Non-linear field theory with supersymmetry

Hydrodynamics and sigma models

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Cover illustration: The figure on the front cover is drawn by Kees Huyser at the National Institute for Nuclear Physics and High Energy Physics (NIKHEF), Amsterdam. It gives an impression of a fluid current moving on sphere, where the scalar fields live. This relativistic fluid mechanics is described in section 2.5. Its supersymmetric extension is treated in section 3.5.1.

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- T.S. Nyawelo, F. Riccioni, J.W. van Holten, and S. Groot Nibbelink "Singular supersymmetric sigma-models", Nucl. Phys. **B663** (2003) 60-78; [hep-th/0302135]
- T.S. Nyawelo "Phenomenological analysis of supersymmetric σ-models on coset spaces $SO(10)/[SU(5)\times U(1)]$ and $E_6/[SO(10)\times U(1)]$", *in preparation*

To my parents

Contents

Chapter 1

Introduction

To understand a science it is necessary to know its history.

Auguste Comt

One of the fundamental new symmetries of nature that has been the subject of intense discussion in particle physics of the past three decades is supersymmetry —the symmetry transformations relating fermions to bosons and vise versa— introduced in the early 1970's by Golfand and Likhtman [1]. An important motivation for the study of supersymmetric theories is that they could bring new insight in the unification of strong, weak and electro-magnetic interactions with gravity and on the difficulties of quantum gravity. This however requires that one finds theories invariant under *local*, and not only *global* supersymmetry. Locally supersymmetric theories are called supergravities, and have been invented by Freedman, Ferrara and van Nieuwenhuizen [2]; see also Deser and Zumino [3]. These supergravity theories are non-renormalizable: quantized supergravity has new divergences from loop contributions. Even though the expectations of solving the problem of quantum gravity with the help of supersymmetry have not materialized in a field theory context, they do describe the low-energy regime of superstring theories which are candidates for a quantum gravity theory.

The Standard model model (SM) of elementary particle physics is the most successful physical theory known. Its particle spectrum, however does not exhibit supersymmetry, certainly not in manifest form. Therefore it is necessary to assume that supersymmetry is broken at energy scales of the standard model and below, i.e. below 1 TeV. At which energy above the Fermi scale supersymmetry is actually broken is a model dependent. If supersymmetry only plays a role in quantum gravity, it may be well be broken at Plank scale (10^{19} GeV). Extrapolation of the running couplings of the standard model indicates, that an approximately supersymmetric particle spectrum at scales as low as the TeV scale would help to make the electro-weak and color gauge couplings unify [4] at an energy near 10^{15-16} GeV. Supersymmetry breaking in the TeV range is the scenario underlying the minimal supersymmetric standard model (MSSM) [5], in which all quarks and leptons supposedly have scalar partners, and all gauge and Higgs bosons (of which there are at least two doublets) are accompanied by fermion partners, with appropriate mass splittings largely adjusted by hand to fit observational constraints.

In the last two decades much more effort has been invested in the construction of supersymmetric models with different particle spectra based on coset models, in which the coset G/H is a Kähler manifolds [60]–[72]. The requirement of Kähler geometry, to be explained in appendix B, is natural in the context of $D = 4$ supersymmetry. In recent years such models based on this construction have been studied in great detail [74, 73, 75, 77, 90], and we now have consistent supersymmetric models with non-linear realizations of groups like $SU(5)$, $SO(10)$, E_6 or E_8, and new scenario's for superunification become possible.

Apart from particle physics, supersymmetry has been applied to a number of areas in physics and mathematics. We mention for instance the use of supersymmetric quantum mechanics to study anomalies in field theory [8] and the application of supersymmetry techniques to prove positivity of asymptotic mass in general relativity [9].

One of the new areas in physics in which supersymmetry can be applied is relativistic fluid mechanics. An understanding of fluids that posses supersymmetric properties may be relevant for cosmological applications as they could be used to describe a supersymmetric phase in the early universe. An other area in physics in which supersymmetric hydrodynamics may be apply is in condensed matter physics, where they might apply to quantum fluids like ^{3}He-^{4}He mixtures, in the regime where terms proportional to the mass-differences of these isotopes can be neglected.

This thesis reports on two topics which at the mathematical level are, related. The first topic is relativistic fluid mechanics and its supersymmetric extension; the second topic deals with the phenomenology of supersymmetric σ-models on coset spaces G/H. The properties that link them are non-linear symmetry and the strong relation between supersymmetry and Kähler geometry. In this chapter we elaborate on this connection, provide the motivation for our research and discusses the context in which the particular problems have been investigated.

1.1 Non-linear symmetries

Non-linear symmetries arise in field theory when a global symmetry group G is broken down to its subgroup H. According to Goldstone's theorem, to each broken generator there corresponds a massless particle. In general, these Goldstone bosons may either be elementary or composite. The non-linear chiral lagrangian which was introduced [28, 29] to give a handy description of soft pion processes, was the first example realizing a non-linear symmetry. The model, based on $G/H = [SU(2)_L \times SU(2)_R]/SU(2)_V$, was soon generalized to arbitrary groups G and H, by Callan, Coleman, Wess and Zumino [30, 31].

Supersymmetric non-linear σ-models have been studied from various points of view; sometimes from a purely formal interest in the extension of the framework of non-linear σ-models, but particularly in connection with the application to the current problems of composite models of quarks and leptons. Zumino [22] was the first to recognize that the scalar fields of supersymmetric non-linear models take their values on a Kähler manifold and gave an explicit expression for the action for the case of the Grassmann manifold $U(m+n)/[U(m) \times U(n)]$. Many authors [32, 33, 34, 35, 36, 37, 38, 39, 40, 41] studied non-linear realizations for more general cases of G/H, where the importance of a complex extension of the group G was pointed out.

In the relatively restricted class of supersymmetric pure σ-models on coset spaces G/H of Kähler type, there is only one free parameter: the scale $\Lambda = 1/f$ at which G breaks to H. This mass parameter gives all the terms in the lagrangian the right mass dimensions; we call it the sigma-model scale. Of course, if one couples the non-linear σ-model to weakly interacting vector fields by promoting some of its symmetries to local gauge symmetries, this may introduce some additional free parameters, such as a gauge coupling constants g. But this is just the usual freedom that one has in all gauge theories. We also note in passing that if one considers the σ-models that arise in supergravity [42, 43, 44, 45], the scale parameter is naturally determined to be the Plank scale (10^{19} GeV), and no independent new parameter is required at all.

1.2 Relativistic fluid mechanics and its supersymmetric extension

In this section, we describe the main ideas of this thesis. The first topic of this thesis deals with relativistic fluid mechanics and its supersymmetric extension. Relativistic fluid mechanics has applications in the laboratory, e.g. in plasma physics and heavy ion collisions, as well as in astrophysics and cosmology [10, 11]. As it is also believed to provide a more accurate description of hydrodynamical phenomena, much work has been invested in its development [51, 52]. Recently, an interesting extension of the theory to include non-abelian charges and currents has been proposed [47, 111]. One of the important aspects of this formalism is that it includes vorticity consistently at the lagrangean level, by developing a non-abelian generalization of the Clebsch decomposition[1] of the vector conjugate to the current; for a review with many references, see [53]. In a related development, Jackiw and Polychronakos [46] have presented a supersymmetric theory of fluid mechanics in (2+1)-dimensional space-time. This model is rather special, as it descents from a supermembrane theory in (3+1) dimensions [50, 54, 55, 56, 57, 58]. It results in a supersymmetric generalization of the non-relativistic planar Chaplygin gas [59]. An interesting result obtained in [46] is, that the vorticity in the theory is generated by the fermion fields, rather then by the bosonic component of the fluid.

The role of space and space-time symmetries has been investigated in [16, 17] and references therein. A rather remarkable result is the existence of an infinite set of conserved currents in 4-D space-time, related to the reparametrization invariance in the space of potentials [13, 14]. This seems to offer an important key to identifying fluid-dynamical phases of 4-D relativistic field theory. In spite of these advances, so far a relativistic and supersymmetric theory of fluid mechanics in (3+1) dimensions is lacking. Chapters 2 and 3 of this thesis are intended to fill this gap.

The main result of this work is an alternative for the Clebsch decomposition of currents in fluid mechanics, in terms of complex potentials taking values in a Kähler manifold. We reformulate classical relativistic fluid mechanics in terms of these complex potentials and rederive the existence of an infinite set of conserved currents. We perform a canonical

[1] Parametrization of any three-dimensional vector field $\mathbf{A}$ in terms of three scalar potential (α, β, γ) is called Clebsch parametrization: $\mathbf{A} = \boldsymbol{\nabla}\alpha + \beta\boldsymbol{\nabla}\gamma$

analysis to find the explicit form of the algebra of conserved charges. The Kähler-space formulation of the theory has a natural supersymmetric extension in 4-D space-time, which we present both in its lagrangian and hamiltonian form. The theory takes the form of a new type of non-linear model for a vector superfield and auxiliary chiral superfields.

1.3 Supersymmetric non-linear σ-models in 4 dimensions

In the second topic of this thesis we study supersymmetric σ-models on homogeneous Kählerian coset spaces G/H in general. It focuses to a large extent on those containing a unification group $SO(10)$ or E_6, like the cosets $SO(10)/SU(5) \times U(1)$ and $E_6/SO(10) \times U(1)$. These models are supposed to described physics beyond the standard model. An important feature of constructing superymmetric extensions of the standard model was renormalizability. Since non-linear σ-models are not renormalizable, there seems to be a problem here. However the non-linear structure we assume to be present at the Plank scale or just below. In this regime supergravity cannot be neglected. Supergravity theories are non-renormalizable by themselves, so non-renormalizable couplings in the matter sector may arise naturally. These non-renormalizable couplings have a very specific structure when they are generated by a non-linearly realized internal symmetry group. Of course we expect that at the standard model energies, the theory reduces to a renomalizable effective theory, but not necessarily supersymmetric.

Constructing supersymmetric σ-models on Kähler manifolds $SO(10)/[SU(5)\times U(1)]$ and $E_6/[SO(10) \times U(1)]$, the fermion partners of the Goldstone bosons —the quasi-Goldstone fermions— have precisely the right quantum numbers to describe one family of quarks and leptons, including a right-handed neutrino. However, supersymmetric coset-models are known to be inconsistent quantum field theories, because of the appearance of anomalies in the holonomy group [83, 84, 85, 86, 87]. The general procedure to construct supersymmetric lagrangians, or equivalently, Kähler potentials for arbitrary Kählerian coset spaces G/H, and how to cancel the anomalies has been described in [19].

In this thesis we focus on phenomenological aspects of the coset-models. In order to discuss various properties of these models, we first review the construction of the lagrangians of anomaly-free models on coset-spaces that are globally consistent. After that we make a first step in the analysis of the phenomenology of those models by discussing first the possible vacuum configurations of these models; in particular the existence of zero of the potential.

We show by studying explicit examples, that upon gauging all of the isometry group G the D-term potential can sometimes force the scalar fields to take vacuum expectation values for which the model becomes singular, in the sense that some of the kinetic terms disappear in the vacuum state, and the space of physical degrees of freedom is reduced. These examples are the supersymmetric σ-models based on coset spaces $SO(10)/[SU(5) \times U(1)]$ and $E_6/[SO(10) \times U(1)]$. In order to gain an understanding; and how to treat supersymmetric field theories in which these types of complications occur we also study in details an anomaly free extension of the supersymmetric $\mathbb{C}P^1$–model, where the scalar fields take values in $SU(2)/U(1)$.

To complete the phenomenological analysis of supersymmetric σ-models, we also consider the gauging of the linear subgroup $SO(10) \times U(1)$ of the E_6 model as well as $SU(5) \times U(1)$ of the $SO(10)$-spinor model. Because this subgroup contains an explicit $U(1)$ factor, we added a Fayet-Iliopoulos term with parameter ξ and we investigate in particular the existence of zeros of the potential, for which the model is anomaly-free, with positive definite kinetic energy. Then we discuss a number of physical aspects of these models, like supersymmetry and internal symmetry breaking, and the resulting mass-spectrum.

1.4 Outline of this thesis

Chapter 2 of this thesis starts with a discussion of the equations of motion of non-dissipative relativistic fluid mechanics. Next we introduce a lagrangian density that reproduces these equations of motion. In chapter 3, we introduce 4-D supersymmetry into the structure, by showing how the fluid current can be naturally incorporated in $N = 1$ superfields. We propose a superfield action and present its component form, which is a generalization of the model proposed in ref. [27]. After that, we study currents and their conservation laws in the supersymmetric model. We show that there exists a regime in which an infinite number of currents is reobtained; this regime we interpret as the description of a supersymmetric fluid. We end with a brief discussion of the interpretation and possible applications of the theory as a model of supersymmetric hydrodynamics.

We review the general features of supersymmetric σ-models on Kähler manifolds in chapter 4. In particular, we discuss the construction of globally anomaly-free supersymmetric σ-models on Kähler manifolds. As chapter 4 is quite general, we illustrate various aspects of this construction by studying supersymmetric anomaly-free models based on $\mathbb{C}P^1$ in chapter 5. Chapters 6 and 7 are devoted to the phenomenology of anomaly-free models based on the coset-spaces $SO(10)/U(5)$ and $E_6/[SO(10) \times U(1)]$ respectively. The conclusions of this work, as well as an outlook on further reasearch that still needs to be done in the field of hydrodynamics and supersymmetric σ-models are given in chapter 8. Finally, there are five appendices. Appendix A, gives the standard notations and conventions that are used throughout this thesis. Appendix B, provides the mathematical background for the geometrical discussion of Kähler manifolds, and a proof of an identity which we used in chapter 3 to show that the isometry currents are conserved. Various more mathematical details of our constructions of Dirac brackets, and supercurrents, which are used mainly in chapter 3 are discussed respectively in the appendices C and D. Appendix E discusses the analysis of solutions corresponding to supersymmetry breaking vacua in the model with gauged $U(5)$ presented in chapter 6.

Chapter 2

Relativistic fluid mechanics

> A theory has only the alternative of being right or wrong. A model has a third possibility: it may be right, but irrelevant.
>
> Manfred Eigen.

2.1 Introduction

The focus of this chapter is on a general lagrangian density for a relativistic fluid that can be used in a supersymmetric extension of relativistic fluid mechanics model building. It provides a basis for the construction of a large set of interesting examples of supersymmetric theories of hydrodynamics of an ideal fluid.

In section 2.2 we recall the basic facts about non-dissipative relativistic fluid mechanics. We propose an alternative to the standard Clebsch parametrization, based on complex potentials taking values in a Kähler manifold, and rederive the fluid equations in this formalism. In section 2.3 we show the existence of a topological invariant (the vortex linking number), and an infinite set of divergence-free currents. This is followed by a discussion of the canonical structure of the theory in terms of Dirac-Poisson brackets in section 2.4. We also compute the algebra of the conserved charges. Section 2.5 discusses a few examples of fluid models.

2.2 Equations of motion

The equations of motion of a perfect (dissipationless) relativistic fluid can be expressed in terms of a conserved and symmetric energy-momentum tensor $T_{\mu\nu}$, derived from Poincaré invariance by Noether's theorem.

The general form of the energy-momentum tensor of a relativistic perfect fluid is (see, e.g. [10, 11]):

$$T_{\mu\nu} = pg_{\mu\nu} + (\varepsilon + p)u_\mu u_\nu, \tag{2.1}$$

where p is the pressure, ε is the energy-density and u^μ is the velocity four-vector, which is a time-like unit vector: $u_\mu^2 = -1$, in natural units ($c = 1$). Local energy-momentum conservation is expressed by the vanishing of the four-divergence of the energy-momentum tensor

$$\partial^\mu T_{\mu\nu} = 0. \tag{2.2}$$

The conserved energy-momentum four-vector is then given in a laboratory inertial frame by

$$P_\mu = \int_{t=t_0} d^3x \, T_{\mu 0}, \quad \frac{dP_\mu}{dt} = 0. \tag{2.3}$$

In addition to the conservation of energy and momentum, the fluid density is conserved during ordinary flow as well. This is expressed by the vanishing divergence of the fluid density current j^μ:

$$\partial_\mu j^\mu = 0, \quad j^\mu = \rho u^\mu, \tag{2.4}$$

where ρ represents the local fluid density in the local rest frame; the normalization of the four velocity then implies that the current satisfies

$$-j_\mu^2 = \rho^2 \geq 0. \tag{2.5}$$

Thus the local fluid density is defined in a Lorentz-invariant manner. In a space-plus-time formulation, equation (2.4) is seen to imply the equation of continuity

$$\partial \cdot j = \partial_t(\rho\gamma) + \nabla_i(\rho\gamma v^i) = 0, \quad \gamma = \left(1 - \mathbf{v}^2\right)^{-1/2}. \tag{2.6}$$

Because of the vanishing divergence, for general fluid flow the current has three independent components. A standard way to express this is to write the current in terms of three scalar potentials (θ, α, β); they are introduced as lagrange multipliers combined in an auxiliary vector field a_μ, with the Clebsch decomposition

$$a_\mu = \partial_\mu \theta + \alpha \partial_\mu \beta. \tag{2.7}$$

In this formalism the component θ describes pure potential flow. Potential flow is the name given to irrotational flow whose current field j_μ is defined in terms of a scalar potential function: $\partial_\mu \theta \equiv j_\mu$. The other two components α and β are necessary to include non-zero vorticity. Vorticity is the circulation in the motion of fluid around a fixed point in fluid; for a review, see [12].

In [18] we have proposed an alternative to the Clebsch decomposition, which is mathematically equivalent but has several advantages: it gives insight into the construction of an infinite set of conserved currents [13, 14], and it allows a straightforward supersymmetric generalization; the latter property leads to a proposal for a 4-d supersymmetric extension of relativistic fluid mechanics.

This approach consists in replacing the real Clebsch potentials (θ, α, β) by one real potential θ and one complex potential z, with its conjugate $\bar{z}$. In terms of these we propose a general lagrangian density for a relativistic fluid, reproducing the conserved energy-momentum tensor (2.1), given by the expression

$$\begin{aligned} \mathcal{L}[j^\mu, \theta, \bar{z}, z] &= -j^\mu a_\mu - f(\rho) \\ &= -j^\mu \left(\partial_\mu \theta + iK_z \partial_\mu z - iK_{\bar{z}} \partial_\mu \bar{z}\right) - f\left(\sqrt{-j^2}\right). \end{aligned} \tag{2.8}$$

Here $K(z, \bar{z})$ is a real function of the complex potentials, which we refer to as the Kähler potential, K_z and $K_{\bar{z}}$ are its partial derivatives w.r.t. z and $\bar{z}$, and f is a function of $\rho = \sqrt{-j^2}$ only.

The equations of motion derived from (2.8) are

$$\begin{aligned} f' \frac{j_\mu}{\sqrt{-j^2}} &= \partial_\mu \theta + iK_z \partial_\mu z - iK_{\bar{z}} \partial_\mu \bar{z}, \quad \partial \cdot j = 0, \\ -2iK_{z\bar{z}}\, j \cdot \partial z &= 2iK_{z\bar{z}}\, j \cdot \partial \bar{z} = 0. \end{aligned} \tag{2.9}$$

Translation invariance of the action constructed from $\mathcal{L}$ implies the conservation of the energy-momentum tensor

$$T_{\mu\nu} = g_{\mu\nu} \left(f' \sqrt{-j^2} - f\right) + f' \,\frac{j_\mu j_\nu}{\sqrt{-j^2}}, \quad \partial^\mu T_{\mu\nu} = 0, \tag{2.10}$$

where f' is the derivative of $f(\sqrt{-j^2})$ w.r.t. its argument $\sqrt{-j^2} = \rho$. With $j_\mu = \rho u_\mu$, this energy-momentum tensor is of the form (2.1) with the pressure and energy density given by

$$\varepsilon = f(\rho), \quad p = \rho f'(\rho) - f(\rho). \tag{2.11}$$

Hence the pressure is the negative of the Legendre transform of the density. A typical equation of state (a linear relations between pressure and specific energy) is obtain from monomial energy densities:

$$\varepsilon = f(\rho) = \alpha \rho^{(1+\eta)} \quad \Rightarrow \quad p = \eta \varepsilon. \tag{2.12}$$

In fact all of the perfect fluids relevant to cosmology are of this type.

2.3 Conservation laws

The essential elements of the class of hydrodynamical models presented above are the existence of a divergence-free density current j_μ and a divergence-free energy-momentum tensor $T_{\mu\nu}$. We now show that there exist further conserved charges, connected with other divergence-free currents in the models defined above.

First we recall the construction of a conserved topological charge, related to the linking number of vortices. Following Carter [15] we define the momentum density

$$\pi_\mu = \left.\frac{\delta \mathcal{L}}{\delta u^\mu}\right|_\rho = \rho \left(\partial_\mu \theta + iK_z \partial_\mu z - iK_{\bar{z}} \partial_\mu \bar{z}\right). \tag{2.13}$$

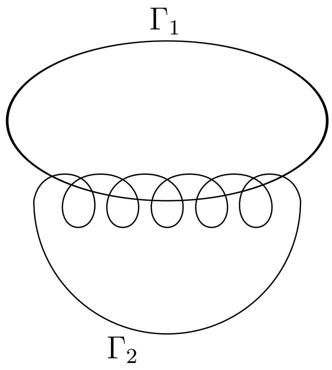

Figure 2.1: Two closed vortices Γ_1 and Γ_2 wind 5 times around each other. The integrals Γ_1 and Γ_2 are taken along some closed fluid contour. The winding number of a closed vortex Γ around a fixed curve in the fluid is given by Ω, as given in (2.16).

Observe that the auxiliary vector potential a_μ is related to the momentum density by $\pi_\mu = \rho a_\mu$:

$$\begin{aligned} a_\mu &= \partial_\mu \theta + iK_z \partial_\mu z - iK_{\bar{z}} \partial_\mu \bar{z} = f' \frac{j_\mu}{\sqrt{-j^2}} = f' u_\mu, \\ \pi_\mu &= f' j_\mu = \rho f' u_\mu = (p + \varepsilon) u_\mu. \end{aligned} \tag{2.14}$$

From the definition it follows, that the current defined by $k^\mu = \varepsilon^{\mu\nu\kappa\lambda} a_\nu \partial_\kappa a_\lambda$ is divergence-free:

$$\partial_\mu k^\mu = \varepsilon^{\mu\nu\kappa\lambda} \partial_\mu a_\nu \, \partial_\kappa a_\lambda = 0. \tag{2.15}$$

The conserved charge is obtained from the divergence-free current (2.15), and it reads

$$\Omega = \int d^3x \, k^0 = \int d^3x \, \varepsilon^{ijk} \, a_i \partial_j a_k = \int d^3x \, \mathbf{a} \cdot \mathbf{w} \tag{2.16}$$

where $\mathbf{w}$ ($w^i = \frac{1}{2} \varepsilon^{ijk} \, w_{jk}$) is the vorticity[1]. (Of course Ω vanishes in the irrotational case.) Notice here that the conserved charge Ω is a topological quantity (the linking number of vortices) whose Chern-Simons term, given by a pure surface term

$$\Omega = \int d^3x \, \partial_i \left[i\varepsilon^{ijk} \theta \, \partial_j (K_{\bar{z}} \partial_k \bar{z} - K_z \partial_k z) \right] = -2i \int d^3x \, \partial_i \left[\varepsilon^{ijk} \theta \, K_{\bar{z}z} \partial_j \bar{z} \, \partial_k z \right] \tag{2.17}$$

measures the quantized winding number of closed vortices around each other as illustrated in figure 2.1

Next we show that there is an infinite set of conserved charges related to the reparametrization of the potentials [13]. As a first step observe that whenever $K_{z\bar{z}} \neq 0$, the equations of motion for the complex potentials z and $\bar{z}$ reduce to

$$-2i \, j^\mu \partial_\mu z = 0, \quad 2i \, j^\mu \partial_\mu \bar{z} = 0. \tag{2.18}$$

[1] In non-relativistic fluid, the vorticity is defined as a curl of velocity $\mathbf{v}$ at a given point in fluid flow: $\mathbf{w} = \boldsymbol{\nabla} \times \mathbf{v} = \frac{1}{2} \varepsilon^{ijk} \, w_{jk} = \varepsilon^{ijk} \partial_j v_k$, where $w_{jk} = \partial_j v_k - \partial_k v_j$. For the relativistic case, the vorticity tensor is defined to be antisymmetric derivative of the momentum density [15]: $w_{\mu\nu} = \partial_\mu \pi_\nu - \partial_\nu \pi_\mu$.

It follows, that any current

$$J_\mu[M] = -2M(\bar{z}, z) j_\mu, \tag{2.19}$$

is divergence-free:

$$\partial \cdot J[M] = -2\Big(M_z j \cdot \partial z + M_{\bar{z}} j \cdot \partial \bar{z} \Big) = 0. \tag{2.20}$$

which allows the construction of infinitely many conserved charges of the form

$$q[M] = \int d^3x \, J^0[M]. \tag{2.21}$$

The non-singularity of the Kähler potential is satisfied in all cases where $K_{z\bar{z}}$ is the metric of a geodesically complete complex manifold. Some examples are discussed in section 2.5.

2.4 Canonical structure

We now pass to the canonical formulation of the theory. First we define the canonical momenta

$$\pi_\theta = \frac{\partial \mathcal{L}}{\partial \dot{\theta}} = j_0, \quad \pi_z = \frac{\partial \mathcal{L}}{\partial \dot{z}} = i K_z j_0, \quad \bar{\pi}_{\bar{z}} = \frac{\partial \mathcal{L}}{\partial \dot{\bar{z}}} = -i K_{\bar{z}} j_0. \tag{2.22}$$

Observe here that j_0 is a canonical momentum, whereas the 3-vector field $\mathbf{j}$ is an auxiliary field, which can be eliminated by its algebraic field equation; in particular in the following we use the identifications

$$j_0 = \pi_\theta, \quad \mathbf{j} = \frac{\rho}{f'(\rho)} \Big(\boldsymbol{\nabla}\theta + i K_z \boldsymbol{\nabla} z - i K_{\bar{z}} \boldsymbol{\nabla} \bar{z} \Big). \tag{2.23}$$

With $\rho = \sqrt{\pi_\theta^2 - \mathbf{j}^2}$ the hamiltonian density reads

$$\mathcal{H} = \frac{f'(\rho)}{\rho} \mathbf{j}^2 + f(\rho). \tag{2.24}$$

Obviously, the last two equations (2.22) are second-class constraints, expressing $(\pi_z, \bar{\pi}_{\bar{z}})$ in terms of the other phase-space variables $(z, \bar{z}, \pi_\theta)$:

$$\chi_z = \pi_z - i K_z \pi_\theta = 0, \quad \chi_{\bar{z}} = \bar{\pi}_{\bar{z}} + i K_{\bar{z}} \pi_\theta = 0. \tag{2.25}$$

To describe the canonical dynamics on the reduced phase-space determined by these equations, we introduce Poisson-Dirac brackets [20, 21]

$$\{A, B\}^* = \{A, B\} - \{A, \bar{\chi}_i\} \, C^{-1}_{ij} \, \{\chi_j, B\}, \tag{2.26}$$

where C^{-1} is the inverse of the matrix of constraint brackets

$$C_{ij} = \{\chi_i(\mathbf{r}, t), \chi_j(\mathbf{r}', t)\} = \begin{pmatrix} 0 & -2i K_{z\bar{z}} \pi_\theta \\ 2i K_{z\bar{z}} \pi_\theta & 0 \end{pmatrix} \delta(\mathbf{r} - \mathbf{r}'). \tag{2.27}$$

From the definition (2.45) it follows, that in the reduced phase space spanned by $(z, \bar{z}, \theta, \pi_\theta)$ the canonical Poisson-Dirac brackets are

$$\begin{aligned}
\{z(\mathbf{r},t), \bar{z}(\mathbf{r}',t)\}^* &= \frac{-i}{2K_{z\bar{z}}\pi_\theta}\,\delta(\mathbf{r}-\mathbf{r}'), \quad \{\theta(\mathbf{r},t), \pi_\theta(\mathbf{r}',t)\}^* = \delta(\mathbf{r}-\mathbf{r}'), \\
\{z(\mathbf{r},t), \theta(\mathbf{r}',t)\}^* &= \frac{K_{\bar{z}}}{2K_{z\bar{z}}\pi_\theta}\,\delta(\mathbf{r}-\mathbf{r}'), \quad \{\bar{z}(\mathbf{r}',t), \theta(\mathbf{r},t)\}^* = \frac{K_z}{2K_{z\bar{z}}\pi_\theta}\,\delta(\mathbf{r}-\mathbf{r}'). \quad (2.28)
\end{aligned}$$

For any on-shell phase-space functional $\Phi(\theta, \pi_\theta, z, \bar{z})$ the time evolution is now determined by

$$\dot{\Phi} = \{\Phi, H\}^*, \quad H = \int d^3x \left[\frac{f'(\rho)}{\rho}\,\mathbf{j}^2 + f(\rho)\right]. \quad (2.29)$$

In particular we note, that the equation

$$\dot{\pi_\theta} = \{\pi_\theta, H\}^* = \boldsymbol{\nabla} \cdot \left[\frac{\rho}{f'(\rho)}\Big(\boldsymbol{\nabla}\theta + iK_z\boldsymbol{\nabla}z - iK_{\bar{z}}\boldsymbol{\nabla}\bar{z}\Big)\right], \quad (2.30)$$

after the identification (2.23) is equivalent with

$$\dot{j_0} = \boldsymbol{\nabla}\cdot\mathbf{j} \quad \Leftrightarrow \quad \partial \cdot j = 0. \quad (2.31)$$

Similarly, the brackets of the fields $(z, \bar{z})$ with the hamiltonian can easily be checked to reproduce the field equations (2.9).

With the help of these rules we can determine the algebra of the conserved charges. It is useful to revert to a geometrical notation in terms of a simple Kähler manifold with metric $g_{z\bar{z}} = g_{\bar{z}z} = K_{z\bar{z}}$ and its inverse $g^{z\bar{z}} = g^{\bar{z}z} = 1/K_{z\bar{z}}$. The action of the $q[M]$ on the potentials then is:

$$\begin{aligned}
\delta_M\theta &= \{q[M], \theta\}^* = \frac{-K_{\bar{z}}M_z - K_zM_{\bar{z}} + 2K_{z\bar{z}}M}{K_{z\bar{z}}} = -g^{z\bar{z}}\left(K_{\bar{z}}M_z + K_zM_{\bar{z}}\right) + 2M, \\
\delta_M z &= \{q[M], z\}^* = \frac{-iM_{\bar{z}}}{K_{z\bar{z}}} = -ig^{z\bar{z}}M_{\bar{z}}, \\
\delta_M \bar{z} &= \{q[M], \bar{z}\}^* = \frac{iM_z}{K_{z\bar{z}}} = ig^{\bar{z}z}M_z. \quad (2.32)
\end{aligned}$$

One may check the closure of the algebra of conserved charges by computing the Poisson-Dirac brackets of such two charges. One finds that the result has the structure of a Poisson bracket on the 2-d manifold spanned by $(\bar{z}, z)$.:

$$\left\{q[M^{(1)}], q[M^{(2)}]\right\}^* = q[M^{(3)}], \quad \text{with} \quad M^{(3)} = ig^{z\bar{z}}\left(M_z^{(1)}M_{\bar{z}}^{(2)} - M_{\bar{z}}^{(1)}M_z^{(2)}\right). \quad (2.33)$$

If $M(\bar{z}, z)$ is taken to transform as a scalar on the complex manifold, the transformations δ_M are seen to take a covariant form and represent a reparametrization of the complex target manifold of the potentials $(\bar{z}, z)$. These transformations have the property that they leave the auxiliary vector potential (one-form) invariant:

$$a = dx^\mu a_\mu = d\theta + iK_z dz - iK_{\bar{z}}d\bar{z} \quad \Rightarrow \quad \delta_M a = 0. \quad (2.34)$$

As $a_\mu^2 = -f'^2(\rho)$, it follows that also $\delta_M \rho = 0$ and $\delta_M j_\mu = 0$. It is clear, that the transformations $\delta_M(\theta, z, \bar{z})$ in eq.(2.32) together with $\delta_M j_\mu = 0$ define an infinite set of global symmetries of the lagrangean (2.8) and the hamiltonian (2.24).:

$$\{q[M], H\}^* = 0, \quad H = \int d^3x\, \mathcal{H}. \tag{2.35}$$

These symmetries imply the invariance of the equations of motion under reparametrization of the auxiliary vector potential.

2.5 Examples: currents on $\mathbf{SU}_\eta(\mathbf{1},\mathbf{1})/\mathbf{U}(\mathbf{1})$

To illustrate various aspects of our general discussion of models of relativistic fluid mechanics described by (2.8), we now study some explicit examples of these models by considering three examples of complex target manifold of the potentials $(z, \bar{z})$. These examples are the complex plane $\mathbb{C}$, the hyperboloid H^2 and the sphere S^2. However, we will discuss the sphere in detail, and give explicit expressions for the transformations (2.32) generated by conserved charges (2.21).

To analyze all three manifolds at once we consider a field theory where the scalars $(z, \bar{z})$ live on the coset space $SU_\eta(1,1)/U(1)$ with $\eta = 0, \pm 1$. Here we have introduced the parameter η to distinguish three different manifolds [19]: $SU(2)/U(1)$ (the two-sphere S^2), the complex plane $\mathbb{C}$ and $SU(1,1)/U(1)$ (the two dimensional hyperboloid); for $\eta = 1, 0$ and -1 respectively. These manifolds are all Kähler so that the metric can locally be obtained from the Kähler potentials

$$K_\eta(z, \bar{z}) = \frac{1}{\eta} \ln(1 + \eta \bar{z} z). \tag{2.36}$$

(The Kähler potential for the case $\eta = 0$ is understood to be obtained by taking the limit $\eta \to 0$.) A comment is in order here about the global definition of models (2.8) with Kähler potentials $K(z, \bar{z})$ given by (2.36). For the flat case $\mathbb{C}$ there is just one coordinate patch so that the local description contains all the information of the model. If the scalar lives on the hyperbolic space, there are two choices depending on whether this space is defined as a single or double sheet. In order to avoid problems with positivity, we use the double sheet geometry.

In the following we only consider the sphere S^2 ($\eta = 1$) with the Kähler potentials

$$K^{(\pm)}(z_\pm, \bar{z}_\pm) = \ln(1 + \bar{z}_\pm z_\pm) \tag{2.37}$$

to be used on the northern and southern hemisphere, respectively, related up to the real part of a holomorphic function by the analytic co-ordinate transformation $z_- = 1/z_+$. In this case, the hermitian metric is obtained from the Kähler potentials $K^{(\pm)}(z_\pm, \bar{z}_\pm)$ in the standard way as the second mixed derivative

$$G^{(\pm)}_{\bar{z}z} = K^{(\pm)}(z_\pm, \bar{z}_\pm) = \frac{1}{(1 + \bar{z}_\pm z_\pm)^2}. \tag{2.38}$$

We will drop the subscript '±' on the fields $(z, \bar{z})$ from now on. The sphere admits the following holomorphic Killing vectors generating the isometries of the sphere corresponding to the coordinate transformations:

$$\delta z = R^z(z) = \epsilon + i\alpha z + \bar{\epsilon} z^2, \quad \delta \bar{z} = \bar{R}^{\bar{z}}(\bar{z}) = \bar{\epsilon} - i\alpha \bar{z} + \epsilon \bar{z}^2. \tag{2.39}$$

Here α is the parameter of $U(1)$ phase transformations, and $(\epsilon, \bar{\epsilon})$ are the complex parameters of the broken off-diagonal $SU(2)$ transformations. The isometries define a Lie algebra with structure constants ${f_{ij}}^k$ via the Lie derivative by:

$$\left(\mathcal{L}_{R_i}[R_j]\right)^z \;=\; R_i^z R_{j,z}^z \;-\; R_j^z R_{i,z}^z \;=\; {f_{ij}}^k\, R_k^z. \tag{2.40}$$

Under the isometry transformations (2.39) the Kähler potential $K(z, \bar{z})$ is invariant up to the real part of a holomorphic function $F(z)$ transforming in the adjoint representation of the algebra (2.40):

$$\delta K(\bar{z}, z) = F(z) + \bar{F}(\bar{z}), \quad F(z; \theta, \bar{\epsilon}) = \frac{i}{2}\,\theta + \bar{\epsilon} z. \tag{2.41}$$

From (B.21) it easy to see that the corresponding Killing potential is given by

$$M(\alpha, \epsilon, \bar{\epsilon}) = i\Big(K_{,z}\, R(z) - F(z)\Big) = \frac{1}{2}\frac{\alpha(1 - \bar{z}z) + 2i(\epsilon\bar{z} - \bar{\epsilon}z)}{1 + \bar{z}z}. \tag{2.42}$$

Indeed, the variations (2.39) are given by

$$\delta z = -iG^{z\bar{z}} M_{,\bar{z}}(\alpha, \epsilon, \bar{\epsilon}). \tag{2.43}$$

We now return to our models, described by (2.8), with the Kähler potential (2.36), but with $\eta = 1$ and a function f defined by

$$f(\rho) = \frac{b}{2}\,\rho^2 \quad \rightarrow \quad p = \varepsilon = \frac{b}{2}\,\rho^2. \tag{2.44}$$

The lagrangian density then becomes

$$\mathcal{L}[j^\mu, \theta, \bar{z}, z] = -j^\mu\Big(\partial_\mu\theta + iK_{,z}\partial_\mu z - iK_{,\bar{z}}\partial_\mu \bar{z}\Big) + \frac{b}{2} j^2. \tag{2.45}$$

This lagrangian is invariant under the infinitesimal transformation generated by the isometries (2.39) provided that the real scalar θ transform as:

$$\delta\theta = \frac{i}{2}\Big(F(z) - \bar{F}(\bar{z})\Big) \tag{2.46}$$

As a result, a conserved charge associated the transformations (2.46) and (2.39) can be derived using Noether's theorem. In summary, one first constructs a conserved Noether current from which the charge is constructed. One finds that the resulting charge is given by (2.21). If we insert the expression (2.42) into the brackets (2.32) we obtain the transformations (2.46) and (2.39).

Chapter 3

Supersymmetric hydrodynamics

Pure logical thinking cannot yield any knowledge of empirical world; all knowledge of reality starts from experience and ends in it.

Albert Einstein

3.1 Introduction

In this chapter we discuss a particular $N = 1$ globally supersymmetric field theory in four-dimensional Minkowski space and its applications to supersymmetric theories of hydrodynamics of an ideal fluid proposed in the previous chapter. In section 3.2, we construct a supersymmetric lagrangean in terms of superfields and work out its component form. In section 3.3 we discuss the internal symmetries of these lagrangeans in terms of Killing vectors, which represent infinitesimal symmetry transformations. Then we construct infinitesimal supersymmetry transformations of the fields appearing in the lagrangean. Using Noether's procedure, we construct the conserved quantities associated to these symmetries such as supercharges, which are the generators of supersymmetry transformations as well as the energy-momentum tensor from which the four-momentum is constructed. A canonical formulation of the theory in terms of a hamiltonian with a corresponding bracket structure is given in section 3.4. In section 3.5 we study currents and their conservation laws in the supersymmetric model. We show that there exists a regime in which an infinite number of currents (2.19) is reobtained; this regime we interpret as the description of a supersymmetric fluid. We finish with a discussion of our results and possible extensions.

3.2 Supersymmetric lagrangians

In this section, we construct $N = 1$ supersymmetric lagrangians, using the tensor calculus as described in [25]. Our aim is to arrive at a recipe which will allow us to write down a general supersymmetric theory, so that later we can apply the results to the special case of supersymmetric extension of relativistic fluid mechanics.

The decomposition of the auxiliary vector in terms of real and complex scalar potentials has a natural supersymmetric extension in 4-d Minkowski space-time. This leads to a proposal for a supersymmetric version of relativistic hydrodynamics in 4-d space-time. The supersymmetric extension is obtained by identifying the current j_μ and the auxiliary vector a_μ with the vector components V_μ and $\mathcal{A}_\mu$ of two real superfields $V = (C, \psi_\pm, B, V_\mu, \lambda_\pm, D)$ and $\mathcal{K} = (\mathcal{C}, \zeta_\pm, \mathcal{H}, \mathcal{A}_\mu, \xi_\pm, \mathcal{D})$ with a general superfield action of the form [18, 26]

$$S[V,\mathcal{K}] = \int d^4x\, \mathcal{L} \quad \mathcal{L} = \Big[V\mathcal{K} - \mathcal{F}(V)\Big]_D, \tag{3.1}$$

where the subscript D is last component, called D-component, of a vector superfield $V\mathcal{K} - \mathcal{F}(V)$ and $\mathcal{F}(V)$ is an analytic function of the real vector multiplet V. In terms of the components the action for V, $\mathcal{K}$ reads

$$\begin{aligned} S[V,\mathcal{K}] \;=\; & \int d^4x \Big\{ C\mathcal{D} + D\mathcal{C} + \frac{1}{2}(\mathcal{H}\bar{B} + \bar{\mathcal{H}}B) - \mathcal{A}_\mu V^\mu - \partial_\mu \mathcal{C}\partial^\mu C - \bar{\psi}_+\xi_+ - -\bar{\psi}_-\xi_- \\ & -\bar{\lambda}_-\zeta_- - \bar{\lambda}_+\zeta_+ - \frac{1}{2}\bar{\psi}_+ \overset{\leftrightarrow}{\not\partial}\, \xi_- - \frac{1}{2}\bar{\xi}_+ \overset{\leftrightarrow}{\not\partial}\, \psi_- - \mathcal{F}'(C)D - \frac{1}{2}\mathcal{F}''(C)\Big[-2\bar{\psi}_+\lambda_+ \\ & -2\bar{\psi}_-\lambda_- + B\,\bar{B} - V\cdot V - \partial C\cdot\partial C - \bar{\psi}_+ \overset{\leftrightarrow}{\not\partial}\, \psi_-\Big] - \frac{1}{8}\mathcal{F}''''(C)\bar{\psi}_+\psi_+\bar{\psi}_-\psi_- \\ & + \frac{1}{4}\mathcal{F}'''(C)\Big[\bar{B}\bar{\psi}_+\psi_+ + B\bar{\psi}_-\psi_- + 2i\bar{\psi}_+ \not{V}\, \psi_-\Big]\Big\} \end{aligned} \tag{3.2}$$

Here the primes are the derivatives of $\mathcal{F}(C)$ w.r.t. its argument. To turn this action into a model for supersymmetric hydrodynamics, we decompose the auxiliary vector superfield in terms of real and/or complex scalar superfields generalizing the potentials $(\theta, \bar{z}, z)$. The simplest way to do this is to introduce $N+1$ sets of complex chiral superfields $(\Lambda, \bar{\Lambda}, \Phi^\alpha, \bar{\Phi}^{\underline{\alpha}})$; $\alpha, \underline{\alpha} = 1, \ldots$,N, and define

$$\mathcal{K} = \Lambda + \bar{\Lambda} + K(\Phi^\alpha, \bar{\Phi}^{\underline{\alpha}}), \tag{3.3}$$

where $K(\Phi^\alpha, \bar{\Phi}^{\underline{\alpha}})$ is a real function of its superfield arguments; below it will become clear that its lowest bosonic component $K(\bar{z}, z)$ is the Kähler potential for the complex potentials $(\bar{z}, z)$ for $\alpha = 1$.

We label the components of the chiral superfields by $\Lambda = (s, \chi, h)$ and $\Phi^\alpha = (z^\alpha, \eta^\alpha, H^\alpha)$. Then the components of the real superfield $\mathcal{K}$ are replaced by the expressions

$$\begin{aligned} \mathcal{C} \;=\; & K(z^\alpha, \bar{z}^{\underline{\alpha}}) + s + \bar{s}, \\ \mathcal{D} \;=\; & 2G_{\alpha\underline{\beta}}\Big(H^\alpha \bar{H}^{\underline{\beta}} - \partial z^\alpha\cdot\partial\bar{z}^{\underline{\beta}} - \bar{\eta}_+^{\underline{\beta}} \overset{\leftrightarrow}{\not\partial}\, \eta_-^\alpha\Big) + 2G_{\alpha\underline{\beta},\gamma\underline{\delta}}\, \bar{\eta}_+^\alpha\, \eta_+^\gamma\, \bar{\eta}_-^{\underline{\beta}}\, \eta_-^{\underline{\delta}} \\ & + 2G_{\alpha\underline{\beta},\gamma}\Big(-\bar{H}^{\underline{\beta}}\,\bar{\eta}_+^\alpha\eta_+^\gamma + \bar{\eta}_+^{\underline{\beta}}\not\partial z^\gamma\eta_-^\alpha\Big) + 2G_{\alpha\underline{\beta},\underline{\gamma}}\Big(-H^\alpha\,\bar{\eta}_-^{\underline{\beta}}\eta_-^{\underline{\gamma}} + \bar{\eta}_-^{\underline{\beta}}\not\partial z^{\underline{\gamma}}\eta_+^\alpha\Big), \\ \mathcal{H} \;=\; & G_{\alpha\underline{\beta}}\,\bar{\eta}_+^{\underline{\beta}}\eta_+^\alpha - K_{,\alpha}H^\alpha - h, \quad \bar{\mathcal{H}} = G_{\alpha\underline{\beta}}\,\bar{\eta}_-^{\underline{\beta}}\eta_-^\alpha - K_{,\underline{\alpha}}\bar{H}^{\underline{\alpha}} - \bar{h} \\ \mathcal{A}_\mu \;=\; & i\Big(K_{,\alpha}\partial_\mu z^\alpha - K_{,\underline{\alpha}}\partial_\mu\bar{z}^{\underline{\alpha}} + \partial_\mu s - \partial_\mu\bar{s} - 2G_{\alpha\underline{\beta}}\,\bar{\eta}_-^{\underline{\beta}}\,\gamma_\mu\,\eta_+^\alpha\Big), \\ \xi_+ \;=\; & -2iG_{\alpha\underline{\beta}}\not\partial z^\alpha\eta_-^{\underline{\beta}} + 2iG_{\alpha\underline{\beta}}\,\bar{H}^{\underline{\beta}}\eta_+^\alpha - 2iG_{\alpha\underline{\beta},\gamma}\,\eta_+^\alpha\,\bar{\eta}_-^{\underline{\beta}}\eta_-^\gamma, \\ \xi_- \;=\; & 2iG_{\alpha\underline{\beta}}\not\partial\bar{z}^{\underline{\beta}}\eta_+^\alpha - 2iG_{\alpha\underline{\beta}}\,H^\alpha\eta_-^{\underline{\beta}} + 2iG_{\alpha\underline{\beta},\underline{\gamma}}\eta_-^{\underline{\gamma}}\,\bar{\eta}_+^{\underline{\beta}}\eta_+^\alpha, \\ \zeta_- \;=\; & 2iK_{\underline{\alpha}}\eta_-^{\underline{\alpha}} + 2i\chi_-, \quad \zeta_+ = -2iK_\alpha\eta_+^\alpha - 2i\chi_+ \end{aligned} \tag{3.4}$$

The component form of the action (3.2) after eliminating the auxiliary fields D, B, h, $H, \chi_\pm, \lambda_\pm$ and their complex conjugates reads

$$\begin{aligned}\mathcal{L} &= V^\mu\Big(2\partial_\mu\theta - iK_{,\alpha}\partial_\mu z^\alpha + iK_{,\underline{\beta}}\partial_\mu\bar{z}^{\underline{\beta}} + 2iG_{\alpha\underline{\beta}}\,\bar{\eta}_-^{\underline{\beta}}\gamma_\mu\eta_+^\alpha + \frac{i}{2}\mathcal{F}'''(C)\bar{\psi}_+\gamma_\mu\psi_-\Big) \\ &\quad - 2CG_{\alpha\underline{\beta}}\Big(\partial_\mu z^\alpha\partial^\mu\bar{z}^{\underline{\beta}} + \bar{\eta}_+^{\underline{\beta}}\overset{\leftrightarrow}{\not{D}}\eta_-^\alpha\Big) + 2CR_{\alpha\underline{\beta}\gamma\underline{\delta}}\,\bar{\eta}_+^{\underline{\beta}}\,\eta_+^\alpha\,\bar{\eta}_-^{\underline{\delta}}\,\eta_-^\gamma \\ &\quad - \frac{1}{2}\mathcal{F}''(C)\Big[\partial_\mu C\partial^\mu C - V_\mu V^\mu + \bar{\psi}_+\overset{\leftrightarrow}{\not{\partial}}\psi_-\Big] - \frac{2}{C}G_{\alpha\underline{\beta}}\,\bar{\psi}_+\,\eta_+^\alpha\,\bar{\psi}_-\,\eta_-^{\underline{\beta}} \\ &\quad + 2iG_{\alpha\underline{\beta}}\Big(\bar{\psi}_+\not{\partial}z^\alpha\eta_-^{\underline{\beta}} - \bar{\psi}_-\not{\partial}\bar{z}^{\underline{\beta}}\eta_+^\alpha\Big) - \frac{1}{8}\mathcal{F}''''(C)\bar{\psi}_+\,\psi_+\,\bar{\psi}_-\,\psi_-.\end{aligned} \tag{3.5}$$

In this expression we used the notation of the geometrical objects for the metric $G_{\alpha\underline{\beta}}$, connection $\Gamma^\alpha_{\beta\gamma}$ and curvature $R_{\alpha\underline{\beta}\gamma\underline{\delta}}$, given in (B.5), (B.7) and (B.10) respectively. Furthermore, we have introduced the notation $\theta = 2\mathrm{Im}\,s$. The Kähler covariant derivative of a chiral spinor and the left-right arrow above the covariant derivative are defined by

$$\begin{aligned}\not{D}\eta_-^{\underline{\alpha}} &= \not{\partial}\eta_-^{\underline{\alpha}} + \bar{\Gamma}^{\underline{\alpha}}_{\underline{\gamma\beta}}\not{\partial}\bar{z}^{\underline{\gamma}}\eta_-^{\underline{\beta}}, \quad \bar{\chi}_\pm\overset{\leftrightarrow}{\not{\partial}}\zeta_\pm = \bar{\chi}_\pm\gamma^\mu\partial_\mu\zeta_\pm - \partial_\mu\bar{\chi}_\pm\gamma^\mu\zeta_\pm \\ \not{D}\eta_+^\alpha &= \not{\partial}\eta_+^\alpha + \Gamma^\alpha_{\beta\gamma}\not{\partial}z^\beta\eta_+^\gamma.\end{aligned} \tag{3.6}$$

It is obvious, that in the absence of fermions $\psi_\pm = \eta_\pm^\alpha = 0$ and for $C = 0$ we reobtain the lagrangean (2.8) with

$$f(\rho) = \frac{1}{2}\mathcal{F}''(0)\rho^2. \tag{3.7}$$

This is of the type (2.44) with $b = \mathcal{F}''(0)$. The additional scalar and spinor fields C, $\psi_\pm$ and $\eta_\pm^\alpha$ describe additional dynamical background fields. As the co-efficient of the kinetic terms of the fields $(\bar{z}, z)$ and $\psi_\pm$, the scalar field C must be non-negative. This can easily be achieved, for example by replacing the real superfield V by another real superfield W such that $V = e^W$. Thus we can take the condition $C \geq 0$ for granted.

In the remainder of this chapter, we construct and discuss all the conserved quantities associated with the symmetries of the action.

3.3 Symmetries and currents

In this subsection, we discuss the symmetries of the theory described by the lagrangean (3.5) and the resulting conserved quantities. We first discuss internal symmetry, then we construct the generators of the supersymmetry transformations. After that we construct the energy-momentum tensor following from the invariance of the action under translations.

The lagrangian (3.5) is invariant under the infinitesimal transformation generated by the isometries [26]

$$
\begin{aligned}
\delta z^\alpha &= \Theta^i R_i^\alpha(z), \quad \delta \bar{z}^{\underline{\alpha}} = \Theta^i \bar{R}_i^{\underline{\alpha}}(\bar{z}) \quad \delta\eta_+^\alpha = \Theta^i R_{i,\beta}^\alpha(z)\eta_+^\beta, \quad \delta\eta_-^{\underline{\alpha}} = \Theta^i \bar{R}_{i,\underline{\beta}}^{\underline{\alpha}}(\bar{z})\eta_-^{\underline{\beta}} \\
\delta H^\alpha &= \Theta^i\Big(R_{i,\beta}^\alpha(z)H^\beta - R_{i,\beta\gamma}^\alpha(z)\bar{\eta}_+^\beta\eta_+^\gamma\Big) \quad \delta\bar{H}^{\underline{\alpha}} = \Theta^i\Big(\bar{R}_{i,\underline{\beta}}^{\underline{\alpha}}(\bar{z})\bar{H}^{\underline{\beta}} - \bar{R}_{i,\underline{\beta\gamma}}(\bar{z})\bar{\eta}_-^{\underline{\beta}}\eta_-^{\underline{\gamma}}\Big), \\
\delta\theta &= \frac{i}{2}\Theta^i\Big(F_i(z) - \bar{F}_i(\bar{z})\Big),
\end{aligned} \tag{3.8}
$$

where Θ^i the parameters of the infinitesimal transformations. A set of conserved currents can be derived using the Noether procedure from the isometry transformations (3.8). The resulting currents written in terms of the Killing potential (B.18) are:

$$
J_\mu(M) = -V_\mu\, M - 2iM_{;\alpha}\Big(C\partial_\mu z^\alpha + \bar{\psi}_-\gamma_\mu\eta_+^\alpha\Big) - 2iCM_{;\underline{\alpha};\beta}\,\bar{\eta}_+^{\underline{\alpha}}\,\gamma_\mu\,\eta_-^{\underline{\beta}} + \text{h.c.} \tag{3.9}
$$

Here the semicolon denotes a covariant derivative using the connection (B.7). For example,

$$
\begin{aligned}
M_{;\underline{\alpha}} &= M_{,\underline{\alpha}}, \quad M_{;\underline{\alpha};\beta} = M_{,\underline{\alpha}\beta}, \quad M_{;\alpha;\beta} = M_{,\alpha\beta} - \Gamma^\gamma_{\alpha\beta}M_{,\gamma} = 0, \\
M_{;\alpha;\underline{\gamma};\beta} &= [D_\beta, D_{\underline{\gamma}}]M_{,\alpha} = R_\alpha{}^\gamma{}_{\beta\underline{\gamma}}\,M_{,\gamma}, \quad M_{;\underline{\alpha};\underline{\beta}} = M_{,\underline{\alpha\beta}} - \bar{\Gamma}^{\underline{\gamma}}_{\underline{\alpha\beta}}M_{,\underline{\gamma}} = 0.
\end{aligned} \tag{3.10}
$$

These currents are divergence free

$$
\partial\cdot J = 0, \tag{3.11}
$$

as it can be verified explicitly using the equations of motion and the following identity

$$
i(R_{\underline{\alpha}\alpha}{}^\gamma{}_\delta\, M^i_{,\gamma})_{;\underline{\beta}} - i(R_{\alpha\underline{\alpha}}{}^{\underline{\gamma}}{}_{\underline{\beta}}\, M^i_{,\underline{\gamma}})_{;\delta} = iR_{\delta\underline{\beta}}{}^{\underline{\gamma}}{}_{\underline{\alpha}}\, M^i_{,\underline{\gamma};\alpha} - iR_{\underline{\beta}\delta}{}^\gamma{}_\alpha\, M^i_{,\gamma;\underline{\alpha}}. \tag{3.12}
$$

Details of the proof of this identity are given in appendix B.2. The field equations derived from lagrangian (3.5) read

$$
\begin{aligned}
\mathcal{F}''(C)\Box C &= 2G_{\alpha\underline{\beta}}\Big(\partial z^\alpha\cdot\partial\bar{z}^{\underline{\beta}} + \bar{\eta}_+^{\underline{\beta}}\overleftrightarrow{\not{D}}\,\eta_-^\alpha\Big) - \frac{1}{2}\mathcal{F}'''(C)\Big[V^2 + (\partial C)^2 - \bar{\psi}_+\overleftrightarrow{\not{\partial}}\,\psi_-\Big] \\
&\quad -\frac{i}{2}\mathcal{F}''''(C)\bar{\psi}_+\not{V}\,\psi_- + \frac{1}{8}\mathcal{F}'''''(C)\bar{\psi}_+\,\psi_+\,\bar{\psi}_-\,\psi_- - \frac{2}{C^2}\,G_{\alpha\underline{\beta}}\bar{\psi}_+\,\eta_+^{\underline{\beta}}\bar{\psi}_-\,\eta_-^\alpha \\
&\quad +2R_{\alpha\underline{\beta}\gamma\underline{\delta}}\,\bar{\eta}_+^{\underline{\beta}}\,\eta_+^\alpha\bar{\eta}_-^{\underline{\delta}}\,\eta_-^\gamma
\end{aligned} \tag{3.13}
$$

$$
\mathcal{F}''(C)V_\mu = i\Big(K_\alpha\partial_\mu z^\alpha - K_{\underline{\alpha}}\partial_\mu\bar{z}^{\underline{\alpha}} - 2G_{\alpha\underline{\beta}}\,\bar{\eta}_-^{\underline{\beta}}\gamma_\mu\eta_+^\alpha\Big) - \frac{i}{2}\mathcal{F}'''\bar{\psi}_+\gamma_\mu\psi_- - 2\partial_\mu\theta \tag{3.14}
$$

$$
\partial\cdot V = 0 \tag{3.15}
$$

$$
\begin{aligned}
-2\mathcal{D}\cdot(C\partial z^\alpha) &= 2i\bar{\psi}_-\not{\partial}\eta_+^\alpha + 2i\Gamma^\alpha_{\beta\gamma}\bar{\psi}_-\not{\partial}z^\beta\,\eta_-^\gamma - 2i\Big(V\cdot\partial z^\alpha + \bar{\eta}_+^\alpha\not{\partial}\psi_-\Big) + 4CR^\alpha{}_{\delta\underline{\beta}\gamma}\,\bar{\eta}_+^{\underline{\beta}}\not{\partial}z^\delta\eta_-^\gamma \\
&\quad +2CR^\alpha{}_{\delta\underline{\beta}\gamma;\underline{\sigma}}\,\bar{\eta}_+^{\underline{\beta}}\eta_+^\delta\bar{\eta}_-^{\underline{\sigma}}\eta_-^\gamma
\end{aligned} \tag{3.16}
$$

$$
\begin{aligned}
\mathcal{F}''(C)\not{\partial}\psi_+ &= -\frac{1}{2}\mathcal{F}'''(C)\Big(\not{\partial}C + i\not{V}\Big)\psi_+ - \frac{1}{4}\mathcal{F}'''(C)\psi_-\,\bar{\psi}_+\,\psi_+ \\
&\quad -2G_{\alpha\underline{\beta}}\Big(\frac{1}{C}\eta_-^{\underline{\beta}}\bar{\zeta}_+ + i\not{\partial}\bar{z}\Big)\eta_+^\alpha,
\end{aligned} \tag{3.17}
$$

$$
4C\not{D}\eta_+^\alpha = -2\Big(\not{\partial}C - i\not{V} + \frac{1}{C}\psi_-\bar{\psi}_+\Big)\eta_+^\alpha - 2i\not{\partial}z^\alpha\psi_+ + 4CR^\alpha{}_{\delta\underline{\beta}\gamma}\,\eta_-^\delta\bar{\eta}_+^{\underline{\beta}}\eta_+^\gamma. \tag{3.18}
$$

In the expression (3.16) we have used the field equation equation (3.18) for η_+^α, and introduced a Kähler-covariant derivative $\mathcal{D}$:

$$\mathcal{D}_\mu(\partial^\mu z^\alpha) = \Box z^\alpha + \partial z^\alpha\, \Gamma^\gamma_{\gamma\delta} \cdot \partial z^\delta. \tag{3.19}$$

All other equations of motion for $(\bar{z}^{\underline{\alpha}}, \psi_-, \eta_-^{\underline{\alpha}})$ are obtained by complex conjugation of (3.16), (3.17) and (3.18) respectively. The conserved charges q are obtained from the currents (3.9)

$$q(M) = \int d^3x J^0(M). \tag{3.20}$$

We now turn the construction of the supercharges. The full lagrangian (3.5) is invariant under the following on-shell supersymmetry transformation, parametrized by anticommuting chiral spinor ϵ_+ and ϵ_-:

$$\begin{aligned}
&\delta C = \tfrac{i}{2}\bar{\epsilon}_+\psi_+ - \tfrac{i}{2}\bar{\epsilon}_-\psi_-, && \delta z^\alpha = \bar{\epsilon}_+\eta_+^\alpha,\\
&\delta\psi_+ = -\tfrac{1}{2}(\not{V} + i\not{\partial}C)\epsilon_- && \delta\psi_- = -\tfrac{1}{2}(\not{V} - i\not{\partial}C)\epsilon_+,\\
&\delta V_\mu = \bar{\epsilon}_+\sigma_{\mu\nu}\partial^\nu\psi_+ + \bar{\epsilon}_-\,\sigma_{\mu\nu}\partial^\nu\psi_- && \delta\bar{z}^{\underline{\alpha}} = \bar{\epsilon}_-\eta_-^{\underline{\alpha}},\\
&\delta\theta = \tfrac{1}{4}\mathcal{F}''(C)(\bar{\epsilon}_+\psi_+ + \bar{\epsilon}_-\psi_-) + \tfrac{i}{2}(\bar{\epsilon}_+K_{,\alpha}\eta_+^\alpha - \bar{\epsilon}_-K_{,\underline{\alpha}}\eta_-^{\underline{\alpha}}),\\
&\delta\eta_+^\alpha = \tfrac{1}{2}\Big[\not{\partial}z^\alpha\epsilon_- + \epsilon_+\tfrac{1}{2CG_{\alpha\underline{\beta}}}\Big(i\bar{\psi}_+\eta_+^\beta + 2CG_{\alpha\underline{\beta},\gamma}\bar{\eta}_+^{\underline{\beta}}\eta_+^\gamma\Big)\Big],\\
&\delta\eta_-^{\underline{\alpha}} = \tfrac{1}{2}\Big[\not{\partial}\bar{z}^{\underline{\alpha}}\epsilon_+ + \epsilon_-\tfrac{1}{2CG_{\underline{\alpha}\beta}}\Big(-i\bar{\psi}_-\eta_-^\beta + 2CG_{\underline{\alpha}\beta,\underline{\gamma}}\bar{\eta}_-^{\underline{\gamma}}\eta_-^\beta\Big)\Big].
\end{aligned} \tag{3.21}$$

Under these transformations the variation of the lagrangean is a total divergence:

$$\delta\mathcal{L} = \frac{i}{2}\partial_\mu(\bar{\epsilon}_+B_+^\mu - \bar{\epsilon}_-B_-^\mu). \tag{3.22}$$

where the vector-spinor fields $B_{\pm\mu}$ are given, modulo equations of motion (for details, see the appendix D.1), by

$$\begin{aligned}
B_+^\mu &\simeq 2G_{\alpha\underline{\beta}}\left(\gamma^\mu\eta_-^{\underline{\beta}}\,\bar{\psi}_+\,\eta_+^\alpha + iC\not{\partial}\bar{z}^{\underline{\beta}}\eta_+^\alpha\right) - \frac{1}{2}\mathcal{F}''(C)\gamma^\mu\left(\not{\partial}C + i\not{V}\right)\psi_+ + \\
&\quad -\frac{1}{2}\mathcal{F}'''(C)\gamma^\mu\psi_-\,\bar{\psi}_+\psi_+,\\
B_-^\mu &\simeq 2G_{\underline{\alpha}\beta}\left(\gamma^\mu\,\eta_+^\beta\,\psi_-\,\eta_-^{\underline{\alpha}} - iC\not{\partial}z^\beta\eta_-^{\underline{\alpha}}\right) - \frac{1}{2}\mathcal{F}''(C)\gamma^\mu\left(\not{\partial}C - i\not{V}\right)\psi_- + \\
&\quad -\frac{1}{2}\mathcal{F}'''(C)\gamma^\mu\psi_+\,\bar{\psi}_-\psi_-.
\end{aligned} \tag{3.23}$$

where the similarity sign $\simeq$ in (3.23) signifies that the vector-spinors are given up to equations of motion. The supercurrents are obtained directly by Noether's procedure and read

$$\begin{aligned} S_{\mu+} &= 4G_{\alpha\underline{\beta}}\Big(\gamma_\mu\,\eta_-^\alpha\,\bar\psi_+\,\eta_+^{\underline{\beta}} - iC\partial\!\!\!/\bar z^{\underline{\beta}}\gamma_\mu\psi_+^\alpha\Big) + \mathcal{F}''(C)(\partial\!\!\!/C + iV\!\!\!/)\gamma_\mu\psi_+ \\ &\quad -\frac{1}{2}\mathcal{F}'''(C)\gamma_\mu\,\psi_-\,\bar\psi_+\,\psi_+ - 2iCG_{\alpha\underline{\beta},\gamma}\,\gamma_\mu\,\eta_-^\alpha\,\bar\eta_+^{\underline{\beta}}\,\eta_+^\gamma \end{aligned} \tag{3.24}$$

$$\begin{aligned} S_{\mu-} &= 4G_{\alpha\underline{\beta}}\Big(\gamma_\mu\,\eta_+^\alpha\,\bar\psi_-\,\eta_-^{\underline{\beta}} + iC\partial\!\!\!/ z^\alpha\gamma_\mu\psi_-^{\underline{\beta}}\Big) + \mathcal{F}''(C)(\partial\!\!\!/C - iV\!\!\!/)\gamma_\mu\psi_- \\ &\quad -\frac{1}{2}\mathcal{F}'''(C)\gamma_\mu\,\psi_+\,\bar\psi_-\,\psi_- + 2iCG_{\alpha\underline{\beta},\gamma}\,\gamma_\mu\,\eta_+^\alpha\,\bar\eta_-^{\underline{\beta}}\,\eta_-^\gamma. \end{aligned} \tag{3.25}$$

The full derivation of the supercurrents $S_{\pm\mu}$ is presented in appendix D.2. The field equations imply that

$$\partial\cdot S_\pm = 0, \tag{3.26}$$

from which the conservation of the supercharges

$$Q_\pm = \int d^3x S_{\pm 0} \tag{3.27}$$

follows.

Since supersymmetric field theories are translationally invariant, the theory described by lagrangean (3.5) conserves energy-momentum. It is derived in two steps. First, by the Noether procedure from lagrangian (3.5) one derives a non-symmetric set of conserved currents

$$\begin{aligned} \Theta_{\mu\nu} &= -\frac{\delta\mathcal{L}}{\delta(\partial_\mu A_i)}\partial^\nu A_i + g_{\mu\nu}\mathcal{L} \\ &= \mathcal{F}''(C)\Big(\partial_\mu C\partial_\nu C + V_\mu V_\nu + \frac{1}{2}\bar\psi_+\gamma_\mu\overleftrightarrow{\partial_\nu}\psi_-\Big) + \frac{i}{2}\mathcal{F}'''(C)V_\mu\bar\psi_+\gamma_\nu\psi_- + \\ &\quad +2CG_{\alpha\underline{\beta}}\Big(\partial_\mu z^\alpha\partial_\nu\bar z^{\underline{\beta}} + \partial_\mu\bar z^{\underline{\beta}}\partial_\nu z^\alpha + \bar\eta_+^{\underline{\beta}}\gamma_\mu\overleftrightarrow{\partial_\nu}\eta_-^\alpha\Big) + 2iG_{\alpha\underline{\beta}}\,\bar\eta_-^{\underline{\beta}}V_\mu\gamma_\nu\eta_+^\alpha \\ &\quad -2\Big[iG_{\alpha\underline{\beta}}\,\bar\psi_+\partial_\nu z^\alpha\,\gamma_\mu\eta_-^{\underline{\beta}} + CG_{\alpha\underline{\beta},\gamma}\bar\eta_+^{\underline{\beta}}\partial_\nu z^\alpha\,\gamma_\mu\eta_-^\gamma + \text{h.c.}\Big] + g_{\mu\nu}\mathcal{L}. \end{aligned} \tag{3.28}$$

Using the field equations (3.14)-(3.18) a straightforward calculation confirms that $\partial^\mu\Theta_{\mu\nu} = 0$. However, it turns out that the symmetric and anti-symmetric part of $\Theta_{\mu\nu}$ are separately conserved. To show this, we write out the anti-symmetric part $2\Omega_{\mu\nu} = \Theta_{\mu\nu} - \Theta_{\nu\mu}$:

$$\begin{aligned} \Omega_{\mu\nu} = -\Omega_{\nu\mu} &= -\frac{1}{4}\mathcal{F}''(C)\bar\psi_+\gamma_{[\mu}\overleftrightarrow{\partial_{\nu]}}\psi_- - \frac{i}{4}\mathcal{F}'''(C)\bar\psi_+V_{[\mu}\gamma_{\nu]}\psi_- \\ &\quad -CG_{\alpha\underline{\beta}}\,\bar\eta_+^{\underline{\beta}}\gamma_{[\mu}\overleftrightarrow{\partial_{\nu]}}\eta_-^\alpha - iG_{\alpha\underline{\beta}}\,\bar\eta_-^{\underline{\beta}}V_{[\mu}\gamma_{\nu]}\eta_+^\alpha + \Big[iG_{\alpha\underline{\beta}}\,\bar\psi_+\partial_{[\nu}z^\alpha\,\gamma_{\mu]}\eta_-^{\underline{\beta}} \\ &\quad +CG_{\alpha\underline{\beta},\gamma}\bar\eta_+^{\underline{\beta}}\partial_{[\nu}z^\alpha\,\gamma_{\mu]}\eta_-^\gamma + \text{h.c.}\Big]. \end{aligned} \tag{3.29}$$

A further application of the field equations then shows that $\partial^\mu \Omega_{\mu\nu} = 0$. It implies, that the symmetric part of the energy-momentum tensor is conserved by itself. Equivalently, one can interpret $\Omega_{\mu\nu}$ as an improvement term to be subtracted from the non-symmetric $\Theta_{\mu\nu}$ so as to construct the symmetric set of conserved energy-momentum currents

$$\begin{aligned} T_{\mu\nu} &= \mathcal{F}''(C)\Big(\partial_\mu C \partial_\nu C + V_\mu V_\nu + \frac{1}{4}\bar{\psi}_+ \gamma_{\{\mu} \overleftrightarrow{\partial}_{\nu\}} \psi_-\Big) + \frac{i}{4}\mathcal{F}'''(C)\bar{\psi}_+ V_{\{\mu}\gamma_{\nu\}}\psi_- + \\ &\quad + 2CG_{\alpha\underline{\beta}}\Big(\partial_\mu z^\alpha \partial_\nu \bar{z}^{\underline{\beta}} + \partial_\mu \bar{z}^{\underline{\beta}} \partial_\nu z^\alpha + \frac{1}{2}\bar{\eta}_+^{\underline{\beta}} \gamma_{\{\mu} \overleftrightarrow{\partial}_{\nu\}} \eta_-^\alpha\Big) + iG_{\alpha\underline{\beta}}\, \bar{\eta}_-^{\underline{\beta}} V_{\{\mu}\gamma_{\nu\}}\eta_+^\alpha \\ &\quad - \Big[iG_{\alpha\underline{\beta}}\, \bar{\psi}_+ \partial_{\{\nu} z^\alpha\, \gamma_{\mu\}}\eta_-^{\underline{\beta}} + CG_{\alpha\underline{\beta},\gamma}\bar{\eta}_+^{\underline{\beta}} \partial_{\{\nu} z^\alpha\, \gamma_{\mu\}}\eta_-^\gamma + \text{h.c.}\Big] + g_{\mu\nu}\mathcal{L}. \end{aligned} \tag{3.30}$$

By construction it has the properties $T_{\mu\nu} = T_{\nu\mu}$ and $\partial^\mu T_{\mu\nu} = 0$, from which the conservation of four-momentum

$$P_\mu = \int d^3x T_{\mu 0} \tag{3.31}$$

follows. In the following section, we construct the explicit expressions for the supercharges (3.27) and four-momentum vector (3.31).

3.4 Canonical analysis

In this section, we show that the supercharges $Q_\pm$ satisfy the supersymmetry algebra and that they generate the supersymmetry transformations (3.21) as well as the space-time translations on the fields. As this action of the supersymmetry algebra in terms of $Q_\pm$ requires the use of canonical variables and hamiltonian equations of motion, we first present a canonical formulation of the theory and describe the dynamics in terms of phase-space coordinates and the hamiltonian. However in this formalism, the fermionic momenta turn out not to be independent degrees of freedom, as they are constrained to fermionic fields themselves. To eliminate these constraints we introduce Poisson-Dirac brackets, defined as the Poisson brackets from which the second class constraints have been projected out.

We now present details of this analysis. The canonical momenta of the theory are defined by

$$\begin{aligned} \pi_C &= \frac{\delta \mathcal{L}}{\delta \dot{C}} = \mathcal{F}''(C)\dot{C}, \quad \pi_\theta = \frac{\delta \mathcal{L}}{\delta \dot{\theta}} = 2V^0, \\ \pi_{z^\alpha} &= \frac{\delta \mathcal{L}}{\delta \dot{z^\alpha}} = iK_{,\alpha}V_0 + 2CG_{\alpha\underline{\beta}}\dot{\bar{z}^{\underline{\beta}}} + 2iG_{\alpha\underline{\beta}}\, \bar{\psi}_+\gamma^0\eta_-^{\underline{\beta}} + 2CG_{\alpha\underline{\beta},\gamma}\, \bar{\eta}_+^{\underline{\beta}}\gamma^0\eta_-^\gamma, \\ \bar{\pi}_{\bar{z}^{\underline{\alpha}}} &= \frac{\delta \mathcal{L}}{\delta \dot{\bar{z}^{\underline{\alpha}}}} = -iK_{,\underline{\alpha}}V_0 + 2CG_{\beta\underline{\alpha}}\dot{z^\beta} - 2iG_{\underline{\alpha}\beta}\, \bar{\psi}_-\gamma^0\eta_+^\beta + 2CG_{\underline{\alpha}\beta,\underline{\gamma}}\, \bar{\eta}_-^{\underline{\gamma}}\gamma^0\eta_+^\beta \\ \pi_{\psi_\pm} &= \gamma_0\, \frac{\delta \mathcal{L}}{\delta \dot{\bar{\psi}}_\mp} = \frac{1}{2}\mathcal{F}''(C)\psi_\pm, \quad \pi_{\eta_\pm^\alpha} = \gamma_0 \frac{\delta \mathcal{L}}{\delta \dot{\bar{\eta}}^\alpha{}_\mp} = 2CG_{\alpha\underline{\beta}}\eta_\pm^{\underline{\beta}}. \end{aligned} \tag{3.32}$$

Here we included γ_0 in the definition of the fermionic momenta so that the momenta of Majorana variables are Majorana themselves as well. Clearly, the last two equations of (3.32) are second-class constraints, expressing the fermionic momenta $(\pi_{\psi_\pm}, \pi_{\eta^\alpha_\pm})$ in terms of fermionic fields:

$$\chi_{\psi_\pm} = \pi_{\psi_\pm} - \frac{1}{2}\mathcal{F}''(C)\psi_\pm \simeq 0, \quad \chi_{\eta^\alpha_\pm} = \pi_{\eta^\alpha_\pm} - 2CG_{\alpha\underline{\beta}}\,\eta^{\beta}_{\mp} \simeq 0 \tag{3.33}$$

The similarity sign $\simeq$ in last equality of (3.33) signifies that the constraints are defined only on a subset (the physical shell) of the full phase space. In this extended phase space, the equal-time Poisson brackets of the theory are defined by

$$\begin{aligned}
\left\{\pi_{\eta^\alpha_\pm}(\mathbf{r}), \bar{\eta}^{\underline{\beta}}_{\mp}(\mathbf{r}')\right\} &= \left\{\eta^\alpha_\mp(\mathbf{r}'), \bar{\pi}_{\eta^{\underline{\beta}}_\pm}(\mathbf{r})\right\} = \delta^\alpha_{\underline{\beta}}\,\gamma^0 P_\mp\,\delta^3(\mathbf{r}-\mathbf{r}') \\
\left\{\pi_{\psi_\pm}(\mathbf{r}), \bar{\psi}_\mp(\mathbf{r}')\right\} &= \left\{\psi_\pm(\mathbf{r}'), \bar{\pi}_{\psi_\mp}(\mathbf{r})\right\} = \gamma^0 P_\mp\,\delta^3(\mathbf{r}-\mathbf{r}') \\
\{C(\mathbf{r}), \pi_C(\mathbf{r}')\} &= \{\theta(\mathbf{r}), \pi_\theta(\mathbf{r}')\} = \delta^3(\mathbf{r}-\mathbf{r}'), \\
\{z^\alpha(\mathbf{r}), \pi_{z^\beta}(\mathbf{r}')\} &= \delta^\alpha_\beta\,\delta^3(\mathbf{r}-\mathbf{r}') \quad \{\bar{z}^{\underline{\alpha}}(\mathbf{r}), \pi_{\bar{z}^{\underline{\beta}}}(\mathbf{r}')\} = \delta^{\underline{\alpha}}_{\underline{\beta}}\,\delta^3(\mathbf{r}-\mathbf{r}'),
\end{aligned} \tag{3.34}$$

where $P_\pm = \frac{1}{2}(1 \pm \gamma_5)$ are the left- and right-handed chiral projection operators. To simplify the notation, from now on we will suppress the fields space dependence and drop the delta function $\delta^3(\mathbf{r}-\mathbf{r}')$.

In order to describe the canonical dynamics on the reduced phase-space determined by the constraint equations (3.33), we introduce Poisson-Dirac brackets defined by (2.26). The corresponding matrix of constraint brackets in this case reads

$$C_{ij} = -\begin{pmatrix} 0 & \mathcal{F}''(C)\gamma^0 P_- & 0 & 0 \\ \mathcal{F}''(C)\gamma^0 P_+ & 0 & 0 & 0 \\ 0 & 0 & 0 & 4CG_{\alpha\underline{\beta}}\gamma^0 P_- \\ 0 & 0 & 4CG_{\alpha\underline{\beta}}\gamma^0 P_+ & 0 \end{pmatrix}. \tag{3.35}$$

Applying this prescription (see appendix C for the details), we obtain the full set of non-zero Poisson-Dirac brackets of our theory:

$$\begin{aligned}
\{z^\alpha, \pi_{z^\beta}\}^* &= \delta^\alpha_\beta, \quad \{\bar{z}^{\underline{\alpha}}, \bar{\pi}_{\bar{z}^{\underline{\beta}}}\}^* = \delta^{\underline{\alpha}}_{\underline{\beta}}, \quad \{C, \pi_C\}^* = \{\theta, \pi_\theta\}^* = 1, \\
\left\{\pi_C, \bar{\psi}_\pm\right\}^* &= \frac{\mathcal{F}'''(C)}{2\mathcal{F}''(C)}\bar{\psi}_\pm, \quad \left\{\pi_{z^\alpha}, \bar{\eta}^{\underline{\beta}}_\pm\right\}^* = \frac{1}{2}\Gamma^\alpha_{\beta\beta}\,\bar{\eta}^{\underline{\beta}}_\pm, \quad \left\{\bar{\pi}_{\bar{z}^{\underline{\alpha}}}, \bar{\eta}^{\underline{\beta}}_\pm\right\}^* = \frac{1}{2}\bar{\Gamma}^{\underline{\alpha}}_{\underline{\gamma\gamma}}\,\bar{\eta}^{\underline{\beta}}_\pm \\
\left\{\eta^\alpha_\pm, \bar{\eta}^{\underline{\beta}}_\mp\right\}^* &= \frac{1}{4C}G^{\alpha\underline{\beta}}\gamma^0 P_\mp, \quad \left\{\psi_\pm, \bar{\psi}_\mp\right\}^* = \frac{1}{2\mathcal{F}''(C)}\gamma^0 P_\mp, \quad \{\psi_\pm, \pi_C\}^* = -\frac{\mathcal{F}'''(C)}{2\mathcal{F}''(C)}\psi_\pm \\
\left\{\eta^\alpha_\pm, \pi_{z^\beta}\right\}^* &= -\frac{1}{2}\Gamma^\alpha_{\gamma\beta}\eta^\gamma_\pm, \quad \{\eta^{\underline{\alpha}}_\pm, \bar{\pi}_{\bar{z}^{\underline{\beta}}}\}^* = -\frac{1}{2}\bar{\Gamma}^{\underline{\alpha}}_{\underline{\gamma\beta}}\eta^{\underline{\gamma}}_\pm.
\end{aligned} \tag{3.36}$$

Subsequently the isometries transformations (3.8) are obtained from the conserved charges (3.20) by the Poisson-Dirac brackets

$$\delta_M A = \{q, A\}^*. \tag{3.37}$$

where the canonical Noether charges (3.20) is

$$
\begin{aligned}
q[M] \;=\; & \int d^3x \Big[\pi_\theta M - G^{\alpha\underline{\beta}}\, M_\alpha \left(i\bar{\pi}_{\underline{\beta}} + \frac{1}{2}\, K_{\underline{\beta}}\pi_\theta\right) + G^{\alpha\underline{\beta}}\, M_{\bar{\beta}}\Big(i\pi_\alpha - \frac{1}{2}\, K_\alpha\pi_\theta\Big) \\
& -2iC\Big(M_\alpha\, \bar{\Gamma}^{\underline{\gamma}}_{\underline{\gamma\beta}} + M_{\underline{\beta}}\, \Gamma^\gamma_{\gamma\alpha} - 2M_{\alpha\underline{\beta}}\Big)\bar{\eta}^{\underline{\beta}}_{-}\gamma_0\eta^\alpha_+\Big]. \qquad (3.38)
\end{aligned}
$$

Again, the closure of the algebra of conserved charges can be checked. Indeed, the result (2.33) is reobtained if one computes the Poisson-Dirac brackets of two charges.

The canonical hamiltonian, obtained from lagrangian (3.5) by Legendre transformation, reads

$$
\begin{aligned}
H \;=\; & \int d^3x \Big[\frac{1}{2\mathcal{F}''(C)}\pi_C^2 + \Big(\frac{1}{8}\mathcal{F}''(C) + G^{\alpha\underline{\beta}}\, \frac{K_\alpha K_{\underline{\beta}}}{8C}\Big)\pi_\theta^2 + G^{\alpha\underline{\beta}}\, \frac{1}{2C}\pi_{z^\alpha}\pi_{\bar{z}^{\underline{\beta}}} \\
& - i\pi_\theta\Big(\frac{1}{4}\mathcal{F}'''(C)\bar{\psi}_+\gamma_0\psi_- + G_{\alpha\underline{\beta}}\, \bar{\eta}^{\underline{\beta}}_-\gamma_0\eta^\alpha_+\Big) + \frac{1}{8}\mathcal{F}''''(C)\bar{\psi}_+\,\psi_+\,\bar{\psi}_-\,\psi_- \\
& + \frac{1}{2}\mathcal{F}''(C)\Big((\boldsymbol{\nabla} C)^2 + \mathbf{V}^2 + \bar{\psi}_+ \overset{\leftrightarrow}{\boldsymbol{\nabla}\!\!\!/}\, \psi_-\Big) + 2CG_{\alpha\underline{\beta}}\left(\boldsymbol{\nabla} z^\alpha\boldsymbol{\nabla}\bar{z}^{\underline{\beta}} + \bar{\eta}^{\underline{\beta}}_+ \overset{\leftrightarrow}{\boldsymbol{\nabla}\!\!\!/}\, \eta^\alpha_-\right) \\
& -2C\Big(R_{\alpha\underline{\beta}\gamma\underline{\delta}} - \Gamma_{\alpha\gamma}{}^{\sigma}\, G_{\sigma\underline{\beta},\underline{\delta}}\Big)\bar{\eta}^{\underline{\beta}}_+\,\eta^\alpha_+\,\bar{\eta}^{\underline{\delta}}_-\,\eta^\gamma_- + \frac{2}{C}\, G_{\alpha\underline{\beta}}\,\bar{\psi}_+\,\gamma^0\,\eta^{\underline{\beta}}_-\,\bar{\psi}_-\,\gamma^0\eta^\alpha_+ \\
& + \frac{2}{C}G_{\alpha\underline{\beta}}\,\bar{\psi}_+\,\eta^{\underline{\beta}}_+\,\bar{\psi}_-\,\eta^\alpha_- + \Big\{\frac{i}{C}\bar{\eta}^{\underline{\alpha}}_-\gamma^0\psi_+\pi_{\bar{z}^{\underline{\alpha}}} + iG_{\alpha\underline{\beta},\underline{\gamma}}\,\bar{\eta}^{\underline{\beta}}_-\,\eta^\alpha_-\,\bar{\psi}_+\,\psi^{\underline{\gamma}}_+ \\
& + G^{\alpha\underline{\beta}}\,\frac{iK_\alpha}{4C}\pi_\theta\pi_{\bar{z}^{\underline{\beta}}} - \frac{K_\alpha}{2C}\pi_\theta\bar{\psi}_-\gamma^0\eta^\alpha_+ + \text{h.c}\Big\} \\
& - \Big\{2iG_{\alpha\underline{\beta}}\,\bar{\eta}^{\underline{\beta}}_-\boldsymbol{\nabla}\!\!\!/\, z^\alpha\psi_+ + 2CG_{\alpha\underline{\beta},\gamma}\bar{\eta}^{\underline{\beta}}_+\boldsymbol{\nabla}\!\!\!/\, z^\alpha\eta^\gamma_- + \bar{\Gamma}_{\underline{\beta}\,\underline{\gamma}}{}^{\underline{\beta}}\pi_{z^\alpha}\,\bar{\eta}^{\underline{\beta}}_-\,\gamma^0\eta^\alpha_+ + \text{h.c}\Big\}\Big]. \qquad (3.39)
\end{aligned}
$$

In this expression we have used for the 3-dimensional contraction $\boldsymbol{\nabla}\!\!\!/ = \boldsymbol{\gamma}\cdot\boldsymbol{\nabla}$ a notation analogous to the 4-dimensional one. After a long and tedious calculation one finds that brackets with the hamiltonian reproduce all the field equations we derived earlier from the lagrangian (3.5):

$$
\partial_0 A = \{A, H\}^*\,. \qquad (3.40)
$$

We now turn to the construction of the canonical super-Poincaré algebra. First we construct the canonical expressions for the energy-momentum vector (3.31) and the supercharges $Q_\pm$ (3.27). For the four-momentum vector we find the result

$$
\begin{aligned}
P_0 \;=\; & \int d^3x\mathcal{H} = H \\
\mathbf{P} \;=\; & \int d^3x\Big[-\pi_C\boldsymbol{\nabla} C - \pi_\theta\,\boldsymbol{\nabla}\theta - \frac{1}{2}\mathcal{F}''(C)\bar{\psi}_+\gamma_0\,\overset{\leftrightarrow}{\boldsymbol{\nabla}}\,\psi_- - 2CG_{\alpha\underline{\beta}}\,\bar{\eta}^{\underline{\beta}}_+\gamma_0\,\overset{\leftrightarrow}{\boldsymbol{\nabla}}\,\eta^\alpha_- \\
& - \Big\{\frac{K_{\underline{\alpha}}}{8C}\pi_\theta\,\bar{\eta}^{\underline{\alpha}}_-\boldsymbol{\gamma}\,\psi_+ + \pi_\alpha\Big(\boldsymbol{\nabla} z^\alpha - \frac{i}{4C}\,\bar{\eta}^\alpha_+\boldsymbol{\gamma}\,\psi_-\Big) \\
& + \frac{i}{2}\,G_{\alpha\underline{\beta},\gamma}\,(\bar{\eta}^\gamma_+\boldsymbol{\gamma}\,\psi_-)\,(\bar{\eta}^{\underline{\beta}}_+\,\gamma_0\,\eta^\alpha_-) + \text{h.c}\Big\}\Big]. \qquad (3.41)
\end{aligned}
$$

It generates space time translations on the fields A:

$$\partial_\mu A = \{A, P_\mu\}^* . \tag{3.42}$$

The phase-space supercharges $Q_\pm$ are obtained directly from the supercurrents (3.25), which reads explicitly

$$\begin{aligned}
Q_+ &= \int d^3\mathbf{r}\Big[\Big(\mathcal{F}''(C)\boldsymbol{\nabla} C - 2i\boldsymbol{\nabla}\theta + K_{\underline{\alpha}}\boldsymbol{\nabla}\bar{z}^{\underline{\alpha}} - K_\alpha\boldsymbol{\nabla} z^\alpha\Big)\gamma_0\psi_+ \\
&\quad + \frac{1}{4}\mathcal{F}''''(C)\gamma_0\psi_-\,\bar{\psi}_+\,\psi_+ + \Big(\pi_C - \frac{i}{2}\mathcal{F}''(C)\pi_\theta\Big)\psi_+ + \Big(K_{,\alpha}\pi_\theta - 2i\pi_{z^\alpha}\Big)\eta_+^\alpha \\
&\quad -4iCG_{\alpha\underline{\beta}}\boldsymbol{\nabla}\bar{z}^{\underline{\beta}}\gamma_0\eta_+^\alpha - G_{\alpha\underline{\beta}}\gamma_0\eta_-^\alpha\bar{\psi}_+\eta_+^{\underline{\beta}}\Big] \qquad (3.43) \\
Q_- &= \int d^3\mathbf{r}\Big[\Big(\mathcal{F}''(C)\boldsymbol{\nabla} C + 2i\boldsymbol{\nabla}\theta + K_\alpha\boldsymbol{\nabla} z^\alpha - K_{\underline{\alpha}}\boldsymbol{\nabla}\bar{z}^{\underline{\alpha}}\Big)\gamma_0\psi_- \\
&\quad + \frac{1}{4}\mathcal{F}''''(C)\gamma_0\psi_+\,\bar{\psi}_-\,\psi_- + \Big(\pi_C + \frac{i}{2}\mathcal{F}''(C)\pi_\theta\Big)\psi_- + \Big(K_{,\underline{\alpha}}\pi_\theta + 2i\pi_{z^{\underline{\alpha}}}\Big)\eta_-^{\underline{\alpha}} \\
&\quad +4iCG_{\underline{\alpha}\beta}\boldsymbol{\nabla} z^\beta\gamma_0\eta_-^{\underline{\alpha}} - G_{\alpha\underline{\beta}}\gamma_0\eta_+^\alpha\bar{\psi}_-\eta_-^{\underline{\beta}}\Big].
\end{aligned}$$

Like the hamiltonian generates the time-evolution, the supercharges generate supersymmetry transformations; explicitly, the results (3.21) are reproduces by the brackets

$$\delta(\epsilon_\pm)A = \left\{A, \frac{i}{2}\bar{\epsilon}_\pm\, Q_\pm\right\}^* . \tag{3.44}$$

The supercharges satisfy the standard super-Poincaré algebra, as is seen from the bracket relations

$$\left\{Q_\pm, \bar{Q}_\mp\right\}^* \;=\; 2\not{P}, \quad \{Q_\pm, P_\mu\}^* = 0. \tag{3.45}$$

The last equation of (3.45) follows immediately from (3.42). The bracket structure shows Poincaré supersymmetry to be realized also in the canonical formulation of the theory.

3.5 The hydrodynamical regime

Having discussed the general formalism for the construction of the lagrangian and conserved quantities, we now discuss the hydrodynamical interpretation of the models described by (3.5). To get the hydrodynamical interpretation of these models, much more work clearly should be done. First, one has to relate the fields in our model to the particle number density ρ and the velocity four-vector u_μ. In particular, in the limit in which all fermion fields vanish, we have to identify the vector component V_μ with the particle number density ρ and u_μ as in (2.4). However, this is not sufficient for this field theory to describe a relativistic model of hydrodynamics. We have to show that this identification is consistent with the field equation (3.15) which is the relativistic equation of continuity. Finally, it should be possible to write the bosonic part of the energy-momentum tensor (3.30) in standard form (2.1).

We now present details of this analysis. A supersymmetric extension of the action for fluid dynamics constructed in section 3.2 generally goes at the expense of most of the infinitely many conservation laws related to reparametrizing the potential, eqs.(2.19), (2.21). This can already be inferred from the bosonic part of the theory. Consider the bosonic terms in the equations of motion for the current (3.15) and the potentials (3.16):

$$\partial \cdot V = 0, \quad -2\mathcal{D} \cdot (C\partial z^{\alpha}) + 2iV \cdot \partial z^{\alpha} = 0, \quad -2\mathcal{D} \cdot (C\partial \bar{z}^{\underline{\alpha}}) - 2iV \cdot \partial \bar{z}^{\underline{\alpha}} = 0. \tag{3.46}$$

Now construct the currents

$$J_{\mu}[\mathcal{G}] = -2\mathcal{G}(\bar{z}^{\underline{\alpha}}, z^{\alpha})V_{\mu} - 2iC\Big(\mathcal{G}_{,\alpha}\partial_{\mu}z^{\alpha} - \mathcal{G}_{,\underline{\alpha}}\partial_{\mu}\bar{z}^{\underline{\alpha}}\Big), \tag{3.47}$$

where $\mathcal{G}(\bar{z}^{\underline{\alpha}}, z^{\alpha})$ is a real function of the complex scalar fields. Using (3.46) it can be seen to satisfy

$$\partial \cdot J[\mathcal{G}] = -2iC\Big(\mathcal{G}_{;\alpha;\beta}\partial z^{\alpha} \cdot \partial z^{\beta} - \mathcal{G}_{;\underline{\alpha};\underline{\beta}}\partial \bar{z}^{\underline{\alpha}} \cdot \partial \bar{z}^{\underline{\beta}}\Big) \tag{3.48}$$

It follows that the divergence of the current vanishes identically only for functions $\mathcal{G}$ such that the homogeneous second derivatives w.r.t. $\bar{z}^{\underline{\alpha}}$ and z^{α} vanish:

$$\mathcal{G}_{;\alpha;\beta} = \mathcal{G}_{;\underline{\alpha};\underline{\beta}} = 0. \tag{3.49}$$

As shown in section 3.3, this happens if $\mathcal{G}$ is the Killing potential for a pair of holomorphic/anti-holomorphic Killing vectors $(R^{\alpha}(z), \bar{R}^{\underline{\alpha}}(\bar{z}))$, corresponding to isometries of the Kähler manifold. This is not surprising, as only isometries leave the kinetic term for the complex fields $(\bar{z}, z)$ invariant. As the number of independent isometries of a finite-dimensional manifold is finite, no infinite set of conserved currents can be generated by Killing vectors.

Still, as anticipated an infinite set of conserved currents $J_{\mu}[M]$ is obtained for all models (3.5) under the restriction $C = 0$. Therefore we identify the manifold of states with $C = 0$ as the hydrodynamical regime of the supersymmetric models constructed here. Observe here that the conserved currents constructed above are precisely the bosonic part of the Noether currents (3.9) for the symmetry transformations (3.8) with $\mathcal{G}(\bar{z}, z) = M(z, \bar{z})$ the Killling potential satisfying the relations (3.10).

For a generic real function $\mathcal{G}(\bar{z}, z)$ which is not a Killing potential, the current $J_{\mu}[\mathcal{G}]$ is not conserved, unless one takes the limit $(C, \eta) \to 0$, such that the spinor field η vanishes as fast as C. Solutions of the model with this property we interpret as describing a supersymmetric fluid.

To analyse this regime, we rescale the fermion fields as follows

$$\psi_{\pm} = \frac{1}{\sqrt{\mathcal{F}''(C)}}\,\Psi_{\pm}, \quad \eta^{\alpha}_{\pm} = C\Omega^{\alpha}_{\pm}. \tag{3.50}$$

Then the lagrangean (3.5) becomes

$$
\begin{aligned}
\mathcal{L} \;=\;& V^\mu\Big(2\partial_\mu\theta - iK_{,\alpha}\partial_\mu z^\alpha + iK_{,\underline{\beta}}\partial_\mu \bar{z}^{\underline{\beta}} + 2iC^2 G_{\alpha\underline{\beta}}\,\bar{\Omega}_-^{\underline{\beta}}\gamma_\mu\Omega_+^\alpha + \frac{i}{2}\frac{\mathcal{F}'''(C)}{\mathcal{F}''(C)}\bar{\Psi}_+\gamma_\mu\Psi_-\Big) \\
& -2CG_{\alpha\underline{\beta}}\Big(\partial_\mu z^\alpha\partial^\mu\bar{z}^{\underline{\beta}} + C^2\bar{\Omega}_+^{\underline{\beta}}\overleftrightarrow{\not{D}}\,\Omega_-^\alpha\Big) + 2C^5 R_{\alpha\underline{\beta}\gamma\underline{\delta}}\,\bar{\Omega}_+^{\underline{\beta}}\,\Omega_+^\alpha\,\bar{\Omega}_-^{\underline{\delta}}\,\Omega_-^\gamma \\
& -\frac{1}{2}\mathcal{F}''(C)\Big[\partial_\mu C\partial^\mu C - V_\mu V^\mu\Big] - 2\frac{C}{\mathcal{F}''(C)}G_{\alpha\underline{\beta}}\,\bar{\Psi}_+\,\Omega_+^\alpha\,\bar{\Psi}_-\,\Omega_-^{\underline{\beta}} - \frac{1}{2}\bar{\Psi}_+\overleftrightarrow{\not{\partial}}\,\Psi_- \\
& + 2i\frac{C}{\sqrt{\mathcal{F}''(C)}}G_{\alpha\underline{\beta}}\Big(\bar{\Psi}_+\not{\partial}z^\alpha\Omega_-^{\underline{\beta}} - \bar{\Psi}_-\not{\partial}\bar{z}^{\underline{\beta}}\Omega_+^\alpha\Big) - \frac{1}{8}\frac{\mathcal{F}''''(C)}{[\mathcal{F}''(C)]^2}\bar{\Psi}_+\,\Psi_+\,\bar{\Psi}_-\,\Psi_-. \qquad (3.51)
\end{aligned}
$$

We observe that in the limit $C = 0$ divergent terms can be avoided, provided $\mathcal{F}''(0) \neq 0$. Then we can always normalize $\mathcal{F}(C)$ such that $\mathcal{F}''(0) = 1$; with this choice the quadratic vector term and the kinetic term of the real scalar C have the canonical normalization.

Next we observe, that there exist many choices of the function $\mathcal{F}(C)$ such that also the coefficients of the bilinear and quadratic terms in Ψ are finite. Indeed, any function such that the second derivative has the expansion

$$
\mathcal{F}''(C) = 1 + \lambda_1 C + \lambda_2 C^2 + \mathcal{O}(C^3) \qquad (3.52)
$$

satisfies the conditions

$$
\mathcal{F}''(0) = 1, \quad \mathcal{F}'''(0) = \lambda_1, \quad \mathcal{F}''''(0) = 2\lambda_2, \qquad (3.53)
$$

and makes the lagrangean finite in the hydrodynamical regime. In the remainder of this chapter, we consider Kähler manifolds of complex dimension one (hence $\alpha, \beta, \cdots = 1$), which is relevant for discussing the supersymmetric version of hydrodynamical models presented in chapter 2

3.5.1 Superhydrodynamics: non-zero vorticity

Having established the existence of regular configurations with $C = 0$, the expression for the current in this regime becomes

$$
V_\mu = iK_z\partial_\mu z - iK_{\bar{z}}\partial_\mu\bar{z} - 2\partial_\mu\theta - \frac{i}{2}\lambda_1\bar{\Psi}_+\gamma_\mu\Psi_-. \qquad (3.54)
$$

The bosonic part has the standard decomposition for a fluid density current; the last term is a fermionic extension required by supersymmetry.

Next we consider the energy-momentum tensor and the equation for C; again in this regime. For $C = 0$, the symmetric energy-momentum tensor and the equation for C derived from

(3.51) reduce to

$$\lambda_1 V_\mu^2 = 4G_{z\bar{z}}\partial_\mu z\partial^\mu \bar{z} - i\left(2\lambda_2 - \lambda_1^2\right)\bar{\Psi}_+ \not{V}\Psi_- - 4iG_{z\bar{z}}\left(\bar{\Psi}_+\not{\partial}z\Omega_- - \bar{\Psi}_-\not{\partial}\bar{z}\Omega_+\right)$$
$$+\frac{1}{2}\left(3\lambda_3 - 2\lambda_1\lambda_2\right)\bar{\Psi}_+\Psi_+\,\bar{\Psi}_-\Psi_- + 4G_{z\bar{z}}\,\bar{\Psi}_+\,\Omega_+\,\bar{\Psi}_-\,\Omega_-$$
$$T_{\mu\nu}(C=0) = V_\mu V_\nu + \frac{1}{4}\,\bar{\Psi}_+\left(\gamma_\mu \overleftrightarrow{\partial}_\nu + \gamma_\nu \overleftrightarrow{\partial}_\mu\right)\Psi_- + \frac{i}{4}\,\lambda_1\,\bar{\Psi}_+\left(\gamma_\mu V_\nu + \gamma_\nu V_\mu\right)\Psi_-$$
$$-g_{\mu\nu}\left[\frac{1}{2}V^2 + \frac{1}{2}\,\bar{\Psi}_+\overleftrightarrow{\not{\partial}}\,\Psi_- + \frac{\lambda_2}{4}\bar{\Psi}_+\Psi_+\bar{\Psi}_-\Psi_-\right]. \tag{3.55}$$

The physical interpretation of these equations is implicit in their bosonic terms. For a hydrodynamical current

$$V_\mu = \rho u_\mu \quad \Rightarrow \quad V_\mu^2 = -\rho^2. \tag{3.56}$$

The bosonic part of the first equation (3.55) becomes

$$\rho^2 = -\frac{4}{\lambda_1}\,G_{z\bar{z}}\partial\bar{z}\cdot\partial z \geq 0. \tag{3.57}$$

In particular, for $\lambda_1 < 0$ it becomes

$$\rho^2 = \frac{4}{|\lambda_1|}\,G_{\bar{z}z}\left(|\boldsymbol{\nabla} z|^2 - |\dot{z}|^2\right), \tag{3.58}$$

which implies that apart from fermionic contributions, the spatial gradient of the complex scalar field determines the fluid density ρ. Similarly, for $\lambda_1 > 0$ the time rate of change of z determines ρ. With the identification (3.56), the bosonic part of the energy-momentum (3.55) is of the form (2.1). The corresponding energy and pressure densities are given by

$$\varepsilon = p = \frac{1}{2}\rho^2 = \frac{2}{|\lambda_1|}\,G_{\bar{z}z}\left(|\boldsymbol{\nabla} z|^2 - |\dot{z}|^2\right). \tag{3.59}$$

It is not difficult to check, that the equation for the complex scalar fields $(\bar{z}, z)$ for $C = 0$ reproduce the conditions

$$2iV\cdot\partial z = -2iV\cdot\partial\bar{z} = 0, \tag{3.60}$$

cf. eqs. (2.18). This establishes the existence of a regime $C = 0$ in which the supersymmetric model allows a fluid-mechanical interpretation with classical vorticity given by (2.17).

3.5.2 Potential flow

We end this chapter by showing that the full energy-momentum tensor (3.55) takes the standard form (2.1), under particular non-trivial conditions specified by

$$\partial_\mu\Psi_\pm = \mp\frac{i}{2}\lambda_1 V_\mu\Psi_\pm - \frac{\lambda_2}{8}\gamma_\mu\Psi_\mp\,\bar{\Psi}_\pm\,\Psi_\pm, \quad \lambda_1^2 = \lambda_2. \tag{3.61}$$

Indeed upon substitution of these expressions into (3.55), the energy-momentum becomes

$$T_{\mu\nu} \;=\; W_\mu W_\nu - \frac{1}{2} g_{\mu\nu} W^2, \quad W_\mu = V_\mu + \frac{i}{2}\lambda_1 \bar{\Psi}_+ \gamma_\mu \Psi_-, \quad \partial\cdot W = 0. \tag{3.62}$$

The vanishing divergence of the current W_μ follows upon using the field equations

$$\partial\!\!\!/ \Psi_\pm = \mp\frac{i}{2}\lambda_1 V\!\!\!\!/\, \Psi_\pm - \frac{\lambda_2}{2}\Psi_\mp\, \bar{\Psi}_\pm\, \Psi_\pm, \quad \partial\cdot V = 0. \tag{3.63}$$

Therefore, we can reinterpret the W_μ as the hydrodynamical current $W_\mu = \rho u_\mu$ with the equation of state:

$$\varepsilon \;=\; p = \frac{1}{2}\rho^2, \quad \rho^2 = -\frac{1}{\lambda_1}\Big[4G_{z\bar{z}}\partial\bar{z}\cdot\partial z + \frac{1}{2}\left(3\lambda_3 - 2\lambda_1^3\right)\bar{\Psi}_+\Psi_+\,\bar{\Psi}_-\Psi_-\Big].$$

Next, we investigate the properties of non-trivial solutions to the Ansatz (3.61). Since (3.61) can be written as

$$\mathcal{D}_\mu\Psi_\pm = \Big(\partial_\mu \pm \frac{i\lambda_1}{2} W_\mu\Big)\Psi_\pm = 0, \tag{3.64}$$

with the axial covariant derivative $\mathcal{D}_\mu = \partial_\mu + \frac{i\lambda_1}{2} W_\mu\,\gamma_5$ using a Fierz identity, the ansatz has the formal solution

$$\Psi_\pm(x) = \mathcal{P}\exp\Big(\mp\frac{i\lambda_1}{2}\int_0^x W\cdot dx\Big)\Psi_\pm(0) \tag{3.65}$$

where, $\mathcal{P}$ is the path-ordering operator. On the other hand, from (3.64) one infers that the field strength associated with the covariant derivative $\mathcal{D}_\mu$ vanishes. This implies that W_μ has to be pure gauge for non–trivial fermion solutions to exist. And because the fermion bilinear is constant, we conclude that the abelian vector field V_μ is pure gauge. This shows that the system in this regime is described by potential flow.

Chapter 4

Supersymmetric σ-models on G/H

Experimentalists think that it is a mathematical theorem while the mathematicians believe it to be an experimental fact.

Gabriel Lippmann

4.1 Introduction

In the remainder of this thesis, we turn to a different class of non-linear supersymmetric field theories: supersymmetric σ-models. These theories may have applications to physics beyond the standard model. For example, supersymmetric extensions of a Grand Unified Theory (GUT) may be relevant for particle physics since they contain less parameters than the Minimal Supersymmetric Standard Model (MSSM). One of the original guidelines of the construction of GUT theories was renormalizability. However, as such GUT models are likely to be realized quite close to the Planck scale, renormalizability is not necessarily an issue as supergravity theories are non-renormalizable by themselves. Moreover, supergravity models often include non-linear coset models such as $SU(1,1)/U(1)$ in $D=4$. Therefore a GUT may be part of a supersymmetric non-linear sigma model based on a coset space G/H, with H a subgroup of G.

This chapter discusses the construction of non-linear supersymmetric σ-models based on homogeneous Kählerian cosets spaces G/H. For the construction of this kind of models the coset space G/H must be a Kähler manifold [60]–[72]. In recent years supersymmetric σ-models on coset spaces, including among others the grassmannian models on $SU(n+m)/[SU(n)\times SU(m)\times U(1)]$, the orthogonal unitary coset models on manifolds $SO(2n)/U(n)$, as well as models on exceptional cosets like $E_6/[SO(10)\times U(1)]$, have been studied in great detail [74, 73, 75, 77, 90]. As E_6 and $SO(10)$ are promising unification groups the coset spaces $E_6/[SO(10)\times U(1)]$ and $SO(10)/[SU(5)\times U(1)]$ are the most interesting for (direct) phenomenology. Anomaly-free supersymmetric σ-models on these two coset spaces have been constructed [74, 73], and we now make a first step in the analysis of the phenomenology of these models.

Before we study the phenomenology of supersymmetric models based on the coset spaces $E_6/[SO(10)\times U(1)]$ and $SO(10)/[SU(5)\times U(1)]$ in later chapters, we start this chapter by reviewing the general construction of anomaly-free supersymmetric non-linear σ-models which describe the Goldstone bosons z^α (and their complex conjugates $\bar{z}^{\underline{\alpha}}$) of spontaneously broken G symmetry to its subgroup H. In section 4.2 the construction of (globally) supersymmetric lagrangians for σ-models with non-linear symmetry is reviewed. As supersymmetric σ–models on homogeneous Kähler cosets are known to be anomalous, in section 4.3 we explain how these anomalies can be cancelled by extending the σ-model lagrangian with additional supermultiplets carrying representations of the original coset G/H [74]. In section 4.4, we describe the gauged supersymmetric σ-models. The coupling of the gauge multiplets to the supersymmetric non-linear σ-models has interesting consequences for the the spectrum. It can induce spontaneous breaking of supersymmetry, and further spontaneous breaking of the internal symmetry at the sigma-model scale $g\Lambda = g/f$, with the Goldstone bosons acting as Higgs fields. Then the Goldstone bosons disappear by being absorbed in massive vector bosons through the Brout-Englert-Higgs mechanism. This is possible because the gauge couplings may explicitly break some of the global symmetry G, hence the Goldstone bosons become pseudo-Goldstone bosons with a mass $\sim g/f$. In most cases, the only remaining massless particles are the gauge bosons and the scalars of the non-broken symmetries, and a set chiral fermions. Section 4.5 discusses an important aspect of supersymmetry breaking and breaking of internal symmetry: mass sum rules. These relations play an important role in constructing realistic supersymmetric gauge theories, containing the standard model. In such a model, each quark and lepton would have a scalar supersymmetric partner (scalar quarks or 'squarks' and scalar leptons or 'sleptons') and each gauge bosons would have a spin 1/2 partner ('photino', 'winos' and 'zinos'). Since no such new particle has been detected, we have from experiment lower limits on the masses of supersymmetric particles. In any case mass inequalities like: $m_{squarks} > m_{quarks}$, $m_{slepton} > m_{leptons}$, must hold in a realistic model.

4.2 Supersymmetric σ–models on Kähler manifolds

In this section we review the construction of $N = 1$ globally supersymmetric lagrangians for non-linear σ-models in 4-D space time. The action is formulated in terms of chiral superfields $\Phi^\alpha = (z^\alpha, \psi_L^\alpha, H^\alpha)$, $\alpha = 1, \ldots, N$ the components of which are complex scalars z^α, an auxiliary field H^α and a (left-handed) chiral fermion [1] ψ_L; and it is defined by two functions of superfields: the real Kähler potential $K(\bar\Phi, \Phi)$, and the holomorphic superpotential $W(\Phi)$. In terms of these functions, a supersymmetric action can be written as

$$S = \int d^4x \, \mathcal{L}_{chiral}, \quad \mathcal{L}_{chiral} = \int \mathrm{d}^2\theta \, \mathrm{d}^2\bar\theta \, K(\bar\Phi, \Phi) + \Big(\int d^2\theta \, W(\Phi) + \text{h.c.}\Big), \tag{4.1}$$

[1] Our conventions for chiral spinors are such, that $\gamma_5\psi_L = +\psi_L$ and $\bar\psi_L\gamma_5 = -\bar\psi_L$; charge conjugations acts as $\psi_R = C\bar\psi_L^T$, where $\bar\psi_L = i\psi_R^\dagger\gamma_0$.

where h.c. denotes hermitian conjugate and $\mathrm{d}^2\theta \equiv -\frac{1}{4}\mathrm{d}\theta^\alpha \mathrm{d}\theta_\alpha$, $\mathrm{d}^2\bar{\theta} \equiv -\frac{1}{4}\mathrm{d}\bar{\theta}^{\underline{\alpha}}\mathrm{d}\bar{\theta}_{\underline{\alpha}}$, such that $\int \mathrm{d}^2\theta(\theta\theta) = \int \mathrm{d}^2\bar{\theta}(\bar{\theta}\bar{\theta}) = 1$. In component fields this expression becomes

$$\begin{aligned}\mathcal{L}_{chiral} \;=\;& -G_{\alpha\underline{\alpha}}(z,\bar{z})\left[\partial^\mu z^\alpha\,\partial_\mu \bar{z}^{\underline{\alpha}} + \bar{\psi}_L^{\underline{\alpha}}\,\overleftrightarrow{\not{\partial}}\,\psi_L^\alpha - H^\alpha \bar{H}^{\underline{\alpha}}\right] + G_{\alpha\underline{\beta},\gamma\underline{\delta}}\,\bar{\psi}_R^\alpha\,\psi_L^\gamma\bar{\psi}_L^{\underline{\beta}}\,\psi_R^{\underline{\delta}} \\ & -\left[G_{\alpha\underline{\beta},\gamma}\partial^\mu z^\gamma - G_{\alpha\underline{\beta},\underline{\gamma}}\partial^\mu \bar{z}^{\underline{\gamma}}\right]\bar{\psi}_L^{\underline{\beta}}\gamma_\mu\psi_L^\alpha - \left[\bar{H}^{\underline{\beta}}G_{\alpha\underline{\beta},\gamma}\bar{\psi}_R^\alpha\,\psi_L^\gamma + H^\alpha G_{\alpha\underline{\beta},\underline{\gamma}}\bar{\psi}_L^{\underline{\gamma}}\,\psi_R^{\underline{\beta}}\right] \\ & -\bar{W}_{,\underline{\alpha}}(\bar{z})\bar{H}^{\underline{\alpha}} - W_{,\alpha}(z)H^\alpha + W_{,\alpha\beta}(z)\bar{\psi}_R^\alpha\psi_L^\beta + \bar{W}_{,\underline{\alpha\beta}}(\bar{z})\bar{\psi}_L^{\underline{\alpha}}\psi_R^{\underline{\beta}}. \end{aligned} \tag{4.2}$$

Here $G_{\alpha\underline{\alpha}}$ is the Kähler metric introduced in (B.5). The theory described by (4.2) has a general isometry group G, with anti-hermitian generators T_i satisfying the Lie algebra

$$[T_i, T_j] = f_{ij}{}^k\, T_k. \tag{4.3}$$

These isometries define infinitesimal symmetry transformations on the Kähler manifold G/H. In components the transformation rules read

$$\begin{aligned}\delta z^\alpha &= \theta^i\, R_i^\alpha(z), \quad \delta\bar{z}^{\underline{\alpha}} = \theta^i\, \bar{R}_i^{\underline{\alpha}}(\bar{z}) \\ \delta H^\alpha &= \theta^i R^\alpha_{i,\beta}(z)\, H_L^\beta - \theta^i R^\alpha_{i,\beta\gamma}(z)\,\bar{\psi}_L^\beta\psi_R^\gamma, \quad \delta\bar{H}^{\underline{\alpha}} = \theta^i \bar{R}^{\underline{\alpha}}_{i,\underline{\beta}}(\bar{z})\,\bar{H}^{\underline{\beta}} - \theta^i \bar{R}^{\underline{\alpha}}_{i,\underline{\beta\gamma}}(\bar{z})\,\bar{\psi}_R^{\underline{\beta}}\psi_L^{\underline{\gamma}} \\ \delta\psi_L^\alpha &= \theta^i R^\alpha_{i,\beta}(z)\,\psi_L^\beta, \quad \delta\bar{\psi}_L^{\underline{\alpha}} = \theta^i \bar{R}^{\underline{\alpha}}_{i,\underline{\beta}}(\bar{z})\,\bar{\psi}_L^{\underline{\beta}}, \end{aligned} \tag{4.4}$$

with θ^i the parameters of the infinitesimal transformations.

As the chiral fermions ψ_L^α couple to the connection and the curvature in $\mathcal{L}_{chiral}$, the consistency of the quantum theory is generally spoiled by anomalies [80, 83, 84, 89]. Anomalies are classical symmetries which are broken at the quantum level (texbook introduction on the subject of anomalies can be found in refs. [108, 109, 110].) Therefore, anomalies have to be removed to allow for gauging the internal symmetries. However as has been shown [88] these anomalies can not be compensated by Wess-Zumino type modifications [106, 107]. A particular solution to this problem has been proposed in [74] by extending the model with additional chiral superfields generically called *matter* superfields.

4.3 Matter coupling and anomaly cancellation

In this section, we summarize the general procedure of cancelling anomalies of supersymmetric σ–models on homogeneous Kähler cosets using additional chiral superfields (matter). For more details on that procedure the reader may consult [64, 75, 74]. The generalization to supergravity was presented in [77]

An important question in the context of supersymmetric matter is how it can be coupled to supersymmetric σ-models on Kähler manifolds without spoiling the (possibly non-linear) invariance of the original theory. This is important for the cancellation of anomalies as shown in [75]. The mathematical framework used to construct matter representations of the isometry group of the Kähler manifold is the theory of complex bundles over Kähler manifolds. These bundles are defined locally on the Kähler manifold by sets of complex fields with specific transformation character under the isometries.

The basic pattern is exhibited by the transformation rule (4.4) for the chiral fermions. This rule shows how a vector (an element of the tangent bundle) transforms under the isometries. Similarly, one can define a matter representation $\Psi^A = (a^A, \chi^A, B^A)$ transforming as a 1-form (an element of the co-tangent bundle):

$$\delta\Psi^A = -R_{i\ ,B}^{\ A}(\Phi)\,\Psi^B. \tag{4.5}$$

More general transformations are obtained by taking tensor products of the tangent or co-tangent bundles. However, for our applications this is not sufficient. The reason is, that the $U(1)$ charges of such representations are completely fixed in terms of the charge of the scalars z^α: a contravariant holomorphic tensor $t^{\alpha_1...\alpha_p}$ of rank p carries a relative charge p, whereas a covariant holomorphic tensor $s_{\alpha_1...\alpha_k}$ of rank k carries a relative $U(1)$ charge $-k$. But in actual models, if one requires anomaly cancellations with a phenomenologically interesting set of matter superfields, one usually needs a different assignment of $U(1)$ charges. Therefore the spectrum of representations must be extended with bundels which differ from tensor bundles by the assignment of $U(1)$ charges. This is achieved for instance by the introduction of complex line bundles [74].

A line bundle is the target space of a single-component complex scalar field over the manifold. We consider line bundles carrying non-trivial representations of the isometry group; these can be defined locally on the Kähler manifold as complex scalar matter fields $S(x)$ coupled to the σ-model, with the infinitesimal transformation law given by

$$\delta_i S = F_i(z)S. \tag{4.6}$$

In the context of supersymmetric field theories such a representation of the isometry group was introduced in [75], and subsequently considered in [68]; it is a representation because of the property (B.23). From the line-bundle S one can obtain other line bundles with different $U(1)$ weights by taking powers:

$$A \equiv S^\lambda \quad \Rightarrow \quad \delta_i A = \lambda F_i(z)A. \tag{4.7}$$

Furthermore, using the line bundle construction, one can modify the transformation rules of fields in tensor representations of the isometry group. For example, defining

$$T^{\alpha_1...\alpha_p} \equiv S^\lambda\, t^{\alpha_1...\alpha_p}, \tag{4.8}$$

the new field T obeys the transformation rule

$$\delta_i T^{\alpha_1...\alpha_p} = \sum_{k=1}^{p} R_{i\ ,\beta}^{\alpha_k} T^{\alpha_1..\beta..\alpha_p} + \lambda F_i\, T^{\alpha_1...\alpha_p}. \tag{4.9}$$

In this way the $U(1)$ charges can be adjusted, be it subject to the charge quantization conditions as we discuss now.

The bundles introduced here are characterized locally on the manifold by their transformation properties. An important question is if these definitions can be extended globally over the manifold. This is always possible for tangent and co-tangent bundles. However, for the line bundles (4.6), this requires in particular that the holomorphic transition functions introduced in (B.14) satisfy the cocycle condition

$$F_{(ij)}(z_j) + F_{(jk)}(z_k) + F_{(ki)}(z_i) = 2\pi i\mathbb{Z}. \tag{4.10}$$

Manifolds with this property are known as Kähler-Hodge [93]; their Kähler forms satisfy the condition

$$\int_{C_2} \omega(K) = 2\pi\, n, \quad \text{with} \quad n \in \mathbb{Z}, \tag{4.11}$$

for any closed two-cycle C_2.

With the above transformations rule, one can construct invariant lagrangians or equivalently the Kähler potentials for matter superfields. A general procedure for constructing a Kähler potential $K_m(\bar{\Phi}, \bar{\Psi}; \Phi, \Psi)$ for these matter superfields was developed by the authors of Ref. [32, 33, 95].

After coupling the coset model to matter fields, the complete Kähler potential is now a sum of the σ-model and matter Kähler potential:

$$\mathcal{K}(\bar{\Sigma}, \Sigma) = K_\sigma(\bar{\Phi}, \Phi) + K_m(\bar{\Phi}, \bar{\Psi}; \Phi, \Psi). \tag{4.12}$$

We have introduced the notation $\Sigma^{\mathcal{A}} = (\Phi^\alpha, \Psi^A)$ for full set of chiral superfields, their physical components are denoted collectively by $(Z^{\mathcal{A}}, \psi_L^{\mathcal{A}}, H^{\mathcal{A}})$, where $Z^{\mathcal{A}} = (z^\alpha, a^A)$ are complex scalars, $\psi_L^{\mathcal{A}} = (\psi_L^\alpha, \chi_L^A)$ are chiral fermions and $H^{\mathcal{A}} = (H^\alpha, B^A)$ are auxiliary complex scalars. With the above notation, the lagrangian for chiral multiplets $\Sigma^{\mathcal{A}}$ is given by (4.2), where one must replace z by Z and the Greek indices $\alpha, \beta \dots$ by curly capital indices $\mathcal{A}, \mathcal{B} \dots$:

$$\mathcal{L}_{chiral}(z \to Z; \alpha, \beta, \dots \to \mathcal{A}, \mathcal{B}, \dots). \tag{4.13}$$

4.4 The gauged σ-models on Kähler manifolds

Having cancelled anomalies by adding additional chiral multiplets, we can now gauge the symmetries (4.4). This can be done by coupling the $\sigma-$model to set of vector multiplets $V^i = (V_\mu^i, \lambda_L^i, D^i)$ [24, 60, 61, 72], where V_μ^i is a gauge field, λ_L^i a gaugino and D^i is an auxiliary complex scalar. First, one introduces the gauge covariant derivatives by simply replacing the infinitesimal parameters (4.4) by gauge fields:

$$D_\mu Z^{\mathcal{A}} = \partial_\mu Z^{\mathcal{A}} - g V_\mu^i R_i^{\mathcal{A}}, \quad D_\mu \psi_L^{\mathcal{A}} = \partial_\mu \psi_L^{\mathcal{A}} - g\, V_\mu^i R_{i,\mathcal{B}}^{\mathcal{A}} \psi_L^{\mathcal{B}}. \tag{4.14}$$

The corresponding covariantizations in the lagrangian then have to be supplemented by Yukawa and scalar couplings in order to restore supersymmetry [24]

$$\Delta\mathcal{L} \;=\; 2\sqrt{2}\, g\, G_{\mathcal{A}\underline{\mathcal{A}}} \Big(\bar{R}_i^{\underline{\mathcal{A}}} \bar{\lambda}_R^i \psi_R^{\mathcal{A}} + R_i^{\mathcal{A}} \bar{\lambda}_L^i \psi_L^{\underline{\mathcal{A}}} \Big) + g D^i (\mathcal{M}_i + \xi_i) \tag{4.15}$$

Here we have added a Fayet-Iliopoulos term with parameter ξ_i in case there is a commuting $U(1)$ vector multiplet [112] and $\mathcal{M}_i$ is the Killing potentials introduced in (B.18)

Finally, one introduces the kinetic terms for the vector multiplets [78, 79], including possible Fayet-Iliopoulos terms:

$$\mathcal{L}_{YM} = -\frac{1}{4}F^i \cdot F^i - \bar{\lambda}^i_R \overleftrightarrow{\not{D}} \lambda^i_R + \frac{1}{2}D^i D^i + \xi_i D^i, \tag{4.16}$$

where

$$F^i_{\mu\nu} = \partial_{[\mu}V^i_{\nu]} - g\, f^{ijk}\, V^j_\mu\, V^k_\nu, \quad D_\mu\lambda^i_R = \partial_\mu\lambda^i_R - g\, f^{ijk}\, V^j_\mu\, \lambda^k_R. \tag{4.17}$$

We close this section by writing the full lagrangian in terms of the geometrical objects for the metric $G_{\mathcal{A}\bar{\mathcal{A}}}$, connection $\Gamma^{\mathcal{A}}_{\mathcal{BC}}$ and curvature $R_{\underline{\mathcal{A}}\mathcal{A}\underline{\mathcal{B}}\mathcal{B}}$. After eliminating the auxiliary fields through their field equations

$$H^{\mathcal{A}} = \Gamma^{\mathcal{A}}_{\mathcal{BC}}\bar{\psi}^{\mathcal{B}}_R\psi^{\mathcal{C}}_L + G^{\mathcal{A}\underline{\mathcal{A}}}\bar{W}_{,\underline{\mathcal{A}}}, \quad D^i = -g\,(\mathcal{M}^i + \xi^i), \tag{4.18}$$

the full lagrangian becomes [70, 75]

$$\begin{aligned}
\mathcal{L} &= \mathcal{L}_{chiral}(\partial_\mu \to D_\mu) + \Delta\mathcal{L} + \mathcal{L}_{YM} \\
&= -G_{\mathcal{A}\underline{\mathcal{A}}}\Big(D\bar{Z}^{\underline{\mathcal{A}}} \cdot DZ^{\mathcal{A}} + \bar{\psi}^{\mathcal{A}}_L \overleftrightarrow{\not{D}}\, \psi^{\underline{\mathcal{A}}}_L\Big) + R_{\mathcal{A}\underline{\mathcal{B}}\mathcal{C}\underline{\mathcal{D}}}\,\bar{\psi}^{\mathcal{A}}_R\,\psi^{\mathcal{C}}_L\bar{\psi}^{\underline{\mathcal{B}}}_L\,\psi^{\underline{\mathcal{D}}}_R \\
&\quad -\Big(G_{\mathcal{A}\underline{\mathcal{B}},\mathcal{C}}D_\mu Z^{\mathcal{C}} - G_{\mathcal{A}\underline{\mathcal{B}},\underline{\mathcal{C}}}D_\mu\bar{Z}^{\underline{\mathcal{C}}}\Big)\bar{\psi}^{\underline{\mathcal{B}}}_L\gamma_\mu\psi^{\mathcal{A}}_L - \frac{g^2}{2}(\mathcal{M}_i + \xi_i)^2 \\
&\quad +2\sqrt{2}\, g\, G_{\mathcal{A}\underline{\mathcal{A}}}\Big(\bar{R}^{\underline{\mathcal{A}}}_i\bar{\lambda}^i_R\psi^{\mathcal{A}}_R + R^{\mathcal{A}}_i\bar{\lambda}^i_L\psi^{\underline{\mathcal{A}}}_L\Big) - \frac{1}{4}F^i \cdot F^i - \bar{\lambda}^i_R \overleftrightarrow{\not{D}}\, \lambda^i_R. \\
&\quad +\bar{W}_{;\underline{\mathcal{A}}\underline{\mathcal{B}}}\bar{\psi}^{\underline{\mathcal{A}}}_L\,\psi^{\underline{\mathcal{A}}}_R + W_{;\mathcal{A}\mathcal{B}}\bar{\psi}^{\mathcal{A}}_R\,\psi^{\mathcal{A}}_L - G^{\mathcal{A}\underline{\mathcal{A}}}\,\bar{W}_{;\underline{\mathcal{A}}}\,W_{;\mathcal{A}}.
\end{aligned} \tag{4.19}$$

4.5 The mass formula

At the end of section 4.1, we mentioned that a realistic supersymmetric model of particle interactions must produce mass inequalities able to justify the absence of any supersymmetric partner of quarks, leptons and gauge bosons in the energy range accessible to present day experiments. This must be achieved by the mechanism of supersymmetry breaking.

A very particular feature of a supersymmetric theories is the existence of a mass formula valid for all possible vacua with spontaneously broken supersymmetry and vacua preserving supersymmetry, relating the masses of all the fields present in the theory. This mass formula is very convenient when discussing realistic models. It is well known that a mass formula holds when supersymmetry is not broken: all states belonging to a given supermultiplet have the same mass. This result has for consequence the following sum rule. The supertrace of the mass matrices squared of all states:

$$\mathrm{STr}\, m^2 = \mathrm{Tr}\Big(m_0^2 + 3m_1^2 - 2m_{1/2}^2\Big), \tag{4.20}$$

where m_1^2, $m_{1/2}^2$ and m_0^2 are respectively the mass matrices squared of spin-1, 1/2 (four component spinors) and 0 (real scalars) states of the theory. For a supersymmetric multiplet of a mass m, $\mathrm{STr}\, m^2$ is defined so that

$$\mathrm{STr}\, m^2 = m^2\,(\text{number of bosons} - \text{number of fermions}) = 0. \tag{4.21}$$

However, the vanishing of the supertrace for a supersymmetric theory is much weaker than statement of the equality of all masses within a supermultiplet. Indeed a formula for $\mathrm{STr}\, m^2$ can be generalized to arbitrary vacua, including those breaking supersymmetry [78]. The standard choice for vacuum configurations is to allow for constant values of Lorentz invariant fields. Thus only scalars $Z^{\mathcal{A}}$ are allowed to have a non-zero vacuum expectation values (v.e.v.), denoted by $\langle Z^{\mathcal{A}}\rangle$. For this configuration, the theory reduces to the scalar potential

$$V = -\mathcal{L}\Big(\partial_\mu \langle Z^{\mathcal{A}}\rangle = \langle \psi_L^{\mathcal{A}}\rangle = \langle \lambda_L^i\rangle = \langle V_\mu^i\rangle = 0\Big) \tag{4.22}$$

In this section we derive the supertrace formula for supersymmetric non-linear σ-models described by (4.19) relevant for later applications. Since in the models we consider in this thesis, the isometry group G does not allow for an invariant trilinear superpotential $W(\Sigma)$, we will not consider here the contributions of $W(\Sigma)$ to the mass formula. For this reason, from now on we take $W(\Sigma) = 0$ (hence the terms involving $W(Z)$ in the full lagrangian (4.19) are absent). In order to calculate $\mathrm{STr}\, m^2$, we need the explicit form of the three mass matrices in (4.20).

We first consider the mass matrix squared for a spin-1 particle. When the scalar fields $Z^{\mathcal{A}}$ acquire a vacuum expectation value, some gauge bosons will become massive in general. From (4.19), the part of the Lagrangian quadratic in spin–1 particles is

$$\mathcal{L}_1 = -G_{\mathcal{A}\underline{\mathcal{A}}}\, D\bar{Z}^{\underline{\mathcal{A}}} \cdot DZ^{\mathcal{A}} - \frac{1}{4}F^i \cdot F^i, \tag{4.23}$$

with the field strength and the covariant derivative defined in (4.17) and (4.14) respectively. Substituting the expressions for the field strength and the covariant derivative the lagrangian (4.23) becomes:

$$\mathcal{L}_1 = -\frac{1}{2}\Big[(\partial_\mu V_\nu^i)(\partial^\mu V^{i\nu} - \partial^\nu V^{i\mu}) - 2g^2 \langle R_i^{\mathcal{A}} \bar{R}_j^{\underline{\mathcal{A}}}\, G_{\mathcal{A}\underline{\mathcal{A}}}\rangle V_\mu^i V^{j\mu}\Big]. \tag{4.24}$$

This expression means that the mass matrix (squared) of spin 1 particles is

$$(m_1^2)_{ij} = 2g^2 \langle R_i^{\mathcal{A}} \bar{R}_j^{\underline{\mathcal{A}}}\, G_{\mathcal{A}\underline{\mathcal{A}}}\rangle \tag{4.25}$$

From (4.25), the trace of the mass matrix squared for gauge fields V_μ^i is

$$3\mathrm{Tr}\, m_1^2 = 6g^2 \langle R_i^{\mathcal{A}} \bar{R}_i^{\underline{\mathcal{A}}}\, G_{\mathcal{A}\underline{\mathcal{A}}}\rangle = 6g^2\, \langle G^{\mathcal{A}\underline{\mathcal{A}}} \mathcal{M}_{i,\mathcal{A}} \mathcal{M}_{i,\underline{\mathcal{A}}}\rangle. \tag{4.26}$$

The last equality follows up on using (B.18).

Turning to the spin-$\frac{1}{2}$ mass matrix, we collect all the terms bilinear in fermionic fields in Lagrangian (4.19) with possible vacuum expectation values $\langle Z^{\mathcal{A}}\rangle$. They read

$$\mathcal{L}_{\frac{1}{2}} = -2\langle G_{\underline{\mathcal{A}}\mathcal{A}}\rangle \bar{\psi}_L^{\underline{\mathcal{A}}} \partial\!\!\!/\, \psi_L^{\mathcal{A}} - 2\bar{\lambda}_L^i \partial\!\!\!/ \lambda_L^i + 2\sqrt{2} i\, g\Big(\mathcal{M}_{i,\mathcal{A}} \bar{\lambda}_R^i\, \psi_L^{\mathcal{A}} - \mathcal{M}_{i,\underline{\mathcal{A}}} \bar{\psi}_L^{\underline{\mathcal{A}}} \lambda_R^i\Big) + \ldots, \tag{4.27}$$

where the dots represent total derivatives terms that do not affect the action. The non-vanishing mass term can be written in a matrix form as

$$\mathcal{L}_{\frac{1}{2}} = -2\langle G_{\underline{\mathcal{A}}\mathcal{A}}\rangle \bar{\psi}_L^{\underline{\mathcal{A}}} \partial\!\!\!/\, \psi_L^{\mathcal{A}} + 2\bar{\lambda}_L^i \partial\!\!\!/ \lambda_L^i + 2\left(\bar{\psi}_L^{\underline{\mathcal{A}}} \quad \bar{\lambda}_R^i\right) M_F \begin{pmatrix} \psi_L^{\mathcal{A}} \\ \lambda_R^j \end{pmatrix} \tag{4.28}$$

with the fermion mass matrix evaluated at the classical minimum of the potential

$$M_F = \begin{pmatrix} 0 & -i\sqrt{2}g\,\mathcal{M}_{i,\underline{\mathcal{A}}} \\ i\sqrt{2}g\,\mathcal{M}_{i,\mathcal{A}} & 0 \end{pmatrix}. \tag{4.29}$$

From this expression we obtain the mass matrix squared of spin $\frac{1}{2}$ particles

$$\left(M^2_{\frac{1}{2}}\right) = \left(M_F\, M_F^\dagger\right) = \begin{pmatrix} 2g^2\,\langle \mathcal{M}_{i,\underline{\mathcal{A}}}\mathcal{M}_{i,\mathcal{A}}\rangle & 0 \\ 0 & 2g^2\,\langle G^{\mathcal{A}\underline{\mathcal{A}}}\,\mathcal{M}_{i,\underline{\mathcal{A}}}\mathcal{M}_{j,\mathcal{A}}\rangle \end{pmatrix}. \tag{4.30}$$

This mass matrix has to be normalized such that the kinetic terms of the fermionic fields take the standard form

$$\mathcal{L}_{\text{Dirac}} = -2\bar{\chi}^I(\not{\partial} - M_{IJ})\chi^J. \tag{4.31}$$

This is achieved by multiplying the mass matrix (4.29) with the inverse metric $G^{\mathcal{A}\underline{\mathcal{A}}}$ and introduces the Dirac fermions as a combination of a left-handed chiral fermions $\psi_L^{\mathcal{A}}$ and the right-handed gauginos λ_R^i. As a result, the trace of the mass matrix squared of spin–$\frac{1}{2}$ particles is then

$$\operatorname{Tr} m^2_{\frac{1}{2}} = \operatorname{Tr} M^2_{\frac{1}{2}} = 4g^2\,\langle G^{\mathcal{A}\underline{\mathcal{A}}}\mathcal{M}_{i,\mathcal{A}}\mathcal{M}_{i,\underline{\mathcal{A}}}\rangle. \tag{4.32}$$

The last thing we need is the scalar mass matrix (squared). The lagrangian has the form

$$\mathcal{L}_0 = -G_{\underline{\mathcal{A}}\mathcal{A}}\,\partial Z^{\mathcal{A}}\cdot\partial\bar{Z}^{\underline{\mathcal{A}}} + V(Z,\bar{Z}). \tag{4.33}$$

By expanding the scalar potential $V(Z,\bar{Z})$ to second order in complex fluctuation $\tilde{Z}^{\mathcal{A}}$ around the minimum $Z^{\mathcal{A}} = \langle Z^{\mathcal{A}}\rangle$, the bilinear terms are

$$\begin{aligned} \mathcal{L}_0 &= -\langle G_{\underline{\mathcal{A}}\mathcal{A}}\rangle\,\partial\tilde{Z}^{\mathcal{A}}\cdot\partial\tilde{\bar{Z}}^{\underline{\mathcal{A}}} + \langle V_{,\underline{\mathcal{A}}\mathcal{A}}\rangle\tilde{Z}^{\mathcal{A}}\,\tilde{\bar{Z}}^{\underline{\mathcal{A}}} + \frac{1}{2}\langle V_{,\mathcal{A}\mathcal{B}}\rangle\tilde{Z}^{\mathcal{A}}\tilde{Z}^{\mathcal{B}} + \frac{1}{2}\langle V_{,\underline{\mathcal{A}\mathcal{B}}}\rangle\tilde{\bar{Z}}^{\underline{\mathcal{A}}}\tilde{\bar{Z}}^{\underline{\mathcal{B}}} \\ &= -\langle G_{\underline{\mathcal{A}}\mathcal{A}}\rangle\,\partial\tilde{Z}^{\mathcal{A}}\cdot\partial\tilde{\bar{Z}}^{\underline{\mathcal{A}}} - \frac{1}{2}\begin{pmatrix}\tilde{Z}^{\mathcal{A}} & \tilde{\bar{Z}}^{\underline{\mathcal{A}}}\end{pmatrix} M_0^2 \begin{pmatrix}\tilde{\bar{Z}}^{\underline{\mathcal{B}}} \\ \tilde{Z}^{\mathcal{B}}\end{pmatrix}, \end{aligned} \tag{4.34}$$

with the spin 0 mass matrix squared M_0^2:

$$M_0^2 = \begin{pmatrix} \langle V_{\mathcal{A}\underline{\mathcal{B}}}\rangle & \langle V_{\mathcal{A}\mathcal{B}}\rangle \\ \langle V_{\underline{\mathcal{A}\mathcal{B}}}\rangle & \langle V_{\underline{\mathcal{A}}\mathcal{B}}\rangle \end{pmatrix}. \tag{4.35}$$

In a similar fashion the bosonic mass eigenstates have to be normalized such that their kinetic lagrangian takes the standard form. This is achieved again by multiplying the mass matrix squared (4.35) with the inverse metric $G^{\underline{\mathcal{A}}\mathcal{A}}$:

$$\operatorname{Tr}\tilde{M}_0^2 = 2\,\langle G^{\underline{\mathcal{A}}\mathcal{A}}\,V_{\underline{\mathcal{A}}\mathcal{A}}\rangle. \tag{4.36}$$

From the scalar potential

$$V = \frac{g^2}{2}(\mathcal{M}_i + \xi_i)^2 \tag{4.37}$$

obtained from our general lagrangian (4.19), one has

$$V_{\underline{\mathcal{A}}\mathcal{A}} = g^2 \Big(\mathcal{M}_{i\underline{\mathcal{A}}}\, \mathcal{M}_{i\mathcal{A}} + (\mathcal{M}_i + \xi_i)\, \mathcal{M}_{i\underline{\mathcal{A}}\mathcal{A}} \Big). \tag{4.38}$$

After substituting the second mixed derivative of the scalar potential (4.38) in (4.36) we obtain the trace of the spin-less mass matrix squared:

$$\mathrm{Tr}\, m_0^2 \;\; = \;\; 2\, g^2\, G^{\underline{\mathcal{A}}\mathcal{A}} \Big(\mathcal{M}_{i\underline{\mathcal{A}}}\, \mathcal{M}_{i\mathcal{A}} + (\mathcal{M}_i + \xi_i)\, \mathcal{M}_{i\underline{\mathcal{A}}\mathcal{A}} \Big). \tag{4.39}$$

Finally, collecting results (4.39), (4.26) and (4.32) leads to the general mass sum rule for non-abelian gauged supersymmetric non-linear sigma models without a superpotential:

$$\mathrm{STr}\, m^2 \;\; = \;\; 2g^2\, G^{\underline{\mathcal{A}}\mathcal{A}}\, (\mathcal{M}_i + \xi_i)\, \mathcal{M}_{i\underline{\mathcal{A}}\mathcal{A}}, \quad D^i = (\mathcal{M}_i + \xi_i) \tag{4.40}$$

which is valid for arbitrary vacuum expectation values $\langle Z^{\mathcal{A}} \rangle$.

The general mass sum rule for Yang-Mills theories with local supersymmetry, was derived by Cremmer, Ferrara, Girardello and van Proeyen [78]. It has also been derived in superspace by considering 1-loop divergences [81, 82, 92] in the (non–singular) field space

$$\mathrm{STr}\, \mathrm{m}^2 = 2iD_i R^{i\mathcal{A}}{}_{;\mathcal{A}} = 2iD_i \Big[R^{i\mathcal{A}}{}_{,\mathcal{A}} + R^{i\mathcal{B}}\, \Gamma^{\mathcal{A}}_{\mathcal{B}\mathcal{A}} \Big]. \tag{4.41}$$

The equivalence of this result (4.41) to ours (4.40) is rather easy to show using (B.18). Observe here, that the first term $R^{i\mathcal{A}}{}_{,\mathcal{A}}$ in (4.41) always vanishes in supersymmetric σ-models on Kähler cosets with anomalies cancelled by matter as in models considered here (non-abelian gauged supersymmetric non-linear sigma models.)

Some comments are in order here about the formula (4.40). It has been derived on the assumption that the Kähler metric $G_{\underline{\mathcal{A}}\mathcal{A}}$ is invertible. However, in some cases as we will discuss in chapter 5, 6 and 7, the Kähler metric $G_{\underline{\mathcal{A}}\mathcal{A}}$ develops a zero mode in the minimum of the potential; and the analysis of the theory becomes complicated by the appearance of the infinities at the classical level. A particular solution to this problem is to shift the minimum of potential away from the position where the singularities occur by adding to the model extra terms which break supersymmetry explicitly. These new terms, which break supersymmetry without generating unwanted quadratic divergences are called soft breaking terms.

Explicit breaking of global supersymmetry has been discussed in [91]. In this thesis we only focus on the scalar soft breaking mass term, relevant for later applications:

$$\mathcal{L}_{break} = |\mu|^2 X(Z, \bar{Z}). \tag{4.42}$$

Here X is real scalar which is invariant under the full set of the isometries G, and μ^2 is real and nonzero.

With the addition the soft breaking terms (4.42), the supertrace formula becomes

$$\mathrm{STr}\, m^2 \;\; = \;\; 2g^2\, G^{\underline{\mathcal{A}}\mathcal{A}}\, (\mathcal{M}_i + \xi_i)\, \mathcal{M}_{i\underline{\mathcal{A}}\mathcal{A}} + 2\mu^2\, G^{\underline{\mathcal{A}}\mathcal{A}}\, X_{\underline{\mathcal{A}}\mathcal{A}}. \tag{4.43}$$

Chapter 5

Singular metrics in supersymmetric σ–models

Do something. If it doesn't work, do something else. No idea is too crazy.

Jim Hightower

5.1 Introduction

In the previous chapter, we have seen that supersymmetric non–linear σ–models in four dimensions are described by a field dependent Kähler metric determining the kinetic terms. However, it is not automatically guaranteed that this metric is always invertible. In some supersymmetric field theories the Kähler metric develops a zero mode; then the model becomes singular in the sense that some of the kinetic terms vanish in the vacuum state, and correspondingly some couplings diverge. In this chapter, we want to investigate how to analyze supersymmetric field theories in which these types of complications may occur. This analysis can be relevant for supersymmetric non-linear σ–model building based on homogeneous Kählerian cosets $E_6/[SO(10) \times U(1)]$ and $SO(10)/[SU(5) \times U(1)]$ which we will discuss in chapter 6 and 7 respectively.

In order to gain an understanding of the situation, in this chapter we study an anomaly-free extension of the $d = 4$ supersymmetric $\mathbb{C}P^1$ [105] model, where the scalar fields take values in $SU(2)/U(1)$, and some of the isometries are gauged. In addition to the chiral multiplet parametrizing the coset manifold, anomaly cancellation requires the inclusion of other chiral multiplets. The simplest choice corresponds to a single supermultiplet with the scalar component defining a section of a complex line bundle. We couple these chiral multiplets to a gauge multiplet, focussing in particular on the case where the full $SU(2)$ isometry group is gauged. If one considers the most general Kähler potential corresponding to this geometry, one realizes that depending on the parameters, the resulting metric can vanish for particular values of the scalars. Moreover, in many cases the potential drives the scalars to a vacuum value exactly at these singular points. At this singularity, some of the four-fermi couplings explode, while the mass terms for the fermions stay in general finite.

Singularities can occur in two places: either the kinetic term of the scalar parametrizing $\mathbb{C}P^1$ or the kinetic term of the scalars parametrizing the section of the line bundle can vanish in the vacuum. When the two singularities occur at the same point, the vacuum preserves both supersymmetry and the whole $SU(2)$ gauge symmetry. The phenomenon of singular kinetic terms has been noted before to occur in the context of the low-energy limit of $N = 2$ super Yang-Mills theory [96].

In section 5.2 we describe the isometry structure of the scalar manifold, and we show how a generic choice of the Kähler potential leads to geometrical singularities. After that we present the off-shell lagrangian. In section 5.3 we describe the gauged version of the anomaly-free $\mathbb{C}P^1$-model. In section 5.4 we classify the possible vacua, discussing general consequences for the gauge symmetries and particle spectra. Subsection 5.4.2 discusses a modification of the model containing a soft supersymmetry breaking mass term which preserves the full non-linear $SU(2)$. The mass term acts as a regulator, as it displaces the vacuum away from the singular point. The particle spectrum in this regulated model is computed and shown to be sensitive to the behaviour of the Kähler potential in the limit of small regulator mass. In section 5.5 we present some examples, showing that the various types of behaviour of the spectra in the limit of small regulator mass can all be realized in actual models. We finish in section 5.6 by gauging the linear subgroup $U(1)$ and the corresponding gauge symmetries and particle spectra.

5.2 Supersymmetric σ–model on SU(2)/U(1)

The simplest supersymmetric non-linear σ–model in which one can study the vanishing of the expectation value of the Kähler metric at the minimum of the D–term potential is an anomaly free extension of the supersymmetric $\mathbb{C}P^1$–model, where the scalar fields take values in $SU(2)/U(1)$ [90]. As the pure $\mathbb{C}P^1$–model in four dimensions is anomalous, consistency imposes the inclusion of another supermultiplet, transforming as a contravariant vector on the $\mathbb{C}P^1$ manifold. The complete field content of the model is therefore specified by a complex scalar superfield $\Phi = (z, \psi_L, H)$ and a second complex scalar superfield $A = (a, \varphi_L, B)$. These superfields define representations of the isometry group $SU(2)$; on the scalar fields they take the infinitesimal form

$$\delta z = \epsilon + i\theta z + \bar{\epsilon} z^2, \quad \delta a = -i\theta a - 2\bar{\epsilon} z a. \tag{5.1}$$

Here θ is the parameter of $U(1)$ phase transformations, and $(\epsilon, \bar{\epsilon})$ are the complex parameters of the broken off-diagonal $SU(2)$ transformations. We take the fields z and a to be dimensionless; dimension-full fields are obtained by introducing a parameter f with the dimension of inverse mass (in natural units in which $c = \hbar = 1$), and making the replacements

$$z \to fz, \quad a \to fa, \tag{5.2}$$

and similarly for other fields to be introduced. Observe, that the opposite linear $U(1)$ transformations of the multiplets are precisely as required for cancellation of the isometry anomalies.

The dimensionless $\mathbb{C}P^1$ Kähler potential

$$K_\sigma(\bar{z}, z) = \ln(1 + \bar{z}z) \tag{5.3}$$

is invariant under the isometry transformations (5.1) up to the real part of a holomorphic function:

$$\delta K_\sigma(\bar{z}, z) = F(z) + \bar{F}(\bar{z}), \quad F(z; \theta, \bar{\epsilon}) = \frac{i}{2}\,\theta + \bar{\epsilon} z. \tag{5.4}$$

Note, that the transformations of the scalar a can therefore be rewritten as

$$\delta a = -2F(z; \theta, \bar{\epsilon})a. \tag{5.5}$$

It follows, that the dimensionless real scalar

$$X = \bar{a}a\, e^{2K_\sigma(\bar{z},z)} = \bar{a}a\,(1 + \bar{z}z)^2, \tag{5.6}$$

is an invariant under the full set of isometries. With this observation in mind, we take as the starting point for our supersymmetric model a Kähler potential

$$\mathcal{K}(\bar{\Phi}, \Phi; \bar{A}, A) = \ln(1 + \bar{\Phi}\Phi) + K_m(\Omega)\ , \quad \Omega = \bar{A}A\, e^{2K_\sigma(\bar{\Phi},\Phi)}\ , \tag{5.7}$$

with $K_m(\Omega)$ some analytic function of the real superfield Ω of which the real scalar quantity X is the lowest component. The kinetic terms of scalars and chiral spinors are given by

$$S = \int d^4x \int d^2\theta d^2\bar{\theta}\, \mathcal{K}(\bar{\Phi}, \Phi; \bar{A}, A) = -\int d^4x\, G_{I\underline{I}} \left(\partial \bar{Z}^{\underline{I}} \cdot \partial Z^I + \bar{\psi}^{\underline{I}}_L \overleftrightarrow{\not{D}}\, \psi^I_L \right) + ... \tag{5.8}$$

Here $Z^I = (z, a)$ and $\psi^I_L = (\psi_L, \varphi_L)$, $I = (1, 2)$, denote the scalar and spinor components of the respective superfields $\Phi^I = (\Phi, A)$, and the dots represent four-fermion interactions. The Kähler metric in field space is obtained from the Kähler potential:

$$G_{I\underline{I}} = \mathcal{K}_{,I\underline{I}} = \begin{pmatrix} G_{z\bar{z}} & G_{z\bar{a}} \\ G_{a\bar{z}} & G_{a\bar{a}} \end{pmatrix} = \begin{pmatrix} \frac{2M(X)+4\bar{z}zXM'(X)}{(1+\bar{z}z)^2} & 2a\bar{z}(1+\bar{z}z)M'(X) \\ 2\bar{a}z(1+\bar{z}z)M'(X) & (1+\bar{z}z)^2 M'(X) \end{pmatrix}, \tag{5.9}$$

with the inverse

$$G^{I\underline{I}} = \begin{pmatrix} \frac{(1+\bar{z}z)^2}{2M(X)} & -\frac{2a\bar{z}(1+\bar{z}z)}{2M(X)} \\ -\frac{2\bar{a}z(1+\bar{z}z)}{2M(X)} & \frac{1}{(1+\bar{z}z)^2}\left(\frac{1}{M'(X)} + \frac{4\bar{z}zX}{2M(X)}\right) \end{pmatrix}. \tag{5.10}$$

Here we have introduced the $SU(2)$-invariant function $M(X)$ defined in terms of $K_m(X)$ as

$$M(X) = \frac{1}{2} + XK'_m(X). \tag{5.11}$$

The primes in the equations denote derivatives w.r.t. X. The determinant of the metric is

$$\det G_{I\underline{I}} = 2M'(X)M(X). \tag{5.12}$$

Positive definite kinetic terms are obtained if both

$$M(X) > 0, \quad M'(X) > 0. \tag{5.13}$$

We then have a standard non-linear field theory, which is well-behaved below a cut-off $\Lambda^2 \sim \mathcal{O}(1/f^2)$, the parameter determining the characteristic scale of the Kähler manifold.

In contrast, if one of the two factors is negative: $\det G < 0$, the theory contains ghosts and is inconsistent. If one of the two factors vanishes, we have critical case and we must resort to some regularization in order to investigate if the model still allows for some physically interesting interpretation. Observe however, that with the given field content it is not possible to construct an $SU(2)$-invariant superpotential $W(\Phi^I)$. With such a flat potential the vacuum expectation values of the scalar fields are undetermined, but it is natural to suppose they are to be fixed in the region where the model is well behaved according to the criterion of (5.13).

We conclude our introduction of the anomaly–free non–linear σ–model based on $\mathbb{C}P^1$, by working out the component form of action (5.8), including four–fermion interactions relevant for general consideration below

$$\begin{aligned}
\mathcal{L}_{\sigma+m} \; = \; & -G_{z\bar{z}} \left(\partial_\mu \bar{z} \cdot \partial_\mu z + \bar{\psi}_L \overleftrightarrow{\not\partial} \psi_L - \bar{H} H \right) - G_{a\bar{a}} \left(\partial_\mu \bar{a} \cdot \partial_\mu a + \bar{\varphi}_L \overleftrightarrow{\not\partial} \varphi_L - \bar{B} B \right) \\
& -G_{z\bar{a}} \left(\partial_\mu \bar{a} \cdot \partial_\mu z + \bar{\varphi}_L \overleftrightarrow{\not\partial} \psi_L - \bar{B} H \right) - G_{a\bar{z}} \left(\partial_\mu \bar{z} \cdot \partial_\mu a + \bar{\psi}_L \overleftrightarrow{\not\partial} \varphi_L - \bar{H} B \right) \\
& -\Big(G_{z\bar{z},z} \partial_\mu z - G_{z\bar{z},\bar{z}} \partial_\mu \bar{z} + G_{z\bar{z},a} \partial_\mu a - G_{z\bar{z},\bar{a}} \partial_\mu \bar{a} \Big) \bar{\psi}_L \gamma^\mu \psi_L \\
& -\Big(G_{a\bar{a},z} \partial_\mu z - G_{a\bar{a},\bar{z}} \partial_\mu \bar{z} + G_{a\bar{a},a} \partial_\mu a - G_{a\bar{a},\bar{a}} \partial_\mu \bar{a} \Big) \bar{\varphi}_L \gamma^\mu \varphi_L \\
& -\Big(G_{z\bar{a},z} \partial_\mu z - G_{z\bar{a},\bar{z}} \partial_\mu \bar{z} + G_{z\bar{a},a} \partial_\mu a - G_{z\bar{a},\bar{a}} \partial_\mu \bar{a} \Big) \bar{\varphi}_L \gamma^\mu \psi_L \\
& -\Big(G_{a\bar{z},z} \partial_\mu z - G_{a\bar{z},\bar{z}} \partial_\mu \bar{z} + G_{a\bar{z},a} \partial_\mu a - G_{a\bar{z},\bar{a}} \partial_\mu \bar{a} \Big) \bar{\psi}_L \gamma^\mu \varphi_L \\
& -\Big(G_{z\bar{z},z} \bar{H} + G_{z\bar{a},z} \bar{B} \Big) \bar{\psi}_R \psi_L - \Big(G_{z\bar{z},\bar{z}} H + G_{a\bar{z},\bar{z}} B \Big) \bar{\psi}_L \psi_R \\
& -\Big(G_{a\bar{z},a} \bar{H} + G_{a\bar{a},a} \bar{B} \Big) \bar{\varphi}_R \varphi_L - \Big(G_{z\bar{a},\bar{a}} H + G_{a\bar{a},\bar{a}} B \Big) \bar{\varphi}_L \varphi_R \\
& -2\Big(G_{z\bar{z},a} \bar{H} + G_{z\bar{a},a} \bar{B} \Big) \bar{\varphi}_R \psi_L - 2\Big(G_{z\bar{z},\bar{a}} H + G_{a\bar{z},\bar{a}} B \Big) \bar{\varphi}_L \psi_R \\
& + G_{z\bar{z},z\bar{z}} \bar{\psi}_R \psi_L \bar{\psi}_L \psi_R + 2\, G_{z\bar{z},a\bar{z}} \bar{\varphi}_R \psi_L \bar{\psi}_L \psi_R + 2\, G_{z\bar{z},z\bar{a}} \bar{\psi}_R \psi_L \bar{\psi}_L \varphi_R \\
& + G_{z\bar{a},z\bar{a}} \bar{\psi}_R \psi_L \bar{\varphi}_L \varphi_R + G_{a\bar{z},a\bar{z}} \bar{\varphi}_R \varphi_L \bar{\psi}_L \psi_R + 4\, G_{z\bar{z},a\bar{a}} \bar{\varphi}_R \psi_L \bar{\psi}_L \varphi_R \\
& + G_{a\bar{a},a\bar{a}} \bar{\varphi}_R \varphi_L \bar{\varphi}_L \varphi_R + 2\, G_{a\bar{a},a\bar{z}} \bar{\varphi}_R \varphi_L \bar{\varphi}_L \psi_R + 2\, G_{a\bar{a},z\bar{a}} \bar{\psi}_R \varphi_L \bar{\varphi}_L \varphi_R .
\end{aligned} \tag{5.14}$$

It is important to stress that the vanishing of the kinetic terms for the scalar fields corresponds to the divergence of some four-fermi couplings in the lagrangian, once one solves the equations for the auxiliary fields. The four–fermion interactions take the form

$$\mathcal{L}_{\text{4-ferm}} = R_{z\bar{z}z\bar{z}}\, \bar{\psi}_R \psi_L\, \bar{\psi}_L \psi_R + R_{a\bar{a}a\bar{a}}\, \bar{\varphi}_R \varphi_L\, \bar{\varphi}_L \varphi_R + \left\{ R_{z\bar{z}a\bar{a}}\, \bar{\psi}_R \psi_L\, \bar{\varphi}_L \varphi_R + \text{perm.} \right\} , \tag{5.15}$$

with the curvature components given by

$$
\begin{aligned}
&R_{z\bar{z}z\bar{z}} = -4M + 8M'X, \quad R_{a\bar{a}a\bar{a}} = M'' + XM''' - X\tfrac{M''^2}{M'} \ , \\
&R_{z\bar{z}a\bar{a}} = 2M' + 2XM'' - 2X\tfrac{M'^2}{M} \ .
\end{aligned}
\tag{5.16}
$$

The last term in eq. (5.15) denotes that four combinations with two ψ's and two φ's appear in the expression of the four-fermion terms. Observe that the a-dependent four-fermion couplings diverge in correspondence with the singularities.

5.3 The gauged $\mathbb{C}P^1$–model

Apart from the pure supersymmetric σ–model determined by this Kähler potential $\mathcal{K}$, we consider models in which (part of) the isometries (5.1) are gauged. As the $SU(2)$ is broken, the Higgs mechanism operates as follows: the Goldstone bosons $(\bar{z}, z)$ are absorbed in the longitudinal component of the charged vector bosons, and if the full $SU(2)$ is gauged we may choose the unitary gauge $\bar{z} = z = 0$. To analyze the model in this gauge, we introduce the covariant derivatives for the dynamical fields. The expressions for the gauge-covariant derivatives of the complex scalar fields read

$$
\begin{aligned}
D_\mu z &= \partial_\mu z - igA_\mu z - \frac{g}{\sqrt{2}}\left(W_\mu^+ + W_\mu^- z^2\right) \simeq -\frac{g}{\sqrt{2}} W_\mu^+, \\
D_\mu a &= \partial_\mu a + igA_\mu a + \sqrt{2}\, gW_\mu^- za \ \simeq \ \partial_\mu a + igA_\mu a,
\end{aligned}
\tag{5.17}
$$

whilst the covariant derivatives of the fermions become

$$
\begin{aligned}
D_\mu \psi_L &= \partial_\mu \psi_L - igA_\mu \psi_L - \sqrt{2}\, gW_\mu^- z\psi_L \simeq \partial_\mu \psi_L - igA_\mu \psi_L \\
D_\mu \varphi_L &= \partial_\mu \varphi_L + igA_\mu \varphi_L + \sqrt{2}\, gW_\mu^- (z\varphi_L + a\psi_L) \\
&\simeq \partial_\mu \varphi_L + igA_\mu \varphi_L + \sqrt{2}\, gW_\mu^- a\psi_L .
\end{aligned}
\tag{5.18}
$$

The last expression on each line is the one in the unitary gauge. We have introduced the notation $W_\mu^\pm$ for the gauge fields corresponding to the broken $SU(2)$ transformations parametrized by $(\epsilon, \bar{\epsilon})$; A_μ is the gauge field of the $U(1)$ transformations.

In this case supersymmetry implies the addition of some scalar and Yukawa couplings of gaugino fields to the quasi-Goldstone and matter fermions:

$$\begin{aligned}
\mathcal{L}_{\mathrm{Yu+sc}} &= 2g\,G_{z\bar{z}}\left[\left(\bar{\lambda}_R^- - i\sqrt{2}\,\bar{z}\bar{\lambda}_R + \bar{z}^2\bar{\lambda}_R^+\right)\psi_L + \bar{\psi}_L\left(\lambda_R^+ + i\sqrt{2}\,z\lambda_R + z^2\lambda_R^-\right)\right] \\
&\quad + 2g\,G_{z\bar{a}}\left[\left(i\sqrt{2}\,\bar{a}\bar{\lambda}_R - 2\bar{z}\bar{a}\bar{\lambda}_R^+\right)\psi_L + \bar{\varphi}_L\left(\lambda_R^+ + i\sqrt{2}\,z\lambda_R + z^2\lambda_R^-\right)\right] \\
&\quad + 2g\,G_{a\bar{z}}\left[\left(\bar{\lambda}_R^- - i\sqrt{2}\,\bar{z}\bar{\lambda}_R + z^2\bar{\lambda}_R^+\right)\varphi_L - \bar{\psi}_L\left(i\sqrt{2}\,a\lambda_R + 2za\lambda_R^-\right)\right] \\
&\quad + 2g\,G_{a\bar{a}}\left[\left(i\sqrt{2}\,\bar{a}\bar{\lambda}_R - 2\bar{z}\bar{a}\bar{\lambda}_R^+\right)\varphi_L - \bar{\varphi}_L\left(i\sqrt{2}\,a\lambda_R + 2za\lambda_R^-\right)\right] \\
&\quad + 2g\,G_{a\bar{a}}\left[\left(i\sqrt{2}\,\bar{a}\bar{\lambda}_R - 2\bar{z}\bar{a}\bar{\lambda}_R^+\right)\varphi_L - \bar{\varphi}_L\left(i\sqrt{2}\,a\lambda_R + 2za\lambda_R^-\right)\right] \\
&\quad - \frac{g}{2}\left(1 + 2XK_m'(X)\right)\frac{D(1-\bar{z}z) + i\sqrt{2}\,(D^+\bar{z} - D^-z)}{1+\bar{z}z}. \qquad (5.19)
\end{aligned}$$

This is the explicit form of Eq. (4.15) in the $SU(2)/U(1)$ case, but without Fayet-Iliopoulos term.

For the D–term scalar potential we need the $SU(2)$ Killing potentials. The $U(1)$ normalizations in (5.4) of the holomorphic function F have been chosen such that these Killing potentials factorize

$$\mathcal{M}(\theta,\epsilon,\bar{\epsilon}) = \frac{\theta(1-\bar{z}z) + 2i(\epsilon\bar{z} - \bar{\epsilon}z)}{1+\bar{z}z}\,M(X). \qquad (5.20)$$

By gauging the full $SU(2)$ isometry group, we obtain the D-term potential

$$V_{SU(2)} = \frac{g^2}{2}\frac{\partial\mathcal{M}}{\partial\epsilon}\frac{\partial\mathcal{M}}{\partial\bar{\epsilon}} + \frac{g^2}{2}\Big(\frac{\partial\mathcal{M}}{\partial\theta}\Big)^2 = \frac{g^2}{2}M^2(X), \qquad (5.21)$$

which is a function of this invariant combination $M(X)$ only. If, instead, only a $U(1)$ subgroup of the $SU(2)$ isometries is gauged, the scalar potential becomes

$$V_{U(1)} = \frac{g^2}{2}\Big(\frac{\partial\mathcal{M}}{\partial\theta} + \xi\Big)^2 = \frac{g^2}{2}\left(\frac{1-\bar{z}z}{1+\bar{z}z}\,M(X) + \xi\right)^2. \qquad (5.22)$$

Here ξ is Fayet-Iliopoulos parameter, which can only be included in the model with gauged linear $U(1)$ (and hence $W^\pm = 0$).

Finally, we also have to introduce kinetic terms for the gauge fields. They are of the canonical form

$$\begin{aligned}
\mathcal{L}_{\mathrm{gauge}} &= -\frac{1}{2}F^+(W)\cdot F^-(W) - \frac{1}{4}F^2(A) - \bar{\lambda}_R^-\,\overset{\leftrightarrow}{\not{D}}\,\lambda_R^+ - \bar{\lambda}_R^+\,\overset{\leftrightarrow}{\not{D}}\,\lambda_R^- \\
&\quad -\bar{\lambda}_R\,\overset{\leftrightarrow}{\not{D}}\,\lambda_R + D^+D^- + \frac{1}{2}D^2 + \xi D. \qquad (5.23)
\end{aligned}$$

Elimination of the auxiliary D-fields from (5.23) with $\xi = 0$ leads to the to the scalar potential (5.21). To obtain this result (5.23), we have decomposed the vector multiplet $V = (W_\mu, \lambda, D)$ as follows

$$V = V^i\tau_i, \quad V^\pm = \frac{1}{\sqrt{2}}(V^1 \pm iV^2), \quad V = V^3, \qquad (5.24)$$

where τ_i are Pauli matrices. In particular, $W^\pm = (W^1 \pm iW^2)/\sqrt{2}$, $A = W^3$, and

$$\begin{aligned} F^{\pm}_{\mu\nu}(W) &= \Big(\partial_\mu \mp igA_\mu\Big)W^\pm_\nu - \Big(\partial_\nu \mp igA_\nu\Big)W^\pm_\mu \\ F_{\mu\nu}(A) &= \partial_\mu A_\nu - \partial_\nu A_\mu - igW^+_\mu W^-_\nu + igW^+_\nu W^-_\mu . \end{aligned} \tag{5.25}$$

In addition to the potential V_D generated by the D-terms, we observe that the equations for the auxiliary fields (H, B) and their complex conjugates become

$$\begin{aligned} G_{z\bar z}H + G_{a\bar z}B &= G_{z\bar z,z}\,\bar\psi_R\psi_L + G_{a\bar z,a}\,\bar\varphi_R\varphi_L + 2G_{z\bar z,a}\bar\varphi_R\psi_L, \\ G_{z\bar a}H + G_{a\bar a}B &= G_{z\bar a,z}\,\bar\psi_R\psi_L + G_{a\bar a,a}\,\bar\varphi_R\varphi_L + 2G_{z\bar a,a}\bar\varphi_R\psi_L, \end{aligned} \tag{5.26}$$

and their conjugates.

In the unitary gauge $z = \bar z = 0$ the full lagrangian becomes

$$\begin{aligned} \mathcal{L}_{\text{full}} = &-2M(X)\left(\frac{g^2}{2}W^-\cdot W^+ + \bar\psi_L \overset{\leftrightarrow}{\not{D}}\psi_L - \bar H H\right) \\ &-M'(X)\Big(D\bar a\cdot Da + \bar\varphi_L \overset{\leftrightarrow}{\not{D}}\varphi_L - \bar B B\Big) - gM(X)D \\ &-2\,(\bar a \overset{\leftrightarrow}{D}_\mu a)\Big(M'(X)\,\bar\psi_L\gamma^\mu\psi_L + M''(X)\,\bar\varphi_L\gamma^\mu\varphi_L\Big) \\ &+\sqrt{2}\,gM'(X)\Big(-aW^-_\mu\,\bar\varphi_L\gamma^\mu\psi_L + \bar a W^+_\mu\,\bar\psi_L\gamma^\mu\varphi_L\Big) \\ &-M''(X)\Big[\bar a\bar B\,\bar\varphi_R\varphi_L + aB\,\bar\varphi_L\varphi_R\Big] - 4M'(X)\Big[\bar a\bar H\,\bar\varphi_R\psi_L + aH\,\bar\varphi_L\psi_R\Big] \\ &-4\Big[M(X) - 2XM'(X)\Big]\bar\psi_R\psi_L\,\bar\psi_L\psi_R + \Big[M''(X) + XM'''(X)\Big]\bar\varphi_R\varphi_L\,\bar\varphi_L\varphi_R \\ &+8\Big[M'(X) + XM''(X)\Big]\bar\varphi_R\psi_L\,\bar\psi_L\varphi_R \\ &+2\sqrt{2}\,g\Big[\sqrt{2}\,M(X)\,(\bar\lambda^-_R\psi_L + \bar\psi_L\lambda^+_R) + iM'(X)\,(\bar a\bar\lambda_R\,\varphi_L - a\bar\varphi_L\lambda_R)\Big] \end{aligned} \tag{5.27}$$

In this expression the covariant derivatives include only the $U(1)$ gauge field. The equations for the auxiliary fields become

$$M(X)\,H = 2\bar a M'(X)\,\bar\varphi_R\psi_L, \quad M'(X)\,B = \bar a M''(X)\,\bar\varphi_R\varphi_L. \tag{5.28}$$

5.4 Analysis of the particle spectrum

In this section we discuss the physical realization and the spectrum of the theory with the gauged $SU(2)$. We begin our analysis by classifying all possible vacua, with manifest or spontaneously broken supersymmetry, which are characterized by minimizing the potential (5.21):

$$\frac{\partial V_{SU(2)}}{\partial \bar a} = M(X)M'(X)\,a = 0. \tag{5.29}$$

It is clear that a priori there may be three ways to solve this equation:

1. $a = 0$ is always a solution.
2. There may be a value $a = a_0$ such that $M_0' = M'(X_0) = 0$; if the potential reaches its minimum here, then the model is critical in the sense discussed above, and we have to be careful in the analysis of the physical realization of the theory.
3. The solution $M_0 = M(X_0) = 0$ is also logically allowed; it implies that the charged vector bosons become massless. It may happen that at the same time $X_0 M'(X_0) = 0$; then all $SU(2)$ gauge bosons become massless and the full gauge symmetry is restored. In that case the complex scalars $(z, \bar{z})$ are no longer Goldstone bosons, and the unitary gauge can not be used to eliminate them.

We also observe, that if the solution $M_0 = 0$ exists, it is necessarily the absolute minimum of the potential: $V_0 = 0$, and supersymmetry is apparently restored as well. Of course, the standard way to describe a situation in which the full gauge symmetry is restored, is to reformulate the physics in terms of linear representations of the gauge symmetry. In the following, we discuss these solutions in turn

5.4.1 Supersymmetry breaking with fully gauged $\mathrm{SU}(2)$ vacuum

In the case $X_0 = 0$ the local $U(1)$ symmetry is not broken, and the corresponding vector field A_μ remains massless. In contrast, the charged vector bosons $W_\mu^\pm$ are massive, having absorbed the Goldstone bosons $(z, \bar{z})$. As $X = 0$, it follows that $M_0 = 1/2$, implying spontaneous breaking of supersymmetry. Indeed

$$V_{SU(2)} = \frac{g^2}{8} > 0. \tag{5.30}$$

The masses of bosons and fermions are no longer degenerate. The D-term potential provides a mass for the complex scalar a. Indeed, we can derive the linearized field equation for fluctuations around the vacuum, by making the expansion

$$a = a_0 + \sqrt{Z_a}\,\tilde{a}, \tag{5.31}$$

where the normalization factor Z_a is still to be determined. With a_0 a solution of (5.29), the quadratic part of the action for the linearized field fluctuations becomes

$$\mathcal{L}(a) = -Z_a\, M_0' |\partial_\mu \tilde{a}|^2 - \frac{g^2}{2}\, M_0^2 - g^2\, Z_a\, M_0\, M_0' |\tilde{a}|^2 + \dots. \tag{5.32}$$

Taking $Z_a = 1/M_0'$ we get a canonically normalized model for the fluctuating field, satisfying in the linearized limit a Klein-Gordon equation

$$\left(-\Box + \mathrm{m}_a^2\right) \tilde{a} = 0, \tag{5.33}$$

with the mass given by

$$\mathrm{m}_{\tilde{a}}^2 = g^2 M_0 = \frac{g^2}{2}. \tag{5.34}$$

scalars

mass	$\mathrm{m}_{\tilde{a}}^2$
value	$\frac{g^2}{2}$

vectors

mass	m_A^2	m_W^2
value	0	$\frac{g^2}{2}$

fermions

mass	m_Ψ^2	$\mathrm{m}_{\varphi_L}^2$	$\mathrm{m}_{\lambda_L^-}^2$	$\mathrm{m}_{\lambda_L}^2$
value	g^2	0	0	0

Table 5.1: Supersymmetry breaking fully gauged $SU(2)$ mass spectrum

The mass term of the $SU(2)$ vector field is generated through the kinetic terms of the Goldstone bosons $(z, \bar{z})$, and reads

$$\mathrm{m}_W^2 = g^2 M_0 = \frac{g^2}{2}. \tag{5.35}$$

Next we turn to the spectrum of fermions. With $U(1)$ not broken, the fermions must fall into charged states. A positively charged Dirac fermion is formed by the quasi-Goldstone fermion ψ_L and the gaugino λ_R^+:

$$\Psi = \sqrt{2M_0}\,\psi_L + \lambda_R^+, \tag{5.36}$$

whereas the remaining fermions $\bar{\varphi}_L$, λ_L^- and λ_R stay massless. The mass of the Dirac fermion (5.36) is $m_\Psi^2 = 2g^2 M_0 = g^2$. A straightforward calculation shows, that in the scenario with manifest non-broken $U(1)$ the standard supertrace formula for the mass spectrum is satisfied:

$$\mathrm{Str}\, m^2 = \sum_J (-1)^{2J} (2J+1)\, m_J^2 = 0. \tag{5.37}$$

We close this subsection by comparing the complete spectrum of the theory summarized in the table 5.1, with the mass sum rule of non-abelian gauged supersymmetric non-linear sigma models (4.40):

$$\mathrm{STr}\,\mathrm{m}^2 = 2g^2\, G^{I\underline{I}}\, \mathcal{M}\, \mathcal{M}_{I\underline{I}} = 2g^2 X_0\, M_0\, \frac{M_0'^2 + M_0 M_0''}{M_0 M_0'}. \tag{5.38}$$

As we have gauged the full $SU(2)$ the linear Fayet–Iliopoulos term is of course absent. To obtain (5.38) we have used the general expressions for the inverse metric $G^{I\underline{I}}$ in Eq. (5.10) and the Killing potential $\mathcal{M}$ Eq. (5.20). Again we see that all physical information is contained in the variable $X = X_0$ only. It is not difficult to confirm that the detailed mass spectrum given in the table 5.1 satisfies this mass sum rule for $X_0 = 0$.

5.4.2 Softly broken supersymmetry

In contrast to the situation discussed in 5.4.1, when $X_0 > 0$ the $U(1)$ symmetry is broken spontaneously. As equation (5.29) in this case implies that $M_0 M_0' = 0$, the phase with spontaneously broken $U(1)$ symmetry is always critical. The analysis of the theory is then complicated by the appearance of infinities at the classical level. However, a completely finite theory is obtained by adding an $SU(2)$-invariant soft supersymmetry breaking scalar mass term $\Delta V(X) = -\mu^2 X$ to the potential. In the following we take the point of view that the critical model is the limit of this regulated theory when the soft mass term is taken to vanish.

With the addition of the regulator mass term, the full potential becomes

$$V(X) = V_{SU(2)} + \Delta V(X) = \frac{g^2}{2} M^2(X) - \mu^2 X. \tag{5.39}$$

As a result the minimum of the potential is shifted to the position where

$$g^2 M' M \, a = \mu^2 a. \tag{5.40}$$

Then either $a_0 = 0$ and $U(1)$ is not broken, as discussed previously; or $U(1)$ is broken, $X_0 = |a_0|^2 > 0$ and

$$g^2 M_0' M_0 = \mu^2. \tag{5.41}$$

Hence the soft supersymmetry breaking term shifts the vacuum of the model away from the critical point. Taking the broken $U(1)$ invariance into account, we parametrize the complex scalar field a as

$$a = \left(\sqrt{X_0} + \sqrt{\frac{Z_h}{2}}\, h \right) e^{i\sqrt{Z_\theta/2}\,\theta}, \tag{5.42}$$

where again Z_h and Z_θ are normalization constants to be fixed such that we obtain canonically normalized kinetic terms. The bosonic terms in the action then become in the unitary gauge

$$\begin{aligned} \mathcal{L}_{bos} &= -g^2 M_0 W^+ \cdot W^- - \frac{Z_h M_0'}{2} (\partial_\mu h)^2 - \frac{Z_\theta X_0 M_0'}{2} \left(\partial_\mu \theta + g \sqrt{\frac{2}{Z_\theta}}\, A_\mu \right)^2 \\ &\quad -\frac{g^2}{2} M_0^2 - \mu^2 X_0 - \frac{g^2 X_0}{M_0'} \left(M_0 M_0'' + M_0'^2 \right) h^2 + ... \\ &= -m_W^2 W^+ \cdot W^- - \frac{m_A^2}{2} \tilde{A}_\mu^2 - V_0 - \frac{1}{2} (\partial_\mu h)^2 - \frac{m_h^2}{2} h^2 + ..., \end{aligned} \tag{5.43}$$

where, except for the irrelevant constant V_0, we have only written out the quadratic terms which determine the linearized field equations for the fluctuating part of the fields. To get the final result, we have taken

$$Z_h = \frac{1}{M_0'}, \quad Z_\theta = \frac{1}{X_0 M_0'}, \tag{5.44}$$

and

$$m_W^2 = g^2 M_0, \quad m_A^2 = 2g^2 X_0 M_0', \quad m_h^2 = \frac{2g^2 X_0}{M_0'} \left(M_0 M_0'' + M_0'^2 \right), \tag{5.45}$$

Also we have redefined the abelian vector field to absorb the Goldstone mode in the usual way:

$$A_\mu \to \tilde{A}_\mu = A_\mu + \frac{1}{g} \sqrt{\frac{Z_\theta}{2}}\, \partial_\mu \theta. \tag{5.46}$$

This is equivalent to the choice of unitary gauge for the broken $U(1)$ symmetry. Finally, as M_0 and M_0' are related by (5.41), we can eliminate M_0, M_0' and M_0'' and the $U(1)$ breaking parameter X_0 in favor of the physical parameters m_W^2, m_A^2, m_h^2 and the soft supersymmetry breaking parameter μ^2:

$$g^2 M_0 = m_W^2, \quad M_0' = \frac{\mu^2}{m_W^2}, \quad X_0 = \frac{m_A^2 m_W^2}{2g^2 \mu^2}, \quad M_0'' = \frac{g^2 \mu^4}{m_A^2 m_W^6} \left(m_h^2 - m_A^2 \right). \tag{5.47}$$

For values $\mu^2 > 0$ the model obviously describes massive charged and neutral vector bosons plus a massive real Higgs scalar h.

In the limit $\mu^2 \to 0$ we can now distinguish various possible scenarios:

1. If there is a number $n > 0$ such that for small μ values $X_0 \sim \mu^{2n}$, then the $U(1)$ symmetry is restored when the regulator mass vanishes; it also follows, that $m_A^2 m_W^2 \sim \mu^{2(n+1)}$. As $X_0 \to 0$ implies that $M_0 \to 1/2$, we find in this limit that $m_W^2 = g^2/2$ is finite and non-zero. Therefore $M'(0) \sim \mu^2$, and $m_A^2 \sim \mu^{2(n+1)}$, which vanishes in the limit $\mu^2 \to 0$ as expected when $U(1)$ is restored. The last relation (5.47) finally implies, that

$$m_h^2 \sim \mu^{2(n-1)} M''(0). \tag{5.48}$$

   For finite $M''(0) < \infty$, the scalar mass then remains finite for $n = 1$, and vanishes for $n > 1$. For $n < 1$ the scalar mass diverges. If $M''(0)$ itself vanishes as μ^p, $p > 0$, these constraints can be further relaxed. The upshot is, that if X_0 vanishes at least as μ^2, then in the limit $\mu^2 \to 0$ we reobtain the results of section 4.

2. If in the supersymmetric limit of vanishing μ^2 the vacuum expectation value $X_0 > 0$, then the $U(1)$ symmetry remains broken. However, in the standard scenario with $m_W^2 > 0$ the $U(1)$ gauge boson is massless in the limit $\mu^2 \to 0$. This apparent contradiction is resolved by looking at the kinetic term for the Goldstone field: it turns out that in the limit its effective charge

$$g_{eff} = g\sqrt{\frac{2}{Z_\theta}} = m_A \to 0. \tag{5.49}$$

   Therefore in this case there is a decoupling: the $U(1)$ symmetry broken by X_0 is a global one, while the $U(1)$ gauge symmetry remains unbroken. Finally the h-scalar mass becomes

$$m_h^2 \sim \frac{M_0''}{\mu^2}. \tag{5.50}$$

   Thus the scalar mass diverges, unless $M_0'' \sim \mu^p$ with $p \geq 2$.

In contrast, if there is a number $k > 0$ such that for small μ values $M_0 \sim \mu^{2k}$, then $m_W^2 \to 0$, and $m_A^2 \sim \mu^{2(1-k)}$. In this case one can not trust the limit $\mu^2 \to 0$ to describe the critical $\mathbb{C}P^1$-model, as the restauration of the $SU(2)$ symmetry is expected to be accompanied by the reappearance of light bosons $(z, \bar{z})$, and it is no longer allowed to use the unitary gauge.

Nevertheless for finite μ^2 the regularized model is well-defined and the mass spectrum can be computed. First of all, m_A^2 becomes large for small μ when $k > 1$; it is finite and μ-independent for $k = 1$, and it vanishes for $k < 1$. Therefore the limit is well-behaved if $k \leq 1$. In that case the Higgs mass behaves as

$$m_h^2 - m_A^2 \sim \mu^{2(2k-1)} M_0''. \tag{5.51}$$

For finite M_0'' this implies that the masses remain finite if $1/2 \leq k \leq 1$. In particular, for $k = 1$ we have $m_h^2 = m_A^2$, both non-zero and finite. For $k = 1/2$, m_h^2 can be finite non-zero whilst $m_A^2 = 0$.

Turning to the fermion sector, the quadratic part of the lagrangean is

$$\begin{aligned} \mathcal{L}_{\text{ferm}} &= -2M_0 \bar{\psi}_L \overset{\leftrightarrow}{\not{\partial}} \psi_L - M_0' \bar{\varphi}_L \overset{\leftrightarrow}{\not{\partial}} \varphi_L - \bar{\lambda}_L \overset{\leftrightarrow}{\not{\partial}} \lambda_L - \bar{\lambda}_L^+ \overset{\leftrightarrow}{\not{\partial}} \lambda_L^- - \bar{\lambda}_L^- \overset{\leftrightarrow}{\not{\partial}} \lambda_L^+ \\ & \quad + 4gM_0 \left(\bar{\lambda}_R^- \psi_L + \bar{\psi}_L \lambda_R^+ \right) + 2ig\sqrt{2X_0}\, M_0' \left(\bar{\lambda}_R \varphi_L - \bar{\varphi}_L \lambda_R \right). \end{aligned} \tag{5.52}$$

This is diagonalized by defining the Dirac spinors

$$\Psi = \sqrt{2M_0}\, \psi_L + \lambda_R^+, \quad \Phi = \sqrt{M_0'}\, \varphi_L - i\lambda_R. \tag{5.53}$$

In terms of these fields, the expression (5.52) becomes

$$\mathcal{L}_{\text{ferm}} = -\bar{\Psi} \overset{\leftrightarrow}{\not{\partial}} \Psi - \bar{\Phi} \overset{\leftrightarrow}{\not{\partial}} \Phi - \bar{\lambda}_L^+ \overset{\leftrightarrow}{\not{\partial}} \lambda_L^- + 2g\sqrt{2M_0}\, \bar{\Psi}\Psi + 2g\sqrt{2X_0 M_0'}\, \bar{\Phi}\Phi. \tag{5.54}$$

Combining the boson and fermion mass spectra as given in table 5.2, we obtain a supertrace formula including soft supersymmetry breaking:

$$\operatorname{Str} m^2 = -\frac{2g^2 M_0}{M_0'} \left(M_0' - X_0 M_0'' \right) = -2m_W^2 - m_A^2 + m_h^2. \tag{5.55}$$

In particular, if in the limit $\mu^2 \to 0$ the $U(1)$ symmetry remains broken: $X_0 > 0$, and if in this limit the model is well-behaved, there are numbers $\omega^2 > 0$ and $1/2 \leq k \leq 1$ such that for small μ^2 to first approximation

$$\begin{aligned} m_W^2 &= \omega^2 \mu^{2k}, \quad m_A^2 = \frac{2g^2 X_0}{\omega^2}\, \mu^{2(1-k)}, \\ m_h^2 &= \frac{2g^2 X_0}{\omega^2}\, \mu^{2(1-k)} \left(1 + \frac{\omega^6}{g^2} \mu^{6k-4} M_0'' \right). \end{aligned} \tag{5.56}$$

Then

$$\operatorname{Str} m^2 = -2\omega^2 \mu^{2k} + 2\omega^4 X_0 M_0''\, \mu^{2(2k-1)} \overset{\mu^2 \to 0}{\longrightarrow} \left\{ \begin{array}{ll} 0, & \text{if } \ 1/2 < k \leq 1; \\ 2\omega^4 X_0 M_0'', & \text{if } \ k = 1/2. \end{array} \right. \tag{5.57}$$

In fact, the supertrace vanishes even for $k > 1$, but that is because the difference $m_h^2 - m_A^2$ then vanishes, even though both masses diverge individually.

The results (5.55) and (5.57) can be compared with the standard result for the supertrace formula [81, 82, 78]. This provides an excellent check on our results, as in the general form the trace of the mass matrix is computed in a gauge independent way. Specially, from the general lagrangian

$$\mathcal{L} = \mathcal{L}_{\sigma+m} + \mathcal{L}_{\text{Yu + Sc}} + \mathcal{L}_{\text{gauge}} \tag{5.58}$$

with $\mathcal{L}_{\sigma+m}$ given by eq. (5.14), where one must replace ordinary derivatives by the covariant ones we obtain the supertrace formula eq. (4.20), with the traces of the mass matrices for various spin given by

$$\operatorname{tr} m_1^2 = 2g^2(M_0 + X_0 M_0'), \tag{5.59}$$

for vector bosons;

$$\operatorname{tr} m_{1/2}^2 = 4g^2(M_0 + X_0 M_0'), \tag{5.60}$$

for fermions; and finally

$$\operatorname{tr} m_0^2 \;=\; 2g^2 G^{I\underline{I}} V_{I\underline{I}} = \frac{M_0 + X_0 M_0'}{M_0 M_0'} V_0' + \frac{2X_0 V_0''}{M_0'}, \tag{5.61}$$

for scalars. To obtain this result, we have used the general expression for the inverse metric $G^{\underline{I}I}$ given by eq. (5.10). Now observing that by definition of the vacuum state $V_0' = 0$, and that

$$\frac{2X_0 V_0''}{M_0'} = \frac{2g^2 X_0}{M_0'}(M_0 + X_0 M_0'), \tag{5.62}$$

the final expression for the supertrace of the mass matrix takes the form (5.55):

$$\operatorname{Str} m^2 = \frac{2g^2 M_0}{M_0'}\,(X_0 M_0'' - M_0')\,. \tag{5.63}$$

The regulator mass μ^2 does not appear explicitly in this expression, because it does not contribute to V_0''. Observe, that the result (5.63) even holds for $X_0 = 0$, due to the equality $g^2 M_0 = \mu^2 / M_0'$. Finally observe, that in this derivation we have not used the unitary gauge at all.

Summarizing, in the limit $\mu^2 \to 0$ the Higgs mass m_h becomes infinite, whilst the vector-boson mass m_A vanishes. Therefore both the longitudinal component of the vector boson (the would-be Goldstone boson) and the Higgs decouple; equivalently, the complete scalar field a decouples from the theory; only its vacuum expectation remains. It could in principle still contribute e.g. to fermion masses and couplings; whether this happens or not depends on the the value of the auxiliary field B in the chiral multiplet of a. The potential happens to be flat in the B-direction, hence the value of B and the equation of $U(1)$-breaking is

scalars

mass	m_h^2
value	$2g^2X_0\frac{M_0M_0''+M_0'^2}{M_0'}$

vectors

mass	m_A^2	m_W^2
value	$2g^2X_0M_0'$	g^2M_0

fermions

mass	m_Ψ^2	m_Φ^2	$m_{\lambda^-}^2$
value	$2g^2M_0$	$2g^2X_0M_0'$	0

Table 5.2: Fully gauged $SU(2)$ mass spectrum in the presence of soft supersymmetry breaking

not decided by the classical field equations; in fact the answer depends on the existence of fermion condensate as it can be seen from equations (5.28) for the F-type auxiliary fields. In the regularized form eqs. (5.28) becomes

$$H_0 = \frac{2g^2\mu^2\sqrt{X_0}}{m_W^4}\,\langle\bar{\varphi}_R\psi_L\rangle, \quad B_0 = \left(\frac{m_h^2}{2\sqrt{X_0}m_W^2} - \frac{2g^2\mu^2\sqrt{X_0}}{m_W^4}\right)\langle\bar{\varphi}_R\varphi_L\rangle. \tag{5.64}$$

It follows that in the limit $\mu^2 = 0$ the v.e.v. $H_0 = 0$, irrespective of whether there is a condensate $\langle\bar{\varphi}_R\psi_L\rangle$. In contrast, the v.e.v. B_0 vanishes only if the condensate $\langle\bar{\varphi}_R\varphi_L\rangle = 0$; (which is what has been assumed in the analysis of the potential). Otherwise in fact the above expression for B_0 diverges as m_h^2 for $\mu^2 = 0$.

5.5 Examples

In this section we provide examples of models with the properties conjectured in subsections 5.4.1 and 5.4.2.

1. Let

$$K_m(X) = \kappa_1 X + \frac{\kappa_2}{2}\,X^2 \tag{5.65}$$

Then

$$M(X) = \frac{1}{2} + \kappa_1 X + \kappa_2 X^2, \quad M'(X) = \kappa_1 + 2\kappa_2 X, \quad M''(X) = 2\kappa_2 \tag{5.66}$$

(a) If $\kappa_1 > 0$ and $\kappa_2 \geq 0$, then $M(X)$ and $M'(X)$ have no zeros, and the potential reaches its minimum for $X = 0$; it follow that the $U(1)$ symmetry is preserved, supersymmetry is broken as described in subsection 5.4.1, eqs. (5.33) and (5.34).

(b) If $\kappa_1 > 0$ and $\kappa_2 < 0$, then $M(X)$ possesses a zero for

$$X_0 = \frac{1}{\kappa_1 \left(\sqrt{1 + \frac{2|\kappa_2|}{\kappa_1^2}} - 1 \right)}. \tag{5.67}$$

However, if we include the soft breaking term (5.39), we find that at the minimum $M < 0$ and $M' < 0$, and the latter condition remains true in the limit $\mu^2 \to 0$; hence the model contains tachyons. We do not consider this case further.

(c) If $\kappa_1 < 0$ and $\kappa_2 > 0$, then $M(X)$ has no zeros for $\kappa_1^2 < 2\kappa_2$; however, $M'(X) = 0$ at

$$X_0 = \frac{|\kappa_1|}{2\kappa_2} < 1 \quad \Rightarrow \quad M_0 = \frac{1}{2}\Big(-\frac{\kappa_1^2}{2\kappa_2}\Big) \tag{5.68}$$

This is the absolute minimum of the D-term potential. If we now include the soft supersymmetry breaking term with small μ^2, we have to first approximation

$$X_1 = X_0 + \Delta X, \quad M_1 = M_0, \quad M_1' = 2\kappa_2 \Delta X, \tag{5.69}$$

with

$$\Delta X = \frac{\mu^2}{2\kappa_2 g^2 M_0} = \frac{2\mu^2}{g^2 \left(2\kappa_2 - \kappa_1^2\right)}. \tag{5.70}$$

In this case the mass spectrum of bosons reads to first approximation

$$m_W^2 = \frac{g^2}{2}\left(1 - \frac{\kappa_1^2}{2\kappa_2}\right), \quad m_A^2 = \frac{4\mu^2|\kappa_1|}{2\kappa_2 - \kappa_1^2}, \quad m_h^2 = \frac{g^4|\kappa_1|}{2\mu^2}\left(1 - \frac{\kappa_1^2}{2\kappa_2}\right)^2. \tag{5.71}$$

For the fermions we obtain the masses

$$m_\Psi^2 = 2m_W^2 = g^2\left(1 - \frac{\kappa_1^2}{2\kappa_2}\right), \quad m_\Phi^2 = m_A^2 = \frac{2\mu^2}{2\kappa_2 - \kappa_1^2}, \quad m_{\lambda^-} = 0. \tag{5.72}$$

(d) If $\kappa_1 < 0$ and $\kappa_2 > 0$ with $\kappa_1^2 > 2\kappa_2$, then $M(X)$ has two zeros, at

$$X_\pm = \frac{1}{|\kappa_1| \left(1 \pm \sqrt{1 - \frac{2\kappa_2}{\kappa_1^2}}\right)}. \tag{5.73}$$

Again, including the soft breaking term the model contains tachyons at X_+. At X_- it is well-behaved, with

$$M_- = 0, \quad M_-' = |\kappa_1|\sqrt{1 - \frac{2\kappa_2}{\kappa_1^2}}. \tag{5.74}$$

The physical minimum of the potential now occurs at

$$X_1 = X_- + \Delta X, \tag{5.75}$$

where to first approximation in μ^2

$$\Delta X = \frac{\mu^2}{g^2 M_-'^2}, \quad M_1 = M_-' \Delta X, \quad M_1' = M_-'. \tag{5.76}$$

It follows, that the bosonic mass spectrum to first approximation in μ^2 reads

$$m_W^2 = \frac{\mu^2}{M_-'}, \quad m_h^2 \approx m_A^2 = 2g^2 X_- M_-' = 2g^2 \frac{\sqrt{1 - \frac{2\kappa_2}{\kappa_1^2}}}{1 - \sqrt{1 - \frac{2\kappa_2}{\kappa_1^2}}}. \tag{5.77}$$

This corresponds to the results (5.56) with $k = 1$ and $\omega^2 = 1/M_-'$. The fermionic mass spectrum becomes

$$m_\Psi^2 = 2m_W^2 = \frac{2\mu^2}{M_-'}, \quad m_\Phi^2 = m_A^2 = 2g^2 X_- M_-', \quad m_{\lambda^-}^2 = 0. \tag{5.78}$$

2. The above analysis is typical for models which do not have simultaneous zeros of $M(X)$ and its derivative $M'(X)$. If such simultanous zeros exist, as in the above model with $2\kappa_2 = \kappa_1^2$, the analysis is changed. As a generic example, consider the model

$$M(X) = \frac{1}{2}\left(\kappa X - 1\right)^n, \quad M'(X) = \frac{n\kappa}{2}\left(\kappa X - 1\right)^{n-1}. \tag{5.79}$$

For this case an n-fold zero of $M(X)$ occurs at $X = 1/\kappa$; it is also an $(n-1)$-fold zero of $M'(X)$. The minimum of the potential, including the soft breaking term, is at

$$\frac{n\kappa g^2}{4}\left(\kappa X - 1\right)^{2n-1} = \mu^2 \quad \Rightarrow \quad (\kappa X - 1) = \left(\frac{4\mu^2}{n\kappa g^2}\right)^{\frac{1}{2n-1}}. \tag{5.80}$$

It follows that at the minimum to lowest order in μ^2:

$$\kappa X_1 = 1 + \left(\frac{4\mu^2}{n\kappa g^2}\right)^{\frac{1}{2n-1}}, \quad M_1 = \frac{1}{2}\left(\frac{4\mu^2}{n\kappa g^2}\right)^{\frac{n}{2n-1}}, \quad M_1' = \frac{n\kappa}{2}\left(\frac{4\mu^2}{n\kappa g^2}\right)^{\frac{n-1}{2n-1}}. \tag{5.81}$$

Then the spectrum of boson masses becomes

$$m_W^2 = \frac{g^2}{2}\left(\frac{4\mu^2}{n\kappa g^2}\right)^{\frac{n}{2n-1}}, \quad m_A^2 = ng^2\left(\frac{4\mu^2}{n\kappa g^2}\right)^{\frac{n-1}{2n-1}}, \tag{5.82}$$

with the Higgs mass to lowest order in μ^2:

$$\begin{aligned} m_h^2 &= \frac{3}{2}\, m_A^2, \quad n = 2, \\ m_h^2 &= m_A^2, \qquad n > 2. \end{aligned} \tag{5.83}$$

In all expressions we have kept only the terms of leading order in μ^2 for small μ^2. The fermion mass spectrum for these models reads

$$m_\Psi^2 = g^2 \left(\frac{4\mu^2}{n\kappa g^2}\right)^{\frac{n}{2n-1}}, \quad m_\Phi^2 = ng^2 \left(\frac{4\mu^2}{n\kappa g^2}\right)^{\frac{n-1}{2n-1}}, \quad m_{\lambda^-}^2 = 0. \tag{5.84}$$

We observe, that in the limit $\mu^2 \to 0$ and for n a positive integer, all masses vanish, even though $X_0 = 1/\kappa$ remains finite and non-zero. For $n = 1$ the massess $m_A^2 = m_\Phi^2$ are finite non-zero, whilst for $1/2 < n < 1$ they divege.

5.6 Supersymmetric gauged $\mathrm{U(1)}$ vacuum

To complete our analysis we consider in this section gauging of the $U(1)$ subgroup of the $SU(2)$ isometries. To determine the physical realization and the spectrum of the theory, we have to minimize the $U(1)$ potential (5.85).

$$V_{U(1)} = \frac{g_1^2}{2}\Big(\frac{\partial \mathcal{M}}{\partial\theta} + \xi\Big)^2 = \frac{g_1^2}{2}\left(\frac{1-\bar{z}z}{1+\bar{z}z}\, M(X) + \xi\right)^2, \tag{5.85}$$

Clearly a zero of the potential requires the condition:

$$\xi = \frac{\bar{z}z - 1}{\bar{z}z + 1}\, M(X) \tag{5.86}$$

There are several types of solutions to the condition (5.86). For all these solutions supersymmetry is preserved. One solution exists for $a = 0$, and values of Fayet-Iliopoulos parameter in the range $-\frac{1}{2} < \xi < \frac{1}{2}$. The value of the potential at this stationary point is zero, indicating that supersymmetry is always preserved, whilst the internal $U(1)$ symmetry is always broken.

We now briefly consider what happens if ξ is outside the range. For $\xi = -\frac{1}{2}$, the minimum of the potential is at origin, $|z|^2 = |a|^2 = 0$. In this case, internal $U(1)$ is symmetry is preserved. For the value $\xi = \frac{1}{2}$, the minimum of the potential moves off to infinity.

The particle spectrum of the theory discussed here depends on whether internal $U(1)$ symmetry is broken or not. The simplest case arise when $U(1)$ invariance is maintained for the special value $\xi = -\frac{1}{2}$. In this case all particles in the theory are massless. On the other hand, if z gets a vacuum expectation value, breaking $U(1)$, the spectrum changes completely. The $U(1)$ gauge field A_μ becomes massive, the corresponding Goldstone scalar disappears from the spectrum as usual and a massive real scalar remains. The quasi-Goldstone fermion ψ_L together with the gaugino λ_R forms a massive Dirac fermion.

To see how this result is obtained in more detail, first notice that the mass term of the $U(1)$ vector field is generated through the kinetic terms by the v.e.v. of z, and reads

$$m_A^2 = 2g^2 v_z^2\, G_{z\bar{z}}, \quad G_{z\bar{z}} = \frac{1}{1+|v_z|^2} \tag{5.87}$$

mass	m_Ψ^2	m_A^2	m_ρ^2
value	$2g_1^2 v_z^2\, G_{z\bar{z}}$	$2g_1^2 v_z^2\, G_{z\bar{z}}$	$2g_1^2 v_z^2\, G_{z\bar{z}}$

(5.92)

Table 5.3: Supersymmetric gauged $U(1)$ mass spectrum.

Next we construct the kinetic terms and potential for the real scalar ρ; it reads

$$\mathcal{L}(\rho) = -\frac{1}{2} G_{z\bar{z}} \left[\partial\rho \cdot \partial\rho - 2g^2 v_z^2\, G_{z\bar{z}} \rho^2 \right] + ..., \tag{5.88}$$

with ρ defined by

$$z = \left(v_z + \frac{\rho}{\sqrt{2}} \right) e^{i\alpha}. \tag{5.89}$$

After appropriate rescaling of ρ to canonical form, we then find that it represents a real scalar of mass $m_\rho^2 = m_A^2$, the vector boson mass. Finally, the kinetic and mass terms for the fermion fields are given by

$$\mathcal{L} \;=\; -\bar{\lambda}_R \overset{\leftrightarrow}{\not{\partial}} \lambda_R - G_{z\bar{z}}\, \bar{\psi}_L \overset{\leftrightarrow}{\not{\partial}} \psi_L + 2i\sqrt{2}\, g\, G_{z\bar{z}} \Big(v_z \bar{\psi}_L \lambda_R - v_{\bar{z}} \bar{\lambda}_R\, \psi_L \Big) \tag{5.90}$$

It follows that the Dirac spinor $\Psi = \lambda_R - i\sqrt{G_{z\bar{z}}}\, \psi_L$ satisfies the massive Dirac equation

$$(\not{\partial} + \mathrm{m}_\Psi)\Psi = 0, \tag{5.91}$$

with $\mathrm{m}_\Psi^2 = \mathrm{m}_A^2 = \mathrm{m}_\rho^2$. This establishes the presence of a massive vector supermultiplet (A_μ, ρ, Ψ) with mass squared given in table 5.6 It is not difficult to see that these masses satisfy the standard mass sum–rule

$$\mathrm{m}_\rho^2 - 4\,\mathrm{m}_\Psi^2 + 3\,\mathrm{m}_A^2 = 0 \tag{5.93}$$

of a supersymmetric theory. This result can be compared with the standard results for the trace formula (4.40):

$$\mathrm{STr}\, m^2 \;=\; 2g^2\, G^{\underline{\mathcal{A}}\mathcal{A}} \,(\mathcal{M} + \xi)\, \mathcal{M}_{\underline{\mathcal{A}}\mathcal{A}} = 0, \tag{5.94}$$

in the supersymmetric minimum $\mathcal{M} + \xi = 0$.

Chapter 6

Phenomenological analysis SO(10)/[SU(5) × U(1)] model

We used to think that if we knew one, we knew two, because one and one are two. We are finding that we must learn a great deal more about 'and'.

Sir Arthur Eddington

6.1 Introduction

The focus of this chapter is on the phenomenology of the $SO(10)/[SU(5) \times U(1)]$-spinor model. From the point of view of unification this is a very interesting coset space as both $SO(10)$ and $SU(5)$ are often used GUT groups. A supersymmetric model built on the $SO(10)/[SU(5) \times U(1)]$ coset is not free of anomalies by itself as all the $\underline{10}$ anti-symmetric complex coordinates z^{ij} $(i, j = 1, \ldots, 5)$ of this manifold carry the same charges. To construct a consistent supersymmetric model on this coset one has to include the fermion partners of the coordinates in an anomaly-free representation. As $SU(5)$ representations are not anomaly free by themselves, we have to use the full $SO(10)$ representations for our additional matter coupling in this case. This has been achieved in [73] by introducing a singlet $\underline{1}$ and completely anti-symmetric tensor with 4 indices which is equivalent to $\underline{\bar{5}}$ to complete the set of complex chiral superfields to form a $\underline{16}$ of $SO(10)$. The anti-symmetric coordinates of the coset is combine into a $\underline{10}$ of $SU(5)$ with a unit $U(1)$ charge. An anomaly free representation is obtained using the branching of the $\underline{16}$. Indeed, its decomposition under $SU(5)$ reads

$$\underline{16} = \underline{10}(1) + \underline{\bar{5}}(-3) + \underline{1}(5), \tag{6.1}$$

where the numbers in parentheses denote the relative $U(1)$ charges.

In section 6.2 we summarize the anomaly-free supersymmetric σ–model on $SO(10)/U(5)$ as described in [73]. In section 6.3 we perform a quite general analysis of gauging the full $SO(10)$ group. We investigate in particular the existence of zeros of the potential. We show that the models with fully gauged $SO(10)$ are singular. Then we extend the model with a soft supersymmetry breaking mass term which preserves the non-linear $SO(10)$. In sections

6.4 and 6.5 we classify the vacua of the model when the linear subgroup $SU(5) \times U(1)$ is gauged. Finally we discuss a number of physical aspects of this model, like supersymmetry and internal symmetry breaking, and the resulting mass-spectrum.

6.2 Supersymmetric σ–model on $SO(10)/U(5)$

The choice for the anomaly-free model on $SO(10)/SU(5)\times U(1)$ is motivated by its fermionic field content, corresponding to one complete family of quarks and leptons, including a right-handed neutrino. This can be seen by looking at the $SU(5)$ representations of the chiral multiplets that the model contains: the target manifold $SO(10)/U(5)$ is parametrized by 10 anti-symmetric complex fields z^{ij} in a chiral superfield $\Phi^{ij} = (z^{ij}, \psi_L^{ij}, H^{ij})$, to which are added $SU(5)$ vector and scalar matter multiplets denoted respectively by $\Psi_i = (k_i, \omega_{L\ i}, B_i)$, and $\Psi = (h, \varphi_L, F)$. The complete Kähler potential of the model is

$$\mathcal{K}(z,\bar{z};k,\bar{k};h,\bar{h}) = \frac{1}{2f^2}K_\sigma(z,\bar{z}) + |h|^2\, e^{-2f^2K_\sigma} + e^{f^2K_\sigma}k\chi^{-1}\bar{k} \tag{6.2}$$

with the submetric $\chi^{-1} = \mathbb{1} + f^2 z\bar{z}$ and $e^{f^2K_\sigma} = (\det\chi)^{-1}$. The dimensionfull constant f is the one introduced before to assign correct physical dimensions to the scalar fields $(z,\bar{z})$. The kinetic terms of scalars $Z^I = (z^{ij}, k_i, h)$ and chiral spinors $\psi_L^I = (\psi_L^{ij}, \omega_{i\,L}, \varphi_L)$

$$\mathcal{L}_{\rm kin} = -G_{I\underline{I}}\left(\partial\bar{Z}^{\underline{I}}\cdot\partial Z^I + \bar{\psi}_L^{\underline{I}}\overset{\leftrightarrow}{\not{D}}\,\psi_L^I\right) \tag{6.3}$$

are given in terms of the Kähler metric $G_{I\underline{I}}$ which is obtained from the full Kähler potential $\mathcal{K}$ by

$$G_{I\underline{I}} = \frac{\partial^2\mathcal{K}}{\partial Z^I\partial\bar{Z}^{\underline{I}}} = \begin{pmatrix} G_{z\bar{z}} & G_{z\bar{k}} & G_{z\bar{h}} \\ G_{k\bar{z}} & G_{k\bar{k}} & 0 \\ G_{h\bar{z}} & 0 & G_{h\bar{h}} \end{pmatrix}. \tag{6.4}$$

This metric possesses a set of holomorphic Killing vectors generating a non-linear representation of $SO(10)$:

$$\begin{aligned}
\delta z &= \frac{1}{f}\,x - u^T z - zu + f\,zx^\dagger z, \\
\delta h &= 2\mathrm{tr}(f\,zx^\dagger - u^T)h, \\
\delta k &= -k\Big(-u^T + f\,zx^\dagger + \mathrm{tr}(-u^T + f\,zx^\dagger)\mathbb{1}\Big),
\end{aligned} \tag{6.5}$$

Here u represents the parameters of the linear diagonal $U(5)$ transformations, and $(x, x^\dagger)$ are the complex parameters of the broken off-diagonal $SO(10)$ transformations.

We observe that for the model (6.2), the isometries (6.5) do not allow for an $SO(10)$ invariant trilinear superpotential. Therefore, the scalar potential only contains D-term contributions from gauging (part of) the symmetries. In the absence of a superpotential, all fields in the action constructed from (6.2) —the Goldstone bosons and their superpartners as well as the chiral superfields defining the matter representations— describe massless spin-0 and chiral spin-1/2 particles.

This situation changes if we add a second family of quarks and leptons, with superfields $\Sigma_{(2)} = (\Phi^{ij}_{(2)}, \Psi_{(2)\,i}, \Psi_{(2)})$. It is then possible to construct an invariant superpotential

$$W(\Sigma) \,=\, \sum_{a=1,2} \lambda_a \,\Psi_{(a)}\, \Psi_{(1)\,i} \Psi_{(2)\,j} \Phi^{ij}_{(2)}. \tag{6.6}$$

The λ_a are coupling constants of dimension $(\text{mass})^{-1}$.

6.3 Gauging of the $\mathrm{SO(10)}$ isometries

As a next step towards a physical interpretation of the fermions as describing quarks and leptons, in this section we introduce gauge interactions. This can have important implications for the spectrum of the theory, as in supersymmetric theories gauge-couplings are accompanied by Yukawa couplings and a D-term potential. We first consider gauging the full $SO(10)$. A local transformation of the form (6.5) then always allows one to go to the unitary gauge $z = \bar{z} = 0$. Thus all Goldstone bosons disappear from the spectrum as a result of the Brout-Englert-Higgs effect; this is confirmed by the finite mass-terms for the gauge fields corresponding to the broken generators of $SO(10)$. However in the presence of matter fields as in (6.2), required for the cancellation of anomalies, the analysis of the D-terms in the potential shows that in the unitary gauge the model becomes singular: in the minimum of the potential the expectation value of the Kähler metric vanishes. Thus the kinetic energy terms of the Goldstone and quasi-Goldstone fields all vanish.

We now present details of this analysis. The theory defined by the lagrangian (6.2) has a global $SO(10)$ symmetry. This global symmetry allows vector bosons to be coupled to the model by turning the $SO(10)$ group, or its subgroup $SU(5) \times U(1)$, into a local gauge group by introducing covariant derivatives into the Lagrangian. The covariant derivatives are defined by:

$$\begin{aligned} D_\mu z &= \partial_\mu z - g_{10}\Big(\frac{1}{f}W_\mu - U_\mu^T z - U_\mu z + f z W_\mu{}^\dagger z\Big) \simeq \partial_\mu z - g_{10}\frac{1}{f}W_\mu, \\ D_\mu k &= \partial_\mu k + g_{10}\, k\,\Big(f W_\mu{}^\dagger z - U_\mu^T + \mathrm{tr}(f W_\mu{}^\dagger z - U_\mu^T)\mathbb{1}\Big) \simeq \partial_\mu k - g_{10}\, k\,\Big(U_\mu^T + \mathrm{tr}(U_\mu^T)\Big), \\ D_\mu h &= \partial_\mu h - 2g_{10}\mathrm{tr}\Big(f W_\mu{}^\dagger z - U_\mu^T\Big) h \simeq \partial_\mu h + 2g_{10}\,\mathrm{tr}(U_\mu^T) h \end{aligned} \tag{6.7}$$

and similarly for the fermions. Again the last expressions are evaluated in the unitary gauge. We have introduced the notation $W_\mu^\dagger$ and W_μ for the 20 gauge fields corresponding to the broken $SO(10)$ transformations parametrized by $(x, x^\dagger)$ and with U_μ, the gauge field of the diagonal transformations parametrized by u.

Next we analyze the scalar potential obtained by elimination of the D-fields for various gaugings. First we recall that the full Killing potential is given by

$$\mathcal{M}(u, x^\dagger, x) = \mathrm{tr}\Big(u\mathcal{M}_u + x^\dagger \mathcal{M}_x + x\mathcal{M}_x^\dagger\Big), \tag{6.8}$$

with the $U(5)$ Killing potentials $\mathcal{M}_u$, and the broken Killing potentials $(\mathcal{M}_x, \mathcal{M}_{x^\dagger})$ given by

$$\begin{aligned} -i\mathcal{M}_u &= M\,(\mathbb{1} - 2f^2\bar{z}\chi z) + e^{f^2K_\sigma}(k^T\bar{k}^T - f^2\bar{z}\bar{k}kz), \\ -i\mathcal{M}_{x^\dagger} &= f\bar{z}\chi\,M + fe^{f^2K_\sigma}\bar{z}\bar{k}k, \\ -i\mathcal{M}_x &= -f\chi z\,M - fe^{f^2K_\sigma}\bar{k}kz, \quad M = \frac{1}{2f^2} - 2|h|^2\,e^{-2f^2K_\sigma} + e^{f^2K_\sigma}k\chi^{-1}\bar{k}. \end{aligned} \tag{6.9}$$

The D-term potential arising from gauging of $SU(5) \times U(1)$ including a Fayet-Iliopoulos parameter ξ

$$\begin{aligned} V &= \frac{g_1^2}{10}(\xi - i\mathcal{M}_Y)^2 + \frac{g_5^2}{2}\mathrm{tr}(-i\mathcal{M}_t)^2 \\ &= \frac{g_1^2}{10}\Big(\xi + (2A_1 - 5)M + 2B_0 - B_1\Big)^2 + \frac{g_5^2}{2}\Big(4(A_2 - \frac{1}{5}A_1^2)M^2 + 4(2B_{-1} \\ &\quad -B_0 - \frac{A_1}{5}[B_0 - B_1])M + \frac{3}{10}(2B_0 - B_1)^2 + \frac{1}{2}B_1^2\Big), \end{aligned} \tag{6.10}$$

with g_1 and g_5 are the $U(1)$ and $SU(5)$ gauge couplings respectively. In this expression we used the notation A_p and B_p introduced in [73]:

$$A_p \equiv \mathrm{tr}(\chi^p), \quad B_p \equiv e^{f^2K_\sigma}k\chi^{-p}\bar{k} \tag{6.11}$$

Furthermore, the $U(1)$ Killing potential $\mathcal{M}_Y$ is defined as the trace of $U(5)$ Killing potential $\mathcal{M}_u$ whereas the remaining $SU(5)$ Killing potential $\mathcal{M}_t$ is defined as a traceless part of $\mathcal{M}_u$:

$$\mathcal{M}_t = \mathcal{M}_u - \frac{1}{5}\mathcal{M}_Y\,\mathbb{1}, \qquad \mathcal{M}_Y = \mathrm{tr}\mathcal{M}_u. \tag{6.12}$$

The case of fully gauged $SO(10)$ is reobtained by taking the coupling constants equal: $g_1 = g_5 = g_{10}$, and the Fayet-Iliopoulos term to vanish: $\xi = 0$. In the unitary gauge the potential for the fully gauged $SO(10)$ model becomes

$$V_{\mathrm{uni}} = \frac{g_{10}^2}{10}\Big(10|h|^2 - \frac{5}{2f^2} - 6|k|^2\Big)^2 + \frac{2}{5}g_{10}^2\Big(|k|^2\Big)^2. \tag{6.13}$$

From this we see that we only have a supersymmetric minimum if

$$|k|^2 = 0, \quad |h|^2 = \frac{1}{4f^2}. \tag{6.14}$$

It can be seen immediately that this solution yields the vanishing of the Kähler metric:

$$G_{z\bar{z}} = G_{\sigma(ij)}{}^{(kl)} = \delta_i^{[k}\delta_j^{l]}\Big(\frac{1}{2f^2} - 2|h|^2 + |k_i|^2\Big) + k^k\delta_i{}^l\bar{k}_j = 0. \tag{6.15}$$

Therefore in this case the kinetic terms of the Goldstone superfield components vanish. Such a solution is unacceptable as we explained in the previous chapter.

6.3.1 Softly broken supersymmetry

To avoid the problem of vanishing of the Kähler metric, we again shift the minimum of the potential away from the singular point by adding $SO(10)$-invariant soft supersymmetry breaking scalar mass term

$$\Delta V(X) = \mu^2 X, \quad X = |h|^2 (\det \chi)^2 \tag{6.16}$$

to the potential. As a result the minimum of the potential is shifted to a position where the expectation value of the Kähler metric is not vanishing; and the scalar h gets a vacuum expectation value

$$|k|^2 = 0, \quad |h|^2 = v^2 = \frac{1}{4f^2} - \frac{\mu^2}{20 g_{10}^2}, \tag{6.17}$$

breaking the linear local $U(1)$ subgroup. The corresponding $U(1)$ vector becomes massive; and the remaining vectors of $SU(5)$ stay massless. In the fermionic sector, two Dirac fermions are realized as a combination of the fermions of the chiral multiplets with the gauginos.

We now present details of the above mass spectrum. Since in general $SO(10)$ is broken in the vacuum, the Goldstone bosons $(\bar{z}, z)$ are absorbed in the longitudinal component of the charged vector bosons, and we may choose the unitary gauge $\bar{z} = z = 0$. In this gauge the Kähler metric in the minimum (6.17) is automatically diagonal:

$$G_{I\underline{I}} = \begin{pmatrix} G_{\sigma(ij)}{}^{(kl)} & 0 & 0 \\ 0 & G^i_j & 0 \\ 0 & 0 & G_{h\bar{h}} \end{pmatrix} = \begin{pmatrix} \delta_i^{[k}\delta_j^{l]}\left(\frac{1}{2f^2} - 2|h|^2\right) & 0 & 0 \\ 0 & \delta^i{}_j & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{6.18}$$

and all the z dependence is removed from the covariant derivatives (6.7). To calculate the bosonic mass spectrum, we consider the bosonic part of the model, which up to the kinetic terms for the gauge bosons is described by the action

$$\begin{aligned} \mathcal{L}_{\rm bos} &= -g_{10}^2\, G_{\sigma(ij)}{}^{(kl)} \bar{W}^{(ij)} \cdot W_{(kl)} - D\bar{k}^i \cdot Dk_i - D\bar{h} \cdot Dh - V_{\rm full} \\ & \quad -\frac{1}{4}\left[\frac{1}{2}\bar{F}_{(ij)}(W) \cdot F^{(ij)}(W) + F^i{}_j(U) \cdot F^i{}_j(U)\right] + \ldots, \end{aligned} \tag{6.19}$$

where $V_{\rm full}$ is given by $V_{\rm uni}$ eq. (6.13) and $\Delta V(X)$ eq. (6.16). In this expression the covariant derivatives include only the $U(5)$ gauge field. To identify the masses of the gauge fields, we decompose the $U(5)$ vector multiplet $U^i{}_j = (U^i_{\mu j}, \Lambda^i_{Rj})$ into a $U(1)$ and $SU(5)$ vector multiplets denoted respectively by $A = (A_\mu, \lambda_R)$ and $V^i_j = (V^i_{\mu j}, \lambda^i_{Rj})$:

$$V = U - \frac{1}{5} A \mathbb{1}_5 \quad \mathrm{tr}(V) = 0, \quad A = \mathrm{tr}(U). \tag{6.20}$$

It follows that the kinetic terms for the $SU(5) \times U(1)$ gauge fields become

$$-\frac{1}{4}\mathrm{tr} F^2_{\mu\nu}(U) - \bar{\Lambda}^i_{Rj} \overset{\leftrightarrow}{\not{\partial}} \Lambda^i_{Rj} = -\frac{1}{5}\left(\frac{1}{4}F^2_{\mu\nu}(A) + \bar{\lambda}_R \overset{\leftrightarrow}{\not{\partial}} \lambda_R\right) + \frac{1}{4}\mathrm{tr}[F^2_{\mu\nu}(V)] - \bar{\lambda}^i_{Rj} \overset{\leftrightarrow}{\not{\partial}} \lambda^i_{Rj}. \tag{6.21}$$

Notice that the kinetic terms for the $U(1)$ multiplet are not canonically normalized. To obtain the standard normalization, we redefine the $U(1)$ multiplet according to

$$A \to \sqrt{5}(\tilde{A}_\mu, \tilde{\lambda}_R). \tag{6.22}$$

With the redefined fields, the kinetic terms for the gauge fields become

$$\begin{aligned}\mathcal{L}_{\text{gauge}} &= -\frac{1}{4}\Big[\frac{1}{2}\bar{F}_{(ij)}(W)\cdot F^{(ij)}(W) + F^2_{\mu\nu}(\tilde{A}) + F^i{}_j(V)\cdot F^i{}_j(V)\Big] \\ &\quad -\bar{\tilde{\lambda}}_R \overleftrightarrow{\partial\!\!\!/}\, \tilde{\lambda}_R - \frac{1}{2}\Big(\frac{1}{2}\bar{\lambda}^{(ij)}_R \overleftrightarrow{\partial\!\!\!/}\, \lambda_{(ij)R} + \frac{1}{2}\bar{\lambda}^{(ij)}_L \overleftrightarrow{\partial\!\!\!/}\, \lambda_{(ij)L}\Big) - \bar{\lambda}^i_{R\,j} \overleftrightarrow{\partial\!\!\!/}\, \lambda^i_{R\,j}.\end{aligned} \tag{6.23}$$

Apart from the scalar h, the masses of the gauge fields can be read off easily form the lagrangean $\mathcal{L}_{\text{bos}}$ given by eq. (6.19); they read:

$$\mathrm{m}^2_W = \frac{4}{f^2}g^2_{10}M_0, \quad \mathrm{m}^2_{\tilde{A}} = 40g^2_{10}v^2, \quad M_0 = \Big(\frac{1}{2f^2} - 2|v|^2\Big) = \frac{\mu^2}{20g^2_{10}} > 0. \tag{6.24}$$

By expanding the potential V_{full} to second order in ρ and $\tilde{k}$ with scalar ρ defined by

$$h = (v + \frac{1}{\sqrt{2}}\rho)e^{\frac{1}{\sqrt{2}v}i\alpha}, \tag{6.25}$$

around the absolute minimum (6.17) we find

$$V_{\text{full}} = V_{uni} + \Delta V(X) = \frac{1}{2}\mathrm{m}^2_\rho\rho^2 + \dots, \quad \text{with} \quad \mathrm{m}^2_\rho = 40g^2_{10}v^2. \tag{6.26}$$

Next we construct the fermionic mass terms. The quadratic part of the lagrangian is

$$\begin{aligned}\mathcal{L}_{\text{ferm}} &= -\bar{\tilde{\lambda}}_R \overleftrightarrow{\partial\!\!\!/}\, \tilde{\lambda}_R - \frac{1}{4}\Big(\bar{\lambda}^{(ij)}_R \overleftrightarrow{\partial\!\!\!/}\, \lambda_{(ij)R} + \bar{\lambda}^{(ij)}_L \overleftrightarrow{\partial\!\!\!/}\, \lambda_{(ij)L}\Big) - \bar{\lambda}^i_{R\,j} \overleftrightarrow{\partial\!\!\!/}\, \lambda^i_{R\,j} \\ &\quad -G_{\sigma\,(ij)}{}^{(kl)}\, \bar{\psi}^{(ij)}_L \overleftrightarrow{\partial\!\!\!/}\, \psi_{(kl)L} - \bar{\omega}^i_L \overleftrightarrow{\partial\!\!\!/}\, \omega_{iL} - \bar{\varphi}_L \overleftrightarrow{\partial\!\!\!/}\, \varphi_L \\ &\quad +2\sqrt{2}g\, G_{\sigma\,(ij)}{}^{(kl)}\Big[\frac{1}{f}\bar{\lambda}^{(ij)}_R\psi_{L(kl)} + \text{h.c.}\Big] + 2\sqrt{2}g\Big[2\sqrt{5}v\bar{\tilde{\lambda}}_R\varphi_L + \text{h.c.}\Big].\end{aligned} \tag{6.27}$$

As a result, two Dirac fermions are formed by combining the quasi-Goldstone fermions $\psi^{[ij]}_L$ and φ_L with the right-handed gauginos $\lambda^{[ij]}_R$ and $\tilde{\lambda}_R$ according to:

$$\Psi = \tilde{\lambda}_R + \varphi_L, \qquad \Lambda^{[ij]} = \sqrt{M_0}\psi^{[ij]}_L + \frac{1}{2}\lambda^{[ij]}_R. \tag{6.28}$$

In terms of these fields, the fermionic lagrangian becomes

$$\mathcal{L}_{\text{ferm}} = -\bar{\Lambda}(\overleftrightarrow{\partial\!\!\!/} - \mathrm{m}_\Lambda)\Lambda - \bar{\Psi}(\overleftrightarrow{\partial\!\!\!/} - \mathrm{m}_\Psi)\Psi - \bar{\lambda}^i_{R\,j} \overleftrightarrow{\partial\!\!\!/}\, \lambda^i_{R\,j} - \frac{1}{4}\bar{\lambda}^{(ij)}_L \overleftrightarrow{\partial\!\!\!/}\, \lambda_{(ij)L} - \bar{\omega}^i_L \overleftrightarrow{\partial\!\!\!/}\, \omega_{iL}, \tag{6.29}$$

with the masses m_Λ and m_Ψ given in table 6.1. The gauginos $\lambda^{[ij]}_L$, λ^i_{Rj} and the chiral fermions ω_{Li} remain massless. From the massive spectrum of the theory as summarized

scalars

mass	m_ρ^2
value	$40\, g_{10}^2 v^2$

vectors

mass	$\mathrm{m}_{\tilde{A}}^2$	m_W^2
value	$40\, g_{10}^2 v^2$	$\frac{\mu^2}{5f^2}$

fermions

mass	m_Ψ^2	m_Λ^2
value	$40\, g_{10}^2 v^2$	$\frac{2\mu^2}{5f^2}$

Table 6.1: Fully gauged $SO(10)$ massive spectrum in the presence of soft supersymmetry breaking.

scalars

mass	$\mathrm{m}_{\tilde{k}}^2$
value	0

vectors

mass	m_V^2
value	0

fermions

mass	$\mathrm{m}_{\lambda_L^{[ij]}}^2$	$\mathrm{m}_{\lambda_j^i}^2$	m_ω^2
value	0	0	0

Table 6.2: Fully gauged $SO(10)$ massless spectrum in the presence of soft supersymmetry breaking.

in the table 6.1 , we obtain the general supertrace formula (4.43)

$$\mathrm{STr}\,\mathrm{m}^2 = \mathrm{m}_\rho^2 + 3\mathrm{m}_{\tilde{A}}^2 + 6\mathrm{m}_W^2 - 4\mathrm{m}_\Psi^2 - 4\mathrm{m}_\Lambda^2 = -\frac{2\mu^2}{5f^2}. \tag{6.30}$$

Of course, the present theory cannot be regarded as complete. On the one hand, extra fermions must be coupled to the lagrangian (6.3) to represent the other families of quarks and leptons. Therefore the model must consist of (at least) three copies of $\underline{10}$, $\underline{\overline{5}}$ and $\underline{1}$ of $SU(5)$ representations in its spectrum, of which one of the $\underline{10}$ are Goldstone bosons of the coset space. On the other hand, since one must require the remaining $SU(5)$ symmetry to break down at lower energy to $SU(3) \times SU(2)_L \times U(1)$, additional interactions are required. For example, we can add the $\underline{24}$ representation of $SU(5)$ to break $SU(5)$ down to smaller symmetry group, which can still accommodate at least unbroken $SU(3) \times U(1)$. However,

Dimension repr.	U(1) charges	Notation	Description of the type of fields
10	1	z^{ij}	$SO(10)/[SU(5)\times U(1)]$ coset coordinates
$\overline{5}$	-3	k_i	Matter additions to $\underline{10}$
1	5	h	to complete the $\underline{16}$
10	1	x^{ij}	
$\overline{5}$	-3	v_i	Second family
1	5	a	
10	1	y^{ij}	
$\overline{5}$	-3	n_i	Third family
1	5	h	
24	0	$s^i{}_j$	Higgs for breaking the $SU(5)$ group to the standard model
$\overline{5}$	-2	c_i	Higgses for breaking the G_{SM}
5	2	c^i	group to the $SU(3)\times U(1)$

Table 6.3: The various $SU(5)$ representations used for our construction of a phenomenological model build around $SO(10)/[SU(5)\times U(1)]$. The first column gives the dimension of the representations, the second column their charges, the third column the notation we use for the scalar components of chiral multiplets. A brief description of what these fields are is given in the last column.

the symmetry breaking in $SU(5)$-GUT via the $\underline{24}$ (Φ) that acquires a v.e.v. of the form

$$\langle\Phi\rangle = \text{diag}\left(v, v, v, -\frac{3}{2}v, -\frac{3}{2}v\right), \tag{6.31}$$

is problematic. This is because the Higgs-doublets and Higgs-triplets, originating from the $\underline{5}$ and $\overline{5}$ representations will naturally have almost the same effective mass. Now these masses should be very large in order to avoid proton decay but on the other hand small, else the standard model Higgses are far too heavy. This inconsistency is called the doublet-triplet-splitting problem. A way out of this problem is provided by the Dimopoulos-Wilczek mechanism [99] as is discussed in ref. [98], and recently by Witten [100]. Such an analysis of including other families of quarks and leptons as well as additional interactions to break $SU(5)$ down to the standard model gauge group is outside the scope of this thesis and requires further development. For the moment we are satisfied with the observation that it is at least possible to cure some of the difficulties mentioned above for the present model with the scalar particle content summarized in table 6.3 in principle.

Dimension repr.	Component structure	Trace property	Branching to representations of $SU_L(2) \times SU(3) \times U(1)$
5	k^i		(2,1)(3) + (1,3)(-2)
$\overline{5}$	k_i		(2,1)(-3) + (1, $\overline{3}$)(2)
10	$a^{[ij]}$		(1,1)(6) + (1, $\overline{3}$)(-4) + (2,3)(1)
15	$s^{(ij)}$		(3,1)(6) + (2,3)(1) + (1,6)(-4)
24	$w^i{}_j$	$w^i{}_i = 0$	(1,1)(0) + (3,1)(0) + (2,3)(-5) + (2, $\overline{3}$)(5) + (1,8)(0)

Table 6.4: This table lists the properties of the $SU(5)$ representations used in this section. In the first column we give the dimension of a representation. The second column gives the explicit index structure of the representation: $i, j, \ldots = 1, \ldots, 5$ are the vector indices. For the conventions concerning (anti-)symmetrization, see tables 7.3 and 6.4. The third column gives additional trace property of the tensors when needed to define them uniquely. The branching rules of these representations to $SU_L(2) \times SU(3) \times U(1)$ representations can be found in the fourth column.

6.4 Supersymmetric gauged $\mathbf{SU(5)} \times \mathbf{U(1)}$ vacua

As an alternative to gauging $SO(10)$, one can gauge only the linear subgroup $SU(5) \times U(1)$ instead. This explicitly breaks the non-linear global $SO(10)$. It is then allowed in principle to construct superpotentials which are invariant only under the local gauge symmetry. In addition, when gauging any group containing the $U(1)$ as a factor, the introduction of a Fayet-Iliopoulos term is allowed. It turns out, that the corresponding models are indeed well-behaved for a range of non-zero values of this parameter.

As the $SU(5) \times U(1)$ subgroup of $SO(10)$ symmetry is not broken in the original σ-model, the Killing vectors corresponding to these symmetries are linear in the fields. The gauge covariant derivatives are then the usual one:

$$\begin{aligned} D_\mu h &= \partial_\mu h - 2\sqrt{5} g_1 \tilde{A}_\mu h, \\ D_\mu \varphi_L &= \partial_\mu \varphi_L - 2\sqrt{5} g_1 \tilde{A}_\mu \varphi_L, \\ D_\mu k &= \partial_\mu k + g_5 (V_\mu^T + \sqrt{5}\tilde{A}_\mu) k, \\ D_\mu \omega_L &= \partial_\mu \omega_L + g_5 (V_\mu^T + \sqrt{5}\tilde{A}_\mu) \omega_L. \end{aligned} \tag{6.32}$$

To determine the physical realization and the spectrum of the theory, we have to minimize the potential (6.10). This potential has absolute minimum at zero if

$$|z|^2 = |k|^2 = 0, \quad |h|^2 = \frac{1}{4f^2} + \frac{1}{10}\xi = v^2, \quad -\frac{5}{2f^2} \leq \xi < 0. \tag{6.33}$$

This solution is supersymmetric and spontaneously breaks $U(1)$, whilst $SU(5)$ is manifestly preserved. As a result, the $U(1)$ gauge field $\tilde{A}_\mu$ become massive with a mass $m^2_{\tilde{A}_\mu} = m^2_\rho$,

scalars

mass	m^2_ρ	$m^2_{\tilde{k}}$	$m^2_{\tilde{z}}$
value	$40\, g_1^2 v^2$	0	0

vectors

mass	$m^2_{\tilde{A}}$	m^2_V
value	$40\, g_1^2 v^2$	0

fermions

mass	m^2_Ψ	m^2_Ω
value	$40\, g_1^2 v^2$	0

Table 6.5: Supersymmetric gauged $SU(5) \times U(1)$ mass spectrum

the mass of the real scalar ρ defined by (6.25). The remaining vectors V_μ of $SU(5)$ stay massless. Of the gauginos, the right-handed components of the $U(1)$ gauge multiplet $\tilde{\lambda}_R$ combine with the left-handed chiral fermions φ_L to become massive Dirac fermions with the same mass as the gauge boson $\tilde{A}_\mu$. However, the left-handed gaugino $\tilde{\lambda}_L$ remain massless together with the Majorana fermions that are the gauginos of unbroken $SU(5)$ symmetry. The full mass spectrum of the theory is summarized in the table 6.5. Clearly, for values of ξ in the range (6.33) it is necessary to perform a finite renormalization of the Goldstone superfields to obtain the canonical value of the kinetic terms; in the Kähler potential this is equivalent to a rescaling of the σ-model scale such that $f^2 \to -5/\xi$. In these models the natural value of the Fayet-Iliopoulos-parameter is therefore the σ-model scale, thereby relating internal and supersymmetry breaking.

We end this section by remarking that one can also consider gauging either the $U(1)$ ($g_5 = 0$) or $SU(5)$ ($g_1 = 0$) symmetry. In the first case when gauging only the $U(1)$ symmetry, the minimum potential is at the same point as in the $SU(5) \times U(1)$ gauging. Therefore the above discussion applies here and one gets the same spectrum with equal masses for the $U(1)$ gauge multiplet. On the other hand, if only $SU(5)$ is gauged, the potential reaches its minimum at $z = k = 0$. Then no supersymmetry breaking or internal symmetry breaking occurs and all particles in the theory are massless.

6.5 Supersymmetry breaking in gauged $\mathrm{U}(5)$ vacua

In this subsection we discuss the breaking of supersymmetry in the model with gauged $SU(5) \times U(1)$. In order for supersymmetry to be broken, the minimum of the potential must be at $\langle V_{\min} \rangle > 0$. By an $SU(5)$ transformation one can always arrange the vacuum expectation value of k_i to be in the first component: $\vec{k}_i = (k, \vec{0}_4)$. With this choice, the vacuum expectation value of the Goldstone scalars can be brought in the following form in which only $z_{\|}$ gets a vacuum expectation value

$$z = z_{\|} = \begin{pmatrix} 0 & \vec{z} \\ -\vec{z} & 0_4 \end{pmatrix}, \quad z_{\perp} = (\mathbb{1} - P_{\|})z(\mathbb{1} - P_{\|}^T), \quad z_{\|} = z - z_{\perp} = (P_{\|}z + zP_{\|}^T), \quad (6.34)$$

with the projection operator $P_{\|} = \frac{\bar{k}k}{|k|^2}$.

To analyse the potential, it is convenient to introduce real variables

$$X = 1 + \frac{1}{|k|^2}kz\bar{z}\bar{k}, \qquad Y = X^2|k|^2, \qquad H = X^{-4}|h|^2 \quad (6.35)$$

In term of these variables, the $U(1)$ and $SU(5)$ scalar potential becomes respectively

$$\begin{aligned} V_1 &= \frac{g_1^2}{10}\Big[\xi + \frac{1}{2}\Big(1 + \frac{4}{X}\Big)\Big(1 - 4H\Big) + 6Y\Big]^2, \quad (6.36) \\ V_5 &= \frac{3}{5}g_5^2\Big[\Big(1 - \frac{1}{X}\Big)^2\Big(1 - 4H + 2XY\Big)^2 + (1 - X + \frac{2}{3}X^2)Y^2 \\ &\quad + Y\Big(X - 2\Big)\Big(1 - \frac{1}{X}\Big)\Big(1 - 4H + 2XY\Big)\Big]. \quad (6.37) \end{aligned}$$

It is convenient to write the complete scalar potential V in the following form

$$V = \alpha + \beta Y + \gamma Y^2 = \gamma\Big(Y + \frac{\beta}{2\gamma}\Big)^2 + \alpha - \frac{\beta^2}{4\gamma}, \quad (6.38)$$

with α, β and γ given below

$$\begin{aligned} \alpha &= \frac{g_1^2}{10}\Big(\xi + \frac{1}{2}\frac{X+4}{X} - 2H\frac{X+4}{X}\Big)^2 + \frac{g_5^2}{10}\Big[6\Big(\frac{X-1}{X}\Big)^2 + 96H^2\Big(\frac{X-1}{X}\Big)^2 \\ &\quad - 48H\Big(\frac{X-1}{X}\Big)^2\Big], \\ \beta &= \frac{6g_1^2}{5}\Big(\xi + \frac{1}{2}\frac{X+4}{X} - 2H\frac{X+4}{X}\Big) + \frac{g_5^2}{10}\Big[H\Big(264 - \frac{120X^2+144}{X}\Big) \\ &\quad + \frac{30X^2+36}{X} - 66\Big], \\ \gamma &= 18\frac{g_1^2}{5} + \frac{g_5^2}{10}\Big(54 + 40X^2 - 90X\Big). \quad (6.39) \end{aligned}$$

We now determine all supersymmetry breaking vacua. Supersymmetry breaking vacua are characterized by the equations

$$\partial_{\bar{h}}V = 0, \quad \partial_{\bar{k}^i}V = 0 \quad \text{and} \quad \partial_{\bar{z}_{mn}}V = 0 \quad (6.40)$$

We take the vacuum expectation values of k, z and h arbitrary and determine the values of X, Y and H. Notice that not all values of X, Y and H are allowed because of the requirement that the potential at the minimum must be positive definite, implying spontaneous supersymmetry breaking by the potential. Therefore from the definition (6.35) we have these inequalities

$$X > 1, \quad H \geq 0 \quad \text{and} \quad Y > 0. \tag{6.41}$$

To discuss the solutions of equations (6.40), it is convenient to write these equations in the following matrix form

$$D = JC \tag{6.42}$$

with D, C and J given by

$$D = \begin{pmatrix} \partial_{\bar{h}} V \\ \partial_{\bar{k}^i} V \\ \partial_{\bar{z}_{mn}} V \end{pmatrix}, \quad C = \begin{pmatrix} \partial_H V \\ \partial_Y V \\ \partial_X V \end{pmatrix}, \quad J = \begin{pmatrix} \partial_{\bar{h}} H & \partial_{\bar{h}} Y & \partial_{\bar{h}} X \\ \partial_{\bar{k}^i} H & \partial_{\bar{k}^i} Y & \partial_{\bar{k}^i} X \\ \partial_{\bar{z}_{mn}} H & \partial_{\bar{z}_{mn}} Y & \partial_{\bar{z}_{mn}} X \end{pmatrix}. \tag{6.43}$$

The condition $JC = 0$ implies that J has right zero-modes, or $C = 0$. Since J is not a squared matrix and hence it can not be diagonalized, we proceed as follows. By multiplying (6.42) from the left by $J^\dagger$ the Hermitian conjugate

$$J^\dagger JC = \tilde{J}C = 0, \tag{6.44}$$

we find $\tilde{J}$ a symmetric 3×3 matrix.

$$\tilde{J} = (X-1) \begin{pmatrix} \frac{H}{(X-1)X^4} + 16\frac{H^2}{X^2} & -8H\frac{Y}{X^2} & -4\frac{H}{X} \\ -8H\frac{Y}{X^2} & \frac{X^2 Y}{(X-1)} + 4\frac{Y^2}{X^2} & \frac{2Y}{X} \\ -4\frac{H}{X} & \frac{2Y}{X} & 1 \end{pmatrix} \tag{6.45}$$

Using this result we can classify the zero-modes. The determinant of $\tilde{J}$ is given by

$$\det \tilde{J} = \frac{X-1}{X^2} HY. \tag{6.46}$$

The case $Y = 0$, and $X = 1$ are not allowed, because of inequalities (6.41). This implies that the matrix $\tilde{J}$ has a zero-modes iff $H = h = 0$ or $X \to \infty$. We can now classify all the solutions of the (6.40). In fact there are three separate solutions to the these equations: $H = 0$, $C = 0$, and $X \to \infty$. We discuss these solutions in turn.

1. $X \to \infty$. This condition implies that $H = 0$ and $Y \to \infty$, as it can be seen from eq. (6.35). It follows that the vacuum expectation value of the scalars $z = \langle z_{||} \rangle \to \infty$. A direct consequence is that the minimum of the potential move to infinity, hence it is not a physical minimum.

2. The second condition $H = 0$ (see the appendix E for the detail), allows two solutions for the minimum of the potential eq. (6.40), with the roots:

$$\begin{aligned} X_1 &= \frac{9}{5g_5^2}\Big(-g_1^2 + g_5^2 - 2g_1^2\xi\Big), \quad Y_1 = \frac{5g_5^2}{27}\Big(-g_5^2 + g_1^2(1+2\xi)\Big)^{-1} \\ X_2 &= \frac{13g_1^2 - 3g_5^2 + 18g_1^2\xi}{8g_1^2 - 3g_5^2 + 16g_1^2\xi} \quad Y_2 = \frac{-3g_1^2\Big(5+2\xi\Big)\Big(-3g_5^2 + 8g_1^2(1+2\xi)\Big)}{\Big(32g_1^2 + 3g_5^2\Big)\Big(-3g_5^2 + g_1^2(13+18\xi)\Big)}. \end{aligned} \tag{6.47}$$

In view of expression (6.35) for H it can be seen that in this case the local $U(1)$ symmetry is always preserved, and the corresponding gauge field $\tilde{A}_\mu$ remains massless. However, have to check for which values ξ the solutions (6.47) are allowed by eqs. (6.41). Because all the parameters (g_1, g_5, ξ) are arbitrary and independent, it hard to make statements without a numerical analysis.

3. $C = 0$, we find that the minimum of the potential has 10 complex solutions. (we again refer to appendix E for full analysis of the these solutions.) The first five roots are:

$$X_1 = 1, \quad X_2 = X_3 = X_-, \quad X_4 = X_5 = X_+, \tag{6.48}$$

with

$$X_\pm = \frac{3(15g_5^2 \pm \sqrt{5}\sqrt{-32g_1^2g_5^2 - 3g_5^4})}{40g_5^2}. \tag{6.49}$$

The corresponding minimum values for Y_1, H_1, $Y_\pm$ and $H_\pm$ are given by (E.10). Again, because of the inequality (6.41), the first root $X_1 = 1$ is not acceptable. As X is a real quantity (see eq.(6.35)), only real solutions can be accepted. It may be observed that the solutions $X_\pm$ are purely imaginary, thus violation the reality condition for the variable X. The other five solutions are obtain from equation

$$aX^5 + bX^4 + cX^3 + dX^2 + eX + f = 0, \tag{6.50}$$

with a, b c d e and f given by eqs. (E.12) respectively. It turn out that this equation can only be solve numerically. Therefore, like for the solutions (6.47), we need a numerical values for the parameters (g_1, g_5, ξ) to complete this analysis.

Chapter 7

Phenomenological analysis of $E_6/SO(10) \times U(1)$ model

The beginning of knowledge is the discovery of something we do not understand.

Frank Herbert

7.1 Introduction

We turn our attention in this chapter to a well known model with a phenomenologically interesting particle spectrum, defined by the homogeneous coset space $E_6/SO(10) \times U(1)$ [70, 71]. The target manifold $E_6/SO(10) \times U(1)$ is parametrized by 16 complex fields z^α in a chiral superfield $\Phi^\alpha = (z^\alpha, \psi_L^\alpha, H^\alpha)$ ($\alpha = 1, ..., 16$), transforming as a Weyl spinor under $SO(10)$. Their chiral fermion superpartners have the quantum numbers of one full generation of quarks and leptons, including a right-handed neutrino. To cancel the $U(1)$-anomaly the model is extended to a complete $\underline{27}$ of E_6. According to the branching rule: $\underline{27} \rightarrow \underline{16}(1) + \underline{10}(-2) + \underline{1}(4)$, where the numbers in parentheses denote the relative $U(1)$ weights. With this choice of matter content, the cancellation of chiral anomalies of the full E_6 isometry group is achieved [74] by introducing a superfield $\Psi_m = (N_m, \chi_{Lm})$ ($m = 1, \ldots, 10$) which is equivalent to a $\underline{10}$ of $SO(10)$ with $U(1)$ charge -2; and finally a singlet $\Lambda = (h, \chi_L)$ of $SO(10)$, with $U(1)$ charge +4. .

In section 7.2, we summarize the results obtained in [70, 71, 74]. Section 7.3 discusses the gauging of internal symmetries in general. In section 7.4, we discuss in some detail the gauging of the full non-linear E_6 symmetry. The qualitative physical aspects of the model are easy to determine in this case. The non-linear realization of E_6 on the coset space $E_6/[SO(10) \times U(1)]$ is an effective description of the breakdown of E_6 to the linearly realized subgroup $SO(10) \times U(1)$. The scalars of the non-linear σ-model represent the Goldstone bosons of this breaking. Since gauging the full E_6 symmetry does not break any of the global symmetries of the model, these observations remain true after gauging. Furthermore, we may choose to study the model in the $SO(10) \times U(1)$ invariant realization by writing the full lagrangian in the unitary gauge $z^\alpha = \bar{z}^{\underline{\alpha}} = 0$. However, because of the Higgs mechanism, the Goldstone bosons are no longer physical degrees of freedom, but are absorbed by the vector bosons corresponding to the broken generators, which become massive. In one of the supersymmetric minima, we find that the D-term potential drives the scalar fields to a singular point of the kinetic terms.

7.2 Phenomenological $E_6/[SO(10)\times U(1)]$ model

In this section, we present a supersymmetric σ–model based on the non-linear realization of E_6 on the coset $E_6/SO(10)\times U(1)$. The anomaly-free supersymmetric σ–model on $E_6/[SO(10)\times U(1)]$, is defined by three chiral superfields $(\Phi^\alpha, \Psi_m, \Lambda)$ with Kähler potential given by

$$\mathcal{K}(\Phi,\bar{\Phi};\Psi,\bar{\Psi};\Lambda,\bar{\Lambda}) = K_\sigma + e^{-6f^2K_\sigma}|h|^2 + g_{mn}\bar{N}_m N_n e^{6f^2K_\sigma}, \tag{7.1}$$

with $K_\sigma = \bar{z}.[Q^{-1}\ln(1+Q)].z$, the σ-model Kähler potential. We have introduced a constant f with the dimension m^{-1}, determining the scale of symmetry breaking $E_6 \to SO(10)\times U(1)$. The positive definite matrix Q is defined as

$$Q_\alpha{}^\beta = \frac{f^2}{4}M^{\beta\delta}_{\alpha\gamma}\bar{z}^\gamma z_\delta, \quad M^{\beta\delta}_{\alpha\gamma} = 3\delta^{+\beta}_\alpha\delta^{+\delta}_\gamma - \frac{1}{2}\Gamma^{+}_{mn\alpha}{}^{\beta}\Gamma^{+}_{mn\gamma}{}^{\delta}. \tag{7.2}$$

Here $\Gamma^+_{mn} = \Gamma_{mn}\delta^+$ are the generators of the $SO(10)$ on positive chirality spinors of $SO(10)$ [70], and δ^+ is the 10-D positive chirality projection operator. Furthermore g_{mn} is the induced metric for the 10-vector representation defined by

$$g_{mn} = \frac{1}{16}\mathrm{tr}\Big(g_T(\Sigma_m C)^\dagger g_T(\Sigma_n C)\Big) \quad \text{and} \quad g_T = (\mathbb{1}_{16}+Q)^{-2}. \tag{7.3}$$

The lagrangian constructed from the Kähler potential (7.1) is invariant under a set of holomorphic Killing vectors generating a non-linear representation of E_6:

$$\begin{aligned}
\delta z_\alpha &= \frac{i}{2}\theta\sqrt{3}z_\alpha - \frac{1}{4}\omega_{mn}(\Gamma^+_{mn}\cdot z)_\alpha + \frac{1}{2}\Big[\frac{i}{f}\epsilon_\beta\delta^\beta_\alpha - \frac{if}{4}\bar{\epsilon}^\beta M^{\gamma\delta}_{\alpha\beta}z_\gamma z_\delta\Big],\\
\delta h &= 2i\Big(\sqrt{3}\theta - 3f\bar{\epsilon}\cdot z\Big)h,\\
\delta N_n &= -i\sqrt{3}\theta N_n - \omega_{nm}N_m - if\bar{\epsilon}\cdot(\Gamma^+_{mn} - 3\delta^+_{mn})\cdot zN_m
\end{aligned} \tag{7.4}$$

where $\delta^+_{mn} = \delta_{mn}\delta^+$, and θ, ω_{mn} and $\epsilon_\alpha, \bar{\epsilon}^\alpha$ are the infinitesimal parameters of the $U(1)$, $SO(10)$ and broken E_6 generators respectively. The corresponding Killing potentials are

$$\mathcal{M}_i = M_i\,E - \frac{1}{8}e^{6K_\sigma}M_{i,\alpha}{}^{\beta}\, g_{T\gamma}{}^{\delta}(C\bar{\Sigma}_m)^{\alpha\gamma}(\Sigma_n C)_{\beta\delta}\bar{N}_m N_n, \tag{7.5}$$

with E and the σ-model Killing potentials $M_i = (M_\theta, M^{(mn)}, \bar{M}^\beta, M_\beta)$ given by

$$\begin{aligned}
M_\theta &= \frac{1}{f^2\sqrt{3}} - \frac{1}{2}\sqrt{3}\bar{z}^\alpha K_{\sigma,\alpha}, \quad M^{mn} = -\frac{i}{2}\bar{z}^\alpha\Gamma^+_{mn\alpha}{}^{\beta}K_{\sigma,\gamma}\\
\bar{M}^\beta &= -\frac{1}{f}K_{\sigma,}{}^{\beta}, \quad M_\beta = -\frac{1}{f}K_{\sigma,\beta}\\
E &= 1 - 6e^{-6K_\sigma}|h|^2 + 6e^{6K_\sigma}g_{mn}\bar{N}_m N_n.
\end{aligned} \tag{7.6}$$

Observe the presence of the constant term in the $U(1)$ Killing potential M^θ which is required to close the Lie algebra on the Killing potentials.

7.3 The gauged model

As explained in chapter 4.4, gauging of the supersymmetric model involves several steps:

1. modification of the kinetic terms by introducing gauge-covariant derivatives,
2. addition of a D-term potential
3. addition of Yukawa couplings for the fermions
4. introduction of kinetic terms for the gauge superfields

The expressions for gauge-covariant derivatives of the complex scalar and fermions fields read

$$\begin{aligned}
D_\mu z_\alpha &= \partial_\mu z_\alpha - g\Big(\frac{i}{2}\sqrt{3} z_\alpha A_\mu + \frac{1}{4}(\Gamma^+_{mn} z)_\alpha A_{\mu(mn)} + \frac{1}{2}(\frac{i}{f} A_{\alpha\mu} - \frac{if}{4}\bar{A}^\beta_\mu\, M^{\gamma\delta}_{\alpha\beta}\, z_\gamma z_\delta\Big), \qquad (7.7)\\
D_\mu h &= \partial_\mu h - 2ig\Big(\sqrt{3}A_\mu - 3f\bar{A}^\alpha_\mu z_\alpha\Big)h,\\
D_\mu N_n &= \partial_\mu N_n + i\sqrt{3} g A_\mu N_n + g A_{\mu(mn)} N_m + ifg\bar{A}_\mu \cdot (\Gamma^+_{mn} - 3\delta^+_{mn}) \cdot z N_m\\
D_\mu \psi_{L\alpha} &= \partial_\mu \psi_{L\alpha} - g\Big(\frac{i}{2}\sqrt{3} A_\mu \psi_{L\alpha} + \frac{1}{4} A_{\mu(mn)}\, \Gamma^+_{mn} \psi_{L\alpha} - \frac{if}{4}\bar{A}^\beta_\mu\, M^{\gamma\delta}_{\alpha\beta}\, z_\gamma \psi_{L\delta}\Big),\\
D_\mu \chi_L &= \partial_\mu \chi_L + 6igf\bar{A}^\alpha_\mu\Big(\psi_{L\alpha} h + \chi_L z_\alpha\Big),\\
D_\mu \chi_{Ln} &= \partial_\mu \chi_{Ln} + g\Big(2i\sqrt{3} A_\mu \chi_{Ln} + A_{\mu(mn)} \chi_{Lm} + if\bar{A}_\mu \cdot (\Gamma^+_{mn} - 3\delta^+_{mn}) \cdot (\psi_L N_m + \chi_{Lm} z)\Big)
\end{aligned}$$

Here we have introduced the notation $(A_{\mu\alpha}, \bar{A}^\alpha_\mu)$ for the 32 charged gauge fields corresponding to the broken E_6 transformations; $A_{\mu(mn)}$ and A_μ are the gauge fields for the remaining $SO(10)$ and $U(1)$ transformations respectively.

We have now to add the kinetic terms for the vector multiplets. They are of the canonical form

$$\begin{aligned}
\mathcal{L}_{\text{gauge}} &= -\frac{1}{2}\Big(\bar{\lambda}^\alpha_R \overleftrightarrow{\not{D}}\, \lambda_{R\alpha} + \bar{\lambda}^\alpha_L \overleftrightarrow{\not{D}}\, \lambda_{L\alpha}\Big) - \frac{1}{2}\bar{\lambda}^{(mn)}_R \overleftrightarrow{\not{D}}\, \lambda^{(mn)}_R - \bar{\lambda}_R \overleftrightarrow{\not{D}}\, \lambda_R\\
&\quad -\frac{1}{4}\Big(F^2_{\mu\nu} + \frac{1}{2}F^{(mn)2}_{\mu\nu} + \bar{F}^\alpha_{\mu\nu} F_{\mu\nu\alpha}\Big) + \frac{1}{2}\Big(\bar{D}^\alpha D_\alpha + \frac{1}{2}D^{(mn)2} + D^2\Big) \qquad (7.8)
\end{aligned}$$

Here we have included a factor $\frac{1}{2}$ to correct for double counting due to anti-symmetry of the indices mn.

Next we couple the gaugino fields to the quasi-Goldstone ψ_L^α and matter fermions (χ_L^m, χ_L) through the Yukawa coupling

$$
\begin{aligned}
\mathcal{L}_{\text{Yuk}} \;=\; & 2\sqrt{2}g\, G_{z_\alpha \bar{z}^\beta}\Big[\Big(-\frac{i}{2}\sqrt{3}\bar{z}^\alpha\lambda_R - \frac{1}{4}(\bar{z}\cdot\Gamma^+_{mn})_\alpha\bar{\lambda}_{R(mn)} + \frac{i}{8}f\bar{z}^\alpha\bar{z}^\delta\, M^{\beta\alpha}_{\delta\gamma}\,\lambda_{R\beta} - \frac{i}{2f}\bar{\lambda}^\alpha_R\Big)\psi_\beta\Big] \\
& + 2\sqrt{2}g\, G_{N_m\bar{N}^n}\Big[\Big(i\sqrt{3}\bar{N}_m\bar{\lambda}_R - \bar{N}_l\bar{\lambda}_{R(ml)} + if\bar{N}_m\bar{z}\cdot(\Gamma^+_{mn} - 3\delta^+_{mn})\cdot\lambda_R\Big)\chi_{Ln}\Big] \\
& + 2\sqrt{2}g\, G_{h\bar{h}}\Big[-2i\bar{h}\Big(\sqrt{3}\bar{\lambda}_R - 3f\bar{z}\cdot\lambda_R\Big)\chi_L\Big] \\
& + 2\sqrt{2}g\, G_{z_\alpha\bar{h}}\Big[-2i\bar{h}\Big(\sqrt{3}\bar{\lambda}_R - 3f\bar{z}\cdot\lambda_R\Big)\psi^\alpha_L + \bar{\chi}_L\Big(\frac{i}{2}\sqrt{3}z^\alpha\lambda_R - \frac{1}{4}\lambda_{R(mn)}(\Gamma^+_{mn}\cdot z)_\alpha\Big) \\
& - \frac{1}{2}\bar{\chi}_L\Big(i\frac{f}{4}\bar{\lambda}^\beta_R\, M^{\gamma\delta}_{\alpha\beta}\, z_\gamma z_\delta - \frac{i}{f}\lambda^\alpha_R\Big)\Big] \\
& + 2\sqrt{2}g\, G_{z_\alpha\bar{N}^m}\Big[\Big(i\sqrt{3}\bar{N}_m\lambda_R + \bar{N}_l\lambda_{R(ml)} + if\bar{N}^n\bar{z}\cdot(\Gamma^+_{mn} - 3\delta^+_{mn})\cdot\lambda_R\Big)\psi^\alpha_L \\
& + \bar{\chi}_{Lm}\Big(\frac{i}{2}\sqrt{3}z^\alpha\lambda_R - \frac{1}{4}\lambda_{R(mn)}(\Gamma^+_{mn}\cdot z)_\alpha\Big)\Big] + \text{h.c.}.
\end{aligned} \tag{7.9}
$$

Here $(G_{z_\alpha\bar{z}^\beta}, G_{N_m\bar{N}^n}, \dots)$ are the second mixed derivatives of the Kähler metric $G_{I\underline{I}} = \mathcal{K}_{,I\underline{I}}$, where $I = (z_\alpha, N_n, h)$ and $\underline{I} = (\bar{z}^\alpha, \bar{N}^n, \bar{h})$.

Finally, elimination of the auxiliary fields $(D^\alpha, D^{(mn)}, D)$ from (7.8) leads to the scalar potential

$$
V_D = \frac{g^2}{2}\sum_i [\mathcal{M}_i]^2 = \frac{g^2}{2}\Big(\mathcal{M}^2_\theta + \frac{1}{2}\mathcal{M}^2_{mn} + \bar{\mathcal{M}}^\beta\mathcal{M}_\beta\Big). \tag{7.10}
$$

7.4 Gauging of the full E_6 symmetry

In this section, we discuss in some detail the gauging of the full non-linear E_6. In this case as already stated, we can choose to study the model in the unitary gauge $z^\alpha = \bar{z}_\alpha = 0$. Then the full potential becomes

$$
\begin{aligned}
V_{\text{unitary}} &= \frac{g^2}{2}\Big(\mathcal{M}^2_\theta + \frac{1}{2}\mathcal{M}^2_{mn}\Big) \\
&= \frac{g^2}{2}\Big(\frac{1}{f^2\sqrt{3}} - 2\sqrt{3}|h|^2 + \sqrt{3}\sum_m |N_m|^2\Big)^2 + \frac{g^2}{2}\sum_{m,n}|\bar{N}_m N_n - \bar{N}_n N_m|^2.
\end{aligned} \tag{7.11}
$$

Observe here that in the unitary gauge, the potential contains only the terms that one also gets in gauging $SO(10)\times U(1)$. Minimization of the potential leads to the following set of supersymmetric minima characterized by the equation

$$
|\bar{N}_m N_n - \bar{N}_n N_m|^2 = 0, \quad |h|^2 = \frac{1}{6f^2} + \frac{1}{2}|N_m|^2. \tag{7.12}
$$

It follows that $|h| \neq 0$ and the $U(1)$ gauge symmetry is always broken; a solution with $|N_m| = 0$ is possible, preserving $SO(10)$. However, solutions with $|N_m| \neq 0$ breaking $SO(10)$ are allowed, and expected in the next stage of the symmetry breaking [104].

7.4.1 Softly broken supersymmetry

In this subsection we discuss the particle spectrum of the theory at the minimum with $SO(10)$ invariant solution:

$$|N_m|^2 = 0, \quad |h|^2 = \frac{1}{6f^2}. \tag{7.13}$$

This shows that the internal symmetry $SO(10) \times U(1)$ is broken to $SO(10)$. However, this solution is not acceptable as it leads to the to the vanishing of the metric of the σ–model fields $G_\alpha{}^\beta = 0$ (and hence the masses of the 32 E_6 gauge fields A_μ^α vanish) To see that in more detail, we first recall that the Kähler metric derived from the Kähler potential $\mathcal{K}$ in the unitary gauge reduces to the form:

$$G_{I\underline{I}} = \mathcal{K}_{I\underline{I}} = \begin{pmatrix} \delta_\alpha{}^\beta\Big(\frac{1}{f^2} - 6|h|^2 + 18|N_m|^2\Big) - 4\bar{N}_m N_n (\Gamma^+_{mn})_\alpha{}^\beta & 0 & 0 \\ 0 & \delta_{mn} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{7.14}$$

It is not difficult to see that at the minimum (7.13) the Kähler metric of the σ-model fields in the upper-left coner of (7.14) vanishes. Clearly, in this domain the model no longer correctly describes the physics of the situation (i.e., the correct vacuum and the corresponding spectrum of small fluctuations). Therefore we add soft breaking terms to shift the minimum a way from the singular point, as discussed in chapters 4, 5 and 6. These terms involve mass terms of the form (4.42) for scalar fields (N_m, h). Since the minimum is at $|N_m| = 0$, we only include an E_6–invariant soft supersymmetry breaking scalar mass term for the singlet h:

$$V_{\text{soft}} = \mu^2 e^{-6K_\sigma}|h|^2 \tag{7.15}$$

The full scalar potential with soft breaking term in the unitary gauge is then:

$$V = V_{\text{unitary}} + \mu^2|h|^2. \tag{7.16}$$

As the complex scalar transforms only under $U(1)$, we choose the unitary gauge for the $U(1)$ symmetry, which allow us to write

$$h = \Big(v + \frac{1}{\sqrt{2}}\rho\Big)e^{\frac{1}{\sqrt{2}v}i\kappa}, \tag{7.17}$$

where κ is the longitudinal component of the massive gauge field A_μ. We now determine the mass spectrum of the theory. Expanding the potential (7.16) to second order in the fluctuations ρ and $\tilde{N}_m$ around the minimum

$$|N_m|^2 = 0, \quad |h|^2 = v^2 = \frac{1}{6f^2} - \frac{\mu^2}{12g^2} \tag{7.18}$$

the bosonic terms in the action then become in the unitary gauge

$$\begin{aligned} \mathcal{L}_{\text{bos}} &= -\frac{1}{4}F^2_{\mu\nu}(\tilde{A}) - \frac{1}{4}\bar{F}^\alpha_{\mu\nu}F_{\alpha\mu\nu} - \frac{1}{8}F^{(mn)2}_{\mu\nu} - \frac{1}{2}\partial\rho\cdot\partial\rho - \frac{1}{2}\partial\tilde{N}_m\cdot\partial\tilde{N}_m - V_0 \\ &\quad -m^2_{A^\alpha}\bar{A}^\alpha\cdot A_\alpha - m^2_{\tilde{A}}\tilde{A}^2_\mu - m^2_{A_{mn}}A^2_{\mu(mn)} - \frac{m^2_\rho}{2}\rho^2 - \frac{1}{2}m^2_{\tilde{N}_m}\tilde{N}^2_m + \ldots, \end{aligned} \tag{7.19}$$

In this expression, the dots represent interactions of the abelian vector field with the scalar ρ. Furthermore, we have absorbed the Goldstone mode κ in the abelian vector by redefining the $U(1)$ gauge field A_μ:

$$A_\mu \to \tilde{A}_\mu = A_\mu - \frac{1}{2\sqrt{6}gv}\partial_\mu \kappa. \tag{7.20}$$

The masses of the bosonic fields read:

$$m^2_{\tilde{A}} = m^2_\rho = 24g^2v^2, \quad m^2_{A^\alpha} = \frac{\mu^2}{4f^2}, \quad m^2_{A_{nm}} = m^2_{\tilde{N}_m} = 0. \tag{7.21}$$

As expected the gauge bosons $A_{\mu[mn]}$ of the non-broken $SO(10)$ symmetry remain massless.

Analyzing the kinetic and mass terms of the fermions

$$\begin{aligned}\mathcal{L}_{\text{ferm}} \;=\;& -G_\alpha^{\;\beta}\bar{\psi}_{L\beta} \overleftrightarrow{\partial\!\!\!/}\, \psi^\alpha_L - -\bar{\chi}^n_L \overleftrightarrow{\partial\!\!\!/}\, \chi^n_L - \bar{\chi}_L \overleftrightarrow{\partial\!\!\!/}\, \chi_L - \frac{1}{2}\Big(\bar{\lambda}^\alpha_R \overleftrightarrow{\partial\!\!\!/}\, \lambda_{\alpha R} + \bar{\lambda}^\alpha_L \overleftrightarrow{\partial\!\!\!/}\, \lambda_{L\alpha}\Big)\\ & -\frac{1}{2}\bar{\lambda}^{(mn)}_R \overleftrightarrow{\partial\!\!\!/}\, \lambda^{(mn)}_R - \bar{\lambda}_R \overleftrightarrow{\partial\!\!\!/}\, \lambda_R + \sqrt{2}g\, G_\alpha^{\;\beta}\frac{i}{f}\Big(\bar{\psi}^\alpha_L\lambda_{R\beta} - \bar{\lambda}^\alpha_R\psi_{L\beta}\Big)\\ & +4i\sqrt{6}vg\Big(\bar{\chi}_L\lambda_R - \bar{\lambda}_R\chi_L\Big).\end{aligned} \tag{7.22}$$

one realizes that two massive Dirac fermions can be formed by combining the fermions of the chiral multiplets with two gauginos:

$$\Psi_\alpha = \frac{1}{\sqrt{2}}\lambda_{R\alpha} - i\sqrt{E_0}\,\psi_{L\alpha}, \qquad \Omega = \lambda_R - i\chi_L \quad \text{with} \quad E_0 = \frac{1}{f^2} - 6v^2 = \frac{\mu^2}{2g^2} > 0. \tag{7.23}$$

In terms of these fields, the expression (7.22) becomes

$$\mathcal{L}_{\text{ferm}} = -\bar{\Psi}^\alpha \overleftrightarrow{\partial\!\!\!/}\, \Psi_\alpha - \bar{\Omega} \overleftrightarrow{\partial\!\!\!/}\, \Omega + 2\frac{\sqrt{E_0}}{f}g\bar{\Psi}^\alpha\Psi_\alpha + 4\sqrt{6}v\, g\, \bar{\Omega}\Omega. \tag{7.24}$$

The masses of these spinors are:

$$m^2_\Psi = \frac{g^2}{f^2}E_0 = \frac{\mu^2}{2f^2}, \qquad m^2_\Omega = 24g^2v^2. \tag{7.25}$$

The $\underline{16}$ of the left-handed gaugino's $\lambda_{L\alpha}$ and quasi-Goldstone fermions χ_{Ln} remain massless, together with the Majorana fermions λ^{mn} that are gauginos of the unbroken $SO(10)$ symmetry. Therefore, in this model the gaugino component $\lambda_{L\alpha}$ are now to be identified with a family of quarks and leptons, rather than the quasi Goldstone fermions themselves. (We have observed a similar thing to happen also in the $SO(10)/U(5)$–spinor model discussed in the previous chapter.) The complete spectrum of the theory is summarized in tables 7.1 and 7.2.

The conclusions that can be drawn from the above analysis may be summarized as follows. Gauging of the full E_6 in the presence of soft supersymmetry breaking may lead to a possibly realistic description of the lightest family of quarks and leptons. To make it fully realistic three important problems must be solved [97]:

scalars

mass	m^2_ρ
value	$24g^2v^2$

vectors

mass	m^2_A	$m^2_{A^\alpha}$
value	$24g^2v^2$	$\frac{\mu^2}{4f^2}$

fermions

mass	$m^2_{\Psi_\alpha}$	m^2_Ω
value	$\frac{\mu^2}{2f^2}$	$24g^2v^2$

Table 7.1: Soft supersymmetry breaking fully gauged E_6 massive spectrum

scalars

mass	$m^2_{\tilde{N}}$
value	0

vectors

mass	$m^2_{A_{[mn]}}$
value	0

fermions

mass	$m^2_{\lambda^{[mn]}}$	$m^2_{\lambda^\alpha_L}$	$m^2_{\chi_n}$
value	0	0	0

Table 7.2: Soft supersymmetry breaking fully gauged E_6 massless spectrum.

1. How to break down the remaining $SO(10)$ symmetry, as required by low-energy phenomenology.

2. It should be possible to include (at least) three generations of quarks and leptons.

3. There should be a source of large Majorana masses, so that the see-saw mechanism provides the explanation for the small neutrino masses.

In the ordinary supersymmetric grand unification models based on the group $SO(10)$, there are several ways to break $SO(10)$ down to the standard model. They are summarized in figure 7.1, in five main routes. There are three left-right symmetric routes. We use the

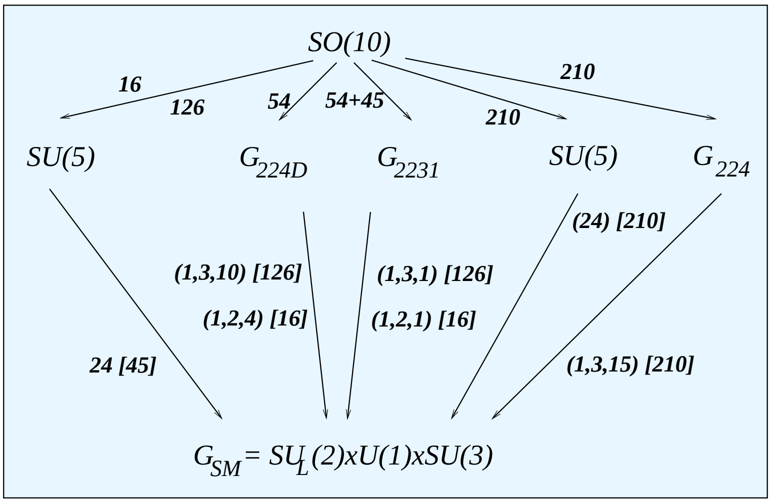

Figure 7.1: Here we give the various symmetry breaking patterns from $SO(10)$ down to the standard model. We have given the representations that are responsible for the symmetry breaking and in brackets [] the $SO(10)$ representations they originate from.

short-hand notations:

$$G_{SM} = SU_L(2) \times U(1) \times SU(3), \quad G_{2231} = SU_L(2) \times SU_R(2) \times SU(3) \times U(1)$$

$$G_{224} = SU_L(2) \times SU_R(2) \times SU(4), \quad G_{224D} = SU_L(2) \times SU_R(2) \times SU(4) \times \mathbb{Z}_2.$$

In one of these left-right symmetric models there is an additional $\mathbb{Z}_2$ symmetry. It is generated by the D-parity operator that corresponds to charge-conjugation in $SO(10)$. This discrete symmetry may lead to domain walls in the early universe and should therefore be broken to avoid this. In figure 7.1 we have also given the representations that can give rise to the indicated symmetry breaking by acquiring a vacuum expectation value, as well as the (smallest) $SO(10)$ representations from which they can originate in brackets [].

Let us explain the information that is contained in figure 7.1 in more detail and see which of these symmetry breaking patterns and Higgs representations could satisfy the first requirement stated above. One should keep in mind however that from the representation structure one can only see whether some symmetry breaking pattern is possible. To show that it indeed happens one has to compute the scalar potential and determine its minima. In the following we rely heavily on the knowledge of representation theory [101] and branching rules of $SO(10)$ to either $SU(5)$ or G_{224}. The reader can find the details of these $SO(10)$ properties in table 7.3. In the second phase of the symmetry breaking one needs the branching rules of $SU(5)$ and $SU(4)$, they can be found in tables 6.4 and 7.4 respectively.

The route via $SU(5)$ goes as follows. The first breaking is achieved by the $\underline{16}$ or the $\underline{126}$. For the second stage of symmetry breaking we need the $\underline{24}$ of $SU(5)$ that fits into the $\underline{45}$ of $SO(10)$. The second two routes both use the $\underline{54}$ to break to a left-right symmetric model. When one in addition includes an $\underline{45}$ adjoint representation then it is possible to avoid the D-parity discrete gauge symmetry and $SU(4)$ is broken to $SU(3)$. To obtain the standard model the group has to be broken down further by either giving the 16 or the 126 a v.e.v. The fourth route has $SU(5)$ as an intermediate symmetry while the fifth route has an intermediate left-right symmetry. But the symmetry breaking in both cases involves the 210 of $SO(10)$ for both stages of symmetry breaking. Hence for the symmetry breaking alone one needs just to add one representation, though it is rather large.

We now turn to the second problem in the list above we would like to impose. The 16 can accommodate one generation of quarks and leptons including the right-handed neutrino. Therefore we need at least three copies of this representation to account for three families. It would be economical (as far as the field content is concerned) to use a 16 both as a representation of quarks and leptons and as the representation that leads to the symmetry breaking $S0(10) \longrightarrow SU(5) \times U(1)$ or $G_{224D}, G_{2231} \longrightarrow G_{SM}$. Therefore a possible solution to the this problem is provided by adding the two other fermion families as additional matter multiplets $\Phi^{\pm}_{\alpha} = (x^{\pm}_{\alpha}, \psi^{\pm}_{\alpha L})$ carrying opposite $U(1)$ charges so that that the internal symmetry is free of anomalies.

The first problem above can be solved by introducing the $SO(10)$ breaking Higgs multiplets A^{mn}, S^{mn} and Q^{mnpq} with $U(1)$ charges taken to be zero. This is not strictly necessary but very convenient in the following. The fermionic partners of the coset coordinates z^{α} form one family of quarks and leptons, the other two family multiplets have scalar components $x^{\pm}_{\alpha}$. We make the charge convention such that x_+ has positive charge $q \geq 0$. We can add more copies of these representations if necessary. To conclude our discussion of the field content we have two additional Higgses E_{mnpqr} and D^{mnpqr} that may also be responsible for symmetry breaking, but in addition are also supposed to give rise to Majorana masses for the right-handed neutrinos. D has charge r and E is it charge conjugate. In addition to all this there should be at least a 10 that can produce the supersymmetric standard model Higgses after symmetry breaking down to the standard model group G_{SM}. We close this section by summarizing the most general scalar field content we consider for phenomenological promising models build on $E_6/[SO(10) \times U(1)]$ in table 7.5.

7.5 Gauging of $SO(10) \times U(1)$ symmetry

The gauging of the $SO(10) \times U(1)$ symmetry instead of the full E_6 gives analogous, but not quite identical, results. Also in this case one finds the potential (7.11), but in general with different values g_1 and g_{10} for the coupling constants of $SO(10)$ and $U(1)$. Except for special values of the parameters, it has a minimum for the $SO(10)$ invariant solution, with $z_{\alpha} = 0$; and again the metric becomes singular. One way to shift the minimum away from this point is by introducing soft breaking terms (7.15). Another option is to add an extra Fayet-Iliopoulos term as the gauge group possesses an explicit $U(1)$ factor. In the first case, the fermionic mass term is given by last line of (7.22). As a result there is now one massive Dirac fermion, from the combination of χ_L with the same gaugino of the broken $U(1)$ as before. The gauginos λ^{mn} that are left over remain unpaired, and hence massless. Furthermore, the chiral fermions ψ^{α}_L and χ^n_L remain massless. The complete spectrum can be read from the table 7.6

In the second case, for special values of the coupling constants g_1 and g_{10}, or the Fayet-Iliopoulos parameter ξ, one can get different results. Since the $SO(10)$ and $U(1)$ coupling

Dimension repr.	Component structure	Branching to repr. of $SU(5) \times U(1)$	Branching to repr. of $SU_L(2) \times SU_R(2) \times SU(4)$
10	V^m	5(2) + $\overline{5}$(-2)	(2, 2, 1) + (1, 1, 6)
16	ψ^α	1(-5) + $\overline{5}$(3) + 10(-1)	(2, 1, 4) + (2, 1, $\overline{4}$)
45	$A^{[mn]}$	1(0) + 10(4) + $\overline{10}$(-4) + 24(0)	(3,1,1) + (1,3,1) + (1,1,15) + (2,2,6)
54	$S^{(mn)}$	15(4) + $\overline{15}$(-4) + 24(0)	(1,1,1) + (3,3,1) + (1,1,20) + (2,2,6)
126	$D^{+[mnpqr]}$	1(-10) + $\overline{5}$(-2) + 10(-6) + 45(2) + $\overline{50}$(-2) +$\overline{15}$(6)	(1,1,6) + (3,1,10) + (1,3,$\overline{10}$) + (2,2,15)
210	$Q^{[mnpq]}$	1(0) + 5(-8) + $\overline{5}$(8) + 10(4) + $\overline{10}$(-4) + 24(0) + 40(-4) + $\overline{40}$(4) + 75(0)	(1,1,1) + (1,1,15) + (2, 2,6) + (3,1,15) + (1,3,15) + (2,2,10) + (2,2, $\overline{10}$)

Table 7.3: This table lists the properties of the $SO(10)$ representations used in this section. In the first column we give the dimension of a representation. The second column gives the explicit index structure of the representation: $m, n, \ldots = 1, \ldots, 10$ are the vector indices and $\alpha, \beta, \ldots = 1, \ldots, 16$ are the spinor index of a positive chirality spinor. The brackets [] denote anti-symmetrization over the indices in between the brackets, while the brackets () denote symmetrization. We do not include a normalization factor to compensate for possible over counting. The + indicates that the tensor $D^{+[mnpqr]}$ is self-dual. The branching rules of these representations to $SU(5) \times U(1)$ representations can be found in the third column while those to $SU_L(2) \times SU_R(2) \times SU(4)$ are given in the fourth column

constants are independent, one may choose to gauge only $SO(10)$ ($g_1 = 0$). In that case both supersymmetry and internal symmetry are preserved, and the particle spectrum of a model contains of a massless $SO(10)$ gauge boson, just like in the usual supersymmetric $SO(10)$ grand unified models.

Dimension repr.	Component structure	Trace property	Branching to $SU(3) \times U(1)$
4	v^a		1(3) + 3(-1)
$\overline{4}$	v_a		1(-3) + $\overline{3}$(1)
6	$a^{[ab]}$		3(2) + $\overline{3}$(-2)
10	$s^{(ab)}$		1(6) + 3(2) + 6(-2)
15	$w^a{}_b$	$w^a{}_a = 0$	1(0) + 3(-4) + $\overline{3}$(4) + 8(0)
20	$c^{([ab][cd])}$	$\epsilon_{abcd} c^{abcd} = 0$	$\overline{6}$(-4) + 6(4) + 8(0)

Table 7.4: This table lists the properties of the $SU(4)$ representations used in this section. In the first column we give the dimension of a representation. The second column gives the explicit index structure of the representation: $a, b, \ldots = 1, \ldots, 4$ are the vector indices. For the conventions concerning (anti-)symmetrization and traces, see tables 7.3 and 6.4. The branching rules of these representations to $SU(3) \times U(1)$ representations can be found in the fourth column.

Dimension repr.	U(1) charges	Notation	Description of the type of fields
16	1	z^α	$E_6/[SO(10) \times U(1)]$ coset coordinates
10	-2	N^m	Matter additions to <u>16</u>
1	4	h	to complete the <u>27</u>
16	q	x^+_α	Two generations
16	-q	x^-_α	
45	0	A^{mn}	Higgses for the unification
54	0	S^{mn}	symmetry breaking
210	0	Q^{mnpq}	
126	r	D^{mnpqr}	Higgses for neutrino Majorana masses
$\overline{126}$	-r	E_{mnpqr}	and symmetry breaking

Table 7.5: The various $SO(10)$ representations used for our construction of a phenomenological model build around $E_6/[SO(10) \times U(1)]$. The first column gives the dimension of the representations, the second column their charges, the third column the notation we use for the scalar components of chiral multiplets. A brief description of what these fields are is given in the last column. The charges q, r will be fixed by dynamical considerations like $SO(10) \times U(1)$ anomaly cancellations and the requirement that various Yukawa couplings can appear in the superpotential.

scalars

mass	m^2_{ρ}	$m^2_{\tilde{N}_m}$	$m^2_{\tilde{z}^{\alpha}}$
value	$24g_1^2$	0	0

vectors

mass	m^2_{A}	$m^2_{A_{mn}}$
value	$24g_1^2v^2$	0

fermions

mass	m^2_{Λ}	$m^2_{\psi_{L\alpha}}$	$m^2_{\chi_{Ln}}$
value	$24g_1^2v^2$	0	0

Table 7.6: Soft supersymmetry breaking gauged $SO(10) \times U(1)$ mass spectrum

Chapter 8

Conclusions

I was born not knowing and have had only a little time to change that here and there.

Richard Feynman

This thesis describes some applications of non-linear supersymmetric field theories. The first one is an extension of relativistic hydrodynamics. Fluid dynamics is usually described by equations of motion. It is highly non-trivial that there exists an action (even for dissipationless fluids) from which these equations can be derived. The Clebsch formalism has the advantage that the equations of motion may be derived from an action principle.

In the first part of this thesis we have proposed a different parameterization in terms of one real and one complex degree of freedom. The complex degree of freedom z takes its values on a Kähler manifold. In this formulation the infinite set of conserved currents $J_\mu[M]$ of fluid dynamics are associated with the set of functions M of this complex variable z and its conjugate $\bar{z}$. A canonical analysis using the Poisson-Dirac bracket shows, that the closure of the algebra of conserved currents leads to a Poisson bracket structure for these functions on the Kähler manifold.

However, the main advantage of our Kähler parameterization of the fluid current is, that it allows for a rather straightforward supersymmetric completion. Understanding of fluids that posses supersymmetric properties may be relevant for cosmological applications as they could be used to describe a supersymmetric phase in the early universe. Therefore we have presented a complete formulation of a particular supersymmetric field theory on an arbitrary Kähler manifold. We have started with the construction of the lagrangeans then, discussed various aspects of these general constructions. The supercurrents and the energy-momentum tensor that follow from the invariance of the action under supersymmetry and translation are constructed. Assuming that there exists an isometry group G which leaves the lagrangean invariant, we have constructed the corresponding conserved isometry currents, in terms of Killing vectors $R^\alpha(z)$. Next, we discussed in details the canonical formulation of the theory in terms of a hamiltonian. We have explicitly constructed the canonical supercharges. We have shown that these supercharges generate the supersymmetry transformations, and satisfy the standard super-Poincaré algebra.

Having completed the construction of the conserved quantities, we found that the supersymmetric lagrangian derived using tensor calculus, does not automatically describe a supersymmetric extension of relativistic mechanics. The langrangian contains not only a conserved current, but also a number of additional fields which complicate the interpretation. However, we have shown that an infinite set of conserved currents emerges in the vacuum sector of the additional fields. This sector can therefore be identified with a regime of supersymmetric fluid mechanics.

The supersymmetric hydrodynamics models discussed in this thesis still leave a lot of room for future research to improve and extend it. For example, we have stayed mostly in the classical framework. Consistent extensions of the models at the quantum level are of potential interest in cosmology, where they could provide an effective description of an early supersymmetric phase of the universe, and in condensed matter physics, where they might apply to quantum fluids like ^{3}He-^{4}He mixtures, in the limit where terms proportional to the mass-differences of these isotopes can be neglected. In quantum theory models involving current of chiral fermions can suffer from anomalies. Therefore, quantization of these models should be studied very carefully.

In the second part of this thesis, we have discussed in detail the phenomenological analysis of supersymmetric σ-models based on homogeneous cosets. We have focused in particular on coset-spaces $E_6/[SO(10)\times U(1)]$ and $SO(10)/U(5)$ which are some of the most interesting for (direct) phenomenology. We have analyzed the possible vacuum configurations of these models. We have investigated in particular the existence of the zeros of the potential, for which the models are anomaly-free, with positive definite kinetic energy. The consequences of this physical requirements have been analyzed for supersymmetric minima, if part of the isometry group is gauged. If the whole isometry group is gauged, the analysis is straightforward as one can employ the unitary gauge to put the Goldstone bosons to zero. We find the kinetic energy of the would-be Goldstone modes and their fermionic partners to vanish. Therefore, the quasi-Goldstone fermions no longer contribute to the cancellation of anomalies.

To gain insight in this problem we have studied this phenomenon for a simple supersymmetric model based on the homogeneous space CP^1. It has two types of metric singularities: either the kinetic terms for the Goldstone scalar z or the matter scalar a (that have fermionic partners needed to cancel the isometry anomalies) become zero. We have shown by general arguments and by regularization based upon the addition of soft supersymmetry breaking terms to the potential, that different types of behaviour are possible. For example, the subset of linear gauge symmetries can be realized manifestly, or in a spontaneously broken mode; this is reflected in the mass of the corresponding vector bosons.

In some cases the singularities imply the vanishing of all vector boson masses, which one expects to be accompanied by the reappearance of light bosons in the physical spectrum. However, this is difficult to show while staying in the original framework, as in particular it invalidates the use of the unitary gauge. Of course, one could abandon the present approach and return to ordinary Yang-Mills theories with matter in linear representations; however, a more sophisticated formulation of the present models is possible and may shed light on this issue [103].

Continuing our line of investigation of the particle spectrum of supersymmetric σ-models on $E_6/[SO(10)\times U(1)]$, and $SO(10)\times U(1)$, we have also studied the possibility of gauging (part of) the linear subgroups, i.e., $SO(10)\times U(1)$ and $U(5)$. In each of these models, we found that the properties of the model investigated depend to a certain extent on the value of parameters (gauge couplings, Fayet-Iliopoulos term) and the presence of extra families and Higgses. We have obtained all supersymmetric minima, of which some are physically problematic as the kinetic terms of the Goldstone multiplets either vanish or have negative values. In addition to zeros of the potentials, there can be also ranges of the parameters, or models with only some proper subgroup gauge for which the minimum of the potential occurs at a positive value: $\langle V \rangle > 0$. In this case supersymmetry is manifestly broken by the potential. In the $SO(10)/U(5)$-spinor model we have classified all the supersymmetry breaking vacua for the case where $U(5)$ is gauged.

In spite of all these nice features, there is still a lot of work needed to improve and extend the anomaly-free supersymmetric σ-models on the coset spaces $SO(10)/U(5)$ and $E_6/[SO(10) \times U(1)]$ discussed in the last two chapters. First, our treatment of the σ-models is based firmly on the classical action, although some features of these models were motivated by quantum aspects like the absence of holonomy and gauge anomalies. Higher order quantum corrections [102] may change the behaviour of the models by renormalization of the Kähler potential.

Another extension of interest would be to study their particle spectrum in presence of extra families and Higgses. In tables 6.3 and 7.5 we have summarized the most general scalar field content we consider for the phenomenological promising models build around the coset spaces $SO(10)/U(5)$ and $E_6/[SO(10) \times U(1)]$.

Finally, in these supersymmetric coset models all or some of the quarks and leptons are quasi-Goldstone fermions, or equivalently set of unpaired chiral gauginos, lacking physical scalar partners. This strong form of spontaneous supersymmetry breaking clearly distinguishes the physical content of these models from the MSSM or standard supersymmetric GUTs. Of course, this difference affects scenarios of gauge unification and the role of supersymmetry in the solution of the hierarchy problem [6, 7], although to what extent is presently not clear.

Samenvatting

Dit proefschrift "Niet–lineaire veldentheorie met supersymmetrie: hydrodynamica en sigma modellen" beschrijft enkele toepassingen van niet-lineaire supersymmetrische veldentheoriën. De eerste toepassing betreft een uitbreiding van relativistische hydrodynamica. Hydrodynamica wordt gewoonlijk beschreven met behulp van de bewegingsvergelijkingen. Het is verre van triviaal dat er een actie bestaat (zelfs voor dissipatieloze vloeistoffen) waar deze bewegingsvergelijkingen van afgeleid kunnen worden. Het formalisme van Clebsch heeft als voordeel dat de bewegingsvergelijkingen kunnen worden afgeleid van het principe van minimale actie.

In het eerste deel van dit proefschrift stellen we een andere parametrisatie met een reële en een complexe vrijheidsgraad voor. De complexe vrijheidsgraad z heeft zijn waarde op een Kähler variëteit. In deze formulering is de oneindige verzameling van behouden stromen $J_\mu[M]$ van de hydrodynamica geassocieerd met de set functies M van deze complexe variabele z en zijn geconjugeerde $\bar{z}$. Een canonieke analyse met behulp van het Poisson-Dirac haakje laat zien dat het sluiten van de algebra van behouden stromen leidt tot een Poisson haakjes structuur voor deze functies op de Kähler variëteit.

Het grote voordeel van onze Kähler parametrisatie van de vloeistofstroom is dat deze een relatief eenvoudige supersymmetrische uitbreiding toestaat. De kennis van vloeistoffen die supersymmetrische eigenschappen hebben, kan relevant zijn voor kosmologische toepassingen in de beschrijving van een supersymmetrische fase van een jong heelal. Daarom hebben we een complete formulering van een bepaalde supersymmetrische veldentheorie op een willekeurig Kähler variëteit gepresenteerd. We zijn begonnen met de constructie van de langrangianen en hebben vervolgens de verschillende aspecten van deze algemene constructies besproken. De superstromen en de energie-impuls tensor die volgen uit invariantie van de actie onder supersymmetrie en translatie zijn geconstrueerd. Aangenomen dat een isometrie groep G bestaat, die de langrangiaan invariant laat, hebben we de bijbehorende behouden stromen in termen van Killing vectoren $R^\alpha(z)$ geconstrueerd. Vervolgens is de kanonieke formulering van de theorie in termen van een hamiltoniaan tot in details besproken en hebben we expliciet de kanonieke superladingen geconstrueerd. Het is aangetoond dat deze superladingen supersymmetrische transformaties genereren en aan de standaard super-Poincaré algebra voldoen.

Na de constructie van de behouden grootheden voltooid te hebben, vonden we dat de supersymmetrische langrangiaan die afgeleid is met behulp van tensor algebra niet automatisch een supersymmetrische uitbreiding van de relativistische vloeistof mechanica beschrijft. De langrangiaan bevat niet alleen een behouden stroom, maar ook een aantal extra velden die de interpretatie compliceren. Daarentegen hebben we aangetoond dat een oneindig aantal behouden stromen in de vacuum sector van de extra velden opduiken.

Deze sector kan daarom geïdentificeerd worden met een regime van supersymmetrische hydrodynamica.

De supersymmetrische hydrodynamica modellen zoals beschreven in dit proefschrift kunnen door verder onderzoek verbeterd en uitgebreid worden. Zo is in dit proefschrift grotendeels binnen een klassiek raamwerk gebleven. Consistente uitbreidingen van deze modellen tot het quantum niveau kunnen interessant zijn voor kosmologie, alwaar ze een effectieve beschrijving van een vroege supersymmetrische fase van het heelal kunnen geven. Binnen het gebied van de vaste stof-fysica kunnen zulke uitbreidingen toepasbaar zijn op quantum vloeistoffen zoals een ^{3}He-^{4}He mengsel, in de limiet waar termen evenredig met het massaverschil van de isotopen verwaarloosd kunnen worden. Modellen met stromen van chirale fermionen kunnen anomaliën vertonen in de quantum theorie, waardoor de quantisatie van zulke modellen nauwkeurig bestudeerd dient te worden.

In het tweede gedeelte van dit proefschrift hebben we de fenomenologische analyse van de supersymmetrische σ-modellen besproken, die op homogene cosets gebaseerd zijn. We hebben de nadruk gelegd op coset-ruimtes $E_6/[SO(10)\times U(1)]$ en $SO(10)/U(5)$, die behoren tot de meest interessante ruimtes voor fenomenologie. We hebben de mogelijke vacuum configuraties van deze modellen geanalyseerd. In het bijzonder hebben we het bestaan van nulpunten van de potentiaal bestudeerd, voor welke de modellen anomalie-vrij zijn, met een positief definiet kinetische energie. De gevolgen van deze fysische eisen zijn geanalyseerd voor supersymmetrische minima, als een gedeelte van de isometrie groep is geijkt. De analyse is triviaal als de gehele groep is geijkt, omdat men de unitaire ijk kan gebruiken om de Goldstone bosonen gelijk aan nul te stellen. We vonden dat de kinetische energie van deze bosonen en hun fermionische partners verdwijnt. Daarom dragen de quasi-Goldstone fermionen niet langer bij tot de uitschakeling van anomaliën.

Om inzicht te verwerven in dit probleem hebben we dit fenomeen bestudeerd voor een eenvoudig supersymmetrisch model, dat gebaseerd is op de homogene ruimte CP^1. Het heeft twee type metrische singulariteiten: een van de kinetische termen voor de Goldstone scalar z en de materie scalar a (die fermioniche partners nodig hebben om de isometrie anomaliën te doen verdwijnen) wordt nul. We hebben aangetoond met algemeen geldende argumenten en met regularisatie (die gebaseerd is op zachte supersymmetrie brekings termen die in de potentiaal zijn toegevoegd) dat verschillende type gedragingen mogelijk zijn. Bijvoorbeeld, de deelverzameling van lineaire ijksymmetriën kan manifest gerealiseerd worden, of op een spontaan gebroken wijze; dit is weerspiegeld in de massa van de corresponderende vector bosonen.

In sommige gevallen houden de singulariteiten het verdwijnen van alle vector boson massa's in, die men verwacht als lichte bosonen in het fysische spectrum voorkomen. Maar dit is moeilijk aan te tonen binnen het orginele raamwerk, omdat het -in het bijzonder- het gebruik van de unitaire ijk ongeldig maakt. Natuurlijk kan men van de huidige benadering afstappen en teruggaan naar gewone Yang-Mills theoriën met materie in lineaire representaties. Maar een meer geavanceerde formulering van de huidige modellen is mogelijk en kan licht werpen op dit onderwerp [103].

Vervolgend met de lijn van het onderzoek naar het deeltjes spectrum van de supersymmetrische σ-modellen op $E_6/[SO(10) \times U(1)]$ en $SO(10) \times U(1)$, hebben we ook de mogelijkheid van ijking van (een gedeelte van) de lineaire subgroepen bestudeerd, dus

$SO(10) \times U(1)$ en $U(5)$. We vonden dat in elk van deze modellen de eigenschappen in zekere mate afhankelijk zijn van de waarde van parameters (ijk koppelingen, Fayet-Iliopoulos termen) en het bestaan van extra families en Higgsvelden. We hebben alle supersymmetrische minima verkregen, waarvan er sommige fysisch problematisch zijn, omdat de kinetische termen van de Goldstone multipletten hetzij verdwijnen, hetzij een negatieve waarde hebben. Naast de nulpunten van de potentiaal kunnen er ook gebieden van parameters of modellen zijn met alleen een bepaalde deelgroep ijking waarvoor het minimum van de potentiaal bestaat met een positieve waarde: $\langle V \rangle > 0$. In dat geval is supersymmetrie manifest gebroken door de potentiaal. In de $SO(10)/U(5)$-spinor model hebben we alle supersymmetrische brekings-vacua geklassificeerd voor het geval dat $U(5)$ geijkt is.

Ondanks al deze goede eigenschappen is er nog veel werk te doen aan verbetering en uitbreiding van de anomalie-vrije supersymmetrische σ-modellen op de coset ruimtes $SO(10)/U(5)$ en $E_6/[SO(10) \times U(1)]$ die beschreven staan in de laatste twee hoofdstukken. Ten eerste is onze aanpak van de σ-modellen gebaseerd op de klassieke actie, terwijl een aantal eigenschappen van deze modellen gemotiveerd zijn door quantum aspecten zoals de afwezigheid van holonomie en ijkanomalieën. Hogere orde quantum correcties [102] kunnen het gedrag veranderen van de modellen via renormalisatie van de Kähler potentiaal.

Een andere interessante uitbreiding zou de studie zijn naar hun deeltjes spectrum bij toevoeging van extra families en Higgsvelden. In tabellen 6.3 en 7.5 hebben we een samenvatting gegeven van de inhoud van het meest algemene scalarveld, dat we hebben beschouwd voor fenomenologisch belovende modellen die gebouwd zijn op de coset ruimtes $SO(10)/U(5)$ en $E_6/[SO(10) \times U(1)]$.

Tenslotte dient gezegd te worden dat in deze supersymmetrische coset modellen alle of sommige quarks en leptonen hetzij quasi-Goldstone fermionen zijn, hetzij een set van chirale gauginos, zonder fysische scalar partners. Vanwege deze sterke variant van spontane symmetrie breking is het fysische spectrum van deze modellen duidelijk anders dan dat van het minimaal supersymmetrische standaard-model of standaard supersymmetrische unificatie theorieën. Het is duidelijk dat dit verschil van invloed is op ijk unificatie modellen en op de rol van supersymmetrie als oplossing voor het hiërarchie probleem [6, 7], hoewel op dit moment niet duidelijk is in welke mate.

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Friendship is almost always the union of a part of one mine with a part of another; people are friends in sport.

George Santayana.

I guess this section of a thesis should be like a 21st or wedding speech. It should be sweet, make me think deeply about my life in last four years.

Basically, writing acknowledgements is not the sort of thing one typically does very often, unless one writes a lot of thesis. I am indebted to many people who helped me in one way or another with the completion of this thesis. Without their help, this thesis would not have been possible.

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Appendix A

Notations and conventions

In the literature one finds enormous variations in the notational conventions. It is often the case that one single object can have two or more different meanings. The purpose of this appendix is to try to disentangle this as much as possible.

For units we use so-called natural units, in which $\hbar = c = 1$. Concerning (anti-)symmetrization of indices we use the following conventions. Symmetrization of objects enclosed is denoted by braces $\{\ldots\}$, anti-symmetrization by the square brackets $[\ldots]$; the total weight of such (anti-)symmetrization is always unity. However, in chapters 2 and 3, we reformulated the theory in a hamiltonian formulation. To avoid a confusion, we have used the braces $\{,\}$ to denote the Poisson brackets, $\{,\}^*$ to denote Poisson-Dirac brackets. The Minkowski metric $g_{\mu\nu}$ has a signature $(-1,+1,+1,+1)$.

A.1 Majorana and Weyl spinors

Spinors are defined as the elements of the representation space of the $SO(4)$-algebra defined by the antisymmetrized product of Dirac-matrices (the irreducible representation of the 4-d Clifford-algebra):

$$\sigma_{\mu\nu} = \frac{1}{4}\left[\gamma_\mu, \gamma_\nu\right], \tag{A.1}$$

which satisfy the commutation relations

$$[\sigma_{\mu\nu}, \sigma_{\kappa\lambda}] = \delta_{\nu\kappa}\sigma_{\mu\lambda} + \delta_{\mu\lambda}\sigma_{\nu\kappa} - \delta_{\nu\lambda}\sigma_{\mu\kappa} - \delta_{\mu\kappa}\sigma_{\nu\lambda}. \tag{A.2}$$

We take the Dirac matrices γ_μ with index $\mu = (1,2,3,4)$ to be euclidean and hermitean, hence the above representation of the $SO(4)$-algebra is anti-hermitean. To use these Dirac-matrices in a Lorentz-covarariant context, one should take $\gamma_4 = i\gamma_0$.

The hermitean γ_5-operator is defined as the pseudoscalar product of the Dirac matrices:

$$\gamma_5 = \frac{1}{4!}\,\varepsilon^{\mu\nu\kappa\lambda}\,\gamma_\mu\gamma_\nu\gamma_\kappa\gamma_\lambda = \gamma_1\gamma_2\gamma_3\gamma_4. \tag{A.3}$$

As $\gamma_5^2 = 1$ the operators

$$P_\pm = \frac{1 \pm \gamma_5}{2}, \tag{A.4}$$

are projection operators; their eigenspinors are called right- and left-handed Weyl spinors, respectively; we use the notation $\theta_\pm$:

$$\gamma_5\, \theta_\pm \,=\, \pm\theta_\pm, \quad \theta_+ \,\equiv\, \theta_R, \quad \theta_- \,\equiv\, \theta_L. \tag{A.5}$$

The Dirac-conjugate spinors $\bar{\theta}$ are defined as

$$\bar{\theta} \,=\, \theta^\dagger \gamma_4 \,=\, i\theta^\dagger \gamma_0. \tag{A.6}$$

This definition is important in the construction of Lorentz-invariant inner products. We use the same labelling for left-eigenspinors of γ_5 as for right-eigenspinors:

$$\bar{\theta}_\pm\, \gamma_5 \,=\, \pm\, \bar{\theta}_\pm, \quad \bar{\theta}_+ \,\equiv\, \bar{\theta}_L, \quad \bar{\theta}_- \,\equiv\, \bar{\theta}_R. \tag{A.7}$$

(The notations (L, R) are current in the literature, but not always convenient in displaying equations). Note, that

$$\bar{\theta}_\pm \,=\, (\theta_\mp)^\dagger\, \gamma_4. \tag{A.8}$$

This is because multiplication with γ_4 flips the γ_5 eigenvalue of Weyl spinor (its handedness): $\gamma_4\gamma_5 = -\gamma_5\gamma_4$.

The charge-conjugation operator C defined in the spinor space is defined by the properties:

$$C \,=\, C^\dagger \,=\, C^{-1} \,=\, -\, C^T, \quad C\gamma_\mu C^{-1} \,=\, -\, \gamma_\mu^T, \tag{A.9}$$

where the superscript T denotes transposition in spinor space. From this it follows, that

$$C\gamma_5 C^{-1} \,=\, \gamma_5^T, \quad C\sigma_{\mu\nu} C^{-1} \,=\, -\sigma_{\mu\nu}^T. \tag{A.10}$$

The charge-conjugate of a spinor θ is defined as

$$\theta^{\,c} \,=\, C\, \bar{\theta}^T. \tag{A.11}$$

A Majorana spinor is equal to its charge conjugate: $\theta^{\,c} \,=\, \theta$. Because of eq.(A.8) a 4-d spinor can never be Weyl and Majorana at the same time. Another way to see this, is by noting that multiplication of a Majorana spinor θ by γ_5 gives an anti-Majorana spinor:

$$\theta_5 = \gamma_5 \theta \quad \Rightarrow \quad \theta_5 \,=\, -C\, \bar{\theta}_5^T. \tag{A.12}$$

Of course $i\gamma_5\theta$ then is a Majorana spinor again. We can convert Majorana and Weyl spinors into each other by applying the chiral projection operators $P_\pm$:

$$\begin{aligned} \theta_\pm \;&=\; \frac{1 \pm \gamma_5}{2}\, \theta, \quad \theta \,=\, \theta^c \\ \Rightarrow \quad \theta \,=\, \theta_+ + \theta_- \quad &\text{and} \quad \theta_\pm^c \,=\, \theta_\mp \,=\, C\, \bar{\theta}_\mp^T. \end{aligned} \tag{A.13}$$

Combining the last two equations, it is easily established that for a Majorana spinor $\theta = \theta_+ + \theta_+^c$, which is manifestly self charge-conjugate. Because of this possibility to switch from

Majorana to Weyl spinors and vice versa, it is obviously a matter of convenience which type of spinors to use as a basis for constructing spinorial expressions in 4-d space-time.

Using the previous definitions one now shows (by taking the transposition of a scalar)

$$\begin{aligned}\bar{\psi}_\pm \mathbb{1} \chi_\pm &= \bar{\chi}_\pm \, \mathbb{1} \, \psi_\pm, \qquad \bar{\psi}_\pm \gamma_\mu \, \chi_\pm = -\bar{\chi}_\pm \, \gamma_\mu \, \psi_\pm, \\ \bar{\psi}_\pm \gamma_5 \chi_\pm &= \bar{\chi}_\pm \, \gamma_5 \, \psi_\pm, \quad \bar{\psi}_\pm \, \sigma_{\mu\nu} \, \chi_\pm = -\bar{\chi}_\pm \, \sigma_{\mu\nu} \, \psi_\pm. \end{aligned} \tag{A.14}$$

Hermitian conjugation on bispinor reverses by definitions the order of the spinors $(\psi \, \chi)^\dagger = \chi^\dagger \, \psi^\dagger$ with no minus sign. Using this one can show that h.c. replaces $i \leftrightarrow -i$ and $+ \leftrightarrow -$. In particular

$$(\bar{\psi}_\pm \, \chi_\pm)^\dagger = \Big(\bar{\psi} \frac{1 \pm \gamma_5}{2} \chi\Big)^\dagger = \chi^\dagger \frac{1 \pm \gamma_5}{2} \gamma_4 \, \psi = \bar{\chi} \frac{1 \mp \gamma_5}{2} \, \psi = \bar{\psi}_\mp \, \chi_\mp, \tag{A.15}$$

$$(\bar{\psi}_\pm \, \gamma^\mu \partial_\mu \, \chi_\mp)^\dagger = \bar{\psi}_\mp \, \gamma^\mu \partial_\mu \, \chi_\pm. \tag{A.16}$$

A.2 Fierz-rearrangements

The 16 matrices $\Gamma_J = (\mathbb{1}, \gamma_\mu, i\sqrt{2}\, \sigma_{\mu\nu}, i\gamma_5\gamma_\mu, \gamma_5)$ define a complete and independent basis for all 4×4-matrices. If X is any such matrix, one can write the following identity due to Fierz:

$$X = \frac{1}{4} \sum_J \mathrm{Tr}\Big(X \Gamma_J\Big) \Gamma_J. \tag{A.17}$$

This relation can be used in particular for a decomposition of the direct product of two spinors. As an illustration, take two anti-commuting Majorana spinors (ψ, χ); the completeness of the basis Γ_J then implies

$$\psi \, \bar{\chi} = \frac{1}{4} \sum_J \mathrm{Tr}\Big(\Gamma_J \psi \, \bar{\chi}\Big) \Gamma_J = -\frac{1}{4} \sum_J \mathrm{Tr}\Big(\bar{\chi} \, \Gamma_J \, \psi\Big) \, \Gamma_J. \tag{A.18}$$

Note the minus sign, which originates in the interchange of the order of the anti-commuting spinors on the r.h.s. of the equation. For chiral Weyl spinors, the relations split into two smaller sets:

$$\begin{aligned} \psi_\pm \bar{\chi}_\pm &= -\frac{1}{2} \, (\bar{\chi}_\pm \psi_\pm) \; P_\pm \; + \; (\bar{\chi}_\pm \sigma^{\mu\nu} \psi_\pm) \; P_\pm \sigma_{\mu\nu}, \\ \psi_\pm \bar{\chi}_\mp &= -\frac{1}{2} \, (\bar{\chi}_\mp \gamma^\mu \psi_\pm) \; P_\pm \gamma_\mu. \end{aligned} \tag{A.19}$$

These equations are easily derived by applying the chiral projection operators $P_\pm$ to eq.(A.18).

Appendix B

Kähler geometry

We mention several times in this thesis that supersymmetric models in four dimensions require the target manifold of scalar fields to be a Kähler manifold. In this appendix we summarize the most important results concerning Kähler manifolds used in this thesis; for a more detailed and complete discussion of Kähler manifolds see [22, 23, 93].

Manifolds are described locally using geometrical objects like the metric and the curvature. A Kähler manifold is a complex manifold, parametrized locally by N complex coordinates Z^α and their complex conjugates $\bar{Z}^{\underline{\alpha}}$ $(\alpha, \underline{\alpha} = 1, \dots, N)$ on which a real line element can be defined by

$$\mathrm{d}s^2 = G_{\alpha\underline{\alpha}}\,\mathrm{d}\bar{Z}^{\underline{\alpha}}\,\mathrm{d}Z^\alpha, \tag{B.1}$$

with a hermitian metric $G_{\alpha\underline{\alpha}}$, and

$$G_{\alpha\beta} = G_{\underline{\alpha}\underline{\beta}} = 0. \tag{B.2}$$

The hermitian metric $G_{\alpha\underline{\alpha}}$ is said to be Kählerian if the corresponding Kähler 2-form $\omega = -iG_{\alpha\underline{\beta}}\,\mathrm{d}\bar{Z}^{\underline{\beta}} \wedge \mathrm{d}Z^\alpha$ is closed,

$$\begin{aligned} \mathrm{d}\omega \;\; = \;\; & -\frac{i}{2}\Big(G_{\alpha\underline{\beta},\gamma} - G_{\gamma\underline{\beta},\alpha}\Big)\mathrm{d}Z^\gamma \wedge \mathrm{d}Z^\alpha \wedge \mathrm{d}\bar{Z}^{\underline{\beta}} \\ & -\frac{i}{2}\Big(G_{\alpha\underline{\beta},\underline{\gamma}} - G_{\alpha\underline{\gamma},\underline{\beta}}\Big)\mathrm{d}Z^\alpha \wedge \mathrm{d}\bar{Z}^{\underline{\beta}} \wedge \mathrm{d}\bar{Z}^{\underline{\gamma}} = 0. \end{aligned} \tag{B.3}$$

As usual, the comma denotes differentiation with respect to Z^α, $\bar{Z}^{\underline{\alpha}}$. The requirement (B.3) is equivalent to the equations

$$G_{\alpha\underline{\beta},\gamma} = G_{\gamma\underline{\beta},\alpha}, \quad G_{\alpha\underline{\beta},\underline{\gamma}} = G_{\alpha\underline{\gamma},\underline{\beta}}. \tag{B.4}$$

Locally, the metric can be derived from a scalar potential $\mathcal{K}(Z, \bar{Z})$, the Kähler potential, as a second mixed derivative with respect to Z^α and $\bar{Z}^{\underline{\alpha}}$

$$G_{\alpha\underline{\alpha}} = \mathcal{K}_{,\alpha\underline{\alpha}}. \tag{B.5}$$

The affine connection is defined through the metric postulate

$$D_\alpha G_{\beta\underline{\gamma}} = G_{\beta\underline{\gamma};\alpha} = G_{\beta\underline{\gamma},\alpha} - \Gamma^\delta_{\alpha\beta} G_{\delta\underline{\gamma}} - \Gamma^{\underline{\delta}}_{\alpha\underline{\gamma}} G_{\beta\underline{\delta}} = 0, \tag{B.6}$$

and the conjugate equation. The properties (B.1)-(B.6) imply the following relations for the connection coefficients:

$$\Gamma^{\alpha}_{\beta\gamma} = G^{\alpha\underline{\alpha}}\, G_{\underline{\alpha}\beta,\gamma}, \quad \bar{\Gamma}^{\underline{\alpha}}_{\underline{\beta}\underline{\gamma}} = G^{\underline{\alpha}\alpha}\, G_{\alpha\underline{\beta},\underline{\gamma}}, \tag{B.7}$$

with $G^{\alpha\underline{\alpha}}$ the inverse of the metric $G_{\alpha\underline{\alpha}}$, while the mixed coefficients vanish:

$$\Gamma^{\alpha}_{\beta\underline{\gamma}} = 0, \qquad \bar{\Gamma}^{\underline{\alpha}}_{\beta\gamma} = 0. \tag{B.8}$$

Moreover, eq. (B.4) implies the vanishing of the torsion:

$$\Gamma^{\alpha}_{\beta\gamma} = \Gamma^{\alpha}_{\gamma\beta}. \tag{B.9}$$

The curvature tensor, has essentially only one independent type of components:

$$R_{\underline{\beta}\alpha\underline{\delta}\gamma} = G_{\gamma\underline{\gamma}}\, \Gamma^{\delta}_{\alpha\gamma,\underline{\beta}} = R_{\alpha\underline{\beta}\gamma\underline{\delta}} = G_{\gamma\underline{\gamma}}\, \bar{\Gamma}^{\underline{\gamma}}_{\underline{\beta}\underline{\delta},\alpha} = G_{\alpha\underline{\beta},\gamma\underline{\delta}} - G_{\alpha\underline{\sigma},\gamma} G^{\sigma\underline{\sigma}} G_{\underline{\beta}\sigma,\underline{\delta}}. \tag{B.10}$$

The other components vanish, for example

$$R_{\alpha\beta}{}^{\delta}{}_{\gamma} = \Gamma^{\delta}_{\beta\gamma,\alpha} - \Gamma^{\delta}_{\alpha\gamma,\beta} - \Gamma^{\delta}_{\gamma[\alpha}\Gamma^{\delta}_{\beta]\gamma} = 0. \tag{B.11}$$

The contracted connection and the Rici tensor are, as always

$$\Gamma^{\beta}_{\beta\alpha} = G^{\beta\underline{\beta}}\, G_{\underline{\beta}\alpha,\beta} = (\ln\det G)_{,\alpha}, \quad R_{\alpha\underline{\alpha}} = G^{\beta\underline{\beta}} R_{\alpha\underline{\alpha}\beta\underline{\beta}} = (\Gamma^{\beta}_{\alpha\beta})_{,\alpha} = (\ln\det G)_{,\alpha\underline{\alpha}}. \tag{B.12}$$

B.1 Isometries of Kähler manifolds

Having discussed the geometry of Kähler manifolds, we now discuss their symmetries. The complex manifolds we are considering are invariant under sets of holomorphic coordinate transformations. Such transformations that leave the metric invariant are called *isometries*. On a Kähler manifold, an isometry of the metric is, in general, an invariance of the Kähler potential modulo holomorphic functions $F(Z)$ and $\bar{F}(\bar{Z})$

$$\mathcal{K}(Z,\bar{Z}) \rightarrow \mathcal{K}'(Z,\bar{Z}) = \mathcal{K}(Z,\bar{Z}) + F(Z) + \bar{F}(\bar{Z}). \tag{B.13}$$

This transformation is called a Kähler transformation. If two complex local coordinate charts $\{Z_i\}$ and $\{Z_j\}$ have non-empty overlap, the Kähler potential in the charts are generally related by

$$\mathcal{K}_i(Z_i,\bar{Z}_i) = \mathcal{K}_j(Z_j,\bar{Z}_j) + F_{(ij)}(Z_j) + \bar{F}_{(ij)}(\bar{Z}_j). \tag{B.14}$$

In an arbitrary coordinate system, the isometry δZ^{α} (and its complex conjugate) of the metric

$$\delta Z^{\alpha} = \Theta^i R_i^{\alpha}(Z), \quad \delta\bar{Z}^{\underline{\alpha}} = \Theta^i \bar{R}_i^{\underline{\alpha}}(\bar{Z}) \tag{B.15}$$

is generated by a Killing vector $R_i^{\alpha}(Z)$, with Θ^i the parameters of the infinitesimal transformations. A necessary and sufficient condition for the transformation to be a Killing vector on a Kähler manifold is the Killing equation:

$$(G_{\beta\underline{\beta}} R_i^{\beta})_{,\alpha} + (\bar{R}_i^{\underline{\alpha}} G_{\alpha\underline{\alpha}})_{,\underline{\beta}} = 0. \tag{B.16}$$

The Killing vectors form a Lie algebra:

$$R^{\beta}_{[i}\,R^{\alpha}_{j],\beta} = R^{\beta}_{i}\,R^{\alpha}_{j,\beta} - R^{\beta}_{j}\,R^{\alpha}_{i,\beta} = f_{ij}{}^{k}R^{\alpha}_{k}. \tag{B.17}$$

Thus, infinitesimal transformations (B.15) define a (generally non-linear) representation of some Lie group G, called the isometry group of the manifold. The $f_{ij}{}^{k}$ are structure constants of the algebra. As it is well-known [24, 70], the holomorphic Killing vectors can be derived from a set of real potentials $M_i(Z,\bar{Z})$ such that

$$R^{\alpha}_{i} = -iG^{\alpha\underline{\alpha}}M_{i,\underline{\alpha}}, \quad \bar{R}^{\underline{\alpha}}_{i} = iG^{\alpha\underline{\alpha}}M_{i,\alpha}. \tag{B.18}$$

From these equations, one sees that the Killing potentials M_i are defined up to an integration constant c_i. It turns out that one can always choose these c_i in such a way that the potentials M_i transform in the adjoint representation of the isometry group:

$$\delta_i M_j = R^{\alpha}_{i}M_{j,\alpha} + \bar{R}^{\underline{\alpha}}_{i}M_{j,\underline{\alpha}} = -iG_{\alpha\underline{\alpha}}\Big(R^{\alpha}_{i}\bar{R}^{\underline{\alpha}}_{j} - R^{\alpha}_{j}\bar{R}^{\underline{\alpha}}_{i}\Big) = f_{ij}{}^{k}M_k. \tag{B.19}$$

Under the transformation (B.15) the Kähler potential itself transforms as

$$\delta_i\mathcal{K} = F_i(Z) + \bar{F}_i(\bar{Z}). \tag{B.20}$$

Now it can be shown that the functions F^i, $\bar{F}^i$ defined by

$$F_i = \mathcal{K}_{,\alpha}R^{\alpha}_{i} + iM_i, \quad \bar{F}_i = \mathcal{K}_{,\underline{\alpha}}\bar{R}^{\underline{\alpha}}_{i} - iM_i, \tag{B.21}$$

are holomorphic:

$$F_{i,\underline{\alpha}} = 0, \qquad \bar{F}_{i,\alpha} = 0. \tag{B.22}$$

From the Lie-algebra (B.17) it follows that one can choose the transformations of the functions $F_i(Z)$ to have the property

$$\delta_i F_j - \delta_j F_i = f_{ij}{}^{k}F_k. \tag{B.23}$$

B.2 Killing identity

In this appendix we prove the identity (3.12), which we have used to show that the currents (3.9) are divergence free. This identity follows upon using first the following result

$$\bar{R}^{i}_{\alpha;\underline{\alpha};\underline{\beta}} = -R_{\alpha\underline{\alpha}\beta\underline{\beta}}\,R^{i\beta}, \tag{B.24}$$

which is obtained from the Killing equation (B.16) by acting upon it with second covariant derivative, use of the Ricci identity

$$[D_{\underline{\beta}}, D_{\alpha}]\,R^{i}_{\underline{\alpha}} = R_{\alpha\underline{\beta}}{}^{\underline{\delta}}{}_{\underline{\alpha}}R^{i}_{\underline{\delta}}, \tag{B.25}$$

and observing that the Killing $R^{i\alpha}(z)$ are holomorphic: $R^{i}_{\underline{\alpha};\beta} = 0$. Furthermore, we have used the metric postulate (B.6). Next we act gain on (B.24) with an other covariant derivative D_δ. By shifting the covariant derivative to let it act on the Killing vectors $\bar{R}^i_\alpha(\bar{z})$ we obtain

$$(R_{\underline{\alpha}\alpha\underline{\gamma}\delta}\,\bar{R}^{i\underline{\gamma}})_{;\beta} + (R_{\alpha\underline{\alpha}\beta\underline{\beta}}\,R^{i\beta})_{;\delta} = -[D_\delta, D_{\underline{\beta}}]\,\bar{R}^i_{\alpha;\underline{\alpha}}. \tag{B.26}$$

Upon further use of the Killing equation (B.16), the right hand side of (B.26) becomes

$$-[D_\delta, D_{\underline{\beta}}]\,\bar{R}^i_{\alpha;\underline{\alpha}} = \bar{\Gamma}^{\underline{\gamma}}_{\underline{\beta}\underline{\alpha},\delta}\,\bar{R}^i_{\alpha,\underline{\gamma}} - \Gamma^{\gamma}_{\delta\alpha,\underline{\beta}}\,\bar{R}^i_{\gamma,\underline{\alpha}} = -R_{\delta\underline{\beta}\gamma\underline{\alpha}}\,R^{i\underline{\gamma}}_{;\alpha} - R_{\underline{\beta}\delta\underline{\gamma}\alpha}\,\bar{R}^{i\underline{\gamma}}_{;\underline{\alpha}} \tag{B.27}$$

As a result, the final expression we obtain is

$$(R_{\underline{\alpha}\alpha\underline{\gamma}\delta}\,\bar{R}^{i\underline{\gamma}})_{;\underline{\beta}} + (R_{\alpha\underline{\alpha}\beta\underline{\beta}}\,R^{i\beta})_{;\delta} = -R_{\delta\underline{\beta}\gamma\underline{\alpha}}\,R^{i\underline{\gamma}}_{;\alpha} - R_{\underline{\beta}\delta\underline{\gamma}\alpha}\,\bar{R}^{i\underline{\gamma}}_{;\underline{\alpha}}. \tag{B.28}$$

Finally, if we use eq. (B.18), we get the expression

$$i(R_{\underline{\alpha}\alpha}{}^{\gamma}{}_{\delta}\,M^i_{,\gamma})_{;\underline{\beta}} - i(R_{\alpha\underline{\alpha}}{}^{\underline{\gamma}}{}_{\underline{\beta}}\,M^i_{,\underline{\gamma}})_{;\delta} = iR_{\delta\underline{\beta}}{}^{\underline{\gamma}}{}_{\underline{\alpha}}\,M^i_{,\underline{\gamma};\alpha} - iR_{\underline{\beta}\delta}{}^{\gamma}{}_{\alpha}\,M^i_{,\gamma;\underline{\alpha}}. \tag{B.29}$$

This proves eq. (3.12).

Appendix C

Computation of the Dirac brackets

In this appendix we construct the Poission-Dirac brackets used in this thesis. The fermionic phase space constraints

$$\begin{aligned}\chi_{\psi_\pm} &= \pi_{\psi_\pm} - \frac{1}{2}\mathcal{F}''(C)\psi_\pm \simeq 0, \quad \bar{\chi}_{\psi_\pm} = \bar{\pi}_{\psi_\pm} - \frac{1}{2}\mathcal{F}''(C)\bar{\psi}_\pm \simeq 0 \\ \chi_{\eta^\alpha_\pm} &= \pi_{\eta^\alpha_\pm} - 2C\, G_{\alpha\underline{\beta}}\eta^\beta_\pm \simeq 0, \quad \bar{\chi}_{\eta^\alpha_\pm} = \bar{\pi}_{\eta^\alpha_\pm} - 2C\, G_{\alpha\underline{\beta}}\,\bar{\eta}^\alpha_\pm \simeq .\end{aligned} \tag{C.1}$$

serve to eliminate the momenta $(\pi_{\psi_\pm}, \pi_{\eta^\alpha_\pm})$ from the hamiltonian dynamics. The Dirac procedure is designed to eliminate the momenta directly from the brackets —by defining new brackets $\{A, B\}^*$— whilst keeping the equations of motion in the form

$$\dot{A} = \{A, H\}^* . \tag{C.2}$$

The starting point is the ordinary Poisson brackets

$$\left\{\pi_{\eta_\pm}, \bar{\eta}_\mp\right\} \;=\; \left\{\pi_{\psi_\pm}, \bar{\psi}_\mp\right\} = \gamma^0 P_\mp \tag{C.3}$$

In the notation we suppress both the space-time dependence and the indices $\alpha, \beta, \ldots$ of chiral spinors . The bracket (C.3) presupposes $(\eta, \psi, \pi_\eta, \pi_\psi)$ to be independent, and is the one which holds in the full phase space.

Using the charge conjugation properties (A.13) for a Majorana spinor the conjugate brackets are:

$$\begin{aligned}\left\{\pi_{\psi_\pm}, \psi^T_\mp\right\} &= -\gamma^0 C^{-1} P_\mp = C^{-1}\gamma^{0\,T} P_\mp = \left\{\psi_\mp, \pi^T_{\psi_\pm}\right\}^T \quad \Rightarrow \left\{\psi_\mp, \bar{\pi}_{\psi_\pm}\right\} = \gamma^0 P_\pm \\ \left\{\pi_{\eta_\pm}, \eta^T_\mp\right\} &= -\gamma^0 C^{-1} P_\mp = C^{-1}\gamma^{0\,T} P_\mp = \left\{\eta_\mp, \pi^T_{\eta_\pm}\right\}^T \quad \Rightarrow \left\{\eta_\mp, \bar{\pi}_{\eta_\pm}\right\} = \gamma^0 P_\pm .\end{aligned} \tag{C.4}$$

Now we compute the brackets of the constraints:

$$\left\{\chi_{\psi_\pm}, \bar{\chi}_{\psi_\mp}\right\} = -\mathcal{F}''(C)\,\gamma^0 P_\mp, \quad \left\{\chi_{\eta^\alpha_\pm}, \bar{\chi}_{\eta^\beta_\mp}\right\} = -4C\, G_{\alpha\underline{\beta}}\gamma^0 P_\mp . \tag{C.5}$$

Using this result, we obtained the matrix of constraint brackets (3.35) with the inverse

$$C^{ij} = \begin{pmatrix} 0 & \frac{1}{\mathcal{F}''(C)}\gamma^0 P_+ & 0 & 0 \\ \frac{1}{\mathcal{F}''(C)}\gamma^0 P_- & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{4C} G^{\alpha\underline{\beta}}\gamma^0 P_+ \\ 0 & 0 & \frac{1}{4C} G^{\alpha\underline{\beta}}\gamma^0 P_- & 0 \end{pmatrix} \tag{C.6}$$

Furthermore we note, that

$$\begin{aligned}
\left\{\pi_C, \bar{\chi}_{\psi_\pm}\right\} &= \frac{1}{2}\mathcal{F}'''(C)\,\bar{\psi}_\pm, \quad \left\{\pi_C, \bar{\chi}_{\eta^\alpha_\pm}\right\} = -2G_{\alpha\underline{\alpha}}\,\bar{\eta}^{\underline{\alpha}}_\pm \\
\left\{\pi_{z^\alpha}, \bar{\chi}_{\eta^{\underline{\beta}}_\mp}\right\} &= 2C\,G_{\alpha\underline{\beta},\gamma}\bar{\eta}^\gamma_\pm, \quad \left\{\bar{\pi}_{\bar{z}^{\underline{\alpha}}}, \bar{\chi}_{\eta^{\underline{\beta}}_\mp}\right\} = 2G_{\alpha\underline{\alpha},\underline{\beta}}\,\bar{\eta}^\alpha_\pm.
\end{aligned} \tag{C.7}$$

The Dirac brackets are now defined as

$$\begin{aligned}
\{A,B\}^* &= \{A,B\} - \frac{1}{\mathcal{F}''(C)}\Big(\left\{A, \bar{\chi}_{\psi_+}\right\}\gamma^0\left\{\chi_{\psi_-}, B\right\} + \left\{A, \bar{\chi}_{\psi_-}\right\}\gamma^0\left\{\chi_{\psi_+}, B\right\}\Big) \\
&\quad -\frac{1}{4C\,G_{\alpha\underline{\beta}}}\Big(\left\{A, \bar{\chi}_{\eta^{\underline{\beta}}_+}\right\}\gamma^0\left\{\chi_{\eta^\alpha_-}, B\right\} + \left\{A, \bar{\chi}_{\eta^{\underline{\beta}}_-}\right\}\gamma^0\left\{\chi_{\eta^\alpha_+}, B\right\}\Big)
\end{aligned} \tag{C.8}$$

From the definition (C.8) one can now compute the elementary Dirac brackets between the effective degrees of freedom:

$$\begin{aligned}
\left\{\pi_C, \bar{\psi}_\pm\right\}^* &= \frac{\mathcal{F}'''(C)}{2\mathcal{F}''(C)}\bar{\psi}_\pm, \quad \left\{\pi_{z^\alpha}, \bar{\eta}^{\underline{\beta}}_\pm\right\}^* = \frac{1}{2}\Gamma^\alpha_{\beta\beta}\,\bar{\eta}^{\underline{\beta}}_\pm, \quad \left\{\bar{\pi}_{\bar{z}^{\underline{\alpha}}}, \bar{\eta}^{\underline{\beta}}_\pm\right\}^* = \frac{1}{2}\bar{\Gamma}^{\underline{\alpha}}_{\underline{\gamma\gamma}}\,\bar{\eta}^{\underline{\beta}}_\pm \\
\left\{\eta^\alpha_\pm, \bar{\eta}^{\underline{\beta}}_\mp\right\}^* &= \frac{1}{4C}G^{\alpha\underline{\beta}}\gamma^0 P_\mp, \quad \left\{\psi_\pm, \bar{\psi}_\mp\right\}^* = \frac{1}{2\mathcal{F}''(C)}\gamma^0 P_\mp, \quad \left\{\psi_\pm, \pi_C\right\}^* = -\frac{\mathcal{F}'''(C)}{2\mathcal{F}''(C)}\psi_\pm \\
\left\{\eta^\alpha_\pm, \pi_{z^\beta}\right\}^* &= -\frac{1}{2}\Gamma^\alpha_{\gamma\beta}\eta^\gamma_\pm, \quad \left\{\eta^{\underline{\alpha}}_\pm, \bar{\pi}_{\bar{z}^{\underline{\beta}}}\right\}^* = -\frac{1}{2}\bar{\Gamma}^{\underline{\alpha}}_{\underline{\gamma\beta}}\eta^{\underline{\gamma}}_\pm.
\end{aligned} \tag{C.9}$$

Appendix D

Variation of the lagrangian density under supersymmetry transformation

D.1 Computation of $\mathrm{B}_{\pm\mu}$

In this appendix we show that the variation $\delta\mathcal{L}$ of the Lagrangian (3.5) caused by supersymmetry transformation (3.21), is a total derivative.

$$\delta\mathcal{L} = \frac{i}{2}\partial_\mu\Big(\bar{\epsilon}_+ B_+^\mu - \bar{\epsilon}_- B_-^\mu\Big). \tag{D.1}$$

Our starting point is the off-shell supersymmetry transformations of the superfield multiplets $(V, \Phi^\alpha, \bar{\Phi}^{\underline{\alpha}}, \Lambda, \bar{\Lambda})$. The components form of supersymmetry variations is:

$$\begin{aligned}
&\delta C = \tfrac{i}{2}\bar{\epsilon}_+\psi_+ - \tfrac{i}{2}\bar{\epsilon}_-\psi_-, && \delta D = \tfrac{i}{2}\bar{\epsilon}_L\partial\!\!\!/\lambda_- + \text{h.c}\\
&\delta\psi_+ = \tfrac{i}{2}\,B\epsilon_+ - \tfrac{1}{2}(V\!\!\!/ + i\partial\!\!\!/C)\epsilon_- && \delta\psi_- = -\tfrac{i}{2}\bar{B}\epsilon_- - \tfrac{1}{2}(V\!\!\!/ - i\partial\!\!\!/C)\epsilon_+,\\
&\delta V_\mu = -\tfrac{1}{2}\bar{\epsilon}_+(\gamma_\mu\lambda_- + \partial_\mu\psi_+) - \tfrac{1}{2}\bar{\epsilon}_-(\gamma_\mu\lambda_+ + \partial_\mu\psi_-) && \\
&\delta\lambda_- = (\sigma^{\mu\nu}\partial_\mu V_\nu - \tfrac{i}{2}D)\epsilon_- && \delta\lambda_+ = (\sigma^{\mu\nu}\partial_\mu V_\nu + \tfrac{i}{2}D)\epsilon_+,\\
&\delta B = -i\bar{\epsilon}_-(\lambda_- + \partial\!\!\!/\psi_+) && \delta\bar{B} = i\bar{\epsilon}_+(\lambda_+ + \partial\!\!\!/\psi_-),\\
&\delta z^\alpha = \bar{\epsilon}_+\eta_+^\alpha, \quad \delta\bar{z}^{\underline{\alpha}} = \bar{\epsilon}_-\eta_-^{\underline{\alpha}} && \delta s = \bar{\epsilon}_+\chi_+, \quad \delta\bar{s} = \bar{\epsilon}_-\chi_-,\\
&\delta\eta_+^\alpha = \tfrac{1}{2}\partial\!\!\!/z^\alpha\epsilon_- + \tfrac{1}{2}H^\alpha\epsilon_+, && \delta\eta_-^{\underline{\alpha}} = \tfrac{1}{2}\partial\!\!\!/\bar{z}^{\underline{\alpha}}\epsilon_+ + \tfrac{1}{2}\bar{H}^{\underline{\alpha}}\epsilon_-,\\
&\delta\chi_+ = \tfrac{1}{2}\partial\!\!\!/s\epsilon_- + \tfrac{1}{2}h\epsilon_+, && \delta\chi_- = \tfrac{1}{2}\partial\!\!\!/\bar{s}\epsilon_+ + \tfrac{1}{2}\bar{h}\epsilon_-,\\
&\delta H^\alpha = \bar{\epsilon}_-\partial\!\!\!/\eta_+^\alpha, \quad \delta h = \bar{\epsilon}_-\partial\!\!\!/\chi_+, && \delta\bar{H}^{\underline{\alpha}} = \bar{\epsilon}_+\partial\!\!\!/\eta_-^{\underline{\alpha}}, \quad \delta\bar{h} = \bar{\epsilon}_-\partial\!\!\!/\chi_-
\end{aligned} \tag{D.2}$$

Here $\epsilon_\pm$ are anti-commuting chiral spinor parameters of supersymmtry. Eliminating of non-dynamical fields $(D, H^\alpha, B, \lambda_+, \chi_+)$ from (D.2) by their field equations:

$$\begin{aligned} H^\alpha &= G^{\alpha\underline{\beta}} G_{\gamma\underline{\beta},\delta} \bar{\eta}_+^\gamma \eta_+^\delta + \frac{i}{C} \bar{\psi}_+ \eta_+^\alpha, \quad D = -\Box C, \quad B = 0, \\ \chi_+ &= \frac{i}{2} \mathcal{F}''(C) \psi_+ - K_\alpha \eta^\alpha, \quad \lambda_+ = -\partial\!\!\!/ \psi_+ \end{aligned} \tag{D.3}$$

and their conjugates lead to the supersymmetry transformations (3.21). We now compute the vector-spinor fields $B_{\pm\mu}$. First we recall, that the Lagrangian (3.5) is the D-term of the real vector multiplet $V(\mathcal{K} - \mathcal{F}(V))$:

$$\mathcal{L} = [V(\mathcal{K} - \mathcal{F}(V))]_D, \quad \mathcal{K} = K(\Phi^\alpha, \bar{\Phi}^{\underline{\alpha}}) + \Lambda + \bar{\Lambda}. \tag{D.4}$$

As it is well known that a D-term transform into a total derivative (see the supersymmetry variation of the D field in eq. (D.2)), we can directly obtain the vector-spinor fields $B_\pm^\mu$. The variation $\delta\mathcal{L}$ in eq. (D.4) is

$$\delta\mathcal{L} = \frac{i}{2} \bar{\epsilon}_+ \partial\!\!\!/ \Big[\Big(V\mathcal{K} - \mathcal{F}(V) \Big) \Big]_{\lambda_-} + \text{h.c}, \tag{D.5}$$

where the components $[\mathcal{F}(V)]_{\lambda_-}$ and $[V\mathcal{K}]_{\lambda_-}$ given by

$$\begin{aligned} [\mathcal{F}(V)]_{\lambda_-} &= \mathcal{F}'(C)\lambda_- + \frac{1}{2}\mathcal{F}''(C)\Big(B\psi_- - iV\!\!\!/ \psi_+ - \partial\!\!\!/ C\psi_+ \Big) - \frac{1}{2}\mathcal{F}'''(C)\psi_- \bar{\psi}_+ \psi_+ & \text{(D.6)} \\ [V\mathcal{K}]_{\lambda_-} &= C\xi_- + \mathcal{C}\lambda_- + \frac{1}{2}\Big(B\zeta_- + \mathcal{H}\psi_- - iV\!\!\!/ \zeta_+ - i\mathcal{A}\!\!\!/ \psi_+ - \partial\!\!\!/ C\zeta_+ - \partial\!\!\!/ \mathcal{C}\psi_+ \Big), & \text{(D.7)} \end{aligned}$$

with $\mathcal{C}, \zeta_+, \mathcal{H}, \mathcal{A}_\mu$, ξ_- are given respectively in (3.4). Upon using the equations of motion (D.3) for (H^α, B, χ_+) and the field equations following from $\delta S(V, \mathcal{K})/\delta V = 0$ ($\delta V = \delta D, \delta V_\mu, \delta \bar{B}$), we get the on-shell expressions

$$\begin{aligned} \mathcal{C} &= K(z^\alpha, \bar{z}^{\underline{\alpha}}) + s + \bar{s} \simeq \mathcal{F}'(C), \\ \mathcal{H} &= G_{\alpha\underline{\beta}}\, \bar{\eta}_+^{\underline{\beta}} \eta_+^\alpha - K_{,\alpha} H^\alpha - h \simeq -\frac{1}{2}\mathcal{F}'''(C)\bar{\psi}_+\psi_+, \\ \mathcal{A}_\mu &= i\Big(K_{,\alpha}\partial_\mu z^\alpha - K_{,\underline{\alpha}}\partial_\mu \bar{z}^{\underline{\alpha}} + \partial_\mu s - \partial_\mu \bar{s} - 2G_{\alpha\underline{\beta}}\, \bar{\eta}_-^{\underline{\beta}} \gamma_\mu \eta_+^\alpha \Big) \simeq \mathcal{F}''(C) V_\mu + \frac{1}{2}\mathcal{F}'''(C)\bar{\psi}_+\gamma_\mu\psi_-, \\ \xi_+ &= -2iG_{\alpha\underline{\beta}}\, \partial\!\!\!/ z^\alpha \eta_-^{\underline{\beta}} + 2iG_{\alpha\underline{\beta}}\, \bar{H}^{\underline{\beta}} \eta_+^\alpha - 2iG_{\alpha\underline{\beta},\gamma}\, \eta_+^\alpha\, \bar{\eta}_-^{\underline{\beta}} \eta_-^\gamma \simeq 2G_{\alpha\underline{\beta}}\Big(\frac{1}{C}(\bar{\eta}_+^{\underline{\beta}}\psi_+)\eta_-^\alpha + \partial\!\!\!/ \bar{z}^{\underline{\beta}}\eta_+^\alpha \Big), \\ \zeta_- &= 2iK_{\underline{\alpha}}\eta_-^{\underline{\alpha}} + 2i\chi_- \simeq \mathcal{F}''(C)\psi_+. \end{aligned} \tag{D.8}$$

By inserting these expressions into (D.7) and after Fierz rearrangement of the term containing three fermions, we obtain the vector-spinor $B_{+\mu}$

$$\begin{aligned} B_+^\mu &= \gamma^\mu \Big([V\mathcal{K}]_{\lambda_-} - [\mathcal{F}(V)]_{\lambda_-} \Big) \\ &\simeq 2G_{\alpha\underline{\beta}}\gamma^\mu \Big(\eta_-^\alpha \bar{\eta}_+^{\underline{\beta}} \psi_+ + iC\partial\!\!\!/ \bar{z}^{\underline{\beta}} \eta_+^\alpha \Big) - \frac{1}{2}\mathcal{F}''(C)\gamma^\mu \Big(\partial\!\!\!/ C + iV\!\!\!/ \Big)\psi_+ + \\ &\quad - \frac{1}{2}\mathcal{F}'''(C)\gamma^\mu \psi_- \bar{\psi}_+ \psi_+. \end{aligned} \tag{D.9}$$

The vector-spinor $B_{-\mu}$ is obtained by the complex conjugate of $B_{+\mu}$.

D.2 Construction of the supercurrents $\mathrm{S}_{\pm\mu}$

Noether's theorem asserts, that a conserved supercurrents $S_{\pm\mu}$ is constructed from

$$\frac{i}{2}\bar{\epsilon}_+ S_{+\mu} - \frac{i}{2}\bar{\epsilon}_- S_{-\mu} = \frac{i}{2}\bar{\epsilon}_+ B_{+\mu} - \frac{i}{2}\bar{\epsilon}_- B_{-\mu} - \sum_{i=\pm} \frac{\delta\mathcal{L}}{\partial_\mu \phi_i}\delta\phi_i, \tag{D.10}$$

where the sum is taken over the various supersymmetry variations $\delta\phi$ of the theory. As we have already computed $B_{\pm\mu}$, we only have to compute the gradient terms. First we consider only the gradient terms with the ϵ_+:

$$\begin{aligned} \delta_+\phi\frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)} &= -\frac{i}{2}\bar{\epsilon}_+\psi_+\mathcal{F}''(C)\partial_\mu C + \frac{1}{2}\mathcal{F}''(C)V_\mu\bar{\epsilon}_+\psi_+ + iK_\alpha\bar{\epsilon}_+\eta_+^\alpha \\ &+\frac{1}{4}\mathcal{F}''(C)\bar{\psi}_+\gamma_\mu\Big(\not{V} - i\not{\partial}C\Big)\epsilon_+ + \bar{\epsilon}_+\eta_+^\alpha\Big(2iG_{\alpha\underline{\beta}}\,\bar{\psi}_+\gamma_\mu\eta_-^{\underline{\beta}} - iK_\alpha V_\mu \\ &+2CG_{\alpha\underline{\beta},\gamma}\bar{\eta}_+^{\underline{\beta}}\gamma_\mu\eta_-^\gamma - 2CG_{\alpha\underline{\beta}}\partial_\mu\bar{z}^{\underline{\beta}}\Big). \end{aligned} \tag{D.11}$$

Substracting of this expression from $B_{+\mu}$ in the form (D.9) then reproduces the result (3.25)

$$\begin{aligned} S_{+\mu} &= 4G_{\alpha\underline{\beta}}\Big(\gamma_\mu\,\eta_-^\alpha\,\bar{\psi}_+\,\eta_+^{\underline{\beta}} - iC\not{\partial}\bar{z}^{\underline{\beta}}\gamma_\mu\psi_+^\alpha\Big) + \mathcal{F}''(C)(\not{\partial}C + i\not{V})\gamma_\mu\psi_+ \\ &-\frac{1}{2}\mathcal{F}'''(C)\gamma_\mu\,\psi_-\,\bar{\psi}_+\,\psi_+ - 2iCG_{\alpha\underline{\beta},\gamma}\gamma_\mu\eta_-^\alpha\,\bar{\eta}_+^{\underline{\beta}}\,\eta_+^\gamma. \end{aligned} \tag{D.12}$$

The computation of $S_{-\mu}$ goes entirely similar.

Appendix E

Analysis of supersymmetry breaking vauca solutions

In this appendix we present the solutions to the equations (6.40) which we have used in section 6.5. We consider first the case $H = 0$. To determine the minimum values for X and Y, we differentiate the scalar potential with respect to X and Y, this gives

$$\frac{\partial V}{\partial Y} = 2\gamma(Y + \frac{\beta}{2\gamma}) = 0, \quad \frac{\partial V}{\partial X} = \frac{\partial \alpha}{\partial X} + Y\frac{\partial \beta}{\partial X} + Y^2\frac{\partial \gamma}{\partial X} = 0, \tag{E.1}$$

with α, β and γ given by the expressions (6.39). The first equation of (E.1) is satisfied if

$$Y = -\frac{\beta}{2\gamma}. \tag{E.2}$$

Substituting of this result into the second equation of (E.1), gives

$$\frac{\partial V}{\partial X} = \mathcal{A}X^2 + \mathcal{B}X + \mathcal{C} = 0, \tag{E.3}$$

where

$$\begin{aligned}
\mathcal{A} &= \frac{1}{125}\Big(200g_1^2g_5^4 - 75g_5^6 + 400\xi g_1^2g_5^4\Big), \qquad (E.4)\\
\mathcal{B} &= \frac{1}{125}\Big(360g_1^4g_5^2 - 820g_1^2g_5^4 + 210g_5^6 + 1400\xi g_1^4g_5^2 - 1400\xi g_1^2g_5^4 + 1400\xi^2g_1^4g_5^2\Big),\\
\mathcal{C} &= \frac{1}{125}\Big(-585g_1^4g_5^2 + 720g_1^2g_5^4 - 135g_5^6 - 1980\xi g_1^4g_5^2 + 1080\xi g_1^2g_5^4 - 1620\xi^2g_1^4g_5^2\Big).
\end{aligned}$$

The equation (E.3) has two zeros, at

$$X_1 = \frac{9}{5g_5^2}(-g_1^2 + g_5^2 - 2g_1^2\xi), \quad X_2 = \frac{13g_1^2 - 3g_5^2 + 18g_1^2\xi}{8g_1^2 - 3g_5^2 + 16g_1^2\xi} \tag{E.5}$$

Inserting this into equation (E.2), we obtain the minimum values of Y, i.e. the roots:

$$Y_1 = \frac{5g_5^2}{27(-g_5^2 + g_1^2(1+2\xi))}, \quad Y_2 = \frac{-3g_1^2(5+2\xi)(-3g_5^2 + 8g_1^2(1+2\xi))}{(32g_1^2 + 3g_5^2)(-3g_5^2 + g_1^2(13+18\xi))} \tag{E.6}$$

We now turn to the second solution of the supersymmety breaking vacuum conditions (6.40), when $H \neq 0$, which exist for $C = 0$. Differentiating the scalar potential with respect to H gives

$$\frac{\partial V}{\partial H} = \mathcal{E}H + \mathcal{F} = 0 \quad \Rightarrow H = -\frac{\mathcal{F}}{\mathcal{E}}, \tag{E.7}$$

with $\mathcal{E}$ and $\mathcal{F}$ given by

$$\begin{aligned}
\mathcal{E} &= -6152g_1^2g2_5 - 52368g_5^4 - \frac{6912g_1^2g_5^2}{X} + \frac{10368g_5^4}{X^2} + \frac{13824g_1^2g_5^2}{X} \\
&\quad -\frac{38016g_5^2}{X} + 520g_1^2g_5^2X + 32160g_5^2X + 160g_1^2g_5^2X^2 + 7440g_5^4X^2 \\
\mathcal{F} &= -190g_1^2g2_5 - 60g_5^4 + 130g_1^2g_5^2X + 120g_5^4X + 40g_1^2g_5^2X^2 + 60g_5^2X^2 \\
&\quad -180g_1^2g_5^2\xi + 220g_1^2g_5^2X\xi - 80g_1^2g_5^2X\xi
\end{aligned} \tag{E.8}$$

Next, we minimize the scalar potential with respect to X and Y. In fact, they are given in eq.(E.1). Substituting (E.7) into

$$\frac{\partial V}{\partial X} = \partial_X\alpha - \frac{\beta}{2\gamma}\partial_X\beta + \frac{\beta^2}{4\gamma^2}\partial_X\gamma = 4\gamma^2\partial_X\alpha - 2\gamma\beta\partial_X\beta + \beta^2\partial_X\gamma = 0, \tag{E.9}$$

gives the minimum value of X. There are 10 separate solutions to this equation. The first five solutions are given by expression (6.48). The corresponding values for H and Y is

$$\begin{aligned}
H_1 &= \frac{1}{20}(5 + 2\xi), \\
H_\pm &= \frac{6g_5^2 + 36g_1^4(1 + 2\xi)}{4(36g_1^4 - 17g_1^2g_5^2 + 6g_5^4)} + \frac{g_1^2\Big(3\sqrt{5}\sqrt{-32g_1^2g_5^2 - 3g_5^4}\,\xi - g_5^2(17 + 27\xi)\Big)}{4(36g_1^4 - 17g_1^2g_5^2 \pm 6g_5^4)}, \\
Y_\pm &= \Big[4g_5^2\Big(24g_5^4\Big(115g_5^2 + 43\sqrt{5}\sqrt{-32g_1^2g_5^2 - 3g_5^4}\Big) \\
&\quad +g_1^2g_5^2\Big(g_5^2(30880 - 22005\xi) + 3\sqrt{5}\sqrt{-32g_1^2g_5^2 - 3g_5^4}\,(352 - 383\xi)\Big) \\
&\quad +17280g_1^6\,\xi + 48g_1^4\Big(9\sqrt{5}\sqrt{-32g_1^2g_5^2 - 3g_5^4}\,(16 + 37\xi) \\
&\quad -40g_5^2(42 + 125\xi)\Big)\Big)\Big]\Big/\Big[3\Big(-15g_5^2 + \sqrt{5}\sqrt{-32g_1^2g_5^2 - 3g_5^4}\Big)\Big(-2304g_1^6 \\
&\quad +9g_5^4\Big(61g_5^2 + 5\sqrt{5}\sqrt{-32g_1^2g_5^2 - 3g_5^4}\Big) - 16g_1^4\Big(23g_5^2 \\
&\quad +15\sqrt{5}\sqrt{-32g_1^2g_5^2 - 3g_5^4}\Big) + 24g_1^4\Big(\pm 73g_5^4 + 5\sqrt{5}g_5^2\sqrt{-32g_1^2g_5^2 - 3g_5^4}\Big)\Big)\Big]
\end{aligned} \tag{E.10}$$

The other five solutions are obtain from equation

$$aX^5 + bX^4 + cX^3 + dX^2 + eX + f = 0, \tag{E.11}$$

with

$$
\begin{aligned}
a &= 320g_1^4 - 6720g_1^2g_5^2 + 3600g_5^4 + 960g_1^4\xi - 12720g_1^2g_5^2\xi + 665g_1^4\xi^2, \\
b &= -15904g_1^4 + 49488g_1^2g_5^2 - 19584g_5^4 - 59456g_1^4\xi + 89136g_1^2g_5^2\xi \\
&\quad -55741g_1^4\xi^2, \\
c &= 70356g_1^4 - 133392g_1^2g_5^2 + 42336g_5^4 + 255440g_1^4\xi - 232560g_1^2g_5^2\xi \\
&\quad +231696g_1^4\xi^2, \\
d &= -118744g_1^4 + 172848g_1^2g_5^2 - 45504g_5^4 - 416384g_1^4\xi + 290064g_1^2g_5^2\xi \\
&\quad -362592g_1^4\xi^2, \\
e &= 88436g_1^4 - 109872g_1^2g_5^2 + 24336g_5^4 + 297072g_1^4\xi - 175392g_1^2g_5^2\xi \\
&\quad +248400g_1^4\xi^2, \\
f &= -22464g_1^4 + 27648g_1^2g_5^2 - 5184g_5^4 - 76032g_1^4\xi + 41472g_1^2g_5^2\xi - 62208g_1^4\xi^2. \quad \text{(E.12)}
\end{aligned}
$$

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