + F(1,z) - F(1,0) + [F(y,z) - F(y,0) - F(1,z) + F(1,0)]\,. \tag{9.7}$$

First three terms are the divergent ones and the rest in finite. One can show that the divergent terms are then given by

$$F(y,0) = \left\{(2-D)\frac{1}{2}\frac{1+y}{1-y} - \frac{2}{z^2}\frac{1+y}{1-y}\right\}|_{z=0},$$

$$F(1,z) = \left\{(2-D)\frac{1}{1-y} - \frac{4(1-z)}{z^2(1-y)}\right\}|_{y=1},$$

$$F(1,0) = \left\{(2-D)\frac{1}{1-y} - \frac{4}{z^2(1-y)}\right\}|_{y=1,z=0}. \tag{9.8}$$

And then the integrand becomes

$$F(y,z) = \frac{1}{(1-y)}(2-D) + \frac{2(2z-1-y)}{z^2(1-y)} + f_1. \tag{9.9}$$

where f_1 is a finite contribution.

At this point the wanted amplitude has the form

$$a_{15}^{(2,0)} = (-ie)g^2 T^a_{ik}T^a_{kj}(\frac{\mu^2}{s})^\epsilon \frac{1}{2(2\pi)^{3-2\epsilon}}(2\pi)^D\delta^{(D)}(P-p_1-p_2) < p_1|\gamma^\alpha|p_2 > \times$$

$$\times\Omega_{(2-2\epsilon)}\int dydz\; z^{1-2\epsilon}(1-y)^{-\epsilon}(1+y)^{-\epsilon}\left[\frac{1}{4(1-y)}(2-D) + \frac{(2z-1-y)}{4z^2(1-y)} + f_1\right] \tag{9.10}$$

The integration over y and z can be done on the limited domain given in Eq. (9.5) but it is not necessary to perform it now. It turns out that another one of the integrals computed here have the same phase-space and we choose to add the integrands and then perform the integration because in this way further simplifications occur.

Let us now go to the amplitude $a_{18}^{(2,0)}$ and use the same parametrization in Eq.(9.1) to compute it.

The phase space limits are obtained in the same manner as above. This time we have two different Heaviside functions, thus more constraints on the limits.

The two constraints coming from the Heaviside functions are

$$\triangle \geqslant |\rho(\mathbf{q_3}, \mathbf{p_1} - \mathbf{q_3})| = \left(z - 1 + \sqrt{1 - 2zy + z^2}\right)$$

and

$$\triangle \geqslant |\rho(\mathbf{q_3}, \mathbf{p_1} - \mathbf{q_3})| = \left(z + 1 - \sqrt{1 - 2zy + z^2}\right). \tag{9.11}$$

The two inequalities above are solved to hold the following limits for z and y :

$$0 \leq z \leq \frac{\triangle}{2} \frac{2 + \triangle}{1 - y + \triangle} \text{ with } -1 \leq y \leq \frac{\triangle}{2},$$

$$0 \leq z \leq \frac{\triangle}{2} \frac{2 - \triangle}{1 + y - \triangle} \text{ with } \frac{\triangle}{2} \leq y \leq 1. \tag{9.12}$$

The same type of manipulations as in the previous calculation should be done here and in the end isolate the divergent part. In this case the integrand is divergent only in the limit $z \to 0$ as the whole expression vanishes in the collinear limit. The divergent part is simply $F(y, 0) = \frac{1}{z^2}|_{z=0}$

The amplitude has the form

$$a_{18}^{(2,0)} = ieg^2 T_{ik}^a T_{kj}^a (\frac{\mu^2}{s})^\epsilon \frac{1}{2(2\pi)^{3-2\epsilon}} (2\pi)^D \delta^{(D)}(P - p_1 - p_2) < p_1|\gamma^\alpha|p_2 > \times$$

$$\Omega_{(2-2\epsilon)} \int dydz\, z^{1-2\epsilon}(1-y)^{-\epsilon}(1+y)^{-\epsilon} \left[\frac{1}{z^2} + f_2\right]. \tag{9.13}$$

where f_2 is finite.

Integrals of this type can be performed making use of the so called 'plus' distribution.

It can be showed that for an integrable function $f(x)$ the following relation holds in the $\epsilon \to 0$ limit

$$\int_a^1 dx(1-x)^{-1-\epsilon} f(x) = -\frac{f(1)}{\epsilon} + \int_a^1 dx f(x) \left[\frac{1}{1-x}\right]_+ \tag{9.14}$$

Where the 'plus' distribution is defined this way

$$\int_a^1 dx f(x) \left[\frac{1}{1-x}\right]_+ = \int_a^1 dx\, (f(x) - f(1)) \frac{1}{1-x} + f(1) ln(1-a). \tag{9.15}$$

This relations are used to solve the integral in Eq.(9.13). The whole calculation will not be presented here being too lenghty. The final results reads

$$2a_{18}^{(2,0)} = \frac{C_F}{2} \frac{\alpha_s}{2\pi} \left(\frac{\mu^2}{s}\right)^\epsilon \left(\frac{1}{\epsilon} + g_2(\triangle) + F_2\right) \times A^{(0)}\left(q_{q1}, \bar{q}_{q2}; \gamma(P)\right), \tag{9.16}$$

where

$$g_2(\triangle) = 2log2 - 2log\left(\frac{\triangle}{2}\right), \tag{9.17}$$

and the finite function

$$F_2 = (2\pi)^{D-1}\delta^{(D-1)}(\vec{P} - \vec{p_1} - \vec{p_2}) < p_1|\gamma^\alpha|p_2 > \times$$

$$\times \int d\tilde{q}_3 \Theta(\triangle - |\rho(\mathbf{q_3}, \mathbf{p_1} - \mathbf{q_3})|)\Theta(\triangle - |\rho(\mathbf{q_3}, \mathbf{p_2})|)\, f_2(p_1, p_2, q_3). \tag{9.18}$$

Next we have to solve the integrals to get the amplitude $a_1^{\{1,1\}}$. Looking at its expression as given in Eq. (5.35) we see that the phase-space is restricted again by a Heaviside function which happens to be the same as in the case of $a_{15}^{\{2,0\}}$. Thus the limits obtained in Eq. (5.5) hold here also.

With the parametrization from Eq.(9.1) and after the singular terms are isolated the amplitude takes the following form

$$a_1^{\{1,1\}} = (-ie)g^2 T^a_{ik} T^a_{kj} \left(\frac{\mu^2}{s}\right)^\epsilon \frac{1}{2(2\pi)^{3-2\epsilon}}(2\pi)^D \delta^{(D)}(P - p_1 - p_2) < p_1|\gamma^\alpha|p_2 > \times$$

$$\times \Omega_{(2-2\epsilon)} \int dydz z^{1-2\epsilon}(1-y)^{-\epsilon}(1+y)^{-\epsilon}\left(\frac{(D-2)z^2 + 4 - 4z}{2(1-y)z^2} + f_3\right). \tag{9.19}$$

We do not need to compute this amplitude at this moment but rather add it to the expression for $a_{15}^{\{2,0\}}$ and get

$$a_1^{\{1,1\}} + a_{15}^{\{2,0\}} = (-ie)g^2 T^a_{ik} T^a_{kj} \left(\frac{\mu^2}{s}\right)^\epsilon \frac{1}{2(2\pi)^{3-2\epsilon}}(2\pi)^D \delta^{(D)}(P - p_1 - p_2) < p_1|\gamma^\alpha|p_2 > \times$$

$$\Omega_{(2-2\epsilon)} \int dydz z^{1-2\epsilon}(1-y)^{-\epsilon}(1+y)^{-\epsilon}\left(\frac{D-2}{4(1-y)} + \frac{3 - 2z - y)}{2z^2(1-y)} + f_3 + f_1\right). \tag{9.20}$$

This last integral is done again using the plus distribution method and the result is

$$2a_{15}^{(2,0)} + 2a_1^{(1,1)} = \frac{C_F}{2}\frac{\alpha_s}{2\pi}\left(\frac{\mu^2}{s}\right)^\epsilon \left(\frac{1}{\epsilon^2} + \frac{1}{2\epsilon} + g_1(\triangle) + F_1\right) \times A^{(0)}\left(q_{q1}, \bar{q}_{q2}; \gamma(P)\right), \tag{9.21}$$

where

$$g_1(\triangle) = \frac{7}{2} - \frac{3}{2}\left(\frac{\triangle}{2}\right)^2 + \left[\frac{1}{2} + 3\frac{\triangle}{2} - \frac{1}{2}\left(\frac{\triangle}{2}\right)^2\right] log\left(\frac{\triangle}{2}\right) - log^2\left(\frac{\triangle}{2}\right) -$$

$$-2log^2\left(1 + \frac{\triangle}{2}\right) - 4Li_2\left(\frac{2}{2+\triangle}\right), \tag{9.22}$$

and the finite function

$$F_1 = (2\pi)^{D-1}\delta^{(D-1)}(\vec{P} - \vec{p_1} - \vec{p_2}) < p_1|\gamma^\alpha|p_2 > \times$$

$$\times \int d\tilde{q}_3 \Theta(\triangle - |\rho(\mathbf{q_3}, \mathbf{p_1} - \mathbf{q_3})|)\Theta(\triangle - |\rho(\mathbf{q_3}, \mathbf{p_2})|) \, (f_1(p_1, p_2, q_3) + f_3(p_1, p_2, q_3)) \, . \qquad (9.23)$$

The last term in Eq.(9.22) is the Spence's function or dilogarithm. For $\triangle \to 0$ its value aproaches $Li_2(1) = \frac{\pi^2}{6}$.

9.2 Appendix chapter 6

We shall compute here the commutators of Eq. (6.48) with the expressions of A_1 and A_2 given in Eq. (6.45) and Eq. (6.46).

Let us start with evaluating $[A_2, A_1]$. Since only the virtual corrections are taken into account there are two terms that contribute. First one is the second order in g :

$$-g^2 \int_{-\infty}^{0} ds \int_{0}^{\infty} dt \, \frac{p_1^\nu}{\omega(p_1)} \frac{p_2^\mu}{\omega(p_2)} \int d\tilde{q} \int d\tilde{k} \, e^{iqp_1 s/\omega(p_1)} e^{-ikp_2 t/\omega(p_2)} \left[T_a a_\mu^a(k), T_b a_\nu^{b\dagger}(q)\right]$$

$$= -g^2 \int_{-\infty}^{0} ds \int_{0}^{\infty} dt \, \frac{p_1^\nu}{\omega(p_1)} \frac{p_2^\mu}{\omega(p_2)} \int d\tilde{q} \int d\tilde{k} \, e^{iqp_1 s/\omega(p_1)} e^{-ikp_2 t/\omega(p_2)} T_a T_b \left[a_\mu^a(k), a_\nu^{b\dagger}(q)\right]$$

$$= -g^2 \int_{-\infty}^{0} ds \int_{0}^{\infty} dt \, \frac{p_1^\nu}{\omega(p_1)} \frac{p_2^\mu}{\omega(p_2)} \int \frac{d^4k}{(2\pi)^4} (-i g_{\mu\nu}) \delta^{ab} \frac{1}{k^2 + i\epsilon} e^{-ik(\frac{p_1 s}{\omega(p_1)} - \frac{p_2 t}{\omega(p_2)})} T_a T_b$$

$$= -g^2 C_F \int d^4k \frac{i}{k^2 + i\epsilon} \frac{p_1 p_2}{p_1 k - i\epsilon} \frac{1}{p_2 k - i\epsilon}.$$

we have used that $\left[T_a a_\mu^a(k), T_b a_\nu^{b\dagger}(q)\right] = T_a T_b \left[a_\mu^a(k), a_\nu^{b\dagger}(q)\right] + [T_a, T_b] \, a_\nu^{b\dagger}(q) a_\mu^a(k)$ and note that the last term is normal ordered so it will not contribute.

For simplicity we denote $e^w = e^{iq_1 p_1 s_1/\omega(p_1) + iq_2 p_1 s_2/\omega(p_1)} e^{-ik_1 p_2 t_1/\omega(p_2) - ik_2 p_2 t_2/\omega(p_2)}$. We shall see that this exponential is common to all g^4 terms.

Second term is of order g^4

$$\frac{1}{4} g^4 \int_{-\infty}^{0} ds_1 \int_{-\infty}^{s_1} ds_2 \int_{0}^{\infty} dt_1 \int_{t_1}^{\infty} dt_2 \frac{p_1^{\nu_1} p_1^{\nu_2}}{(\omega(p_1))^2} \frac{p_2^{\mu_1} p_2^{\mu_2}}{(\omega(p_2))^2} \int d\tilde{q}_1 d\tilde{q}_2 \int d\tilde{k}_1 d\tilde{k}_2 e^w \times$$

$$\times \left[[T_{a_2}, T_{a_1}] \, a_{\mu_1}^{a_1}(k_1) a_{\mu_2}^{a_2}(k_2), [T_{b_1}, T_{b_2}] \, a_{\nu_1}^{b_1\dagger}(q_1) a_{\nu_2}^{b_2\dagger}(q_2) \right]$$

$$= \frac{1}{4} g^4 \int_{-\infty}^{0} ds_1 \int_{-\infty}^{s_1} ds_2 \int_{0}^{\infty} dt_1 \int_{t_1}^{\infty} dt_2 \int d\tilde{q}_1 d\tilde{q}_2 \int d\tilde{k}_1 d\tilde{k}_2 \frac{p_1^{\nu_1} p_1^{\nu_2}}{(\omega(p_1))^2} \frac{p_2^{\mu_1} p_2^{\mu_2}}{(\omega(p_2))^2} e^w \times$$

$$\times [T_{a_2}, T_{a_1}] [T_{b_1}, T_{b_2}] \left\{ \left[a_{\mu_1}^{a_1}(k_1), a_{\nu_1}^{b_1\dagger}(q_1)\right] \left[a_{\mu_2}^{a_2}(k_2), a_{\nu_2}^{b_2\dagger}(q_2)\right] + \left[a_{\mu_1}^{a_1}(k_1), a_{\nu_2}^{b_2\dagger}(q_2)\right] \left[a_{\mu_2}^{a_2}(k_2), a_{\nu_1}^{b_1\dagger}(q_1)\right] \right\}$$

$$= \frac{1}{4} g^4 (-C_A C_F) \int_{-\infty}^{0} ds_1 \int_{-\infty}^{s_1} ds_2 \int_{0}^{\infty} dt_1 \int_{t_1}^{\infty} dt_2 \frac{p_1 p_2}{(\omega(p_1))^2} \frac{p_1 p_2}{(\omega(p_2))^2} \int \frac{d^4k_1}{(2\pi)^4} \frac{d^4k_2}{(2\pi)^4} \frac{i}{k_1^2 + i\epsilon} \frac{i}{k_2^2 + i\epsilon} \times$$

$$\times \{ -\exp \, (ik_1 p_1 s_1/\omega(p_1) + ik_2 p_1 s_2/\omega(p_1)) \exp(-ik_1 p_2 t_1/\omega(p_2) - ik_2 p_2 t_2/\omega(p_2)) +$$

$$+ \exp \, (ik_2 p_1 s_1/\omega(p_1) + ik_1 p_1 s_2/\omega(p_1)) \exp(-ik_1 p_2 t_1/\omega(p_2) - ik_2 p_2 t_2/\omega(p_2)) \}$$

$$= \frac{1}{4} g^4 (-C_A C_F) \int \frac{d^4k_1}{(2\pi)^4} \frac{d^4k_2}{(2\pi)^4} \frac{i}{k_1^2 + i\epsilon} \frac{i}{k_2^2 + i\epsilon} (p_1 p_2)^2 \times$$

$$\times\left\{-\frac{1}{(k_2p_1-i\epsilon)(k_2p_2-i\epsilon)(p_1(k_1+k_2)-i\epsilon)(p_2(k_1+k_2)-i\epsilon)}+\frac{1}{(k_1p_1-i\epsilon)(k_2p_2-i\epsilon)(p_1(k_1+k_2)-i\epsilon)(p_2(k_1+k_2)-i\epsilon)}\right\}.$$

Next commutator to be computed is of the form $\frac{1}{2}\left[A_1,\left[A_1,A_2\right]\right]$. The argument of keeping only the virtual corrections is used again to find

$$\frac{1}{2}\left[A_1,\left[A_1,A_2\right]\right]=\frac{1}{2}\left[\left[A_2,A_1\right],A_1\right]=$$

$$=\frac{1}{4}g^4\int_0^\infty dt_1\int_{t_1}^\infty dt_2\int_{-\infty}^0 ds_1\int_{-\infty}^0 ds_2\frac{p_1^{\nu_1}p_1^{\nu_2}}{(\omega(p_1))^2}\frac{p_2^{\mu_1}p_2^{\mu_2}}{(\omega(p_2))^2}\int d\tilde{q}_1d\tilde{q}_2\int d\tilde{k}_1d\tilde{k}_2\times$$

$$\times e^w\left[\left[\left[T_{a_2},T_{a_1}\right]a_{\mu_1}^{a_1}(k_1)a_{\mu_2}^{a_2}(k_2)\,,\,T_{b_1}a_{\nu_1}^{b_1\dagger}(q_1)\right],\,T_{b_2}a_{\nu_2}^{b_2\dagger}(q_2)\right]$$

$$=\frac{1}{4}g^4\int_0^\infty dt_1\int_{t_1}^\infty dt_2\int_{-\infty}^0 ds_1\int_{-\infty}^0 ds_2\frac{p_1^{\nu_1}p_1^{\nu_2}}{(\omega(p_1))^2}\frac{p_2^{\mu_1}p_2^{\mu_2}}{(\omega(p_2))^2}\int d\tilde{q}_1d\tilde{q}_2\int d\tilde{k}_1d\tilde{k}_2\left[T_{a_2},T_{a_1}\right]T_{b_1}T_{b_2}e^w\times$$

$$\times\left\{\left[a_{\mu_1}^{a_1}(k_1),a_{\nu_1}^{b_1\dagger}(q_1)\right]\left[a_{\mu_2}^{a_2}(k_2),a_{\nu_2}^{b_2\dagger}(q_2)\right]+\left[a_{\mu_1}^{a_1}(k_1),a_{\nu_2}^{b_2\dagger}(q_2)\right]\left[a_{\mu_2}^{a_2}(k_2),a_{\nu_1}^{b_1\dagger}(q_1)\right]\right\}$$

$$=\frac{1}{4}g^4(-\frac{C_AC_F}{2})\int_0^\infty dt_1\int_{t_1}^\infty dt_2\int_{-\infty}^0 ds_1\int_{-\infty}^0 ds_2\int\frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4}\frac{i}{k_1^2+i\epsilon}\frac{i}{k_2^2+i\epsilon}(\frac{p_1p_2}{\omega(p_1)\omega(p_2)})^2\times$$

$$\times\left\{\exp\left(ik_1p_1s_1/\omega(p_1)+ik_2p_1s_2/\omega(p_1)\right)\exp\left(-ik_1p_2t_1/\omega(p_2)-ik_2p_2t_2/\omega(p_2)\right)-\right.$$

$$\left.-\exp\left(ik_2p_1s_1/\omega(p_1)+ik_1p_1s_2/\omega(p_1)\right)\exp\left(-ik_1p_2t_1/\omega(p_2)-ik_2p_2t_2/\omega(p_2)\right)\right\}$$

$$=-\frac{1}{4}g^4(-\frac{C_AC_F}{2})\int\frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4}\frac{i}{k_1^2+i\epsilon}\frac{i}{k_2^2+i\epsilon}(p_1p_2)^2\times$$

$$\times\left\{\frac{1}{(p_1k_2-i\epsilon)(p_2k_2-i\epsilon)(p_1k_1-i\epsilon)(p_2(k_1+k_2)-i\epsilon)}-\frac{1}{(p_1k_1-i\epsilon)(p_2k_2-i\epsilon)(p_1k_2-i\epsilon)(p_2(k_1+k_2)-i\epsilon)}\right\}=0.$$

And in a simmilar manner

$$\frac{1}{2}\left[A_2,\left[A_2,A_1\right]\right]=0.$$

And last commutator to evaluate is the one with four fields

$$\frac{1}{4}[A_2,[A_1,[A_1,A_2]]]=\frac{1}{4}g^4\int_{-\infty}^0 ds_1\int_{-\infty}^0 ds_2\int_0^\infty dt_1\int_0^\infty dt_2\int d\tilde{k}_1d\tilde{k}_2\int d\tilde{q}_1d\tilde{q}_2\frac{p_1^{\nu_1}p_1^{\nu_2}}{(\omega(p_1))^2}\frac{p_2^{\mu_1}p_2^{\mu_2}}{(\omega(p_2))^2}\times$$

$$\times e^w\left[T_{a_1}a_{\mu_1}^{a_1}(k_1),\left[T_{b_1}a_{\nu_1}^{b_1\dagger}(q_1),\left[T_{b_2}a_{\nu_2}^{b_2\dagger}(q_2),T_{a_2}a_{\mu_2}^{a_2}(k_2)\right]\right]\right]=$$

$$=\frac{1}{4}g^4\int_{-\infty}^0 ds_1\int_{-\infty}^0 ds_2\int_0^\infty dt_1\int_0^\infty dt_2(\frac{p_1p_2}{\omega(p_1)\omega(p_2)})^2\int d\tilde{k}_1d\tilde{k}_2e^w\times$$

$$\times T_{a_1}\left[T_{a_2},T_{b_2}\right]T_{b_1}\left[a^{a_1}_{\mu_1}(k_1),a^{b_2\dagger}_{\nu_2}(q_2)\right]\left[a^{a_2}_{\mu_2}(k_2),a^{b_1\dagger}_{\nu_1}(q_1)\right]=$$

$$=\frac{1}{4}g^4\left(-\frac{C_AC_F}{2}\right)\int\frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4}\frac{i}{k_1^2+i\epsilon}\frac{i}{k_2^2+i\epsilon}(p_1p_2)^2\frac{1}{(p_1k_1-i\epsilon)(p_1k_2-i\epsilon)\,(p_2k_1-i\epsilon)\,(p_2k_2-i\epsilon)}=$$

$$=\frac{1}{2}g^4\left(-\frac{C_AC_F}{2}\right)\int\frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4}\frac{i}{k_1^2+i\epsilon}\frac{i}{k_2^2+i\epsilon}(p_1p_2)^2\times$$

$$\times\left\{\frac{1}{(k_2p_1+i\epsilon)(k_2p_2+i\epsilon)(p_1(k_1+k_2)+i\epsilon)(p_2(k_1+k_2)+i\epsilon)}+\frac{1}{(k_1p_1+i\epsilon)(k_2p_2+i\epsilon)(p_1(k_1+k_2)+i\epsilon)(p_2(k_1+k_2)+i\epsilon)}\right\}.$$

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