Resummation

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Outline

Resummation

"Resummation" is as vague a word as "perturbation theory". It is in addition misleading: it implies you sum again what was summed before.

Resummation:

> *The art of constructing, from a subset of terms in a finite order perturbation series, an all-orders expression whose expansion gives at least those terms back.*

So to be clear about resummation, we must specify more.

Our context shall be QCD observables at hadron colliders, and their perturbative description.

Perturbation theory

One describes an observable (cross section) by

$$O = \sum_n c_n \alpha_s^n + R_n$$

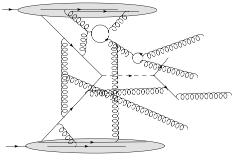
The c_n are computed using Feynman diagrams. Lowest order usually only needs tree graphs, next-to-leading order has 1-loop graphs etc. State of the art: 3-loop.

This approach requires

- α_s small enough
- R_n very small
- (an infrared safe O)
- c_n should not misbehave as n increases

QCD FACTORIZATION THEOREM

With quantum QCD, things look bad



This does not look at all like the parton model. However, one can show that for QCD, to all orders in perturbation theory

$$\sigma_{AB\to X}(Q) = \sum_{ab} \int d\xi_1 \int d\xi_2 \phi_{a/A}(\xi_1,\mu)\, \phi_{b/B}(\xi_2,\mu) \times \hat{\sigma}_{ab\to X}(\xi_1,\xi_2,\mu,Q,\alpha_s(\mu)) + \underbrace{\mathcal{O}\left(\frac{1}{Q^p}\right)}_{\text{power corrections}}$$

Collins, Soper, Sterman; Bodwin

Parton model formula survives! Note

- it separates short-distance from long-distance physics
- the α_s dependendence of the partonic cross section
- the μ dependence of the parton distributions function $\phi_{a/A}(\xi,\mu)$
- the $\phi_{a/A}(\xi,\mu)$ are *shown* to be process-independent

QCD OBSERVABLES AT HADRON COLLIDERS

In summary

$$O = \phi \otimes \hat{O} + P_0$$

- $\hat{O}$: partonic version of observable
- ϕ: universal parton distribution/fragmentation functions
- P_O: power corrections $p_n(\Lambda_{QCD}/Q)^n$

This relation can be used in various ways:

- take ϕ from WWW, assume $P_O = 0$, calculate $\hat{O}$, compare with O_{exp}.
- calculate $\hat{O}$ to certain approximation (assume $P_O = 0$), measure O, fit ϕ, F.

The game: find the *weakest link*, and update.

Need best possible calculation of $\hat{O} = \sum_n c_n \alpha_s^n$

But c_n often misbehave...

Large logarithms

Two generic situations:

1. Single log:

$$\hat{O}_1 = 1 + \alpha(L+1) + \alpha^2(L^2+L+1) + \alpha^3(L^3+L^2+L+1) + \ldots$$

2. Double log:

$$\hat{O}_2 = 1 + \alpha(L^2+L+1) + \alpha^2(L^4+L^3+L^2+L+1) + \ldots$$

"1": constants ($\pi^2 \ldots$). Effective expansion parameters: O_1: αL O_2: αL^2

Resummation When good logs go bad

Resummation = organization of large logarithms in perturbative expansions:

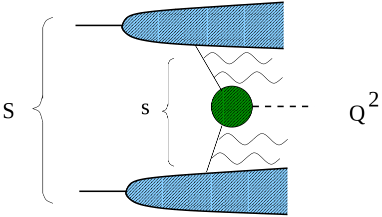
$$
\begin{aligned}
\hat{O} &= 1 + \alpha_s(L^2 + L + 1) + \alpha_s^2(L^4 + L^3 + L^2 + L + 1) + \ldots \\
&= \exp\left(\underbrace{\underbrace{Lg_1(\alpha_s L)}_{LL} + g_2(\alpha_s L)}_{NLL} + \alpha_s g_3(\alpha_s L) + \ldots\right) \underbrace{C(\alpha_s)}_{\text{constants}} \\
&\quad + \text{ suppressed terms}
\end{aligned}
$$

$L = \ln(?)$. Argument differs per observable. Benefits/hopes:

- Restore predictive power
- Better description of physics
- Increase theoretical accuracy

Threshold double logs

Arguments of recoil log was "visible". Invisible logs can also plague PT, e.g. in Drell-Yan ($p + \bar{p} \rightarrow \gamma^*(Q) + X$)

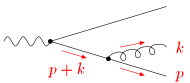


"Threshold" double logs: $L^2 = \ln^2\left(1 - \frac{Q^2}{s}\right)$ with $S > s > Q^2$, then

$$S \gtrsim Q^2 \quad \rightarrow \quad \frac{Q^2}{s} \simeq 1 \quad \rightarrow \quad \ln^2\left(1 - \frac{Q^2}{s}\right) \gg 1$$

NOTE: argument of L contains s parton cms energy, to be integrated over $[Q^2, S]$. But L can be large in whole integration region.

Origin of double logs

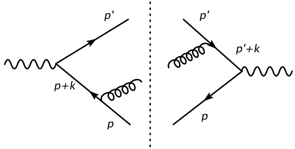


- propagator with $p^2 = k^2 = 0$. Singularities: $E_{\rm g} = 0 \longrightarrow$ *soft*; $\theta_{\rm qg} = 0 \longrightarrow$ *collinear*.

$$\frac{1}{(p+k)^2} = \frac{1}{2p\cdot k} = \frac{1}{2E_{\rm g}E_{\rm q}(1-\cos\theta_{\rm qg})}\,.$$

- phase space integration

$$\alpha_s \int \frac{d^4k}{(2\pi)^4}\,\frac{p\cdot p'}{p\cdot k\; p'\cdot k} \sim \alpha_s \int \frac{dE_{\rm g}}{E_{\rm g}} \int \frac{d\theta_{\rm qg}}{\theta_{\rm qg}} \sim \alpha_s \ln^2(\ldots)\,.$$



Logs and divergences

When dimensionally regulating this integral

$$\alpha_s \int \frac{d^{4-2\epsilon}k}{(2\pi)^4} \frac{p \cdot p'}{p \cdot k \, p' \cdot k} \sim \alpha_s \int^K \frac{dE_{\mathrm{g}} E_{\mathrm{g}}^{-\epsilon}}{E_{\mathrm{g}}} \int \frac{d\theta_{\mathrm{qg}} \sin^{-\epsilon} \theta_{\mathrm{qg}}}{\theta_{\mathrm{qg}}}$$

$$\sim \alpha_s \left(\frac{1}{\epsilon^2} + \ln^2(K)\right). \quad (1)$$

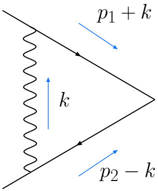
we see that double logs follow from the same integral as IR and COL divergences

So, how to analyse IR and COL divergences to all orders?

Landau equations

We follow the approach of Collins, Soper, Sterman

For a general Feynman diagram, find its infrared and collinear divergences, by solving the *Landau equations*. Triangle diagram



$$\int d^4k \int d\alpha_1 d\alpha_2 \alpha_3 \frac{N\delta(1-\alpha_1-\alpha_2-\alpha_3)}{D^3},$$

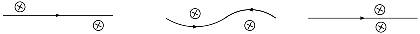
$$D = \alpha_1(p_1-k)^2 + \alpha_2(p_2+k)^2 + \alpha_3 k^2 + i\epsilon \quad (2)$$

Integrals are in complex plane. Divergence when $D=0$, *unavoidably*.

Landau equations

$$0 = D = \alpha_1(p_1 - k)^2 + \alpha_2(p_2 + k)^2 + \alpha_3 k^2 + i\epsilon$$

when, for each line, either $\alpha_i = 0$ or the line is on-shell. Quadratic equation in k^μ $\rightarrow$ 2 solutions. $D = 0$ can be *avoided* unless 2 solutions *pinch* the k^μ contours.



Landau equations:

$$D = 0, \quad \frac{\partial}{\partial k^\mu} D = 0, \quad \text{and} \quad \alpha_i = 0 \text{ or } l_i^2 = 0$$

Here

$$-\alpha_1(p_1 - k)^\mu + \alpha_2(p_2 + k)^\mu + \alpha_3 k^\mu = 0$$

$$-\alpha_1(p_1 - k)^\mu + \alpha_2(p_2 + k)^\mu + \alpha_3 k^\mu = 0$$

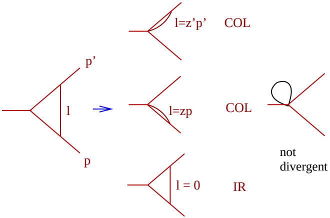
Solutions

$$k^\mu = z p_1^\mu, \quad \alpha_2 = 0, \quad \alpha_1(1-z) = \alpha_3 z \qquad (\mathrm{COL}-1)$$
$$k^\mu = -y p_2^\mu, \quad \alpha_1 = 0, \quad \alpha_2(1-z) = -\alpha_3 y \qquad (\mathrm{COL}-2)$$
$$k^\mu = 0, \quad \alpha_1/\alpha_3 = 0, \quad \alpha_2/\alpha_3 = 0 \qquad (\mathrm{IR})$$

Do-able for this case, but solving equation for multiloop diagram seems very hard.

Reduced diagrams, physical pictures

Trick: it can be done graphically, the divergences can be represented as reduced diagrams.



Reduced diagrams represent solutions to Landau equations.

S. Coleman, R. Norton '65: they must represent physically possible scatterings, with free propagation between collision points.

Separation of divergences

In this way, one can categorize all IR and COL divergences, and write, schematically

$$\sigma = J(\mathrm{col}) \times J(\mathrm{col}) \times S(\mathrm{soft}) \times H(\mathrm{off-shell})$$

- each function a series in α_s
- $J's$ a result of collinear integration regions, S from soft ones, H the rest
- a factorization of degrees of freedom!

Fine, but how does this help us resum?

From factorization to resummation, UV case

Consider a bare Green's function and its multiplicative renormalization

$$G_B(p, \Lambda, g_B) = Z\left(\frac{\mu}{\Lambda}, g(\mu)\right)\; G_R\left(\frac{p}{\mu}, g_R(\mu)\right)$$

- μ: renormalization scale, Λ: UV cut-off for loop momenta
- G_R, g_R renormalized (finite) Green function and coupling
- $L = \ln(\mu/p)$
- Z contains divergence ("effects of highly virtual modes") when $\Lambda \to \infty$

From factorization to resummation, UV case

Act with $d/d\mu$ on

$$G_B(p, \Lambda, g_B) = Z\left(\frac{\mu}{\Lambda}, g(\mu)\right)\; G_R\left(\frac{p}{\mu}, g_R(\mu)\right)$$

Gives 0 on LHS, so

$$\begin{aligned} \mu\frac{d}{d\mu}\ln G_R\left(\frac{p}{\mu}, g_R(\mu)\right) &= -\mu\frac{d}{d\mu}\ln Z\left(\frac{\mu}{\Lambda}, g(\mu)\right) \\ &\equiv \gamma(g_R(\mu)) \end{aligned}$$

(γ cannot depend on Λ). Hence

$$G_R\left(\frac{p}{\mu}, g_R(\mu)\right) = G_R\left(1, g_R(p)\right) \underbrace{\exp\left[\int_p^\mu \frac{d\lambda}{\lambda}\gamma(g_R(\lambda))\right]}_{\text{resummed}}$$

One can do something analogous for the IR + COL case, and obtain resummed cross sections Contopanagos, EL, Sterman:

Threshold resummation in Drell-Yan

The threshold-resummed cross section is G. Sterman; S. Catani, L. Trentadue:

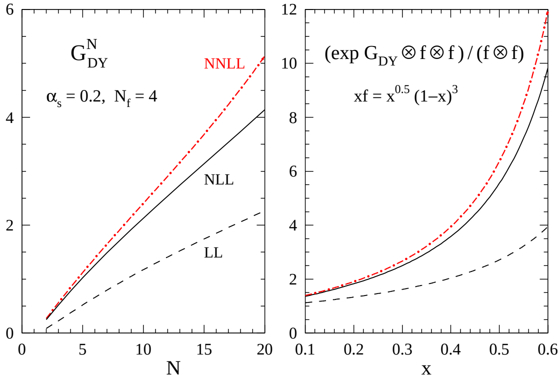
$$\begin{aligned}\hat{\sigma} &= \int_C \frac{dN}{2\pi i}\, z^{-N}\, \hat{\sigma}(N) \\ \sigma(N) &= \exp\left[-\int_0^1 dx\, \frac{x^{N-1}-1}{1-x} \left\{ \int_{Q^2}^{Q^2(1-x)^2} \frac{d\mu}{\mu} A(\alpha_s(\mu)) \right.\right. \\ &\qquad \left.\left. D(\alpha_s((1-x)Q) \right\} \right] \times (1 + \alpha_s(Q^2)\frac{C_F}{\pi} + \ldots)\end{aligned}$$

- Exponential form, two logaritmic integrals
- Integrands only function of α_s
- Residue series contains no logs
- (Inverse “Mellin” transform)

One can carry out the integrals and find

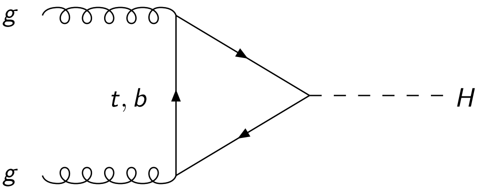
$$\hat{\sigma}_{DY}(N, Q^2) = g_0(Q^2) \exp\left[G^N_{DY}(Q^2)\right]$$
$$G^N_{DY} = \ln N g_1(\lambda) + g_2(\lambda) + \alpha_s g_3(\lambda) + \ldots, \qquad \lambda = \beta_0 \alpha_s \ln N$$

One sees good convergence both in the exponent and for the hadronic cross section Vogt



Threshold resummation in Higgs production

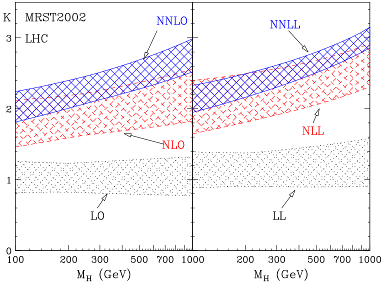
This is now among the best-known cross sections in QCD (apart from one minor matter..). It is known to NNLO C. Anastasiou, K. Melnikov '02; R. Harlander, W. Kilgore '02; W. van Neerven, V. Ravindran, J. Smith '03, whereas for the threshold resummation formula $g_{1,2,3}$ (NNLL) is known.
At lowest order the process happens via heavy quarks



HIGGS THRESHOLD RESUMMATION

NNLL resummed Higgs cross section S. Catani, D. de Florian, M. Grazzini, P. Nason '03 including extra terms M. Krämer, EL, M. Spira '98

K-factor:



- Scale uncertainties reduced by resummation, pert. theory well-behaved
- At the Tevatron:
  NLL + NLO = NLO(1+ 30%)
  NNLL + NNLO = NLO(1+ 16%)
- At the LHC:
  NLL + NLO = NLO(1+ 20%)
  NNLL + NNLO = NLO(1+ 9%)

→ we are ready to use the Higgs signal, after discovery, as an excellent QCD calibration.

Resumming constants in Drell-Yan

T. Eynck, EL, L. Magnea '03. We saw that a refactorization of the partonic cross section leads to resummation of logarithms.

This approach allows resummation also of large *constants* because they have same origin

$$\mathrm{Re}\left[\ln^2(-Q^2)\right] = \ln^2(Q^2) + \pi^2$$

Resummed constants

$$
\begin{aligned}
\hat{\sigma}(N) &= \left|\frac{\Gamma(Q,\epsilon)}{\phi_V(Q,\epsilon)}\right|^2 \exp\left[F_{\overline{\mathrm{MS}}}(\alpha_s)\right] \\
&\times \exp\left[\int_0^1 dz\, \frac{z^{N-1}-1}{1-z}\left\{4\int_Q^{(1-z)Q}\frac{d\mu}{\mu}\, A\left(\alpha_s(\mu)\right)\right.\right. \\
&+ \left.\left. D\left(\alpha_s\left((1-z)Q\right)\right)\right\}\right] + \mathcal{O}(1/N)\,.
\end{aligned}
$$

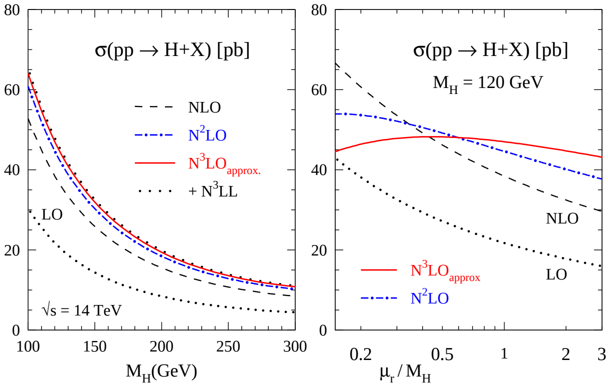
- *All* N-independent terms in Drell-Yan cross section exponentiate, including large π^2 terms.
- Less predictive than logarithm resummation, but non-trivial patterns emerge $\rightarrow$ new equations relating radiation functions to purely virtual diagrams

NNNLL threshold DY from DIS

Including the "constants" we find for Drell-Yan

$$\underbrace{D(\alpha_s)}_{\text{Single logs}} = 4 \underbrace{B_\delta(\alpha_s)}_{\text{virtual split.fn.}} - 2 \underbrace{G(\alpha_s)}_{\text{form factor}} + \beta(\alpha_s)\frac{d}{d\alpha_s}F_{\overline{\text{MS}}}(\alpha_s)$$

EL, Magnea; S. Moch, J. Vermaseren, A. Vogt Approximate *third* order results for Drell-Yan and Higgs production



Recoil resummation

Consider the cross section $d\sigma/dQ_T$

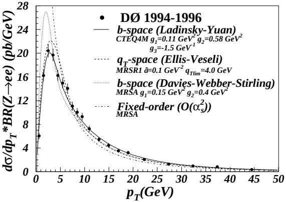
$$h + h \to V(Q_T) + X,$$

The Q_T results from recoil against the rest of the final state (soft gluons and quarks)

In recoil resummation one sums, for small Q_T, $\ln^2(Q/Q_T)$ effects to all orders.

In threshold resummation one deals mainly with normalizations (size, uncertainty).

Recoil resummation affects shapes of cross sections.

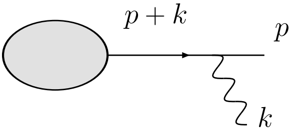


- finite order gets its wrong at small Q_T
- resummation gets it essentially right

The "eikonal" approximation

Consider the bit of Feynman diagram

$$M(\frac{i(\not{p}+\not{k})}{(p+k)^2}(-ig_s\gamma^\mu)\,u(p)$$



and assume $k \ll p$ so that

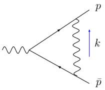
$$\not{p}+\not{k} \to \not{p}, \qquad (p+k)^2 \to 2\,p\cdot k,$$

Then, using Dirac equation, find

$$g_s\Big(M\,u(p)\Big)\frac{p_\mu}{p\cdot k}$$

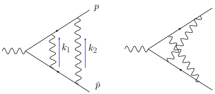
With these approximations, the 1st order virtual contribution is

$$\sigma_0 \int d^n k \frac{1}{k^2} \frac{p \cdot \bar{p}}{(p \cdot k)(\bar{p} \cdot k)}$$



The second order one:

$$\sigma_0 \int d^n k_1 d^n k_2 \frac{(p \cdot \bar{p})^2}{k_1^2 k_2^2} \frac{1}{\bar{p} \cdot (k_1 + k_2) \bar{p} \cdot k_2} \left[\frac{1}{p \cdot (k_1 + k_2)\, p \cdot k_2} + \frac{1}{p \cdot (k_1 + k_2)\, p \cdot k_1} \right]$$



Using the identity

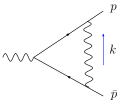
$$\frac{1}{p \cdot (k_1 + k_2)\, p \cdot k_2} + \frac{1}{p \cdot (k_1 + k_2)\, p \cdot k_1} = \frac{1}{p \cdot k_1\, p \cdot k_2}$$

and symmetrization yields the beginning of an exponential series!

$$\sigma_0 \frac{1}{2} \left(\int d^n k \frac{1}{k^2} \frac{p \cdot \bar{p}}{(p \cdot k)(\bar{p} \cdot k)} \right)^2$$

WEBS

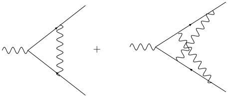
So we find the *algebraic* exponentiation

ln(QED eikonal perturbation theory) = 

In QCD, there is a color matrix at every vertex. It still holds that J. Gatherall '83; J. Frenkel, J. Taylor '84.

$$Y = \exp X$$

where X is subset of the Feynman graphs in Y, with different color factors: webs

ln(QCD eikonal perturbation theory) = 

Note: only for 2 color lines. Powerful approach.
Recent progress: also for 4 color lines there are surprising patterns Aybat, Dixon, Sterman.

Joint resummation

Resums consistently both threshold ($\ln^2(1-z)$) and recoil ($\ln^2(Q/Q_T)$) double logs H.-N. Li '99; EL, G. Sterman, W. Vogelsang '00,'01.

After Mellin $\int_0^1 dz z^N$ and Fourier $\int d^2 k_T \exp(ib \cdot k_T)$ transforms the logs are

$$\ln^2 N, \qquad \ln^2 b$$

For joint resummation the expression is similar

$$d\sigma_{AB}(z, Q_T) = \sigma_0 \sum_a C_{a/A}(N,b) e^{E_{a\bar{a}}(N,b)} C_{\bar{a}/B}(N,b)$$

Typical (NLL) exponent:

$$E(N,b,Q) = \left[\int_{Q/\chi(N,b)}^{Q} \frac{d\mu}{\mu} A(\alpha_s(\mu)) \ln \frac{N\mu}{Q} \right], \quad \chi(N,b) = N + bQ$$

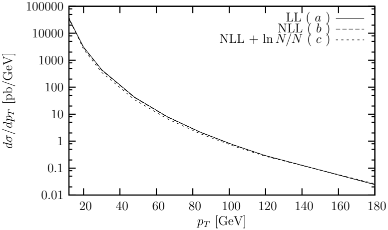
SUBLEADING $\ln^i N/N$ TERMS

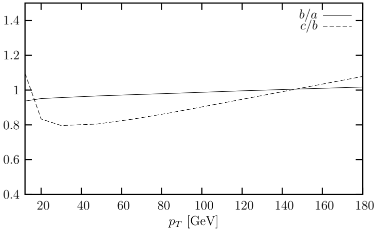
Basu, EL, Misra, Motylinski 207

How to include terms "beyond the constants"?

$$\alpha_s^i \sum_j^{2i-1} \frac{\ln^j N}{N}$$

Important for Higgs production! Can do this for joint and threshold resummation





Summary

- Resummation can extend the validity of perturbation theory
- Improves normalizations and shapes of key cross sections
- Resummation just as systematic at normal perturbation theory
- Factorization of soft, collinear gluons key
- New technology promises further improvements in accuracy