NIKHEF Academic Lectures 2006: The Standard Model

Lecture 2: Construction of (quantum) Standard Model

Eric Laenen

NIKHEF and Utrecht University

Todays outline

1. Recap
2. Local non-abelian symmetry
    - Local non-abelian symmetry
    - Lie groups and algebras
3. Chiral fermions
    - Chirality
    - Chiral matter and local symmetry
    - SM Fermions and $SU(2) \times U(1)$ gauge fields
    - Gauge field mixing
4. Scalar fields
    - Higgs doublet
5. Quantum aspects
    - Scattering amplitudes, S-matrix
    - Loops in Feynman diagrams
    - Renormalizability
    - Unitarity

Yesterday's messages

- Important concept: fields
- QFT = QM + Special Relativity
- Dynamics via path-integral in field space (functional integral)
- Phase-weight of path dictated by action
- Global symmetries of action constrain possible interactions
- Local symmetries lead to new interactions with gauge fields

Local non-abelian symmetry

Start with

$$\mathcal{L} = \partial_\mu \phi^\dagger \partial^\mu \phi - m^2 \phi^\dagger \phi - \lambda (\phi^\dagger \phi)^2, \quad \phi^\dagger = (\phi_1^*, \ldots, \phi_N^*)$$

$\mathcal{L}$ has a global matrix-phase invariance

$$\phi(x) \to U\phi(x), \quad U^\dagger = U^{-1}, \quad \det U = 1$$

Radical demand: can we also let symmetry hold *locally*?

$$\phi(x) \to U(x)\phi(x)$$

Ok for 2nd and 3rd term, but not for first, because

$$\partial_\mu \phi(x) \to U(x)\partial_\mu \phi(x) + (\partial_\mu U(x))\phi(x)$$

Can we define better, covariant derivative D_μ such that

$$D_\mu \phi(x) \to U(x) D_\mu \phi(x)$$

Covariant derivative

To make covariant derivative work, use new matrix field W_μ

$$D_\mu = \mathbf{1}\partial_\mu - W_\mu, \quad W'_\mu(x) = U(x)W_\mu(x)U^{-1}(x) + (\partial_\mu U(x))U^{-1}(x)$$

Then all we have to do: $\partial_\mu \to D_\mu$ in $\mathcal{L}$

$$\mathcal{L} = D_\mu\phi^\dagger D^\mu\phi - m^2\phi^\dagger\phi - \lambda(\phi^\dagger\phi)^2$$

Interactions! Also

$$[D_\mu, D_\nu]\phi = -G_{\mu\nu}\phi, \quad G_{\mu\nu} = \partial_\mu W_\nu - \partial_\nu W_\mu - [W_\mu, W_\nu]$$

$G_{\mu\nu}$ is locally covariant: $G'_{\mu,\nu}(x) = U(x)G_{\mu,\nu}(x)U^{-1}(x)$.

To understand the matrix field W_μ better, we need a little bit more group theory.

Lie groups, $SU(N)$

Lie group: infinite number of group elements, parametrized by n parameters

$$g(\xi^1, \ldots, \xi^n) = \exp(i\xi^a t_a) = \sum_{n=0}^{\infty} \frac{1}{n!}(i\xi^a t_a)^n$$

The t_a are also $N \times N$ matrices: the *generators* of the group, *hermitian*.
Only need to vary ξ's to reach all elements.
Product of two group elements must have the same form $\rightarrow$

$$[t_a, t_b] = if_{abc} t_c$$

The f_{abc} are *structure constants*, they more or less define the group, and give the rules for the Lie algebra. E.g. $SU(2)$

$$t_a^F = \tfrac{1}{2}\sigma_a, \quad [t_a^F, t_b^F] = i\epsilon_{abc} t_c^F$$

Also certain 3×3 matrices obey same Lie algebra $[t_a^A, t_b^A] = i\epsilon_{abc} t_c^A$

$$t_1^A = i \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}, \; t_2^A = i \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \; t_3^A = i \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Representations of $SU(N)$

We just saw 2 representations of $SU(N)$, obeying the same $SU(2)$ Lie algebra

- Fundamental/vector/defining: $N \times N$
- Adjoint $(N^2-1) \times (N^2-1)$ [why?]

Clever way to represent fields that transform in adjoint representation: don't make (N^2-1)-entry vectors, but rather make combination $\chi^a t_a^F$

$$SU(2): \qquad \chi \equiv \chi^a t_a^F = \frac{1}{2}\begin{pmatrix} \chi^3 & \chi^1 - i\chi^2 \\ \chi^1 + i\chi^2 & \chi^3 \end{pmatrix}$$

Transformation rule

$$\chi' = U \chi U^{-1}$$

Useful because then

$$Tr(\chi_1' \chi_2') = Tr(U\chi_1 U^{-1}\, U\chi_2 U^{-1}) = Tr(\chi_1 \chi_2)$$

Products of Lie groups

A straightforward procedure to construct bigger (more parameters) groups from smaller ones is by products $G = G_1 \times G_2$.

$$g = g_1 \times g_2, \quad g \cdot h = (g_1 \cdot h_1) \times (g_2 \cdot h_2)$$

In matrix representation, if $G_1 = SU(2)$ and $G_2 = U(1)$

$$\begin{pmatrix} \phi_1' \\ \phi_2' \end{pmatrix} = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} e^{iy\alpha} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix} = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} \begin{pmatrix} e^{iy\alpha} & 0 \\ 0 & e^{iy\alpha} \end{pmatrix} \begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$$

Gauge field representation

Returning to our non-abelian covariant derivative $D_\mu = \mathbf{1}\partial_\mu - W_\mu(x)$ one finds as matrix

$$W_\mu(x) = W_\mu^1(x) t_1 + \ldots + W_\mu^{(N^2-1)}(x) t_{(N^2-1)}$$

So there are as many separate gauge fields as are there are group generators. *Gauge fields transform in adjoint representation.* Another manifestion of how strong forces and symmetry are linked.

Now easy to prove:

$$\mathrm{Tr}\left[G'_{\mu\nu}(x) G'^{\mu\nu}(x)\right] = \mathrm{Tr}\left[U(x) G_{\mu\nu}(x) G^{\mu\nu}(x) U^{-1}(x)\right]$$

Non-abelian gauge theory

Again thanks to covariant derivative we can construct a NAGT of scalars and gauge fields

$$\mathcal{L} = (D_\mu \phi)^\dagger D^\mu \phi - m^2 \phi^\dagger \phi - \lambda(\phi^\dagger \phi)^2 - \frac{1}{4g^2} \mathrm{Tr}\,[G_{\mu,\nu} G^{\mu\nu}]$$

From this point of view non-abelian gauge theories are not so hard to construct, just have to deal with matrices. Notice

- Different number of gauge fields than "matter" fields
- $G_{\mu\nu}$ has both order W_μ and order $W_\mu W_\nu$ terms, nonlinear!

$$G_{\mu\nu} = \partial_\mu W_\nu - \partial_\nu W_\mu - [W_\mu, W_\nu]$$

- In $\mathrm{Tr}\,[G_{\mu,\nu} G^{\mu\nu}]$ therefore cubic and quartice terms in gauge field.

g is coupling, normal form after rescaling $W_\mu \to g W_\mu$, $\xi^a \to g \xi^a$.

Chiral fermions

To continue our construction effort, we step back from local symmetries and gauge fields, and consider fermionic fields. Represented by spinor fields

$$\psi(x) = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \\ \psi_3(x) \\ \psi_4(x) \end{pmatrix}$$

Special 4×4 γ matrix, γ_5, allows us to define projection matrices $P_{L,R}$

$$\gamma_5 = \begin{pmatrix} -\mathbf{1}_2 & 0 \\ 0 & \mathbf{1}_2 \end{pmatrix}, \; P_L = \tfrac{1}{2}(1 - \gamma_5) = \begin{pmatrix} \mathbf{1}_2 & 0 \\ 0 & 0 \end{pmatrix}, \; P_R = \tfrac{1}{2}(1 + \gamma_5) = \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{1}_2 \end{pmatrix}$$

They project out the *left- and righthanded* parts of a spinor.

$$\psi_L(x) = P_L\psi(x) = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \\ 0 \\ 0 \end{pmatrix}, \quad \psi_R(x) = P_R\psi(x) = \begin{pmatrix} 0 \\ 0 \\ \psi_3(x) \\ \psi_4(x) \end{pmatrix}$$

Chiral fermions, cont'd

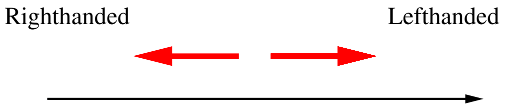
Already the basic Dirac action has something interesting to say about chirality. Two lines of calculation show

$$\overline{\psi}(\not\partial - m)\psi = \ldots = \overline{\psi_L}(\not\partial)\psi_L + \overline{\psi_R}(\not\partial)\psi_R - m(\overline{\psi_L}\psi_R + \overline{\psi_R}\psi_L)$$

We conclude

- Mass terms link L and R projections of ψ
- Without mass terms, they behave independently, as if they were different particles.

Meaning of chirality:
for massless fermions, chirality is same as helicity



The Standard Model treats left- and righthanded fermions very differently.

Chiral fermions and abelian gauge fields

Consider a Lagrangian for only one left-handed massless fermion field

$$\mathcal{L} = \overline{\psi_L} \partial\!\!\!/ \psi_L$$

There is a global phase symmetry ($U(1)$)

$$\psi_L \to e^{i y_L \alpha} \psi_L$$

with α a parameter, and y_L some kind of charge. Now make it local

$$\partial_\mu \to \partial_\mu - i y_L B_\mu$$

For its right-handed partner, could do the same with *different* y_R.

Since we are dealing with massless fermions, why not.
(Not consistent at quantum level - anomaly, see Lecture 4).

In fact, the Standard Model has such a local phase symmetry :
Hypercharge.

Chiral fermions and non-abelian gauge fields

To build a NAGT with chiral fermions, we group different fermions together

$$\begin{pmatrix} \psi_{L,1}(x) \\ \vdots \\ \psi_{L,N}(x) \end{pmatrix}, \quad \begin{pmatrix} \psi_{R,1}(x) \\ \vdots \\ \psi_{R,M}(x) \end{pmatrix}$$

In Standard Model: $N = 2, M = 1$. If the group is a product, such as $SU(2) \times U(1)$, then L,R can transform differently under each factor. In Standard Model: left-handed up and down quarks transform in fundamental representation of $SU(2)$, i.e. as doublet, and have hypercharge

$$\begin{pmatrix} u_L(x) \\ d_L(x) \end{pmatrix}, \quad y = 1/3$$

The right-handed projections u_R and d_R do *not* transform under $SU(2)$, but have hypercharges $4/3$ and $-2/3$.

Consequence: cannot write $m\overline{\psi_L}\psi_R$! No direct mass terms allowed in SM!

SM Fermions and $SU(2) \times U(1)$ gauge fields

We can now give the full listing of SM fermions and how they transform.

				$SU(2)$	$U(1)$
$Q_L =$	$\begin{pmatrix} u_L \\ d_L \end{pmatrix}$	$\begin{pmatrix} c_L \\ s_L \end{pmatrix}$	$\begin{pmatrix} t_L \\ b_L \end{pmatrix}$	2	1/3
$u_R =$	u_R	c_R	t_R	1	4/3
$d_R =$	d_R	s_R	b_R	1	$-2/3$
$L_L =$	$\begin{pmatrix} \nu_L^e \\ e_L \end{pmatrix}$	$\begin{pmatrix} \nu_L^\mu \\ \mu_L \end{pmatrix}$	$\begin{pmatrix} \nu_L^\tau \\ \tau_L \end{pmatrix}$	2	-1

We call the two symmetries

- "$U(1)$ of hypercharge"
- "$SU(2)$ of weak isospin"

$SU(2)$ has 3 generators, $t_3 = \sigma_3/2 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is 3rd component of weak isospin. Electric charge:

$$Q = t_3 + y/2$$

Need for gauge field mixing

To build a NAGT for the symmetry product group $SU_L(2) \times U_Y(1)$, we need the appropiate covariant derivative

$$D_\mu = \mathbf{1}_X \partial_\mu - igW_\mu^a t_a^X - ig' B_\mu \mathbf{Y}_X$$

Remarks

- The X indicates that the precise form depends on what D_μ is acting on.
- W_μ^a, $a = 1, 2, 3$ do not have hypercharge, B_μ is invariant under $SU(2)$ (does not change)

If we'd like to interpret hypercharge as simply electric charge, we have a problem: the B_μ does not couple to W_μ, so it cannot be photon field.

Need for gauge field mixing, cont'd

Consider

$$\mathrm{Tr}\left[G_{\mu,\nu} G^{\mu\nu}\right] = \left(\partial_\mu W^a_\nu - \partial_\nu W^a_\mu - -ig\epsilon_{abc} W^b_\mu W^c_\nu\right)^2$$

The cubic term looks like $g\epsilon_{abc}\left(\partial_\mu W^a_\nu\right) W^{\mu,b} W^{\nu,c}$ We can achieve our goal if

- out of W^1_μ, W^2_μ we make $W^\pm_\mu$
- out of W^3_μ, B_μ we make A_μ, Z_μ

$$W^3_\mu = \cos\theta_W Z_\mu + \sin\theta_W A_\mu, \quad B_\mu = \cos\theta_W A_\mu - \sin\theta_W Z_\mu$$

- Hence $e = g\sin\theta_W$
- θ_W: weak (or Weinberg) mixing angle.

We'll see later if this also works for fermion and scalar fields.

Scalar fields

We have seen

- multiplets of fermion fields
- having both non-abelian and abelian charge
- interacting with gauge fields (thanks to covariant derivative)

Now consider one *doublet* of complex scalars. Why a doublet we will see later.

$$\Phi = \begin{pmatrix} \phi_1(x) \\ \phi_2(x) \end{pmatrix}, \quad y = 1$$

via Lagrangian

$$\mathcal{L}_\Phi = (\partial_\mu \Phi)^\dagger \partial^\mu \Phi - \mu^2 \Phi^\dagger \Phi - \lambda (\Phi^\dagger \Phi)^2$$

This is a bigger leap than so far: no fundamental scalar boson has yet been seen. Note the global $SU(2) \times U(1)$ symmetry.

Scalar fields, cont'd

Since we specified the way it transfroms under $SU(2)$ (as a simple doublet), and what hypercharge it has ($y = 1$), we can "go local" by $\partial_\mu \rightarrow$

$$D_\mu = \mathbf{1}_2 \partial_\mu - igW^a_\mu t^F_a - ig'/2 B_\mu \mathbf{1}_2, \quad t^F_a = \frac{1}{2}\sigma_a$$

Remarks

- We have interactions of scalar fields (4 of them!) with gauge fields
- No interactions of Φ with chiral fermions yet, soon.

What is electric charge? Using $Q = t_3 + y/2$

$$\Phi = \begin{pmatrix} \phi^+(x) \\ \phi^0(x) \end{pmatrix}, \quad \Phi^\dagger = \begin{pmatrix} \phi^-(x) \\ \phi^0(x) \end{pmatrix} \quad (y = -1 \text{ now})$$

YUKAWA INTERACTIONS

We had not yet added possible bits of action that link the chiral fermions with the scalar fields. Ingredients:

- Scalar doublet field $\begin{pmatrix} \phi_1 \\ \phi_2 \end{pmatrix}$
- Left-handed fermion doublet $\begin{pmatrix} \nu_L^e \\ e_L \end{pmatrix}$
- Right-handed fermion singlet e_R

Our mission

- combine all three
- such that invariant under $SU(2) \times U(1)$

Two possibilities

$$\mathrm{I}: \quad \overline{\psi_R}\left((\Phi^*)^a \psi_L^b\right) \delta_{ab}, \qquad \mathrm{II}: \quad \overline{\psi_R}\left(\Phi^a \psi_L^b\right) \epsilon_{ab}, \quad \epsilon_{12} = -\epsilon_{21} = 1$$

Not a bug, a nice feature!

The Standard Model Lagrangian

We're done! We have all the bits. In summary

$$\mathcal{L}_{matter} = i\overline{Q^i_L}\not{\partial}Q^i_L + i\overline{u^i_R}\not{\partial}u^i_R + i\overline{d^i_R}\not{\partial}d^i_R + i\overline{L^i_L}\not{\partial}L^i_L + i\overline{e^i_R}\not{\partial}e^i_R$$

Replace

$$\partial_\mu \to D_\mu = \partial_\mu + ig_s G^a_\mu T_a + igW^i_\mu T_i + ig' B_\mu Y$$

with the generators and corresponding gauge fields

$$\begin{aligned} T_a &: \quad 8 \ SU_{color}(3) \text{ generators} \sim \text{gluons } G^a_\mu \\ T_i &: \quad 3 \ SU_{I_W}(2) \text{ generators} \sim \text{weak bosons } W^i_\mu \\ Y &: \quad 1 \ U_Y(1) \text{ generator} \sim B_\mu \end{aligned}$$

Mix W^3_μ and B_μ to Z_μ and A_μ:

$$B_\mu = -\sin\theta_w Z_\mu + \cos\theta_w A_\mu, \qquad W^3_\mu = \sin\theta_w A_\mu + \cos\theta_w Z_\mu,$$

Add to $\mathcal{L}_{matter}$ the kinetic and self-interaction terms for the gauge fields

$$\mathcal{L}_{gauge} = -\frac{1}{4}B^{\mu\nu}B_{\mu\nu} - \frac{1}{4}G^{\mu\nu}_a G^a_{\mu\nu} - \frac{1}{4}W^{\mu\nu}_i W^i_{\mu\nu}$$

The Standard Model Lagrangian, cont'd

Add complex scalar doublet

$$\mathcal{L}_\Phi = (\partial_\mu \Phi)^\dagger \partial^\mu \Phi - \mu^2 \Phi^\dagger \Phi - \lambda (\Phi^\dagger \Phi)^2$$

Finally, add scalar-fermion interactions

$$\mathcal{L}_{Yukawa} = y_u^{ij} \ \overline{Q_L^i} \sigma_2 \Phi^* u_R^j + y_d^{ij} \ \overline{Q_L^i} \Phi d_R^j + \ldots$$

That's it.

There is still much more to say, but for now we put $\mathcal{L}_{SM}$ aside, and think about quantum consistency.

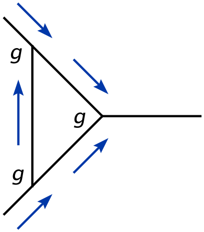
Scattering amplitudes and cross sections

From the path integral with $S_{SM} = \int d^4x \mathcal{L}_{SM}$ we can

- extract Feynman rules
- compute scattering amplitudes $M(i + j \rightarrow k + l + ..)$
- compute in principle M to arbitrary high order
- to compute cross section $\sigma \propto \int d(PS) |M|^2$ to higher order

Loop diagrams

With a diagram



is associated an expression:

$$\int d^4 l \frac{g^3}{[l^2 - m^2][(p-l)^2 - m^2][(k+l)^2 - m^2]}$$

A diagram with V vertices is order g^V. To compute a cross section e.g. draw and compute all possible diagrams of desired order, sum them coherently, and square the result for the cross section.

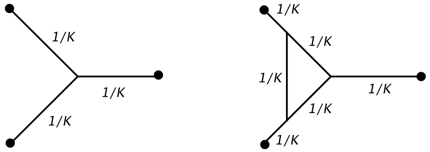
PATH INTEGRALS PRODUCE HIGHER ORDER FEYNMAN DIAGRAMS

Three-point function in ϕ^3 theory, 0-dimensional

$$\left(\frac{\partial}{\partial J}\right)^3 \int d\phi e^{\phi K \phi + J\phi + \lambda\phi^3}\Big|_{J=0} = \left(\frac{\partial}{\partial J}\right)^3 \int d\phi e^{\lambda(\partial/\partial J)^3} e^{\phi K \phi + J\phi}\Big|_{J=0}$$

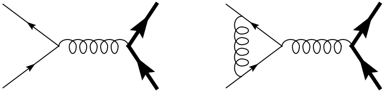
$$= \left(\frac{\partial}{\partial J}\right)^3 \left[1 + \lambda(\partial/\partial J)^3 + \tfrac{1}{2}(\lambda(\partial/\partial J)^3)^2 + \dots\right] e^{J\frac{1}{K}J}\Big|_{J=0}$$

$$= \lambda \frac{1}{K}\frac{1}{K}\frac{1}{K} + \lambda^3 \frac{1}{K}\frac{1}{K}\frac{1}{K}\frac{1}{K}\frac{1}{K}\frac{1}{K} + \dots$$

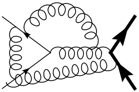


Perturbation theory

If g is small, we can keep only the simplest diagrams. E.g. for top quark pair production, keep only



and neglect



Therefore, to increase precision one must compute more complicated Feynman diagrams.
In a sense, the quantum physics sits in the loops: an L-loop diagram is proportial to $\hbar^L$.

Problems in loops

All well and good, but small problems might occur:

- The higher-order contribution are sometimes ∞
- Probability is perhaps no longer conserved
- Local symmetry might be destroyed

Any of these problems is fatal for the Standard Model as a predictive theory.

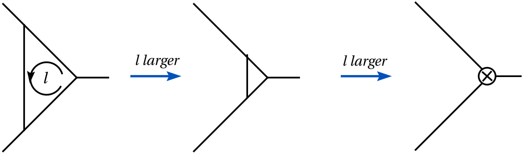
These problems are, more or less, related.

Renormalization

Why do some (*not all*!) loop contributions sometimes become infinite?

$$g^3 \int d^4 l \frac{l^2}{[l^2 - m^2][(p-l)^2 - m^2][(k+l)^2 - m^2]} \overset{|l|\to\infty}{\simeq} g^3 \int \frac{d^4 l}{l^4}$$

Logarithmic integral! Large momentum $\leftrightarrow$ short distance



Looks like basic vertex, with new (infinite strength). What to do?
Redefine:

$$g + g^3 \frac{K}{\epsilon} = g_R$$

We realize that original g was infinite!

Renormalization, cont'd

Seems like cheating. But

- must show that same *renormalization* of g is good to fix any Green function, cross section, etc
- to all orders in perturbation theory!
- no important symmetries are destroyed after renormalization
- only the original quantities (couplings, masses) get renormalized

Moreover, some features to this bug:

Renormalization procedure → renormalization group evolution equation; can relate Green's functions at 1 TeV to those at 1 GeV.

The Standard Model has been shown to be renormalizable to all orders (`'t Hooft, Veltman`).

Unitarity: conservation of probability

Unphysical degrees of freedom should not become physical.
The case of the photon.

- Massless photon has two spin states
- Yet described by $A_\mu(x)$, four fields
- Two must be unphysical.
- One is put to zero by equation of motion, the other is gauge degree of freedom

Propagator for photon

$$\frac{1}{k^2}\eta^{\mu\nu}, \text{ or } \quad \frac{1}{k^2}\left(\eta^{\mu\nu} + \frac{n^\mu k^\nu + n_\nu k_\mu}{k \cdot n}\right)$$

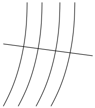
- First propagator (Feynman gauge) nice, but includes unphysical degrees of freedom
- Second propagator (Physical gauge), physical degrees of freedom, but less nice

Path integrals, gauge fixing and unitarity

In

$$\int \mathcal{D}A_\mu e^{iS[A]}$$

one should not integrate A_μ field configurations that differ only by $A_\mu(x) = A_\mu(x) + \partial_\mu \alpha(x)$. So take a slice



To properly account for this in the non-abelian case, one must add to $\mathcal{L}_{SM}$ a term involving fictitious fields: *ghosts*. Ghostly message:

- No physical consequence
- Good for bookkeeping
- Nice theoretical tool

The Standard Model has been shown to be unitary theory (`'t Hooft, Veltman`).