WIGNER TRANSFORMATION
FOR THE DETERMINANT OF DIRAC OPERATORS

L.L. Salcedo[1] and E. Ruiz Arriola[1,2]

[1]Departamento de Física Moderna, Universidad de Granada
E-18071 Granada, Spain

[2]National Institute for Nuclear Physics and High Energy Physics, (NIKHEF-K)
1009-DB Amsterdam, The Netherlands

ABSTRACT

We use the ζ-function regularization and an integral representation of the complex power of a pseudo differential operator, to give an unambiguous definition of the determinant of the Dirac operator. We bring this definition to a workable form by making use of an asymmetric Wigner representation. The expression so obtained is amenable to several treatments of which we consider in detail two, the inverse mass expansion and the gradient expansion, with concrete examples. We obtain explicit closed expressions for the corresponding Seeley-DeWitt coefficients to all orders. The determinant is shown to be vector gauge invariant and to posses the correct axial and scale anomalies. The main virtue of our approach is that it is conceptually simple and systematic and can be extended naturally to more general problems (bosonic operators, gravitational fields, etc). In particular, it avoids defining the real and imaginary parts of the effective action separately. In addition, it does not reduce the problem to a bosonic one to apply heat kernel nor performs further analytical rotations of the fields to make the Dirac operator Hermitian. We illustrate the flexibility of the method by studying some interesting cases.

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1. Introduction

The success of the Lagrangian formulation of relativistic field theories is due to the easy implementation of symmetries and in particular of Lorentz invariance. The same is true for the associated covariant quantization procedure, the Feynman functional integral approach. In a typical setting, as will be ours in this work, one considers Dirac fermions in D space-time dimensions in the presence of arbitrary non gravitational bosonic background fields coupled to Lorentz and to internal symmetry indices of the fermions. Generally speaking, the background fields can be treated as external. This is no restriction since they can be quantized introducing the corresponding functional integration over them. The effective action of the system is then obtained by integrating out the fermion fields. Formally the Grassmann integral gives the determinant of the Dirac operator, $\mathrm{Det}\,\mathbf{D}$, and the effective action is just its logarithm. As a consequence this determinant plays an important role in the functional integral formulation [1].

In the relativistic case the naive determinant is ultraviolet divergent and one has to define a renormalized determinant by introducing counterterms or some other equivalent technique. In this process some classical symmetries can be lost. Of course, all this parallels the diagrammatic approach where the determinant is represented by the one fermion loop graphs.

There has been a number of ways to address the problem of defining a finite determinant or equivalently a finite effective action [2]. Most of the methods developed in the literature try to reduce the problem to a bosonic one. This is because second order differential operators are algebraically simpler and better studied in this context. In addition, the important ultraviolet problem can be treated within an inverse mass expansion (i.e. a simultaneous weak and smooth field expansion) with the heat kernel technique, and this applies to second order definite positive Hermitian differential operators. The reduction is usually achieved by considering $\mathbf{D}^2$ or $\mathbf{D}\mathbf{D}^\dagger$ and $\mathbf{D}^\dagger\mathbf{D}$ [3-8]. In the first case a further analytical rotation of the fields (besides the Euclidean rotation) is assumed to make $\mathbf{D}$ antihermitian. This requires to extend the internal symmetry group too and it is known that the rotation back of concrete subgroups can be ambiguous [2]. Furthermore, although $\mathbf{D}^2$ looks like a bosonic theory, some of the efficient methods developed to go beyond the inverse mass expansion [9], may not be straightforwardly applied to fermions. This is because such methods assume the unrestricted validity of the formal relation $\mathrm{Det}\,(AB) = \mathrm{Det}\,(A)\mathrm{Det}\,(B)$, which does not hold under regularization. In the bosonic case the use of this formal relation just redefines the counterterms, but for fermions it makes the determination of the chiral anomaly ambiguous. Other approaches use formal relations to define independently the real and imaginary parts of the effective action, using $\mathbf{D}$ and $\mathbf{D}^\dagger$ and are rather ad hoc [8,10]. Similar approaches used to define an ad hoc Jacobian of the functional measure under chiral transformations are known to lead to results inconsistent with the Bardeen anomaly [11].

A different approach is that of Leutwyler [12] and Ball [2]. It uses a formal definition of the variation of the effective action in Euclidean space which is regularized in a chirally invariant way using $\mathbf{D}^\dagger\mathbf{D}$ in a proper time representation. For the real part of the action this defines a true variation. The regularized variation of the imaginary part satisfies the integrability condition only after adding suitable polynomial counterterms which introduce the chiral anomaly. This approach is both mathematically impeachable and also computationally convenient in heat kernel-like expansions. However it is rather sophisticated and relies heavily on the previous knowledge of the subtleties involved in $\mathrm{Det}\,\mathbf{D}$ obtained after years of deeply original insights [13-20]. For instance the operator $\mathbf{D}^\dagger$, which in principle is unrelated to the problem and formally cancels in the definition, is needed in the construction. It is not clear to us how this approach can be properly extended to more general theories, such as non local theories, string theories, and so on.

We think that it can be of interest to pursue an alternative statement of the problem where the definition of the determinant is given at the very beginning and then every other quantity can be defined and calculated unambiguously without the need of new prescriptions. A good example of the latter is provided by the functional Jacobian under variations of $\mathbf{D}$. Although such an object is not needed in this kind of approach, it is perfectly well defined and can be computed if desired [21]. The only remaining freedom is of course the addition of counterterms, polynomials of degree D in the number of derivatives plus external fields, which allows to reproduce any other renormalization prescription or enforce particular symmetries. A suitable definition, given long ago [22], is the ζ-function regularization which has the advantage of preserving automatically a large class of the classical symmetries, namely those which are implemented by similarity transformations of $\mathbf{D}$. This includes relativistic invariance, vector gauge invariance and so on. Other virtue is that it is well grounded mathematically [23]. Using a typical Cauchy integral representation, the ζ-function can be related to the resolvent of $\mathbf{D}$ which is a more tractable object and is suitable to perform systematic expansions. Finally, a Wigner representation [24,25] allows us to treat properly the ultraviolet divergences appearing in the diagonal elements of the resolvent. In this way a well defined workable form for the effective action and other quantities is obtained. Although by no means is it intended in this work to achieve strict mathematical rigor, the troublesome ultraviolet sector is treated carefully, and only in the infrared we proceed formally by assuming that $\mathbf{D}$ is well behaved in this sector, that is, effective boundaryless boundary conditions and no zero modes.

Because the subject has been extensively studied in the past, we do not intend to present truly new theorems, rather our emphasis is on introducing a conceptually simple scheme to make the subject more easily graspable with more systematic, and sometimes simpler, proofs of known results and also with an eye put on the generalization to worse known systems where our intuition is less developed.

The paper is organized as follows. In Section 2, we present our conventions and set up the basic formalism for the ζ-function regularized determinant of the Dirac operator, as well as the corresponding consistent currents. Actually, we find that the effective action can be reconstructed from the current without loss of information in this particular regularization. Section 3 deals with one special definition of the Wigner transformation, particularly adapted to the Dirac operator, and rather convenient from a computational point of view. In Section 4 we consider an inverse mass expansion for the Dirac operator and among other things, we obtain a direct, i.e. non recursive, determination of the corresponding Seeley-DeWitt coefficients. These can be profitably used to write down inverse mass expansions both for the effective action and the effective currents. A further application of the previous results can be found in Section 5, when computing chiral and scale anomalies within the ζ-function regularization. Moreover, we establish the general form of the counterterms needed to bring the chiral anomaly to its minimal (Bardeen) form. To do so the Wigner transformation method turns out to be very useful. In Section 6, some aspects of the so-called gradient or derivative expansion are studied within the present formalism, i.e. Wigner transformation technique. We consider for instance a direct calculation of the gauged Wess-Zumino-Witten term from the Dirac determinant. We do so without using better known methods, like e.g. trial and error, differential geometry or integration of the chiral anomaly. Along similar lines, we study fermionic currents and their relation to effective actions in two less dimensions, again with the help of the Wigner transformation method. Finally, parity anomalies in odd dimensions are revisited in Section 7, where the gauge invariant Chern-Simons action is obtained. In Appendix A, we establish a explicit closed formula for the Seeley-DeWitt coefficients of the Dirac operator to all even orders. In Appendix B, the connection between the well-known Heat Kernel expansion for the squared Dirac operator, $\mathbf{D}^2$, and our inverse mass expansion for the Dirac operator, $\mathbf{D}$, is developed in the even-dimensional case. Appendices C and D collect explicit formulas for the chiral anomaly and the counterterms for the most general Dirac operator in four dimensions.

2. ζ-function regularization

Let $\mathbf{D}$ be the Dirac operator in D dimensional Euclidean space for a fermion in the presence of arbitrary (non gravitational) external fields

$$\mathbf{D} = i\partial\!\!\!/ + \mathbf{Y}(x) \tag{2.1}$$

Here $\mathbf{Y}(x)$ is a matrix in the internal degrees of freedom, i.e. spinor and flavor, but does not contain derivative operators. Our conventions regarding gamma matrices are as follows

$$\begin{aligned} &\gamma_\mu^\dagger = -\gamma_\mu\,, \qquad \{\gamma_\mu, \gamma_\nu\} = -2\delta_{\mu\nu} \\ &\gamma_5 = -i^{D/2}\gamma_0\gamma_1\cdots\gamma_{D-1}\,, \qquad \epsilon_{01\ldots D-1} = +1 \end{aligned} \tag{2.2}$$

Whenever needed we will assume the standard hermiticity for the external fields, that is, such that if they transform covariantly under the Wick rotation, $\gamma_0\mathbf{D}$ is Hermitian in Minkowski space. This implies that the Euclidean effective action is real in the pseudoparity even sector (containing no Levi-Civita pseudotensor) and imaginary in the pseudoparity odd one (containing a Levi-Civita pseudotensor). We will often use the object $\mathbf{D}$ itself rather than $\mathbf{Y}(x)$ because this produces more compact formulas. Another important point is that $\mathbf{D}$ transforms homogeneously under the classical symmetry transformations

$$\begin{aligned} \mathbf{D} \to \mathbf{D}^\Omega &= \Omega_2 \mathbf{D}\Omega_1^{-1} \\ \psi^\Omega(x) &= \Omega_1\psi(x') \\ \bar\psi^\Omega(x) &= \bar\psi(x')\Omega_2^{-1} \end{aligned} \tag{2.3}$$

whereas $\mathbf{Y}$, in general, transforms inhomogeneously. This transformation corresponds to a classical symmetry if it maintains the structure (2.1), i.e. $\mathbf{D}^\Omega = i\partial\!\!\!/ + \mathbf{Y}^\Omega(x)$, but otherwise Ω_1, and Ω_2 can depend on x and contain derivative operators. In this paper we will consider explicitly two classical symmetries, namely chiral gauge rotations and scale transformations. Notice that the form of the Dirac operator does not include general coordinate transformations as classical symmetries. This would require to extend the Dirac operator by properly coupling gravitational fields, and subsequent generalization of our computational procedure to curved space-time. Such a study will not be undertaken here and is left for future research.

For definiteness we can think of $\mathbf{D}$ as admitting a complete set of left and right eigenvectors

$$\mathbf{D}\phi_n(x) = \lambda_n \phi_n(x)\,, \qquad \xi_n(x)\mathbf{D} = \lambda_n \xi_n(x) \tag{2.4}$$

which can be normalized (in a box) so that $\langle \xi_n | \phi_k \rangle = \delta_{nk}$. More generally $\mathbf{D}$ can have a general Jordan form, i.e. completeness of the eigenvectors will not be required. The Euclidean partition function is a functional of the external fields $\mathbf{Y}(x)$ given by

$$Z = \int \mathcal{D}\bar\psi \mathcal{D}\psi \exp\left\{ -\int \mathrm{d}^D x\, \bar\psi(x) \mathbf{D} \psi(x) \right\} \tag{2.5}$$

and formally $Z = \mathrm{Det}\,\mathbf{D}$. Hence the Euclidean effective action is formally

$$W = -\log Z = -\mathrm{Tr}\,\log \mathbf{D} = -\sum_n \log(\lambda_n) \tag{2.6}$$

The sum in (2.6) is ultraviolet divergent and must be regularized. To do so we shall adopt the ζ-function regularization prescription [1,22]. We shall assume that $\mathbf{D}$ has no eigenvalues in some neighborhood of zero, otherwise some infrared regularization would be needed too. If there is only a finite number of zero modes, isolated from the rest of the spectrum, one can define a restricted determinant excluding the zero modes.

Let us consider the pseudo differential operator $(\mathbf{D}/\mu)^s$, where μ is a scale introduced for dimensional bookkeeping and the complex number s is the regulator. If $\mathbf{D}$ admits a complete set of eigenvectors, this operator is characterized by its eigenvectors and eigenvalues $\{\phi_n, (\lambda_n/\mu)^s\}$. In any case a convenient representation is given by [23]

$$\left(\frac{\mathbf{D}}{\mu}\right)^s = -\int_\Gamma \frac{\mathrm{d}z}{2\pi i} \left(\frac{z}{\mu}\right)^s \frac{1}{\mathbf{D} - z} \tag{2.7}$$

where Γ is a path that starts at infinity, follows a ray of minimal growth (i.e. no eigenvalue of $\mathbf{D}$ lies on it), encircles the origin clockwise and goes back to infinity along the ray. No eigenvalues are encircled by Γ. If the set of eigenvalues is bounded, Γ can be deformed to an anticlockwise oriented closed path containing the eigenvalues of $\mathbf{D}$ but excluding the origin. This construction assumes that no eigenvalue lies in a neighborhood of the origin and hence that $\mathbf{D}$ is non singular. This representation is meaningful for $\mathrm{Re}\, s < 0$, and can be analytically continued to the complex plane of s. With this prescription, $\langle x|\mathbf{D}^s|y\rangle$ is an entire function of s for $x \neq y$ and has simple poles at $s = -1, -2, \ldots, -D$, for $x = y$ [23]. (Here $|x\rangle$ is an eigenvector of the multiplicative operator $\hat{x}_\mu$ in D-dimensional Euclidean space. In what follows we will not use a different notation for a function of x_μ and the corresponding multiplicative operator, and similarly we will identify the momentum operator and $i\partial_\mu$). In particular $\mathbf{D}^s$ is analytic at $s = 0$ and this allows for the definition of a regularized logarithm of $\mathbf{D}$

$$\log\left(\frac{\mathbf{D}}{\mu}\right) = \frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\mathbf{D}}{\mu}\right)^s\Big|_{s=0} \tag{2.8}$$

and of the regularized determinant of $\mathbf{D}$ [1,22]

$$W(\mathbf{D}) = -\log \mathrm{Det}\left(\frac{\mathbf{D}}{\mu}\right) = -\mathrm{Tr}\,\log\left(\frac{\mathbf{D}}{\mu}\right) = -\frac{\mathrm{d}}{\mathrm{d}s}\int \mathrm{d}^D x\, \mathrm{tr}\langle x|\left(\frac{\mathbf{D}}{\mu}\right)^s|x\rangle\Big|_{s=0} \tag{2.9}$$

The determinant so defined is completely finite but depends on the arbitrary scale μ introduced by the regularization.

Using previous formulas it is easy to prove the following identity satisfied by the action in this regularization

$$W(\mathbf{D}) = -\frac{\mathrm{d}}{\mathrm{d}s}\frac{\Gamma(s+1)}{\Gamma(s+n)}\int_\Gamma \frac{\mathrm{d}z}{2\pi i} z^{s+n-1}(-\frac{\partial}{\partial z})^n W(\mathbf{D}-z)|_{s=0} \tag{2.10}$$

In fact in the r.h.s. we can use any other regularization because all of them differ by a polynomial in z and its contribution cancels in the integral. On the other hand using a value $n > D$, $\partial_z^n W(\mathbf{D}-z)$ is ultraviolet finite and independent of the regularization and can be used to reconstruct the action.

It is also of interest to obtain expressions for the variation of the action under a generic infinitesimal transformation of the fields in the Dirac operator. Let $\delta_X \mathbf{D} = \mathbf{X}$ be such a variation of $\mathbf{D}$, where the only restriction is that $\mathbf{X}$ is a multiplicative operator, that is, without derivatives. Hence it corresponds to an infinitesimal classical symmetry transformation. Correspondingly

$$\delta_X \mathbf{D} = \mathbf{X}, \qquad \delta_X W = \int \mathrm{d}^D x\, \mathrm{tr}(\mathbf{X}\mathbf{J}), \qquad \mathbf{J} = \frac{\delta W}{\delta \mathbf{D}} \tag{2.11}$$

We will refer to $\mathbf{J}$ as the current associated to W, although usually this name is reserved for the variation under gauge fields and otherwise $\mathbf{J}$ is called a density. The current so defined satisfies consistency conditions [17] which reflect that it is a true variation and hence it is known as the consistent current. Other definitions of the current are sometimes more convenient, in particular the so called (chiral) covariant current [26], which will be considered in Section 5. They differ by a polynomial in the fields and derivatives.

An explicit expression for the current can be obtained by using the technique introduced in [21,27]

$$\delta_X W = -\delta_X \frac{\mathrm{d}}{\mathrm{d}s} \mathrm{Tr}\, \frac{\mathbf{D}^s}{\mu^s}\Big|_{s=0} = -\frac{\mathrm{d}}{\mathrm{d}s} s \mathrm{Tr}\,(\mathbf{X}\frac{\mathbf{D}^{s-1}}{\mu^s})\Big|_{s=0} \tag{2.12}$$

We have used the cyclicity of the trace in presence of the regulator s. Because $\mathbf{X}$ is local, we have

$$\mathbf{J}(x;\mathbf{D}) = -\frac{\mathrm{d}}{\mathrm{d}s} s \langle x|\frac{\mathbf{D}^{s-1}}{\mu^s}|x\rangle\Big|_{s=0} \tag{2.13}$$

The current can be used to recover the action by applying Eq. (2.10) for $n=1$,

$$W(\mathbf{D}) = -\int \mathrm{d}^D x\, \mathrm{tr} \frac{\mathrm{d}}{\mathrm{d}s} \int_\Gamma \frac{\mathrm{d}z}{2\pi i} z^s \mathbf{J}(x;\mathbf{D}-z)|_{s=0} \tag{2.14}$$

Again, we can use any other regularization for the current in the r.h.s., moreover there is no need to use the consistent current, it can be for instance the covariant current, and the z integral will take care of introducing the proper chiral anomaly into the action.

The ζ-function regularized action enjoys all the classical symmetries which are also symmetries of the quantum Dirac equation i.e. the eigenvalue equation $\mathbf{D}\phi_n = \lambda_n \phi_n$. This includes in particular vector gauge invariance, $\mathbf{D} \to \Omega(x)\mathbf{D}\Omega^{-1}(x)$, with $\Omega(x)$ a matrix valued function acting on flavor space but not in Dirac space. On the other hand scale and axial transformations are symmetries only of the classical equation $\mathbf{D}\phi = 0$, and the corresponding currents are anomalous. Note that the classical equation does not imply a zero mode in the quantum equation due to the different boundary conditions of both equations. Actually the solutions of the classical and the quantum equations are normalizable in a spatial box and space-time box respectively. The scale and axial anomalies will be considered in Section 5.

3. Wigner transformation

Seeley's representation (2.7) requires to invert the operator $\mathbf{D}-z$. This can be conveniently accomplished by means of an asymmetric version of the Wigner representation [24,25]. For any operator A, let

$$A(x,p) = \int \mathrm{d}^D y\, e^{iyp} \langle x|A|x-y\rangle = \frac{\langle x|A|p\rangle}{\langle x|p\rangle} \tag{3.1}$$

be its (asymmetric) Wigner representation. $|p\rangle$ is the momentum eigenstate with $\langle x|p\rangle = e^{-ixp}$. When A is a pseudo differential operator [28], the quantity so defined is closely related to the so called symbol of A. For a pseudo differential operator A of complex order m, $A(x,p)$ is of the form $\sum_{j=0}^{\infty} a_{m-j}(x,p)$ where $a_k(x,p)$ are homogeneous functions of degree k in p. In particular $\mathbf{D}^s$ is of order s [23].

From this definition

$$\langle x|A|x\rangle = \int \frac{\mathrm{d}^D p}{(2\pi)^D} A(x,p)\,, \quad \mathrm{Tr}\, A = \int \frac{\mathrm{d}^D x \mathrm{d}^D p}{(2\pi)^D} \mathrm{tr} A(x,p)\,, \tag{3.2}$$

where tr acts on internal and Dirac spinor degrees of freedom only, and the product of two operators satisfy the following formula

$$(AB)(x,p) = \exp(i\partial_p^A \cdot \partial_x^B) A(x,p) B(x,p)\,, \tag{3.3}$$

where ∂_p^A acts only on the p-dependence in $A(x,p)$ and ∂_x^B on the x-dependence in $B(x,p)$.

Let the propagator or resolvent of $\mathbf{D}$, be

$$G(z) = (\mathbf{D} - z)^{-1} \tag{3.4}$$

and $G(x,p;z)$ its Wigner representation. A convenient expression for $G(x,p;z)$ can be obtained using the trick of ref. [9]. Recalling $\hat{x}_\mu|x\rangle = x_\mu|x\rangle$

$$\begin{aligned} \langle x|G(z)|p\rangle &= \langle x|e^{-ik\hat{x}} e^{+ik\hat{x}} (\mathbf{D} - z)^{-1}|p\rangle \\ &= e^{-ikx}\langle x|(\not{k} + \mathbf{D} - z)^{-1} e^{+ik\hat{x}}|p\rangle = \langle x|k\rangle\langle x|(\not{k} + \mathbf{D} - z)^{-1}|p-k\rangle \end{aligned} \tag{3.5}$$

where we have made use of $e^{+ik\hat{x}} i\partial_\mu e^{-ik\hat{x}} = i\partial_\mu + k_\mu$. Choosing $k = p$ we obtain the following compact expression for the Wigner representation of the propagator

$$G(x,p;z) = \langle x|(\not{p} + \mathbf{D} - z)^{-1}|0\rangle \tag{3.6}$$

where $|0\rangle$ is the state of zero momentum, $\langle x|0\rangle = 1$. In practice this implies that $i\partial_\mu$ derivates every x dependence at is right, until it annihilates $|0\rangle$. At this point we have separated from the standard approach, which uses the product formula (3.3) to set up a recurrence relation to compute the symbol of the resolvent [21]. Our method is more efficient for it computes directly, that is non recursively, each of the terms.

Several expansions can be devised to compute $G(x,p;z)$. Two of them will be considered in the next sections. Let us point out that the definition given for $A(x,p)$ is not gauge covariant because $|p\rangle$ is not. It would be very interesting to have a gauge invariant version of the Wigner representation. In fact the problem of finding non local covariant expansions, as required for instance in massless theories, is still open [2, p. 128]. In what follows we will consider only local objects of the form $\langle x|f(\mathbf{D})|x\rangle$ as given by the formula

$$\langle x|f(\mathbf{D})|x\rangle = -\int \frac{\mathrm{d}^D p}{(2\pi)^D} \int_\Gamma \frac{\mathrm{d}z}{2\pi i} f(z) G(x,p;z) \tag{3.7}$$

We will assume that the function $f(z)$ is sufficiently convergent at infinity or else that it can be obtained as a suitable analytical extrapolation from a parametric family $f(z,s)$ in the variable s. In either case the integration over z should be performed in the first place, to yield the Wigner representation of the operator $f(\mathbf{D})$. Afterwards, the p integration is carried out, corresponding to take the diagonal matrix elements of $\langle y|f(\mathbf{D})|x\rangle$, hence restoring gauge covariance.

Because the subject of this paper has been considered extensively in the past [2], and to some extent it has been taken up by mathematicians, we must make some comment on the validity of equations such as (3.2) from a more rigorous point of view. This equation is based on an assumption, namely that there exists an operator $\hat{x}_\mu$ such that it satisfies Heisenberg commutation relations $[i\partial_\mu, \hat{x}_\nu] = i\delta_{\mu\nu}$. Of course this is not the case for manifolds without a global chart and in particular for the nice compact manifolds without boundary, usually considered in the literature. For instance if the previous formalism is applied to compute $\mathrm{Tr}\,(\mathbf{D}^s)$ with $\mathbf{D} = i\partial_x + m$ on the interval $[0,T]$ compactified to a circle, we will find that the sum on the eigenvalues of the exact formula has been replaced by an integral on a continuum momentum label, thus introducing an approximation which becomes exact as $T \to \infty$. On the other hand there is no problem for R^D if the compactification comes from the nature of the external fields in the infinity. For instance an equation similar to (3.2) for the one dimensional second order differential operator $A = -\partial_x^2 + x^2$ would still be exact, including detailed information on the discrete spectrum of A. However such a formula would not be very useful unless one makes some expansion, typically a gradient expansion (to be considered in Section 6), and this kind of expansion substitutes the discrete spectrum by a smoothed continuous density of states. This is a common feature of any asymptotic expansion such as heat kernel [29] or the Wigner-Kirkwood [30] expansions.

4. Inverse mass expansion for Dirac operators

4.1 Seeley-DeWitt coefficients for Dirac operators

In this section we will consider a Dirac operator $\mathbf{D} = \mathbf{D}_0 + m$, with m a mass term and $\mathbf{D}_0$ of the general form (2.1). This will allow us to obtain an expansion for the effective action in powers of $\mathbf{D}_0$ or equivalently in inverse powers of m. This expansion is closely related in spirit to the Seeley-DeWitt expansion for the heat kernel of positive definite second order operators. Both are asymptotic expansions in the number of external fields and their derivatives. We will consider the quantity $\langle x|(\mathbf{D}/\mu)^s|x\rangle$ from which the effective action is easily obtained, Eq. (2.9). From Eq. (3.6) it is straightforward to derive the following expansion

$$G(x,p;z) = -\langle x| \sum_{N=0}^{\infty} \frac{((\not p + z - m)\mathbf{D}_0)^N (\not p + z - m)}{(p^2 + (z-m)^2)^{N+1}} |0\rangle \tag{4.1}$$

where recall that ∂_μ inside $\mathbf{D}_0$ annihilates $|0\rangle$.

The series will generally be asymptotic, giving only the analytical part of $\langle x|(\not p + m - z + \lambda \mathbf{D}_0)^{-1}|0\rangle$ as a function of λ about $\lambda = 0$. This expansion is thus relevant for large p_μ, z and m or equivalently for sufficiently weak and smooth external fields.

Inserting the series for $G(x,p;z)$ in Eq. (3.7) for $f(z) = (z/\mu)^s$, the following expansion will be found

$$\langle x| \left(\frac{\mathbf{D}}{\mu}\right)^s |x\rangle = \sum_{N=0}^{\infty} \frac{1}{(4\pi)^{D/2}} \frac{m^{s+D-N}}{\mu^s} c_N(s) \, \langle x|\mathcal{O}_N(\mathbf{D}_0)|0\rangle \tag{4.2}$$

where D is the space-time dimension, $c_N(s)$ is a numerical function of s, N and D, and $\mathcal{O}_N(\mathbf{D}_0)$ are polynomials of degree N in $\mathbf{D}_0$ which can be written in a D independent form and are matrices in internal and Dirac spinor space. The precise factorization between $c_N(s)$ and $\mathcal{O}_N$ is, to some extent, a matter of convenience.

To obtain explicit expressions for $\mathcal{O}_N$ it is best to work out the numerator in Eq. (4.1) using the object $\mathbf{A}_\mu = \frac{1}{2}\{\gamma_\mu, \mathbf{D}_0\}$, and the property $[p_\mu \mathbf{A}_\mu, \not p] = 0$. Afterwards, an angular average over p_μ is done so that the following integral applies:

$$I_1(k,a,b,D,s,m) = \int \frac{d^D p}{(2\pi)^D} \int_\Gamma \frac{dz}{2\pi i} z^s p^{2k} (z-m)^a (p^2 + (z-m)^2)^{-b} \tag{4.3}$$

where $m > 0$, Γ goes along the real negative axis, $D,k,a,b \in \mathrm{Z}$, $D,a \geq 0$, $D + 2k > 0$ and $\mathrm{Re}\,(s) < 2b - D - 2k - a - 1$. Explicitly

$$I_1 = \frac{(-1)^D}{(4\pi)^{D/2}} \frac{\Gamma(\frac{1}{2}D + k)\Gamma(b - \frac{1}{2}D - k)\Gamma(s+1) m^{s+1+D+2k+a-2b}}{\Gamma(\frac{1}{2}D)\Gamma(b)\Gamma(2b - D - 2k - a)\Gamma(s + 2 + D + 2k + a - 2b)} \tag{4.4}$$

If m is allowed to be negative an additional factor $\epsilon(m)^D$ has to be included, $\epsilon(x)$ being the sign of x. This circumstance will be relevant in the odd dimensional case (see Section 7).

In this way one obtains for $c_N(s)$

$$\begin{aligned} c_N(s) &= \zeta_{N-D}\frac{\Gamma(s+1)}{\Gamma(s+1-N+D)} \\ \zeta_{N-D} &= (-1)^D\frac{\Gamma(\bar{N}-\frac{1}{2}D)}{\Gamma(N-D+1)}\,, \qquad \bar{N}=\lfloor\frac{N}{2}+1\rfloor \end{aligned} \tag{4.5}$$

where $\lfloor x\rfloor$ stands for largest integer not exceeding x. This explicit expression for ζ_K holds for $N\geq D$ and an analytical continuation in D is understood for $N<D$. Note that ζ_K depends also on D (mod 2). These quantities can be obtained more conveniently from the recurrence

$$\begin{aligned} &\zeta_{K-1}=\epsilon_K\zeta_K\,, \\ &\zeta_0=1 \quad (\text{even } D), \qquad \zeta_0=-\Gamma(\frac{1}{2}) \quad (\text{odd } D), \\ &\epsilon_K=2 \quad (\text{even } D+K), \qquad \epsilon_K=K \quad (\text{odd } D+K) \end{aligned} \tag{4.6}$$

In particular ζ_K vanishes for negative K if D is odd. Some useful particular values of the coefficients are given in Table 4.1.

On the other hand, for the operators $\mathcal{O}_N$ at lower orders, one obtains

$$\begin{aligned} \mathcal{O}_0(\mathbf{D}) &= 1 \\ \mathcal{O}_1(\mathbf{D}) &= \mathbf{D}+\mathbf{A}_\mu\gamma_\mu \\ \mathcal{O}_2(\mathbf{D}) &= \mathbf{D}^2+\mathbf{A}_\mu^2 \\ \mathcal{O}_3(\mathbf{D}) &= \mathbf{D}^3+\frac{1}{2}\mathbf{D}^2\mathbf{A}_\mu\gamma_\mu+\frac{1}{2}\mathbf{A}_\mu\gamma_\mu\mathbf{D}^2+\mathbf{A}_\mu\mathbf{D}\mathbf{A}_\mu \\ &\quad +\frac{1}{3}(\mathbf{A}_\mu^2\mathbf{A}_\nu\gamma_\nu+\mathbf{A}_\mu\mathbf{A}_\nu\mathbf{A}_\mu\gamma_\nu+\mathbf{A}_\mu\mathbf{A}_\nu^2\gamma_\mu) \\ \mathcal{O}_4(\mathbf{D}) &= \frac{1}{2}\mathbf{D}^4+\frac{1}{3}(\mathbf{D}^2\mathbf{A}_\mu^2+\mathbf{A}_\mu\mathbf{D}^2\mathbf{A}_\mu+\mathbf{A}_\mu^2\mathbf{D}^2) \\ &\quad +\frac{1}{6}(\mathbf{A}_\mu^2\mathbf{A}_\nu^2+(\mathbf{A}_\mu\mathbf{A}_\nu)^2+\mathbf{A}_\mu\mathbf{A}_\nu^2\mathbf{A}_\mu) \end{aligned} \tag{4.7}$$

where $\mathbf{A}_\mu=\frac{1}{2}\{\gamma_\mu,\mathbf{D}\}$. We want to emphasize that these expressions do not make any assumption on $\mathbf{D}$ other than Eqs. (2.1) and (2.2). Note also that they have been written in a D independent fashion. The expression for the coefficients $\mathcal{O}_N(\mathbf{D})$ for all even orders, is given in Appendix A.

In the expansion (4.2), what actually appears are the matrix valued functions $\mathcal{O}_N(x) = \langle x|\mathcal{O}_N|0\rangle$, rather than the operators $\mathcal{O}_N$ themselves. Because the regularization is vector gauge invariant, we expect that $\mathcal{O}_N(x)$ will be covariant under vector gauge transformations of $\mathbf{D}$, that is, under $\mathbf{D} \to \Omega(x)\mathbf{D}\Omega^{-1}(x)$, $\mathcal{O}_N(x)$ will transform as $\mathcal{O}_N(x) \to \Omega(x)\mathcal{O}_N(x)\Omega^{-1}(x)$, even if $|0\rangle$ itself is not gauge invariant. Let us show this explicitly for $\mathcal{O}_2$. Letting $\mathbf{D} = i\not{\mathcal{D}} + \mathbf{X}$ with $i\mathcal{D}_\mu = i\partial_\mu + V_\mu(x)$, the operator $\mathcal{O}_2$ can be written as

$$\mathcal{O}_2 = \mathbf{X}^2 + \frac{1}{4}\{\gamma_\mu, \mathbf{X}\}^2 + \frac{1}{2}\sigma_{\mu\nu} iF_{\mu\nu} + \frac{1}{2}[\gamma_\mu, [i\mathcal{D}_\mu, \mathbf{X}]], \quad \sigma_{\mu\nu} = \frac{1}{2}[\gamma_\mu, \gamma_\nu], \quad iF_{\mu\nu} = [i\mathcal{D}_\mu, i\mathcal{D}_\nu] \tag{4.8}$$

Therefore all the derivatives appear inside commutators and $\mathcal{O}_2$ is a purely multiplicative operator. In this case taking the matrix element $\langle x|\mathcal{O}_2|0\rangle$ does not break gauge invariance. The same thing can be shown for higher orders. A more economical manner of establishing the gauge covariance of an expression is the following. Let $\hat{f}$ be an operator formed algebraically out of $i\mathcal{D}_\mu$ and other gauge covariant non derivative operators $\mathbf{X}_i(x)$. Then of course $\hat{f}$ is itself a gauge covariant operator. The matrix valued function $f(x) = \langle x|\hat{f}|0\rangle$ will be gauge covariant if $\hat{f}$ is a multiplicative operator, that is, if all the $i\mathcal{D}_\mu$ in $\hat{f}$ appear as covariant derivatives inside commutators. In turn this will be only the case if $\hat{f}$ is invariant under $V_\mu(x) \to V_\mu(x) + a_\mu$, with a_μ an arbitrary constant c-number, that is if

$$\delta_g \hat{f} = 0 \qquad \text{under} \qquad \delta_g \mathbf{D} = \not{a}, \quad \delta_g \mathbf{X}_i = 0 \tag{4.9}$$

For the expressions listed above one checks, using $\delta_g \mathbf{A}_\mu = -a_\mu$, that in fact $\delta_g \mathcal{O}_N = 0$. Note that this rule refers only to vector gauge invariance of local objects. For instance in the Chern-Simons action $\frac{1}{16\pi} i \int \mathrm{d}^3x\, \epsilon_{\mu\nu\alpha}\, \mathrm{tr}(V_\mu \partial_\nu V_\alpha - \frac{2}{3} i V_\mu V_\nu V_\alpha)$, (to be discussed in Section 7), the integrand is only invariant up to a total derivative but the action itself is invariant.

Another remark about gauge invariance is that it follows from formal integration by parts over p [9] in Eq. (3.7). In the case of the integral $I_1(k, a, b, D, s, m)$ in Eq. (4.4), only the relationship

$$\frac{I_1(k+1, a-2, b, D, s, m)}{I_1(k, a, b, D, s, m)} = \frac{\frac{1}{2}D + k}{b - \frac{1}{2}D - k - 1} \tag{4.10}$$

is required to form the gauge covariant operators $\mathcal{O}_N$. Because this ratio is independent of s, the same quantities will appear in the expansion of $\langle x|f(\mathbf{D})|x\rangle$ for other analytic functions $f(z)$ whenever the z integrals involved are convergent. This argument is made rigorous in [23] for $f(A)$, A being a zeroth order pseudo differential operator and $f(z)$ analytic in a region containing the spectrum of A.

It is worth noticing that the operators $\mathcal{O}_N$ are related among themselves in a simple way. This follows from the observation that $\mathbf{D}$ is invariant under

$$\delta_m \mathbf{D}_0 = -\delta m, \qquad \delta_m m = \delta m \tag{4.11}$$

Imposing $\delta_m \langle x|\mathbf{D}^s|x\rangle = 0$ in Eq. (4.2), one finds

$$\delta_m \mathcal{O}_N = -\epsilon_{N-D}\mathcal{O}_{N-1}\delta m \tag{4.12}$$

where the quantities ϵ_K were introduced in Eq. (4.6). Therefore from $\mathcal{O}_N$ one can obtain algebraically $\mathcal{O}_M, \quad M < N$. It is interesting to notice that the variations δ_g (see Eq. (4.9)) and also δ_m (see Eq. (4.11)) greatly restrict, but do not completely fix, the actual values of the coefficients appearing in the expression for $\mathcal{O}_N$ as given by Eq. (4.7). It turns out that the even order coefficients can be deduced entirely as combinatorial factors, the odd orders being obtained by taking the variation with respect the mass, as given by Eq. (4.12). Further details can be looked up at Appendix A.

The coefficient $c_N(s)$ has simple poles at $s = -1, -2, -3, \ldots, N - D$ for $N < D$. This is in agreement with general theorems for pseudo differential operators [23]. On the other hand $c_N(s)$ vanishes for $s = 0, 1, 2, \ldots, N - D$ if $N \geq D$. Thus for non negative integer values of s, the inverse mass expansion is exact [23] and is just a polynomial in m. Setting $m = 0$, one finds:

$$\langle x|\mathbf{D}^n|x\rangle = \frac{1}{(4\pi)^{D/2}}\zeta_n\Gamma(n+1)\mathcal{O}_{D+n}(x;\mathbf{D}), \qquad n = 0, 1, 2, \ldots \tag{4.13}$$

This result is regularization dependent, since a naive evaluation of the matrix element would diverge. In fact, the l.h.s. stands for $\langle x|\mathbf{D}^{n+s}|x\rangle|_{s=0}$ through an analytical continuation in s, and thus it is specific of the ζ-function regularization.

4.2 Effective action

For the effective action we have, using (2.9) and (4.2),

$$\begin{aligned} W(\mathbf{D}) &= \sum_{N=0}^{\infty}\left(\alpha^W_{N-D} + \beta^W_{N-D}\log\left(\frac{m}{\mu}\right)\right)m^{D-N}\langle\mathcal{O}_N(\mathbf{D}_0)\rangle \\ \alpha^W_K &= -\frac{\mathrm{d}}{\mathrm{d}s}c_{D+K}(0), \qquad \beta^W_K = -c_{D+K}(0) \end{aligned} \tag{4.14}$$

In the r.h.s. we have used the shorthand notation

$$\langle f(\mathbf{D})\rangle = \frac{1}{(4\pi)^{D/2}}\int \mathrm{d}^D x \mathrm{tr}\,\langle x|f(\mathbf{D})|0\rangle = \frac{1}{(4\pi)^{D/2}}\mathrm{tr}\langle 0|f(\mathbf{D})|0\rangle \tag{4.15}$$

The symbol $\langle\ \ \rangle$ enjoys some of the properties of the trace and in particular the trace cyclic property, but only for multiplicative operators,

$$\langle AB\rangle = \langle BA\rangle \qquad\qquad \text{if} \quad \delta_g A = \delta_g B = 0 \tag{4.16}$$

If one deals, however, with non multiplicative operators, like e.g. differential operators, an additional commutator has to be added. Also note that unlike Tr , the symbol $\langle\ \ \rangle$ has dimension of m^{-D}.

More explicit expressions for α_K^W and β_K^W are given in Table 4.2. Note that α_K^W, β_K^W depend only on D (mod 2).

In particular for $D=4$

$$\begin{aligned} W = &\left\langle \frac{25}{24}m^4\mathcal{O}_0 - \frac{11}{9}m^3\mathcal{O}_1 - \frac{3}{2}m^2\mathcal{O}_2 + 2m\mathcal{O}_3 - \frac{1}{m}\mathcal{O}_5 + \cdots \right\rangle \\ &+ \log\left(\frac{m}{\mu}\right)\left\langle -\frac{1}{2}m^4\mathcal{O}_0 + \frac{2}{3}m^3\mathcal{O}_1 + m^2\mathcal{O}_2 - 2m\mathcal{O}_3 - \mathcal{O}_4 \right\rangle \end{aligned} \tag{4.17}$$

We can distinguish three contributions to the effective action, according to their m dependence, namely, 1) the contribution from α_K^W and $\beta_K^W \log\mu$ for $N \le D$, which is a polynomial of degree D in m, 2) a logarithmic part from $\beta_K^W \log m$ with $N \le D$ and 3) an inverse mass expansion from α_K^W and $N > D$. The polynomial part is regularization dependent as it can be modified by adding suitable local polynomial counterterms to the action. These terms are such that the action depends on $\mathbf{D}$ and not on $\mathbf{D}_0$ and m separately and they vanish for odd dimensions. The logarithmic and inverse mass parts are regularization independent. In fact by applying the operator $(\mathrm{d}/\mathrm{d}m)^n$, $n > D$ at both sides of Eq. (4.14), the l.h.s. becomes $(-1)^n\mathrm{Tr}\,(\mathbf{D}^{-n})$ which is ultraviolet finite and hence independent of the renormalization prescription. In the r.h.s. the polynomial part as well as the dependence on μ disappears whereas the other terms remain, yielding a pure inverse mass expansion without logarithms. The dependence on $\log\mu$ was a trivial additive constant for the unregularized action. This is no longer the case after renormalization, indicating that the action has developed a scale anomaly. The practical interest of the former arguments lies in the possibility of reconstructing the action by dealing with explicitly convergent and hence regularization independent objects, namely $(-1)^n\mathrm{Tr}\,(\mathbf{D}^{-n})$, $n > D$, and integrating back in the mass parameter. By properly fixing the arbitrary integration constants one might reproduce a given renormalized action. This idea is already contained in Eq. (2.10) for the specific case of the ζ-function.

4.3 Effective currents

Finally, we can also obtain an inverse mass expansion for the current $\mathbf{J}$ introduced in Section 2. Rather than computing $\mathbf{J}$ through the variation of $\langle \mathcal{O}_N \rangle$ in Eq. (4.14), we will use the closed expression (2.13). Recalling that $\langle x|\mathbf{D}^s|x\rangle$ has a simple pole at $s=-1$, we obtain the following expansion for the current

$$\mathbf{J}(x) = \frac{1}{(4\pi)^{D/2}} \sum_{N=0}^{\infty} \left(\alpha^J_{N-D} + \beta^J_{N-D} \log\left(\frac{m}{\mu}\right)\right) m^{D-N-1} \mathcal{O}_N(x;\mathbf{D}_0) \tag{4.18}$$

where

$$\begin{aligned} \alpha^J_K &= -\frac{\mathrm{d}}{\mathrm{d}s}(s c_{D+K}(s-1))\Big|_{s=0} = \epsilon_{K+1}\alpha^W_{K+1} \\ \beta^J_K &= -s c_{D+K}(s-1)\Big|_{s=0} = \epsilon_{K+1}\beta^W_{K+1} \end{aligned} \tag{4.19}$$

Again we can check that $\delta_m \mathbf{J} = 0$. Also comparing with the expansion for the action, one finds

$$\frac{\delta\langle \mathcal{O}_N\rangle}{\delta \mathbf{D}} = \frac{1}{(4\pi)^{D/2}} \epsilon_{N-D} \mathcal{O}_{N-1}(x) \tag{4.20}$$

which is consistent with Eq. (4.12). A similar relation holds for the Seeley-DeWitt coefficients [2]. It is instructive to try to obtain the same coefficients $\mathcal{O}_N(x)$ from a heat kernel approach. The main result is that in even dimensions, the Seeley-DeWitt coefficients of the second order bosonic differential operator $-\mathbf{D}^2$ coincide with the inverse mass expansion coefficients of even order $\mathcal{O}_{2n}(\mathbf{D})$ of the first order fermionic operator $\mathbf{D}$. This result has interesting consequences and will be proven in Appendix B.

5. Anomalies within the ζ-function regularization

5.1 Chiral and scale anomalies

As it is well known the effective action does not share all the symmetries of the classical action. In the ζ-function regularization approach, this is because not all the symmetry transformations of the classical equation $\mathbf{D}\phi = 0$ are also symmetries of the eigenvalue equation $\mathbf{D}\phi_n = \lambda_n \phi_n$. Symmetries which are broken by a mass term classically, develop an anomaly at the quantum level. This is the case of chiral and scale transformations.

As already mentioned under the combined set of vector and axial transformations only the axial ones present an anomaly, within a ζ-function regularization. Vector gauge symmetry remains unbroken. The axial anomaly is defined [15] as the variation of the effective action under an axial gauge transformation of $\mathbf{D}$, that is

$$\mathbf{D} \underset{A}{\longrightarrow} e^{-i\alpha\gamma_5}\mathbf{D}e^{-i\alpha\gamma_5}, \qquad \delta_A\mathbf{D} = -\{i\alpha\gamma_5, \mathbf{D}\}, \qquad \mathcal{A}_A = \delta_A W(\mathbf{D}) \tag{5.1}$$

where $\alpha(x)$ is an infinitesimal matrix valued function in flavor space only. This transformation is consistent with the general structure assumed for $\mathbf{D}$, i.e. $\mathbf{D} = i\partial\!\!\!/ + \mathbf{Y}$, with $\mathbf{Y}$ a local function. $\mathcal{A}_A$ is the axial anomaly, which is ultraviolet finite and of dimension D in the external fields [15]. The best way of computing $\mathcal{A}_A$ in the ζ-function context is to use the same trick as in Eq. (2.12):

$$\begin{aligned} \mathcal{A}_A &= -\delta_A \mathrm{Tr}\,\log\mathbf{D} = -\delta_A\frac{\mathrm{d}}{\mathrm{d}s}\mathrm{Tr}\left(\frac{\mathbf{D}}{\mu}\right)^s\Big|_{s=0} \\ &= \frac{\mathrm{d}}{\mathrm{d}s}\mathrm{Tr}\left(s\{i\alpha\gamma_5,\mathbf{D}\}\frac{\mathbf{D}^{s-1}}{\mu^s}\right)\Big|_{s=0} = \mathrm{Tr}\left(2i\alpha\gamma_5\mathbf{D}^s\right)\Big|_{s=0} := \mathrm{Tr}\left(2i\alpha\gamma_5\mathbf{D}^0\right) \end{aligned} \tag{5.2}$$

Here we have used the trace cyclic property and also that $\langle x|\mathbf{D}^s|x\rangle$ is analytic at $s=0$. Since formally $\mathbf{D}^0$ would correspond to the identity operator, the previous result can be interpreted as a regularization of this operator. Note that the anomaly is independent of the scale μ. Now using the expression for $\langle x|\mathbf{D}^n|x\rangle$, $n = 0,1,2,\ldots$ in Eq. (4.13) (for even D), we have

$$\mathcal{A}_A = \langle 2i\alpha\gamma_5\mathcal{O}_D\rangle \tag{5.3}$$

which is manifestly ultraviolet finite. With the expressions for $\mathcal{O}_2$ and $\mathcal{O}_4$ in Eq. (4.7) and after some algebra using only defining properties of the gamma matrices, we obtain simpler explicit forms for $\mathcal{A}_A$ in two and four dimensions

$$\begin{aligned} &(D=2) \qquad\qquad \mathcal{A}_A = \langle 2i\alpha(x)\gamma_5\mathbf{D}^2\rangle \\ &(D=4) \qquad\qquad \mathcal{A}_A = \langle 2i\alpha(x)\gamma_5(\frac{1}{2}\mathbf{D}^4 + \frac{1}{3}\mathbf{D}\mathbf{A}_\mu^2\mathbf{D})\rangle \end{aligned} \tag{5.4}$$

These expressions are more easily arrived at by going back to Eq. (4.1) to compute $\mathrm{Tr}\,(2i\alpha\gamma_5\mathbf{D}^s)|_{s=0}$ directly. For illustration purposes we quote in Appendix C more elaborated versions of Eq. (5.4) for a Dirac operator with the most general spinorial structure in two and four dimensions, after explicit evaluation of the Dirac traces.

The scale invariance can be treated in a completely similar way in this regularization. The trace anomaly is the variation of the effective action under a scale transformation of $\mathbf{D}$ [1].

$$\mathbf{D} = i\partial\!\!\!/ + \mathbf{Y}(x) \underset{S}{\longrightarrow} i\partial\!\!\!/ + e^{-\epsilon}\mathbf{Y}(e^{-\epsilon}x), \qquad\qquad \mathcal{A}_S = \delta_S W \tag{5.5}$$

We can best compute the trace anomaly by rewriting the scale transformation as a homogeneous transformation for $\mathbf{D}$

$$\delta_S \mathbf{D} = -\epsilon(\mathbf{D} + [x_\mu \partial_\mu, \mathbf{D}]), \tag{5.6}$$

and using the ζ-function regularization of the action

$$\begin{aligned} \mathcal{A}_S &= -\delta_S \frac{\mathrm{d}}{\mathrm{d}s} \mathrm{Tr} \left(\frac{\mathbf{D}}{\mu} \right)^s \Big|_{s=0} \\ &= \epsilon \frac{\mathrm{d}}{\mathrm{d}s} s \mathrm{Tr} \left((\mathbf{D} + [x_\mu \partial_\mu, \mathbf{D}]) \frac{\mathbf{D}^{s-1}}{\mu^s} \right) \Big|_{s=0} = \epsilon \frac{\mathrm{d}}{\mathrm{d}s} s \mathrm{Tr} \left(\mathbf{D} \frac{\mathbf{D}^{s-1}}{\mu^s} \right) \Big|_{s=0} \\ &= \epsilon \mathrm{Tr}\left(\mathbf{D}^0\right) \end{aligned} \tag{5.7}$$

Likewise the axial case, the zeroth power of $\mathbf{D}$ appears in the final expression for the anomaly. Using Eq. (4.13) one finally obtains

$$\mathcal{A}_S = \epsilon \zeta_0 \langle \mathcal{O}_D \rangle \tag{5.8}$$

for the trace anomaly.

From the last line of Eq. (5.7), the scale anomaly can alternatively be written as

$$\mathcal{A}_S = \epsilon \mu \frac{\partial W}{\partial \mu} \tag{5.9}$$

which is consistent with Eq. (4.14). Because the axial anomaly $\delta_A W = \mathcal{A}_A$ does not depend on μ, and using the cross variation condition $[\delta_A, \delta_S] = 0$, we conclude that $\mathcal{A}_S$ is chirally invariant. In addition it is also scale invariant. Moreover the scale anomaly vanishes for odd D because from Eq. (4.20) $\delta_X \langle \mathcal{O}_D \rangle = 0$ for an arbitrary variation.

Another issue is that of the anomalous breaking of parity in odd dimensions and the related Chern-Simons action [31,32]. It will be considered in Section 7.

5.2 Minimal form of the anomaly

The presence of the anomaly indicates that the effective action has terms which are not chirally invariant. As it is well known different regularizations in principle produce effective actions which differ in local polynomial terms. Given the fact that such different actions are related by counterterms, whose parameters are to be fixed anyway by some renormalization prescription, they are considered physically equivalent. These differences reflect in turn in the form of the anomaly. The different regularizations give anomalies differing by so called unessential terms, that is, terms which can be obtained as the variation of local polynomial actions. Bardeen [15] worked out the four dimensional case, including vector, axial-vector, scalar and pseudoscalar fields in the Dirac operator. He showed that the scalar and pseudoscalar fields did not contribute to the essential anomaly and moreover that the only essential terms were those of abnormal pseudoparity, i.e. containing the Levi-Civita pseudotensor and thus purely imaginary in Euclidean space. This Bardeen's or minimal anomaly were later shown to derive from the Wess-Zumino-Witten action [17,20], which of course is not a local polynomial. Since the work of Bardeen, it has been shown that nongauge fields [33,34], and internal gauge fields (i.e. transforming homogeneously under chiral transformations) [35] do not contribute to the essential anomaly. This is also suggested by the fact that if the formal variation of the action is regularized in a chirally covariant way, there is an obstruction to the integrability conditions which depends on the vector and axial gauge fields only [2,12].

Let us restate this result with our formalism in a way which is easily extended to higher dimensions and more general theories, i.e. relying only on algebraic transformation properties but not on the detailed coupling structure of the external fields. That is, let us show that in fact for a completely general Dirac operator in two and four dimensions in Euclidean flat space one can write enough counterterms to bring the anomaly to Bardeen's form. To be precise, let $\mathbf{D}$ be of the form $\mathbf{D} = \mathbf{D}_0 + \mathbf{X}$ where $\mathbf{X}$ is a local function, and $\mathbf{D}_0$ and $\mathbf{X}$ transform independently under axial transformations

$$\mathbf{D}_0 \underset{A}{\longrightarrow} e^{-i\alpha\gamma_5}\mathbf{D}_0 e^{-i\alpha\gamma_5}, \qquad \mathbf{X} \underset{A}{\longrightarrow} e^{-i\alpha\gamma_5}\mathbf{X} e^{-i\alpha\gamma_5} \tag{5.10}$$

$\mathbf{D}_0$ contains the derivative part, and hence the corresponding fields transform inhomogeneously. We want to show that all the contributions to the anomaly coming from $\mathbf{X}$ can be removed by counterterms. The construction is more easily presented by following the approach of Ref. [35]. Let $\mathbf{D}_t = \mathbf{D}_0 + t\mathbf{X}$, then one can write the identity

$$W(\mathbf{D}) - W(\mathbf{D}_0) = \int_0^1 \mathrm{d}t\, \delta_X W(\mathbf{D}_t) = \int_0^1 \mathrm{d}t \int \mathrm{d}^D x\, \mathrm{tr}\,(\mathbf{X}\mathbf{J}(\mathbf{D}_t)) \tag{5.11}$$

Here $\mathbf{J}(\mathbf{D})$ is the current introduced in Eq. (2.11). The axial anomalous contribution to the action containing $\mathbf{X}$ will be local polynomials if and only if the current has the form

$$\mathbf{J}(\mathbf{D}) = \mathbf{J}_c(\mathbf{D}) + \mathbf{P}(\mathbf{D}) \tag{5.12}$$

where $\mathbf{P}$ is a local polynomial and $\mathbf{J}_c$ is a chiral covariant current, transforming as

$$\mathbf{J}_c \underset{A}{\longrightarrow} e^{i\alpha\gamma_5}\mathbf{J}_c e^{i\alpha\gamma_5} \tag{5.13}$$

so that $\delta_A \mathrm{Tr}\,(\mathbf{X}\mathbf{J}_c) = 0$. Notice the opposite sign for the axial transformation as compared to the Dirac operator, Eq. (5.1). Indeed, the above decomposition of the current yields an analogous separation for the action, namely

$$W(\mathbf{D}) = W(\mathbf{D}_0) + \int_0^1 \mathrm{d}t \int \mathrm{d}^D x\, \mathrm{tr}(\mathbf{X}\mathbf{J}_c(\mathbf{D}_t)) + \int_0^1 \mathrm{d}t \int \mathrm{d}^D x\, \mathrm{tr}(\mathbf{X}\mathbf{P}(\mathbf{D}_t)) \qquad (5.14)$$

The first term gives $\mathcal{A}_A(\mathbf{D}_0)$, the second is chiral invariant and the last is the local polynomial counterterm. The observation that the current is of the form (5.12) was already made in Ref. [26] for chiral fermions in the presence of gauge fields. As already pointed out by Bardeen and Zumino, the fact that the total current admits such a decomposition is not obvious and requires a constructive proof for each case.

Let us construct $\mathbf{P}$ explicitly for the two dimensional case. Clearly $\mathbf{P}$ must satisfy the conditions

$$\delta_A \mathbf{P} = \delta_A \mathbf{J} = \frac{\delta \mathcal{A}_A}{\delta \mathbf{D}}\,, \qquad \delta_g \mathbf{P} = 0 \qquad (5.15)$$

as a consequence of Eq. (5.12) and vector gauge invariance. Notice that since the anomaly involves the symbol $\langle\ \rangle$ and $\mathbf{D}$ is not a multiplicative operator (see the remark to Eq. (4.16)), cyclic property might not be applied in principle to compute $\delta\mathcal{A}_A/\delta\mathbf{D}$. Nevertheless, the vector gauge invariance of the anomaly and the fact that $\delta\mathbf{D} = \mathbf{X}$ is a multiplicative operator, allows to do so, yielding $\delta_X \mathcal{A}_A = \langle \mathbf{X}\{2i\alpha\gamma_5, \mathbf{D}\}\rangle$. One can check that the first relation is then satisfied by

$$\mathbf{P}_0 = -\frac{1}{4\pi}\langle x|\mathbf{D}|0\rangle \qquad (5.16)$$

Unfortunately this solution breaks vector gauge invariance. In order to construct $\mathbf{P}$ we should subtract a new polynomial $\mathbf{P}_1$ from $\mathbf{P}_0$ to reestablish vector gauge invariance. Also $\mathbf{P}_1$ must be axial covariant in order not to modify the already correct axial transformation of $\mathbf{P}_0$

$$\delta_g \mathbf{P}_1 = \delta_g \mathbf{P}_0\,, \qquad \mathbf{P}_1 \underset{A}{\longrightarrow} e^{i\alpha\gamma_5}\mathbf{P}_1 e^{i\alpha\gamma_5} \qquad (5.17)$$

The only object algebraically made out of $\mathbf{D}$, transforming axially as $\mathbf{P}_1$ should, is $\gamma_\mu \mathbf{D}\gamma_\mu$, which however is also vector gauge invariant and hence cannot match $\delta_g \mathbf{P}_0$. Therefore we must resort to new objects or use more information on $\mathbf{D}$ to write enough counterterms. Let $\bar{\mathbf{D}}$ be other Dirac operator with

$$\delta_g \bar{\mathbf{D}} = \delta_g \mathbf{D} = \not{d}, \qquad \bar{\mathbf{D}} \underset{A}{\longrightarrow} e^{i\alpha\gamma_5}\bar{\mathbf{D}} e^{i\alpha\gamma_5} \qquad (5.18)$$

If $\mathbf{D}$ has the standard hermiticity, (i.e. the Hamiltonian is Hermitian in Minkowski space), a solution is provided by $\bar{\mathbf{D}} = -\mathbf{D}^\dagger$. Now it is straightforward to obtain $\mathbf{P}_1$ by writing all the possible objects of second order with the correct axial transformation, constructed with $\mathbf{D}$, $\bar{\mathbf{D}}$ and γ_μ and adjusting their coefficients to match $\delta_g \mathbf{P}_0$. We finally find the solution

$$\mathbf{P} = -\frac{1}{4\pi}\langle x|(\mathbf{D} - \bar{\mathbf{D}})|0\rangle \tag{5.19}$$

In the previous construction $\mathbf{P}_0$ has been obtained by trial and error. In more complicated cases, the best way to proceed is to introduce a polynomial action in the Dirac operator, $W_0(\mathbf{D})$, from which $\mathbf{P}_0$ formally derives, since the number of possible terms decreases substantially. The fact that the anomaly cannot be subtracted by local and polynomial counterterms, prevents the existence of such an action in a literal sense. Nevertheless one can impose $\delta_X W_0(\mathbf{D}) = \int \mathrm{d}^D x\, \mathrm{tr}(\mathbf{X}\mathbf{P}_0(\mathbf{D}))$ and $\delta_A W_0(\mathbf{D}) = \mathcal{A}_A$ modulo commutator terms, which would vanish if cyclic property were valid, i.e. if all the operators involved were multiplicative. That is what we mean by formal in this context. There are two key observations. First, that actions $W_0(\mathbf{D})$ which are algebraically made out of $\mathbf{D}$, uniquely determine the current $\mathbf{P}_0(\mathbf{D})$, namely, by substituting $\mathbf{D} \to \mathbf{D}+\mathbf{X}$ in $W_0(\mathbf{D})$, keeping terms with just one $\mathbf{X}$ and freely using the cyclic property to bring all the $\mathbf{X}$ say to the left. The relation is unique even if $\mathbf{P}_0(\mathbf{D})$ is only the formal variation of $W_0(\mathbf{D})$. And second, if the current $\mathbf{P}_0(\mathbf{D})$ reproduces the variation of the axial anomaly, i.e. $\delta_A \mathbf{P}_0(\mathbf{D}) = \delta\mathcal{A}_A/\delta\mathbf{D}$ (see Eq. (5.15)), the action must reproduce the anomaly $\delta_A W_0(\mathbf{D}) = \mathcal{A}_A$ at the formal level. It is therefore advantageous to solve this latter equation and hence to derive $\mathbf{P}_0(\mathbf{D})$.

One can check that a solution in the two-dimensional case, is given by the action

$$W_0(\mathbf{D}) = -\left\langle \frac{1}{2}\mathbf{D}^2 \right\rangle = -\frac{1}{8\pi}\int \mathrm{d}^2 x\, \mathrm{tr}\langle x|\mathbf{D}^2|0\rangle \tag{5.20}$$

which of course is not gauge invariant. $W_0(\mathbf{D})$ formally gives the axial anomaly, and its current is the same $\mathbf{P}_0(\mathbf{D})$ found previously, Eq. (5.16). In summary, $\mathbf{P}_0$ is a local polynomial current with the same anomaly as $\mathbf{J}$. Once $\mathbf{P}_0$ is available one can proceed as explained above to obtain $\mathbf{P}$.

The previous method can be applied to the four dimensional case. We find for W_0

and $\mathbf{P}$ the following expressions

$$\begin{aligned} W_0(\mathbf{D}) &= -\langle \frac{1}{24}\mathbf{D}^4 + \frac{1}{24}\mathbf{D}^2\gamma_\mu\mathbf{D}^2\gamma_\mu + \frac{1}{12}\mathbf{D}^3\gamma_\mu\mathbf{D}\gamma_\mu\rangle \\ \mathbf{P} &= -\frac{1}{12}\frac{1}{(4\pi)^2}\langle x|(2(\mathbf{D}^3 - \bar{\mathbf{D}}\mathbf{D}\bar{\mathbf{D}}) + \mathbf{D}\gamma_\mu\mathbf{D}^2\gamma_\mu - \bar{\mathbf{D}}\gamma_\mu\bar{\mathbf{D}}\mathbf{D}\gamma_\mu \\ &\quad + \gamma_\mu\mathbf{D}^2\gamma_\mu\mathbf{D} - \gamma_\mu\mathbf{D}\bar{\mathbf{D}}\gamma_\mu\bar{\mathbf{D}} + \mathbf{D}\gamma_\mu\mathbf{D}\gamma_\mu\mathbf{D} - \bar{\mathbf{D}}\gamma_\mu\bar{\mathbf{D}}\gamma_\mu\bar{\mathbf{D}} \\ &\quad + \gamma_\mu\mathbf{D}^3\gamma_\mu - \gamma_\mu\mathbf{D}\bar{\mathbf{D}}\mathbf{D}\gamma_\mu + \mathbf{D}^2\gamma_\mu\mathbf{D}\gamma_\mu - \bar{\mathbf{D}}\mathbf{D}\gamma_\mu\mathbf{D}\gamma_\mu \\ &\quad + \gamma_\mu\mathbf{D}\gamma_\mu\mathbf{D}^2 - \gamma_\mu\mathbf{D}\gamma_\mu\mathbf{D}\bar{\mathbf{D}})|0\rangle \end{aligned} \tag{5.21}$$

The terms without $\bar{\mathbf{D}}$ are those coming from $\mathbf{P}_0$. This polynomial generalizes that found in Ref. [26]. This completes the proof that all the fields in $\mathbf{D}$ transforming homogeneously under chiral rotations do not contribute to the essential anomaly in two and four dimensions. If one keeps only external vector and axial fields in $\mathbf{D}_0$, $\mathcal{A}_A(\mathbf{D}_0)$ is Bardeen's anomaly (up to some normal pseudoparity terms which can again be removed by counterterms). Explicit expressions for the counterterms after having worked out the Dirac algebra can be looked up at Appendix D.

An interesting aspect of the previous results is that the counterterms needed to reproduce the Bardeen form of the anomaly in ζ-function regularization requires introducing, besides $\mathbf{D}$, a new Dirac operator $\bar{\mathbf{D}}$ transforming in the same way as the current under the chiral group. As already mentioned, an operator transforming in that way in Euclidean space is given by $-\mathbf{D}^\dagger$. This agrees with similar findings in other regularization schemes, using ad hoc prescriptions like e.g. to separate the action into real and imaginary parts $W(\mathbf{D}) = W^+(\mathbf{D}) + W^-(\mathbf{D})$, by means of the formula

$$\begin{aligned} W^+(\mathbf{D}) &= -\frac{1}{4}\left\{\mathrm{Tr}\,\log(\mathbf{D}\mathbf{D}^\dagger) + \mathrm{Tr}\,\log(\mathbf{D}^\dagger\mathbf{D})\right\} \\ W^-(\mathbf{D}) &= -\frac{1}{4}\left\{\mathrm{Tr}\,\log(\mathbf{D}^2) - \mathrm{Tr}\,\log(\mathbf{D}^{\dagger 2})\right\} \end{aligned} \tag{5.22}$$

and similar ones [2,10,36,37].

In contrast to the axial anomaly, the scale anomaly $\mathcal{A}_S$ contains no unessential terms: by dimensional counting the possible local polynomials would be scale invariant or else would have to include external functions, thus introducing an anomaly in the Poincaré symmetry.

6. Gradient expansion

6.1 General considerations

In Section 4 we considered an inverse mass expansion for the effective action of $\mathbf{D}$. It was both an expansion in the number of external fields and the number of derivatives. Here we shall consider an expansion in the number of derivatives and the number of fields with Lorentz indices. That is, we take

$$\mathbf{D} = \mathbf{M} + \mathbf{D}_0 \,, \qquad \mathbf{M}(x) = S(x) + i\gamma_5 P(x) \tag{6.1}$$

where $S(x)$ and $P(x)$ are scalar and pseudoscalar fields and $\mathbf{D}_0$ includes $i\not\partial$ as well as vector, axial vector, tensor fields, etc, and we expand in powers of $\mathbf{D}_0$. This is a resummation of the inverse mass expansion, relaxing the restriction that S and P should be weak fields. Another standard resummation, complementary to this one, is the perturbative expansion which assumes weak but not necessarily smooth fields. The gradient expansion is a semiclassical expansion similar to that used in quantum mechanics and many body physics [38,39]. This means that its starting point approximates the spectrum of $\mathbf{D}$ by a continuum. Discretization effects are averaged and cannot be recovered in detail by resummation, hence the expansion is at most only asymptotic. The same is true of course for the inverse mass expansion.

Expanding eq. (3.6) and using (3.7), we have

$$\begin{aligned} \mathrm{Tr}\,\mathbf{D}^s &= \sum_{N=0}^{\infty} [\mathrm{Tr}\,\mathbf{D}^s]_N \\ [\mathrm{Tr}\,\mathbf{D}^s]_N &= (-1)^{N+1} \int \frac{\mathrm{d}^D x \mathrm{d}^D p}{(2\pi)^D} \int_\Gamma \frac{\mathrm{d}z}{2\pi i} z^s \mathrm{tr}\langle x|(G_0 \mathbf{D}_0)^N G_0|0\rangle \end{aligned} \tag{6.2}$$

$$\begin{aligned} G_0(x,p;z) &= (\not p - z + \mathbf{M})^{-1} \\ &= -(p^2 + (z-S)^2 + P^2 + i\gamma_5[S,P])^{-1}(\not p + z - S + i\gamma_5 P) \end{aligned} \tag{6.3}$$

Let us remark that for $N > D$ the integrals are ultraviolet finite, yet we cannot proceed formally by simply taking $\langle \log(\mathbf{M} + \not p + \mathbf{D}_0)\rangle$ and expanding in powers of $\mathbf{D}_0$ because this would require trace cyclicity and in fact gives wrong results (also we cannot expand formally $\mathrm{Tr}\,\log(\mathbf{M} + \mathbf{D}_0)$ before using the Wigner transformation method because every term would diverge). However it can be done for the current or, equivalently, we can take $s = -1$ above and perform the z integral first. Thus, we obtain for the current in N-th order, with $N > D$

$$[J(x)]_N = (-1)^{N+1} \int \frac{\mathrm{d}^D p}{(2\pi)^D} \mathrm{tr}\langle x| \left(\frac{1}{\mathbf{M}+\not p}\mathbf{D}_0\right)^N \frac{1}{\mathbf{M}+\not p}|0\rangle \,, \qquad N > D \tag{6.4}$$

The action can then be reconstructed with the Eq. (2.14). Because z no longer appears in Eq. (6.4), $\mathbf{J}_{N>D}$ enjoys all the classical symmetries of $\mathbf{D}$; so long as the symmetry transformation does not mix different orders of the expansion, which is the case for the usual Poincaré invariant internal symmetries.

If $\mathbf{D}_0$ contains only fields with an odd number of Lorentz indices, the terms with N odd vanish after performing the p integration in even dimension.

6.2 Effective action in 1+1 dimensions

The problem with gradient expansions is that it is in general difficult to work out explicitly the inversion of matrices implied in $G_0(x,p;z)$, which is necessary to perform the z and p integrals [9]. To be concrete and keep the computations simple, we will consider the 1+1 dimensional case with SU(2) flavor symmetry, that is

$$\mathbf{D} = i\partial\!\!\!/ + V\!\!\!/ + A\!\!\!/\,\gamma_5 + S + i\gamma_5 P \tag{6.5}$$

where $S(x)$ is a Hermitian c-number and $P(x)$, $V_\mu(x)$ and $A_\mu(x)$ are matrices in the fundamental representation of su(2). Because of the two dimensional identity $\gamma_\mu\gamma_5 = i\epsilon_{\mu\nu}\gamma_\nu$, we can simplify the amount of algebra without loss of generality by reabsorbing $A_\mu(x)$ in $V_\mu(x)$ with $\tilde{V}_\mu = V_\mu - i\epsilon_{\mu\nu}A_\nu$.

Let $\mathcal{L}$ be the matrices in the fundamental representation of sl(2,C), i.e. of the form $a_i\tau_i$ where τ_i are the Pauli matrices and a_i are c-numbers. The following properties of $\mathcal{L}$ will be widely used later:

$$\text{for } A, B \in \mathcal{L}, \qquad \mathrm{tr}A = 0, \quad [A,B] \in \mathcal{L}, \quad \{A,B\} = \text{c-number} \tag{6.6}$$

In particular $G_0(x,p;z)$ can be computed to obtain

$$[\mathrm{Tr}\,\mathbf{D}^s]_N = \int \frac{\mathrm{d}^2x\mathrm{d}^2p}{(2\pi)^2}\int_\Gamma \frac{\mathrm{d}z}{2\pi i} z^s \mathrm{tr}\langle x| \left(\left(\frac{p\!\!\!/ + \sigma + i\gamma_5 P}{\Delta} i\tilde{D\!\!\!\!/} \right)^N \frac{p\!\!\!/ + \sigma + i\gamma_5 P}{\Delta} \right) |0\rangle \tag{6.7}$$

where we have defined

$$\sigma = z - S \qquad \Delta = p^2 + \sigma^2 + P^2\,, \qquad i\tilde{\mathcal{D}}_\mu = i\partial_\mu + \tilde{V}_\mu \tag{6.8}$$

and σ, P^2 and Δ are c-numbers. Odd orders vanish. Let us compute explicitly the first two terms, $N = 0, 2$. One finds integrals of the following form

$$I_2(k,N,D,s;S,P) = \int \frac{\mathrm{d}^D p}{(2\pi)^D}\int_\Gamma \frac{\mathrm{d}z}{2\pi i}\frac{z^s p^{2k}}{\Delta^N} \tag{6.9}$$

Because the integral is a meromorphic function of D, we can perform first the integral in p as in dimensional regularization, and then the z integral gives the hypergeometric function ${}_2F_1$ [40, p. 555]:

$$I_2 = -\frac{1}{(4\pi)^{D/2}} \frac{\Gamma(N-n)\Gamma(2n-s-1)}{\Gamma(\frac{D}{2})\Gamma(N)\Gamma(n-s)} \times \left(e^{+i\pi n} z_+^{s-2n+1} \, {}_2F_1(n, 2n-s-1; n-s; \frac{z_-}{z_+}) \right.$$
$$\left. + \, e^{-i\pi n} z_-^{s-2n+1} \, {}_2F_1(n, 2n-s-1; n-s; \frac{z_+}{z_-}) \right) \tag{6.10}$$

with $n = N - \frac{1}{2}D - k$, $z_\pm = S \pm i|P|$ and $|P| = (P^2)^{\frac{1}{2}}$. For even D it is convenient to write the same result as

$$I_2 = \frac{1}{(4\pi)^{D/2}} \frac{\Gamma(N-n)}{\Gamma(\frac{D}{2})\Gamma(N)} \sum_{z=z_\pm} \left(\frac{\partial}{\partial z}\right)^{n-1} \frac{z^s}{(z-z^*)^n} \tag{6.11}$$

with the prescription

$$\text{for } \alpha, \beta \in \mathrm{C}, \qquad \left(\frac{\partial}{\partial z}\right)^{\alpha} z^\beta = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} z^{\beta-\alpha}, \qquad \left(\frac{\partial}{\partial z}\right)^{\alpha} z^{*\beta} = 0 \tag{6.12}$$

In this form eq. (6.11) holds too if z^s is replaced by any function analytic in $\mathrm{C}-\mathrm{R}^-$ and sufficiently convergent at ∞. After taking the trace in Dirac space we have

$$[\mathrm{Tr}\,\mathbf{D}^s]_{N=0} = 2\left\langle \sum_{z=z_\pm} \left(\frac{\partial}{\partial z}\right)^{-1} z^s(z-S) \right\rangle = 4\,\mathrm{Re}\left\langle \frac{z_+^{s+2}}{s+2} - S\frac{z_+^{s+1}}{s+1} \right\rangle, \quad s \in \mathrm{R} \tag{6.13}$$

And hence for the action at zeroth order

$$W_0 = \left\langle M^2 \left(\log(\frac{M^2}{\mu^2}) - 3\right) + 2P^2 \right\rangle \tag{6.14}$$

$\langle\ \rangle$ was defined in Eq. (4.15) and in these formulas tr acts on flavor space only, and $M^2 = S^2 + P^2$. A similar calculation in four dimensions gives

$$W_0 = -\left\langle M^4 \left(\log(\frac{M^2}{\mu^2}) - \frac{25}{6}\right) + \frac{4}{3}M^2P^2 + \frac{4}{3}P^4 \right\rangle \tag{6.15}$$

The polynomial chiral breaking terms yield the spurious axial anomaly discussed in the previous section.

Let us compute the second order term. After taking the trace in Dirac space, we can always proceed by carrying the covariant derivatives $i\tilde{\mathcal{D}}_\mu$ to the right, for instance

$$i\tilde{\mathcal{D}}_\mu \Delta^{-1} = -\Delta^{-2}(-2\sigma[i\tilde{\mathcal{D}}_\mu, S] + \{P, [i\tilde{\mathcal{D}}_\mu, P]\}) + \Delta^{-1} i\tilde{\mathcal{D}}_\mu \tag{6.16}$$

so that the various Δ and σ are collected to the left and the integral I_2 applies. This is not the most efficient strategy in this case but it is systematic and allows for algebraic manipulator implementation in more complicated cases. Due to vector covariance, one finds that $i\tilde{\mathcal{D}}_\mu$ appears only covariantly, that is in the form $[i\tilde{\mathcal{D}}_\mu, X]$. The vanishing of the non covariant terms follows after the p integration and using the $\mathcal{L}$ identities, eq. (6.6). The result can be brought to a more symmetric form by using integration by parts of the covariant derivatives and trace cyclicity:

$$W_2 = \Big\langle -\frac{1}{12}\frac{(\tilde{D}_\mu M^2)^2}{M^4} + \frac{1}{2}\frac{(\tilde{D}_\mu S)^2 + (\tilde{D}_\mu P)^2}{M^2} \\ + \arctan(\frac{P}{S}) i\epsilon_{\mu\nu}\tilde{F}_{\mu\nu} + \frac{1}{2}(\frac{1}{P^2}\frac{S}{M^2} - \frac{1}{P^3}\arctan(\frac{P}{S}))\epsilon_{\mu\nu}P\tilde{D}_\mu P\tilde{D}_\nu P\Big\rangle \tag{6.17}$$

Here $\tilde{D}_\mu X = [\tilde{\mathcal{D}}_\mu, X]$, $i\tilde{F}_{\mu\nu} = [i\tilde{\mathcal{D}}_\mu, i\tilde{\mathcal{D}}_\nu]$.

6.3 Analysis of the effective action. Wess-Zumino-Witten action

In order to obtain a separation $W_2 = W_2^+ + W_2^-$ in two terms with well defined hermiticity and pseudoparity, we have to reexpand $i\tilde{\mathcal{D}}_\mu$ as $\tilde{D}_\mu X = D_\mu X - \epsilon_{\mu\nu}[A_\nu, X]$. The real part, W_2^+, comes from terms without explicit $\epsilon_{\mu\nu}$ and even number of axial fields or else with explicit $\epsilon_{\mu\nu}$ and odd number of axial fields, whereas W_2^- is the other way around. In principle, the effective action up to second order in the gradient expansion should saturate the anomaly equations. This can be trivially checked for the scale anomaly. Although the resulting expressions are manifestly vector gauge invariant, the corresponding check for the axial anomaly requires more work. A convenient method to do so is to explicitly establish that the anomalous terms of the computed effective action coincide with the Wess-Zumino-Witten action plus additional polynomial counterterms.

To analyze the chiral transformation properties of W_2, let $M(x)$, $U(x)$ and $\phi(x)$ be defined by

$$S + iP = MU, \qquad U = e^{i\phi} = \cos\phi + i\sin\phi \tag{6.18}$$

hence $U \in \mathrm{SU}(2)$, $\phi, \sin\phi \in \mathrm{su}(2)$ and M, $\cos\phi$ are c-numbers. $\phi(x)$ is a multivalued function of S and P, well defined everywhere by assuming explicitly that $M(x) > 0$, or equivalently that $\mathbf{M}$ is non singular, as required by Seeley's formula (2.7).

Under the previous assumptions, the terms in W_2 with explicit $\epsilon_{\mu\nu}$ turn out not to depend on $M(x)$, as it is readily shown. That is, this part of W_2 can be written as

$$W_2^{\epsilon_{\mu\nu}} = \Big\langle 2i\tilde{F}\varphi + \frac{1}{4}\frac{\sin(2\phi) - 2\varphi}{\sin^3\phi}\sin\phi(\tilde{D}\sin\phi)^2\Big\rangle \tag{6.19}$$

$$\varphi = \phi + n\pi\frac{\phi}{|\phi|}\,, \qquad n \in \mathrm{Z} \tag{6.20}$$

φ is such that $\tan\varphi = \tan\phi$ and takes into account explicitly the multivaluation. A standard exterior algebra notation has been used [26],

$$\tilde{F} = \frac{1}{2}F_{\mu\nu}\mathrm{d}x_\mu\mathrm{d}x_\nu\,, \qquad \tilde{D}X = \tilde{D}_\mu X\mathrm{d}x_\mu \tag{6.21}$$

and $\mathrm{d}x_\mu$ are anticommuting variables. For more general symmetry groups, $M(x)$, defined by the factorization $S + iP = MU$ with M Hermitian and U unitary, will not be a c-number and will not cancel in W^-. Nevertheless, the symmetry under local rescaling $\mathbf{M}(x) \to \lambda(x)\mathbf{M}(x)$, where $\lambda(x)$ is a c-number local function, still holds. This general fact will be established below when dealing with the fermionic current.

In order to work out the expressions it is convenient to use the following identities, particularly useful for computing derivatives of the fields:

$$\text{for } a \in \mathcal{L}, \quad \delta f_+(a) = \frac{f'_+(a)}{2a}\{a,\delta a\}, \quad \delta f_-(a) = \frac{f'_-(a)}{2a}\{a,\delta a\} + \frac{f_-(a)}{2a^2}[a,\delta a] \tag{6.22}$$

where $f_\pm$ are arbitrary even and odd functions respectively, and $f'_\pm$ their derivatives. As a consequence

$$\begin{aligned} iU^\dagger\delta U &= -\frac{1}{\cos\phi}\delta\sin\phi + \frac{1}{2}(\tan\phi + i)[\sin\phi, \delta\sin\phi] \\ iU\delta U^\dagger &= +\frac{1}{\cos\phi}\delta\sin\phi - \frac{1}{2}(\tan\phi - i)[\sin\phi, \delta\sin\phi] \end{aligned} \tag{6.23}$$

After some algebra, the real part of W_2 can be written as

$$W_2^+ = \left\langle \frac{1}{24}\left(\frac{\partial_\mu M^2}{M^2}\right)^2 + \frac{1}{2}\mathcal{R}_\mu^2 - 2A_\mu^2 \right\rangle \tag{6.24}$$

where we have defined

$$\mathcal{R}_\mu = U^\dagger i\partial_\mu U - V_\mu^R + U^\dagger V_\mu^L U\,, \qquad V_\mu^{R,L} = V_\mu \pm A_\mu \tag{6.25}$$

$\mathcal{R}_\mu$ transforms covariantly under chiral gauge transformations

$$\Omega_{R,L}(x) \in \mathrm{SU}(2), \qquad U \to \Omega_L U \Omega_R^\dagger\,, \qquad V_\mu^{R,L} \to \Omega_{R,L}(i\partial_\mu + V_\mu^{R,L})\Omega_{R,L}^\dagger \tag{6.26}$$

Vector gauge transformations correspond to the diagonal subgroup. M is chiral invariant and the only chiral breaking term in W_2^+ is A_μ^2, which is a polynomial.

Similarly, the imaginary part can be brought to the following form

$$\begin{aligned} W_2^- &= \Gamma_{\rm WZW}(U) + \Gamma_G(U, V^R, V^L) - \Gamma_G(I, V^R, V^L) \\ \Gamma_{\rm WZW}(U) &= \frac{1}{12\pi}\int_{D_3} {\rm tr}(U^\dagger i{\rm d}U)^3 \\ \Gamma_G(U, V^R, V^L) &= \langle V^R U^\dagger {\rm d}U - V^L U {\rm d}U^\dagger - iV^R U^\dagger V^L U\rangle \end{aligned} \tag{6.27}$$

where ${\rm d}X = \partial_\mu X {\rm d}x_\mu$. $\Gamma_{\rm WZW}$ is the correctly normalized Wess-Zumino-Witten action in two dimensions. $\Gamma_G(U, V^R, V^L)$ is obtained by chirally gauging $\Gamma_{\rm WZW}$ [20]. The last term is known as Bardeen's subtraction, a counterterm which reestablishes vector gauge invariance. There I stands for the identity of SU(2).

The way $\Gamma_{\rm WZW}$ appears deserves some comment. The field configuration $U(x)$ defines a two dimensional manifold M_2 without boundary inside SU(2), and D_3 is a three dimensional manifold such that $\partial D_3 = M_2$. Up to a quantized multivaluation [20], $\Gamma_{\rm WZW}$ does not depend on the choice of D_3 because $\omega_3 = \frac{1}{12\pi}{\rm tr}(U^\dagger i{\rm d}U)^3$ is closed. Hence, there is a 2-form Ω_2 such that locally $\omega_3 = {\rm d}\Omega_2$. However Ω_2 cannot be regular everywhere, because ω_3 is not exact (in fact it is just the volume element in SU(2)). Choosing D_3 to avoid the Dirac singularity one has

$$\Gamma_{\rm WZW} = \int_{D_3} \omega_3 = \int_{M_2} \Omega_2 \tag{6.28}$$

To arrive to eq. (6.27), what one can show is that a valid choice for Ω_2 is

$$\Omega_2 = \frac{1}{16\pi}{\rm tr}\left(\frac{\sin(2\phi) - 2\varphi}{\sin^3\phi}\sin\phi({\rm d}\sin\phi)^2\right) \tag{6.29}$$

which is just $W_2^{\epsilon_{\mu\nu}}$ in eq. (6.19) for $V_\mu = A_\mu = 0$, and arbitrary n. The singularity is at $U = (-I)^{n+1}$. In fact the solution of $\omega_3 = {\rm d}\Omega_2$ is unique imposing global vector invariance and regularity at $\phi = 0$. Global chiral invariance is not manifest for any choice of Ω_2. Also one can check that by varying n, $\int\Omega_2$ changes by an integer multiple of $2\pi i$. Another interesting point is that vector gauge invariance would be achieved by minimal coupling $i{\rm d} \to i{\rm d} + \tilde{V}$ in Ω_2. However such an action would not be single valued modulo $2\pi i$. The new vector gauge invariant term $\langle 2i\tilde{F}\varphi\rangle$ in $W_2^{\epsilon_{\mu\nu}}$ reestablishes the one-valuedness of the action.

W_2^- can be written in other interesting form by applying Stokes' theorem to Γ_G, namely

$$W_2^- = \frac{1}{4\pi}\int_{D_3} {\rm tr}(\frac{1}{6}\mathcal{R}^3 - i\mathcal{R}\mathcal{F}_R - 2iA\mathcal{F}_R + \frac{4}{3}A^3) - {\rm p.c.} \tag{6.30}$$

where $\mathcal{R} = U^\dagger i\mathrm{d}U - V_R + U^\dagger V_L U$, $\mathcal{F}_R = \mathrm{d}V_R - iV_R^2$ are chiral covariant, and p.c. means to subtract the parity conjugate terms, i.e. the same terms exchanging $U \leftrightarrow U^\dagger$, $V_R \leftrightarrow V_L$ and $A \leftrightarrow -A$. This form is manifestly vector gauge invariant and the axial anomalous terms (those with A_μ) are independent of U. These terms are polynomial, yet they cannot be removed by counterterms because they do not form a closed 3-form by themselves.

We would like to emphasize that ours is an ab initio calculation of the action. Once the ζ-function prescription is adopted, there is no more freedom nor ambiguity in the calculation. This is in contrast to derivations of Γ_{WZW} by integration of the chiral anomaly, which have to assume that $S + iP$ lies on the chiral circle, i.e. $M(x)$ constant [41-44].

6.4 The fermionic number current

As another illustration of the Wigner transformation technique, let us consider the fermionic number current $-\psi\gamma_\mu\psi$ in four dimensions with arbitrary internal GL(n,C) symmetry. It is obtained from

$$\delta_\omega \mathbf{D} = -\not{\omega}(x), \qquad \delta_\omega W = \int \mathrm{d}^D x\, \omega_\mu(x) j_\mu(x) \tag{6.31}$$

where $\omega_\mu(x)$ is a c-number field. As an example, for $\mathbf{D} = i\not{\partial} + \not{V}(x) + m$, a simple calculation from Eqs. (4.7) and (4.17) gives

$$W = -\frac{2}{3}\langle F_{\mu\nu}^2\rangle \log\left(\frac{m}{\mu}\right) + O(m^2) \tag{6.32}$$

which gives a contribution to $j_\mu(x)$. Here we will consider the anomalous part of the current, $j_\mu^-(x)$ in the presence of scalar and pseudo scalar gl(n,C)-matrix valued fields

$$\mathbf{D} = i\not{\partial} + \mathbf{M}(x)\,, \qquad \mathbf{M}(x) = S(x) + i\gamma_5 P(x) \tag{6.33}$$

Because the pseudoparity odd (p.o.) part of the current is ultraviolet finite we will use directly the expression

$$\delta_\omega W^- = \mathrm{Tr}\left(\not{\omega}\frac{1}{\mathbf{D}}\right)_{\mathrm{p.o.}} \tag{6.34}$$

On the other hand, the lowest order contribution appears at fourth order in a gradient expansion of W, due to the presence of the Levi-Civita pseudotensor. Thus to lowest order, the anomalous current is obtained from

$$\delta_\omega W^-_{\mathrm{lowest}} = -\int \frac{\mathrm{d}^4x\mathrm{d}^4p}{(2\pi)^4}\mathrm{tr}\langle x|\not{\omega}\frac{1}{\not{p}+\mathbf{M}}\left(i\not{\partial}\frac{1}{\not{p}+\mathbf{M}}\right)^3|0\rangle_{\mathrm{p.o.}} \tag{6.35}$$

It is convenient to introduce the definitions

$$\begin{aligned} M_{R,L} &= S \pm iP\,, \qquad P_{R,L} = \frac{1}{2}(1\pm\gamma_5) \\ G_{LR} &= (p^2 + M_L M_R)^{-1}\,, \qquad G_{RL} = (p^2 + M_R M_L)^{-1} \end{aligned} \tag{6.36}$$

so that

$$\begin{aligned} (\not p + \mathbf{M})^{-1} &= -(p^2 + \mathbf{M}^\dagger \mathbf{M})^{-1}(\not p - \mathbf{M}^\dagger) = \\ &= P_R G_{LR} M_L P_R + P_L G_{RL} M_R P_L - P_R G_{LR} \not p\, P_L - P_L G_{RL} \not p\, P_R \end{aligned} \tag{6.37}$$

Substituting in $\delta_\omega W^-$, and keeping only terms with γ_5, one obtains

$$\begin{aligned} \delta_\omega W^-_{\text{lowest}} = -2i \int \frac{\mathrm{d}^4 p}{(2\pi)^4} \omega \mathrm{tr}\big(& G_{LR} M_L \mathrm{d} G_{RL} M_R \mathrm{d} G_{LR} M_L \mathrm{d} G_{RL} M_R \\ &+ \frac{1}{2} p^2 (G_{LR} \mathrm{d} G_{LR} \mathrm{d} G_{LR} M_L \mathrm{d} G_{RL} M_R - G_{LR} \mathrm{d} G_{LR} M_L \mathrm{d} G_{RL} M_R \mathrm{d} G_{LR}) \\ &+ \frac{1}{2} p^2 (G_{LR} M_L \mathrm{d} G_{RL} \mathrm{d} G_{RL} \mathrm{d} G_{RL} M_R + G_{LR} M_L \mathrm{d} G_{RL} M_R \mathrm{d} G_{LR} \mathrm{d} G_{LR})\big) - \text{p.c.} \end{aligned} \tag{6.38}$$

The parity conjugate (p.c.) is obtained by exchanging the labels R and L everywhere, tr no longer includes Dirac trace, and an exterior algebra notation has been used with $\omega = \omega_\mu \mathrm{d}x_\mu$ and $\mathrm{d} = \mathrm{d}x_\mu \partial_\mu$. d derivates to the right until it finds another d or gives zero if it reaches the right end. The integral over p is convergent, but it cannot be done in closed form for arbitrary $M_{R,L}$ fields.

An interesting property of $\delta_\omega W^-$, which can already be derived without explicitly performing the momentum integral, is its invariance under the local rescaling $S(x) \to \lambda(x) S(x)$ and $P(x) \to \lambda(x) P(x)$, with $\lambda(x)$ an arbitrary local c-number function. Due to the close relation between the fermionic number current and the imaginary part of the effective action [45,46], to be addressed below, this property holds for W^- at lowest nonvanishing order as well. As a consequence, in the particular case of $M_R(x) = M(x) U(x)$, $M(x)$ being a Hermitian c-number, not necessarily constant, the expression simplifies: $G_{LR} = G_{RL} = (p^2 + M^2)^{-1}$ is also a c-number, all the terms containing $\mathrm{d}M(x)$ are readily shown to cancel, the terms with p^2 vanish and the integral over p is immediate, with the following result

$$\delta_\omega W^-_{\text{lowest}} = -\frac{1}{24\pi^2} \int \omega \mathrm{tr}(R^3)\,, \qquad R = U^\dagger i \mathrm{d} U \tag{6.39}$$

This gives the correctly normalized Goldstone-Wilczek current [19]

$$j^-_{\mu,\text{lowest}}(x) = -\frac{1}{24\pi^2} \epsilon_{\mu\nu\rho\sigma} \mathrm{tr}(R_\nu R_\rho R_\sigma)\,, \qquad R_\mu = U^\dagger i \partial_\mu U \tag{6.40}$$

Comparing Eqs. (6.27) and (6.40) we find that the 3-form ω_3 is the integrand of the action in two dimensions and also it is the fermionic number density in four dimensions, in both cases at lowest order and in the pseudoparity odd sector.

6.5 Effective action from the fermionic number

The relation pointed out in the last paragraph of the previous subsection is much more general as noted by several authors [45,46]. We will show subsequently that our formalism can handle the proof of this general relation quite naturally. The general statement is as follows. Let $\mathbf{D}$ be a Dirac operator in D dimensions and W^- the pseudoparity odd part of its action. Consider the extension of $\mathbf{D}$ to a family of Dirac operators $\mathbf{D}(v)$, $v \in [0,T]$ which interpolates between $\mathbf{D}$ at $v = T$ and $\mathbf{D}_0$ at $v = 0$, where $\mathbf{D}_0$ will be characterized below. Similarly consider the extension of the space-time with two more dimensions (u, v), u playing the role of Euclidean time, and of $\mathbf{D}(v)$ to a $D + 2$ dimensional Dirac operator $\hat{\mathbf{D}}'$:

$$
\begin{aligned}
&\hat{\mathbf{D}}' = \hat{\gamma}_u i\partial_u + \hat{\gamma}_v i\partial_v + \hat{\mathbf{D}} \\
&\hat{\gamma}_u = iI \times \sigma_1\,, \quad \hat{\gamma}_v = iI \times \sigma_2\,, \quad \hat{\gamma}_\mu = \gamma_\mu \times \sigma_3\,, \quad \hat{\gamma}_5 = \gamma_5 \times \sigma_3
\end{aligned}
\tag{6.41}
$$

The $D+2$ matrices $\hat{\gamma}_a$ satisfy the Dirac algebra $\{\hat{\gamma}_a, \hat{\gamma}_b\} = -2\delta_{ab}$, $a, b = u, v, 0, \dots, D-1$, and $\hat{\gamma}_5 = i\hat{\gamma}_u\hat{\gamma}_v\gamma_5$ for even D. The operator $\hat{\mathbf{D}}$ is defined by making the substitutions $\gamma_{\mu,5} \to \hat{\gamma}_{\mu,5}$ in $\mathbf{D}(v)$. Thus it has the form

$$
\hat{\mathbf{D}} = \begin{pmatrix} \mathbf{D}(v) & 0 \\ 0 & \mathbf{D}^\dagger(v) \end{pmatrix}
\tag{6.42}
$$

where $\mathbf{D}^\dagger$ appears because the hermiticity of the fields in $\mathbf{D}$ is such that the Hamiltonian $\hat{\gamma}_u(\hat{\gamma}_v i\partial_v + \hat{\mathbf{D}})$ of the $D+2$ dimensional extended Dirac operator, is Hermitian in Minkowski metric, and hence $\hat{\mathbf{D}}'$ has the standard hermiticity in Euclidean space.

The theorem establishing the relation between the fermion number and the effective action can be stated in the following manner. Let B be the $D+1$-dimensional spatial region defined by (v, x_μ), $v \in [0,T]$ and $x_\mu \in \mathrm{R}^D$, and F the fermion number enclosed in B in the $D + 2$ dimensional system. Then for a suitable choice of the interpolating path $\mathbf{D}(v)$, to be considered below, $2\pi F = -W^-$ in Euclidean metric, W^- being the D-dimensional pseudoparity odd effective action. In fact, one can state this relation in a more general form, namely

$$
\int_B \mathrm{d}^D x \mathrm{d}v\, \mathrm{tr}\langle u, v, x|\hat{\gamma}_u f'(\hat{\mathbf{D}}')|u, v, x\rangle = \frac{1}{4\pi}\left(\mathrm{Tr}\, f(\mathbf{D}) - \mathrm{Tr}\, f(\mathbf{D}^\dagger)\right)\Big|_{v=0}^{T}
\tag{6.43}
$$

for a sufficiently convergent function $f(z)$ analytic in $\mathrm{C}-\mathrm{R}^-$, and $f'(z)$ its derivative. The relation between F and W^- is obtained with $f(z) = \log z$ defined as the analytical extrapolation of $\mathrm{d}z^s/\mathrm{d}s$ in s at $s=0$.

Let us consider under which conditions this relation will be valid. The contribution from the value $v=0$ to the r.h.s. of Eq. (6.43) will vanish if $\mathbf{D}_0$ is related to $\mathbf{D}_0^\dagger$ by a similarity transformation. This can always be achieved for even D by taking $\mathbf{D}_0$ without pseudotensor fields, i.e. without explicit γ_5, since in this case $\mathbf{D}_0^\dagger = \gamma_5 \mathbf{D}_0 \gamma_5$. Observe that $\hat{\mathbf{D}}'$ is time independent and does not contain fields with Lorentz index u, hence u in the l.h.s. must come from the Levi-Civita pseudotensor and it is automatically pseudoparity odd as the r.h.s. Regarding the choice of the interpolating path $\mathbf{D}(v)$, note that the l.h.s. can be obtained from

$$\delta_\omega \hat{\mathbf{D}}' = -\not{\omega}\,, \qquad \delta_\omega \mathrm{Tr}\, f(\hat{\mathbf{D}}') = \int \mathrm{d}^D x\, \mathrm{d}u\, \mathrm{d}v\, \omega_a J_a \tag{6.44}$$

Where $\not{\omega} = \gamma_u\omega_u + \gamma_v\omega_v + \gamma_\mu\omega_\mu$ and $\mu = 1,\dots,D$. Because of vector gauge invariance, ω_a can only appear in the form $\partial_a\omega_b - \partial_b\omega_a$, hence locally $J_a = \partial_b J_{ab}$ with J_{ab} antisymmetric. Note that the l.h.s. of Eq. (6.43) can also be written as $-\int_B \mathrm{d}^D x \mathrm{d}v J_u = -\int_B \mathrm{d}^D x \mathrm{d}v(\partial_v J_{uv} + \partial_\mu J_{u\mu})$. The terms containing $J_{u\mu}$ cancel because we are assuming $\mathbf{D}$ boundaryless in R^D. The non vanishing contribution comes from J_{uv} at the boundary, i.e. $v=0$ and $v=T$. However, for even dimensions the $v=0$ contribution vanishes since J_{uv} must contain a $D+2$-dimensional Levi-Civita pseudotensor, which cannot occur if $\mathbf{D}_0$ is chosen as explained above. If we take an adiabatic path, that is, a smooth path with $\partial_v \mathbf{D}(v) = O(1/T)$ and $T\to\infty$, $J_{uv}|_{v=T}$ will depend only on $\mathbf{D}$ and $f(z)$ and not on the particular adiabatic path chosen. For arbitrary paths J_{uv} at $v=T$ would also depend on $\partial_v^n \mathbf{D}|_{v=T}$, $n>0$, hence the result would be path dependent. Let us remark that at lowest order J_{ab} exists only locally if $f(z)$ is a multivalued function such as $\log z$. In this case different homotopy classes of adiabatic paths pick different branches of $f(z)$. An example is again provided by ω_3 which is closed and hence locally exact. Indeed one of the interesting aspects of the relation $2\pi F = -W^-$ is that F is computed as the integral of a single valued expression.

To proof Eq. (6.43) let us rewrite the l.h.s. using Seeley's representation and the Wigner transformation trick in the space spanned by $|u,v\rangle$

$$-\mathrm{Tr}\int_0^T \mathrm{d}v \int \frac{\mathrm{d}^2\omega}{(2\pi)^2}\int_\Gamma \frac{\mathrm{d}z}{2\pi i} f'(z) \langle u,v|\hat{\gamma}_u \frac{1}{\hat{\gamma}_u(i\partial_u+\omega_u)+\gamma_v(i\partial_v+\omega_v)+\hat{\mathbf{D}}-z}|0\rangle \tag{6.45}$$

$|0\rangle$ is the state with zero momentum in the space $|u,v\rangle$, i.e. $\langle u,v|0\rangle = 1$, and ∂_u and ∂_v act to the right until annihilating $|0\rangle$. Tr includes trace in the space $|x_\mu\rangle$ as well as spinor and internal degrees of freedom in $D+2$ dimensions. The ultraviolet limit corresponds to integrate over ω which is finite after the z integration. To proceed let us introduce the definitions

$$\begin{aligned} \Gamma = i\hat{\gamma}_u\hat{\gamma}_v = I \times \sigma_3\,, \quad & \Gamma = \Gamma^\dagger\,, \quad \Gamma^2 = I \\ iH = \hat{\gamma}_u(\hat{\mathbf{D}} - z)\,, \quad & \{H,\Gamma\} = 0 \end{aligned} \tag{6.46}$$

Manipulating the denominator of Eq. (6.45) as in (6.37) the formula can be written as

$$\begin{aligned} -\mathrm{Tr} \int_0^T \mathrm{d}v \int \frac{\mathrm{d}^2\omega}{(2\pi)^2} \int_\Gamma \frac{\mathrm{d}z}{2\pi i} f'(z) \\ \times \langle u,v| \frac{1}{(i\partial_u+\omega_u)^2 + (i\partial_v+\omega_v)^2 + H^2 - i\Gamma(\partial_v H)} (i\partial_u + \omega_u + \Gamma\partial_v - i\Gamma\omega_v + iH)|0\rangle \end{aligned} \tag{6.47}$$

We can set $\partial_u = 0$ everywhere and also $\partial_v|0\rangle = 0$. Furthermore in an expansion in powers of ∂_v, one can check that only odd powers survive after doing an angular average over ω and taking trace using that H is off-diagonal. In particular for an adiabatic interpolating path, only the term linear in ∂_v survives. Then the expression takes the form

$$-\mathrm{Tr} \int_0^T \mathrm{d}v \int \frac{\mathrm{d}^2\omega}{(2\pi)^2} \int_\Gamma \frac{\mathrm{d}z}{2\pi i} f'(z)\Gamma \frac{1}{\omega^2+H^2} \left(-(\partial_v \frac{\omega^2}{\omega^2+H^2}) - (\partial_v H)H\frac{1}{\omega^2+H^2} \right) \tag{6.48}$$

Now we have the trace of an operator in the space $|x_\mu\rangle$ only. Because $f(z)$ is convergent the trace exists and the cyclic property can be freely used in the two terms in Eq. (6.48). The first term can be put in the form $\partial_v(\Gamma g(H^2,\omega^2))$, and it vanishes after taking the trace because H^2 is block diagonal with blocks $(\mathbf{D}^\dagger - z)(\mathbf{D}-z)$ and $(\mathbf{D}-z)(\mathbf{D}^\dagger - z)$ which have the same spectrum. In the second term we can integrate ω, thus

$$\begin{aligned} -\frac{1}{4\pi}\mathrm{Tr} \int_0^T \mathrm{d}v \int_\Gamma \frac{\mathrm{d}z}{2\pi i} f'(z)\Gamma\partial_v\hat{\mathbf{D}}\frac{1}{\hat{\mathbf{D}}-z} &= \frac{1}{4\pi}\int_0^T \mathrm{d}v\partial_v \mathrm{Tr}\,(\Gamma f(\hat{\mathbf{D}})) \\ &= \frac{1}{4\pi}\mathrm{Tr}\,(\Gamma f(\hat{\mathbf{D}}))\Big|_{v=0}^T \end{aligned} \tag{6.49}$$

which coincides with the r.h.s of Eq. (6.43). This completes the proof. Let us remark that the staircase relation (6.43) is valid beyond a gradient expansion (in D dimensions), and in case this expansion applies the leading terms provide a relationship between a topological action and a topological fermion current. Furthermore Eq. (6.43) does not rely on the chiral circle constraint and holds for any space time dimension.

A rather amusing illustration of the relation comes from the zero dimensional problem. In $D=0$ the most general Dirac operator is $\mathbf{D}=S+i\gamma_5 P$, S, P being Hermitian matrices in flavor space and γ_5 a 1×1 matrix which equals -1 in our convention (see Eq. (2.2)). Certainly, the Grassmann integral (2.5) can be carried out and $W=-\mathrm{tr}\log(\mathbf{D}/\mu)$ exactly, without regularization. Remarkably, the effective action is ultraviolet finite yet it possesses both scale anomaly and (essential) axial anomaly (even if there are no gauge fields). This is also in agreement with the general formulas (5.3) and (5.8).

In the Abelian case $\mathbf{D}=Me^{i\gamma_5\phi}$, $M,\phi\in$R, the action is simply $W=-\log(M/\mu)+i\phi$. On the other hand, the pseudoparity odd singlet current in two dimensions $(0+2)$ is

$$j^-_\mu(x)=-\frac{i}{2\pi}\epsilon_{\mu\nu}\partial_\nu(\phi+\frac{1}{6M^2}\partial^2_\alpha\phi+\cdots) \tag{6.50}$$

With $\mu,\nu,\alpha=u,v$. In the notation used in the proof of the theorem stated above, this formula corresponds to identify $J_{uv}=-(i/2\pi)(\phi+\partial_v^2\phi/(6M^2)+\cdots)$, since ϕ does not depend on u. As expected the fermionic current at lowest order is a closed 1-form whereas the higher orders are total derivatives of v. $j^-_\mu(x)$ checks the general Eq. (6.43), the higher orders vanishing only in the adiabatic limit. Note that in the non adiabatic case the fermion number depends on the v-derivatives of ϕ at $v=0$ and $v=T$, and hence does not agree with the zero dimensional pseudoparity odd effective action.

7. Odd dimensions

In this section we will consider the odd dimensional case in greater detail. The main issue here is the quantum realization of parity which is a symmetry at the classical level [31]. For odd D and up to a similarity transformation, parity corresponds to the transformation $\mathbf{D}=i\partial\!\!\!/+\mathbf{Y}(x)\to i\partial\!\!\!/-\mathbf{Y}(-x)$. It turns λ_n into $-\lambda_n$ in the eigenvalue equation (2.4). As a consequence we can expect that divergent terms in the action can break this symmetry.

In an inverse mass expansion of the action we must look for invariance under $m\to -m$ and $\mathbf{D}_0\to-\mathbf{D}_0$, however the formula (4.14) is not appropriate because it was derived for $m>0$. The cut $\Gamma=\mathrm{R}^-$ in the integral I_1 breaks parity explicitly, and indeed for negative m the free spectrum $m\pm i|p|$ would intersect Γ. In fact, there is no ray of minimal growth for both positive and negative masses as required from Seeley's representation, Eq. (2.7). To deal with this problem one can use some infrared regulator such as including a factor $\theta(|p|-m_0)$ in the p integral and letting $m_0\to 0^+$. However, a simpler procedure is to use dimensional regularization instead of ζ-function, which also maintains gauge invariance. As always, both regularizations will coincide up to counterterms. The procedure in this case is to compute the current $\mathbf{J} = -\langle x|\mathbf{D}^{-1}|x\rangle$ in a formal inverse mass expansion for a generic dimension and afterwards to reconstruct the action. Of course we do this using the Wigner transformation. One easily obtains the result

$$\mathbf{J}(x) = \frac{1}{(4\pi)^{D/2}}\epsilon(m)^D \sum_{N=0}^{\infty} a_N^J\, m^{D-N-1}\mathcal{O}_N(x;\mathbf{D}_0) \tag{7.1}$$

where $\mathbf{D} = \mathbf{D}_0 + m$, $\mathcal{O}(x)$ are the same coefficients as in Section 4, and

$$\begin{aligned} \epsilon(m) &= \frac{m}{|m|}, \qquad m \in \mathrm{R} \\ a_N^J &= (-1)^{N+1}\Gamma(\bar{N} - \frac{1}{2}D), \qquad \bar{N} = \lfloor \frac{N}{2} + 1 \rfloor \end{aligned} \tag{7.2}$$

The terms of $\mathbf{J}$ in (7.1) which are ultraviolet finite coincide with (4.18) (including the factor $\epsilon(m)^D$ which comes from integral I_1 by making the momentum integral first). On the other hand, the terms $N < D$ of the current depend on the regularization. The coefficients $a_{N<D}^J$ have poles at even integer values of D. Such poles give rise to the logarithmic terms in (4.18). For odd D, the terms $N < D$ in (7.1) give a finite contribution proportional to $\epsilon(m)$, whereas in ζ-function they vanish for positive m. A more careful treatment would show that in fact there is a step function factor $\theta(-m)$ in (4.18), so that both regularizations differ only by a polynomial of m, since $m^k\theta(-m) = m^k\frac{1}{2} - \frac{1}{2}m^k\epsilon(m)$. The sign function $\epsilon(m)$ does not necessarily vanish at $m = 0$; its precise value is ambiguous and further information is required.

The current in (7.1) is manifestly gauge covariant and also parity covariant (for odd D) thanks to the factor $\epsilon(m)$, since both symmetries are respected by dimensional regularization and the current turns out to be finite. In contrast, to compute the action directly in dimensional regularization one would need a Seeley-like representation $\mathrm{Tr}\,\log(\mathbf{D}) = -(2\pi i)^{-1}\mathrm{Tr}\int_\Gamma \mathrm{d}z\log(z)(\mathbf{D}-z)^{-1}$, thus manifestly breaking parity invariance. This situation resembles the method of Leutwyler [12] and Ball [2], based on the integrability obstruction of the covariant regularization of the formal variation of the action. We must then consider whether $\mathbf{J}$ is a consistent current and independent of the choice of the splitting of $\mathbf{D}$ into $\mathbf{D}_0$ and m, that is, whether $\delta_m\mathbf{J} = 0$. To answer these questions let us consider separately the ultraviolet finite and divergent parts, $\mathbf{J} = \mathbf{J}_f + \mathbf{J}_d$, respectively. The finite part coincides with the ζ-function current and hence it is consistent, indeed it is the variation of the action (4.14) (after including the $\epsilon(m)$ factor). Also it is invariant under δ_m by the same arguments as in Section 4, with the prescription of defining $\epsilon(m)m^{D-N+1}$ as a derivative of the distribution $\epsilon(m)\log|m|$ in the m variable.

Consider now the divergent part of the current, $\mathbf{J}_d$ corresponding to $N < D$, and odd D. From (4.12) it is immediate to check that $\delta_m \mathbf{J}_d = 0$ if $m \neq 0$, that is, except for Dirac delta terms coming from $\delta_m \epsilon(m)$. Hence we can resum all these terms by pulling out the factor $\epsilon(m)$ and setting $\mathbf{D}_0 \to \mathbf{D}$ and $m \to 0$ in the remainder, thus

$$\mathbf{J}_d = -\frac{\epsilon(m)}{2^D \pi^{(D-1)/2}} \mathcal{O}_{D-1}(x; \mathbf{D}) \,. \tag{7.3}$$

Because $\langle \mathcal{O}_D \rangle$ vanishes for odd D, it is not obvious whether $\mathbf{J}_d$ is consistent. We can check that up to the $\epsilon(m)$ factor, to be discussed below, the current appears to be consistent for $D = 1$ and 3. Indeed, this can be accomplished by writing the possible terms in the action and adjusting their coefficients. In the case $D > 1$, one can see that this cannot be done with just terms of the form $\langle P(\mathbf{D}) \rangle$, P being a polynomial, and one must resort to polynomials on $\mathbf{D} = i\partial\!\!\!/ + \mathbf{Y}$ and $\bar{\mathbf{D}} = i\partial\!\!\!/ - \mathbf{Y}$. For given $\mathbf{J}$ the solution is unique up to an absolute constant. We find

$$\begin{aligned}
&(D=1) \qquad && W_d = -\frac{\epsilon(m)}{2} \int \mathrm{d}x \, \mathrm{tr}\,(\mathbf{Y}) \\
&(D=3) \qquad && W_d = -\frac{\epsilon(m)}{8\pi} \int \mathrm{d}^3x \, \mathrm{tr}\,(\frac{1}{12}\mathbf{Y}^3 + \frac{1}{4}\mathbf{Y}\gamma_\mu \mathbf{Y}\gamma_\mu \mathbf{Y} + \frac{1}{2}\mathbf{Y} i\partial\!\!\!/ \, \mathbf{Y})
\end{aligned} \tag{7.4}$$

In the $D = 3$ case $\gamma_0\gamma_1\gamma_2 = \sigma = \pm 1$ distinguishes the two inequivalent irreducible representation of γ_μ, and $\mathbf{Y} = A\!\!\!/ + M$, A_μ, M being Hermitian flavor matrices. Working out the Dirac space algebra, the divergent parts of the current and the action can be written more explicitly as

$$\begin{aligned}
\mathbf{J}_d &= \frac{\epsilon(m)}{8\pi}(2M^2 + \frac{1}{2}\sigma\epsilon_{\mu\nu\alpha} i F_{\mu\nu}\gamma_\alpha) \\
W_d &= \frac{\epsilon(m)}{8\pi} \int \mathrm{d}^3x \, \mathrm{tr}\,(\frac{4}{3}M^3 - i\sigma\epsilon_{\mu\nu\alpha}(A_\mu \partial_\nu A_\alpha - \frac{2}{3} i A_\mu A_\nu A_\alpha))
\end{aligned} \tag{7.5}$$

Here tr no longer acts on Dirac space. Up to the factor $\epsilon(m)$, the pseudoparity odd term of the action, W_d^-, is the correctly normalized Chern-Simons action [31] and it is gauge invariant.

Finally let us consider the issue of whether $\mathbf{J}_d$ is invariant under δ_m, i.e. whether it is independent on the choice of the expansion point m. The invariance is broken by the factor $\epsilon(m)$ in (7.3), and $\delta_m \mathbf{J}_d = -(4\pi)^{-(D-1)/2}\mathcal{O}_{D-1}\delta(m)\delta m$ does not vanish as a distribution. Still one can consider adding counterterms to cancel the variation. The rationale is as follows. The ultraviolet divergent integrals become well defined by applying $\partial/\partial m$

a sufficient number of times, hence the ambiguity in the integral must be a polynomial in m. By the same token, expressions which diverge as $m \to 0$ become well defined after multiplying by m a sufficient number of times, hence the infrared ambiguities, i.e. the admissible counterterms, consist of the distribution $\delta(m)$ and its derivatives. Unlike the ultraviolet case, however, there are infrared divergences to all orders in the inverse mass expansion.

For instance in the case $D = 1$, if we change $\mathbf{J}_d = -\frac{1}{2}\epsilon(m)$ to $\mathbf{J}_d = -\frac{1}{2}\langle x|\epsilon(\mathbf{D})|0\rangle$, and expand in powers of $\mathbf{D}_0$, the difference are just infrared counterterms and now $\delta_m \mathbf{J}_d = 0$ explicitly. Unfortunately this current is no longer gauge invariant nor consistent. So we must consider simultaneously the three conditions of δ_m invariance, gauge invariance and consistency of the current after introducing infrared counterterms. Considering again the $D = 1$ case one can see that these conditions are not compatible. Indeed let $\gamma_0 = i\sigma$, $\mathbf{Y} = M + \gamma_0 A_0$, $M, A_0 \in$R. δ_m invariance implies that $\mathbf{J}_d$ must depend only on the complex number $\mathbf{Y}$, consistency requires that $\mathbf{J}_d$ must be an analytic function of $\mathbf{Y}$ and finally gauge invariance requires $\mathbf{J}$ to be independent of $\mathrm{Im}\,\mathbf{Y}$, therefore $\mathbf{J}_d$ must be a constant, which contradicts $\mathbf{J}_d = -\frac{1}{2}\epsilon(m)$ for $\mathbf{Y} = m$. Also, this solution would break parity, which requires the current to be an odd function of $\mathbf{Y}$.

In conclusion, if we insist that the current and the action must be gauge invariant we must choose between two possibilities. First, take m and $\mathbf{D}_0$ as independent variables. In this case the actions (7.4) are simultaneously gauge and parity invariant. And second, that the action depends only on $\mathbf{D}$. In this case we obtain two different actions by taking m positive or negative both of which are gauge invariant but have a parity anomaly. This is what happens also in the massless case [31]. The choice between the two possibilities depends on the physical problem at hand, for instance whether the problem admits a natural definition of m or not. Note that the ultraviolet finite part of the action was independent of m.

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Appendix A. Seeley-DeWitt coefficients for Dirac operators to all orders

In the case of even order, the Seeley-DeWitt coefficients for $\mathbf{D}$ admit a simple form. The general pattern can be illustrated by inspection of the first coefficients up to $\mathcal{O}_6(\mathbf{D})$

$$\begin{aligned}
\mathcal{O}_0(\mathbf{D}) &= 1 \\
\mathcal{O}_2(\mathbf{D}) &= \mathbf{D}^2 + \mathbf{A}^2 \\
\mathcal{O}_4(\mathbf{D}) &= \frac{1}{2}\mathbf{D}^4 + \frac{1}{3}(\mathbf{D}^2\mathbf{A}^2 + \mathbf{A}\mathbf{D}^2\mathbf{A} + \mathbf{A}^2\mathbf{D}^2) + \frac{1}{6}\mathbf{A}^4 \\
\mathcal{O}_6(\mathbf{D}) &= \frac{1}{6}\mathbf{D}^6 + \frac{1}{12}(\mathbf{D}^4\mathbf{A}^2 + \mathbf{D}^2\mathbf{A}\mathbf{D}^2\mathbf{A} + \mathbf{D}^2\mathbf{A}^2\mathbf{D}^2 + \mathbf{A}\mathbf{D}^4\mathbf{A} + \mathbf{A}\mathbf{D}^2\mathbf{A}\mathbf{D}^2\mathbf{D}^4\mathbf{A}^2\mathbf{D}^4) \\
&\quad + \frac{1}{30}(\mathbf{D}^2\mathbf{A}^4 + \mathbf{A}\mathbf{D}^2\mathbf{A}^3 + \mathbf{A}^2\mathbf{D}^2\mathbf{A}^2 + \mathbf{A}^3\mathbf{D}^2\mathbf{A} + \mathbf{A}^4\mathbf{D}^2) + \frac{1}{90}\mathbf{A}^6
\end{aligned} \tag{A.1}$$

Here, each monomial of degree $2m$ in $\mathbf{A}$, stands for the $(2m-1)!!$ terms obtained by all possible contractions in the Lorentz indices involving the $2m$ $\mathbf{A}_\mu$ vectors. (Compare with Eq. (4.7) for $\mathcal{O}_2$ and $\mathcal{O}_4$). Note that $\mathbf{D}^2$ does not commute with the symbol $\mathbf{A}$ and also that the cyclic property cannot be applied.

One realizes that these coefficients depend only on $\mathbf{A}_\mu$ and $\mathbf{D}^2$. This is in fact a general rule which can be inferred from a direct comparison of inverse mass and Heat Kernel expansions (see Eq. (B.7)). Moreover, we note that the $\binom{n+2m}{n}$ monomials of degree n in $\mathbf{D}^2$ and $2m$ in $\mathbf{A}$, with $2n+2m=N$, appear in all combinations and with the same coefficient $a_{N,n}$, namely

$$a_{N,n} = \frac{2^{N/2-n}}{(N-n)!} \tag{A.2}$$

In summary, the final formula for $\mathcal{O}_N(\mathbf{D})$ reads

$$\mathcal{O}_N(\mathbf{D}) = \sum_{n=0}^{N/2} a_{N,n}[\mathbf{D}^{2n}\mathbf{A}^{N-2n}] \tag{A.3}$$

Where $[\mathbf{D}^{2n}\mathbf{A}^{N-2n}]$ stands for the sum of the $\binom{N-n}{n}$ terms obtained by all possible monomials of degree n in $\mathbf{D}^2$ and $N-2n$ in $\mathbf{A}$. As expected, the coefficients do not depend explicitly on the space time dimension D. In fact, if one assumes the validity of Eq. (A.3), the coefficients can be completely determined by taking particular cases. For instance, the relative weight of the coefficients is determined by gauge invariance, and the global normalization can be fixed by either going to the $D=0$ case or by taking $\mathbf{D}$ to be a c-number. Finally, we mention that the odd order coefficients can be derived by direct use of the formula $\delta_m \mathcal{O}_N = -\epsilon_{N-D}\mathcal{O}_{N-1}\delta m$.

The previous result can straightforwardly be applied to compute the chiral anomaly in D dimensions for a purely vector external field. Defining the field strength tensor $iF_{\mu\nu} = [i\mathcal{D}_\mu, i\mathcal{D}_\nu]$, one finds

$$\mathcal{A}_A = -\frac{1}{(D/2)!}\epsilon_{\mu_1\cdots\mu_D}\left\langle 2i\alpha F_{\mu_1\mu_2}\cdots F_{\mu_{D-1}\mu_D}\right\rangle \tag{A.4}$$

where $\langle\ \rangle$ was defined in Eq. (4.15) and tr includes flavor space only.

Appendix B. Inverse mass and Heat Kernel expansions for Dirac operators in even dimensions

In even dimensions a straightforward relation between the well-known Heat Kernel expansion and the inverse mass expansion of Section 4 can be established. Our result is a reminiscent of the formal determinantal relation $\mathrm{Det}\,(\mathbf{D})\mathrm{Det}\,(-\mathbf{D}) = \mathrm{Det}\,(-\mathbf{D}^2)$. Actually, what is found is that this relation holds for the ζ-function regularization of the determinants at each order in an inverse mass expansion, where the l.h.s. stands for the Wigner transformation result and the r.h.s. for the Heat Kernel expansion.

Let Δ be a positive definite second order differential operator of the form

$$\Delta = -\hat{\mathcal{D}}_\mu^2 + Y(x), \qquad i\hat{\mathcal{D}}_\mu = i\partial_\mu + B_\mu(x) \tag{B.1}$$

The heat kernel $\langle x|e^{-\tau\Delta}|x\rangle$ is ultraviolet finite for $\tau > 0$ and it admits an asymptotic expansion [29] around $\tau = 0$

$$\langle x|e^{-\tau\Delta}|x\rangle = (4\pi\tau)^{-D/2}\sum_{n=0}^{\infty}\tau^n a_n(x) \tag{B.2}$$

where the Seeley-DeWitt coefficients $a_n(x)$ are covariant local polynomials in $i\hat{\mathcal{D}}_\mu$ and Y of degree p, q respectively with $2p+q=n$. This coefficients can be written in a dimension independent way [2]. If we proceed formally, relying on the good behavior of the required analytical continuations, we find

$$\begin{aligned}\langle x|\mathbf{D}^s|x\rangle &= \frac{(-1)^{-s/2}}{\Gamma(-\frac{s}{2})}\int_0^\infty d\tau\tau^{-s/2-1}e^{\tau m^2}\langle x|e^{-\tau\Delta}|x\rangle \\ \Delta &= -\mathbf{D}^2 + m^2 = -\mathbf{D}_0^2 - 2m\mathbf{D}_0\end{aligned} \tag{B.3}$$

Now, by using the heat kernel expansion, the τ integral can be carried out order by order (for $m^2 < 0$ and then analytically continued to $m^2 > 0$). In this way we obtain another representation of the ζ-function

$$\langle x|\mathbf{D}^s|x\rangle = \sum_{n=0}^{\infty}(-1)^{\frac{D}{2}-n}\frac{m^{D+s-2n}}{(4\pi)^{D/2}}\frac{\Gamma(n-\frac{D+s}{2})}{\Gamma(-\frac{s}{2})}a_n(x) \tag{B.4}$$

This is to be compared to the expansion obtained from the Wigner transformation approach

$$\langle x|\mathbf{D}^s|x\rangle = \sum_{N=0}^{\infty}\frac{m^{D+s-N}}{(4\pi)^{D/2}}\frac{\Gamma(s+1)}{\Gamma(s+1-N+D)}\zeta_{N-D}\mathcal{O}_N(x) \tag{B.5}$$

The comparison however is not immediate because we must first of all reexpand $a_n(x)$ in powers of m, or equivalently in powers of $i\mathcal{D}_\mu$ and $\mathbf{X}$, where $\mathbf{D}_0 = i\not{\mathcal{D}} + \mathbf{X}$. For a given order N, $\mathcal{O}_N(x)$ generically gets contributions from all the terms $a_n(x)$ with $n \geq \frac{1}{2}N$. This is because $a_n(x)$ is made out of $i\hat{\mathcal{D}}_\mu$ and $Y(x)$ which are of zeroth order in $i\mathcal{D}_\mu$ and $\mathbf{X}$, namely

$$\begin{aligned} i\hat{\mathcal{D}}_\mu &= i\mathcal{D}_\mu - \frac{1}{2}\{\gamma_\mu, \mathbf{X}+m\} \\ Y &= m^2 - (\mathbf{X}+m)^2 - \frac{1}{2}[\gamma_\mu,[i\mathcal{D}_\mu,\mathbf{X}]] - \frac{1}{4}\{\gamma_\mu,\mathbf{X}+m\}^2 - \frac{1}{2}\sigma_{\mu\nu}[i\mathcal{D}_\mu, i\mathcal{D}_\nu] \end{aligned} \tag{B.6}$$

Nevertheless, whenever $\mathcal{O}_N(x)$ requires only a finite number of heat kernel terms we can check that both expressions coincide. In particular choosing D even, $s = m = 0$, we find the following identity $(4\pi)^{D/2}\langle x|\mathbf{D}^0|x\rangle = \mathcal{O}_D(x;\mathbf{D}) = a_{D/2}(x;\Delta = -\mathbf{D}^2)$, which holds regardless of the way the coefficients are written. If both coefficients are expressed in a dimension independent form, one has the further identity, for even dimensions

$$\mathcal{O}_{2n}(x;\mathbf{D}) = a_n(x;\Delta = -\mathbf{D}^2), \qquad n = 0,1,2,\ldots \tag{B.7}$$

This can be better checked for lowers orders rewriting Eq. (B.6) (for $m = 0$) as

$$i\hat{\mathcal{D}}_\mu = -\mathbf{A}_\mu, \qquad Y = -\mathbf{D}^2 - \mathbf{A}_\mu^2 \tag{B.8}$$

Using well known expressions for a_0, a_1, a_2 [2], we reproduce $\mathcal{O}_0$, $\mathcal{O}_2$ and $\mathcal{O}_4$. An interesting consequence of $\mathcal{O}_D(x;\mathbf{D}) = a_{D/2}(x;\Delta = -\mathbf{D}^2)$ is that, although an anomaly calculation using $\mathbf{D}^2$ as regulator might be questionable in principle for non normal $\mathbf{D}$ or non positive $-\mathbf{D}^2$, such a procedure turns out to be justified a posteriori.

Appendix C. Explicit form of the axial anomaly in four dimensions

The most general spinor structure for the Dirac operator in four Euclidean dimensions is given by

$$\mathbf{D} = i\not{\mathcal{D}} + S + i\gamma_5 P + \not{A}\gamma_5 + \frac{1}{2}i\sigma_{\mu\nu}T_{\mu\nu} - \frac{1}{2}\sigma_{\mu\nu}\gamma_5 T'_{\mu\nu} \tag{C.1}$$

with $i\mathcal{D}_\mu = i\partial_\mu + V_\mu$, $\sigma_{\mu\nu} = \frac{1}{2}[\gamma_\mu, \gamma_\nu]$ and $T^*_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}T_{\rho\sigma}$. If the Hamiltonian $\gamma^0\mathbf{D}$ is Hermitian in Minkowski space, and assuming that the fields transform as usual under the Wick rotation, the terms S, $\not{A}\gamma_5$ and $\frac{1}{2}i\sigma_{\mu\nu}T_{\mu\nu}$ are Hermitian whereas $i\not{\mathcal{D}}$, $i\gamma_5 P$ and $\frac{1}{2}\sigma_{\mu\nu}\gamma_5 T'_{\mu\nu}$ are antihermitian. This allows to distinguish between the two tensor terms. However, as our expressions will not depend on the hermiticity properties of $\mathbf{D}$ we can reabsorb $T'_{\mu\nu}$ into $T_{\mu\nu}$, i.e., $T_{\mu\nu} - iT'^*_{\mu\nu} \to T_{\mu\nu}$.

In the following expressions, $\langle\quad\rangle$ stands for $\int d^4x \,\mathrm{tr}\, \frac{1}{4\pi^2}\alpha(x)$ and tr refers to flavor only. The axial anomaly is obtained adding up all the terms.

$$\begin{aligned}
\mathcal{A}[V^4] &= \frac{1}{4}i\epsilon_{\mu\nu\rho\sigma}\langle\ [i\mathcal{D}_\mu, i\mathcal{D}_\nu][i\mathcal{D}_\rho, i\mathcal{D}_\sigma]\ \rangle \\
\mathcal{A}[A^4] &= -\frac{1}{3}i\epsilon_{\mu\nu\rho\sigma}\langle\ A_\mu A_\nu A_\rho A_\sigma\ \rangle \\
\mathcal{A}[V^2A^2] &= -i\epsilon_{\mu\nu\rho\sigma}\langle\ \frac{1}{6}\{[i\mathcal{D}_\mu, i\mathcal{D}_\nu], A_\rho A_\sigma\} + \frac{2}{3}A_\mu[i\mathcal{D}_\nu, i\mathcal{D}_\rho]A_\sigma - \frac{1}{3}[i\mathcal{D}_\mu, A_\nu][i\mathcal{D}_\rho, A_\sigma]\ \rangle \\
\mathcal{A}[V^3A] &= i\langle\ [i\mathcal{D}_\mu,[i\mathcal{D}_\mu,[i\mathcal{D}_\nu, A_\nu]]] - [i\mathcal{D}_\mu,[i\mathcal{D}_\nu,[i\mathcal{D}_\mu, A_\nu]]] + \frac{1}{3}[i\mathcal{D}_\nu,[i\mathcal{D}_\mu,[i\mathcal{D}_\mu, A_\nu]]]\ \rangle \\
\mathcal{A}[VA^3] &= i\langle\ [i\mathcal{D}_\mu, A_\nu]A_\nu A_\mu + A_\mu A_\nu[i\mathcal{D}_\mu, A_\nu] + \frac{1}{3}A_\nu A_\nu[i\mathcal{D}_\mu, A_\mu] \\
&\quad + \frac{1}{3}[i\mathcal{D}_\mu, A_\mu]A_\nu A_\nu - \frac{1}{3}[i\mathcal{D}_\mu, A_\nu]A_\mu A_\nu - \frac{1}{3}A_\nu A_\mu[i\mathcal{D}_\mu, A_\nu] \\
&\quad + \frac{2}{3}A_\nu[i\mathcal{D}_\mu, A_\mu]A_\nu + \frac{2}{3}A_\mu[i\mathcal{D}_\mu, A_\nu]A_\nu + \frac{2}{3}A_\nu[i\mathcal{D}_\mu, A_\nu]A_\mu\ \rangle \\
\mathcal{A}[S^3P] &= \langle\ -S\{S,P\}S + \frac{5}{3}\{S^3, P\}\ \rangle \\
\mathcal{A}[SP^3] &= \langle\ P\{S,P\}P + \{S, P^3\}\ \rangle \\
\mathcal{A}[SPV^2] &= \langle\ \{[i\mathcal{D}_\mu, P],[i\mathcal{D}_\mu, S]\} + \frac{1}{3}\{[i\mathcal{D}_\mu,[i\mathcal{D}_\mu, S], P\} + \{[i\mathcal{D}_\mu,[i\mathcal{D}_\mu, P]], S\}\ \rangle \\
\mathcal{A}[S^2VA] &= i\langle\ \{S^2,[i\mathcal{D}_\mu, A_\mu]\} + SA_\mu[i\mathcal{D}_\mu, S] + [i\mathcal{D}_\mu, S]A_\mu S \\
&\quad + \frac{5}{3}[i\mathcal{D}_\mu, S]SA_\mu + \frac{5}{3}A_\mu S[i\mathcal{D}_\mu, S] + \frac{4}{3}S[i\mathcal{D}_\mu, S]A_\mu + \frac{4}{3}A_\mu[i\mathcal{D}_\mu, S]S\ \rangle \\
\mathcal{A}[P^2VA] &= i\langle\ [[A_\mu, P],[i\mathcal{D}_\mu, P]] + \{[i\mathcal{D}_\mu, A_\mu], P^2\}\ \rangle \\
\mathcal{A}[SPA^2] &= \langle\ [A_\mu A_\mu, [S,P]] + \{\{A_\mu, S\},\{A_\mu, P\}\}\ \rangle
\end{aligned}$$

$$\mathcal{A}[T^4] = i\langle\ \frac{1}{12}T_{\mu\nu}T_{\alpha\beta}T_{\alpha\beta}T^*_{\mu\nu} - \frac{1}{4}T_{\mu\nu}T_{\alpha\beta}T^*_{\alpha\beta}T_{\mu\nu} - \frac{1}{3}T_{\mu\nu}[T_{\alpha\beta}, T_{\mu\beta}]T^*_{\nu\alpha}\ \rangle$$

$$\mathcal{A}[ST^3] = \frac{1}{3}\langle\ T^*_{\mu\nu}T_{\mu\alpha}T_{\nu\alpha}S + ST_{\mu\alpha}T_{\nu\alpha}T^*_{\mu\nu} + T_{\mu\alpha}\{S, T^*_{\mu\nu}\}T_{\nu\alpha}\ \rangle$$

$$\mathcal{A}[PT^3] = i\langle\ T_{\mu\alpha}\{P, T_{\mu\nu}\}T_{\nu\alpha} - \frac{1}{3}T_{\mu\nu}T_{\mu\alpha}T_{\nu\alpha}P - \frac{1}{3}PT_{\mu\alpha}T_{\nu\alpha}T_{\mu\nu}\ \rangle$$

$$\mathcal{A}[S^2T^2] = i\langle\ \frac{5}{6}T_{\mu\nu}S^2T^*_{\mu\nu} - \frac{1}{2}ST_{\mu\nu}T^*_{\mu\nu}S + \frac{1}{6}S\{S, T^*_{\mu\nu}\}T_{\mu\nu} + \frac{1}{6}T_{\mu\nu}\{S, T^*_{\mu\nu}\}S\ \rangle$$

$$\mathcal{A}[P^2T^2] = \frac{1}{2}i\langle\ T_{\mu\nu}P^2T^*_{\mu\nu} + PT_{\mu\nu}T^*_{\mu\nu}P + P\{P, T_{\mu\nu}\}T^*_{\mu\nu} + T^*_{\mu\nu}\{P, T_{\mu\nu}\}P\ \rangle$$

$$\begin{aligned}\mathcal{A}[SPT^2] = \langle\ &\frac{1}{6}ST_{\mu\nu}T_{\mu\nu}P + \frac{1}{6}PT_{\mu\nu}T_{\mu\nu}S - \frac{1}{2}T_{\mu\nu}\{S, P\}T_{\mu\nu} + \frac{1}{6}P\{S, T_{\mu\nu}\}T_{\mu\nu} \\ &+ \frac{1}{6}T_{\mu\nu}\{S, T_{\mu\nu}\}P - \frac{1}{2}S\{P, T_{\mu\nu}\}T_{\mu\nu} - \frac{1}{2}T_{\mu\nu}\{P, T_{\mu\nu}\}S\ \rangle\end{aligned}$$

$$\begin{aligned}\mathcal{A}[V^2T^2] = i\langle\ &\{[i\mathcal{D}_\mu, [i\mathcal{D}_\nu, T_{\nu\alpha}]], T^*_{\mu\alpha}\} + \frac{1}{3}\{[i\mathcal{D}_\mu, [i\mathcal{D}_\nu, T^*_{\nu\alpha}]], T_{\mu\alpha}\} \\ &+ \frac{1}{3}\{[i\mathcal{D}_\mu, T_{\mu\alpha}], [i\mathcal{D}_\nu, T^*_{\nu\alpha}]\} + \frac{1}{3}[i\mathcal{D}_\mu, T_{\alpha\beta}][i\mathcal{D}_\mu, T^*_{\alpha\beta}]\ \rangle\end{aligned}$$

$$\begin{aligned}\mathcal{A}[A^2T^2] = i\langle\ &\frac{1}{2}T_{\alpha\beta}A_\mu A_\mu T^*_{\alpha\beta} + \frac{1}{2}A_\mu T_{\alpha\beta}T^*_{\alpha\beta}A_\mu + \frac{1}{3}T_{\mu\alpha}[A_\mu, A_\nu]T^*_{\nu\alpha} \\ &+ \frac{1}{3}A_\mu(T_{\mu\alpha}T^*_{\nu\alpha} - T_{\nu\alpha}T^*_{\mu\alpha})A_\nu - \frac{1}{3}T_{\nu\alpha}\{A_\mu, T^*_{\mu\alpha}\}A_\nu \\ &- \frac{1}{3}A_\nu\{A_\mu, T^*_{\mu\alpha}\}T_{\nu\alpha} - \frac{1}{3}T^*_{\nu\alpha}[A_\mu, T_{\mu\alpha}]A_\nu + \frac{1}{3}A_\nu[A_\mu, T_{\mu\alpha}]T^*_{\nu\alpha}\ \rangle\end{aligned}$$

$$\begin{aligned}\mathcal{A}[VAT^2] = -i\langle\ &\frac{1}{6}\{T_{\alpha\beta}T_{\alpha\beta}, [i\mathcal{D}_\mu, A_\mu]\} - \frac{1}{3}\{\{T_{\mu\alpha}, T_{\nu\alpha}\}, [i\mathcal{D}_\mu, A_\nu]\} + \frac{1}{3}[i\mathcal{D}_\mu, T_{\mu\alpha}]T_{\nu\alpha}A_\nu \\ &+ \frac{1}{3}A_\nu T_{\nu\alpha}[i\mathcal{D}_\mu, T_{\mu\alpha}] - \frac{1}{6}[i\mathcal{D}_\mu, T_{\alpha\beta}]T_{\alpha\beta}A_\mu - \frac{1}{6}A_\mu T_{\alpha\beta}[i\mathcal{D}_\mu, T_{\alpha\beta}] \\ &- \frac{1}{3}[i\mathcal{D}_\mu, T_{\nu\alpha}]T_{\mu\alpha}A_\nu - \frac{1}{3}A_\nu T_{\mu\alpha}[i\mathcal{D}_\mu, T_{\nu\alpha}] - \frac{4}{3}T_{\mu\alpha}[i\mathcal{D}_\nu, T_{\nu\alpha}]A_\mu \\ &- \frac{4}{3}A_\mu[i\mathcal{D}_\nu, T_{\nu\alpha}]T_{\mu\alpha} - \frac{1}{3}[i\mathcal{D}_\mu, T_{\mu\alpha}]A_\nu T_{\nu\alpha} - \frac{1}{3}T_{\nu\alpha}A_\nu[i\mathcal{D}_\mu, T_{\mu\alpha}] \\ &+ \frac{1}{3}[i\mathcal{D}_\mu, T_{\nu\alpha}]A_\nu T_{\mu\alpha} + \frac{1}{3}T_{\mu\alpha}A_\nu[i\mathcal{D}_\mu, T_{\nu\alpha}] + \frac{1}{6}[i\mathcal{D}_\mu, T_{\alpha\beta}]A_\mu T_{\alpha\beta} \\ &+ \frac{1}{6}T_{\alpha\beta}A_\mu[i\mathcal{D}_\mu, T_{\alpha\beta}] + \frac{2}{3}T_{\alpha\beta}[i\mathcal{D}_\mu, A_\mu]T_{\alpha\beta}\ \rangle\end{aligned}$$

$$\begin{aligned}\mathcal{A}[SV^2T] = -\langle\ &\frac{2}{3}S[i\mathcal{D}_\mu, i\mathcal{D}_\nu]T^*_{\mu\nu} + \frac{2}{3}T^*_{\mu\nu}[i\mathcal{D}_\mu, i\mathcal{D}_\nu]S \\ &+ \frac{1}{6}\{\{S, T^*_{\mu\nu}\}, [i\mathcal{D}_\mu, i\mathcal{D}_\nu]\} + \frac{1}{3}[[i\mathcal{D}_\mu, T^*_{\mu\nu}], [i\mathcal{D}_\nu, S]]\ \rangle\end{aligned}$$

$$\begin{aligned}\mathcal{A}[SA^2T] = -\frac{1}{3}\langle\ &T^*_{\mu\nu}A_\mu A_\nu S + SA_\mu A_\nu T^*_{\mu\nu} + A_\mu\{S, T^*_{\mu\nu}\}A_\nu - 3T^*_{\mu\nu}\{S, A_\nu\}A_\mu \\ &+ 3A_\mu\{S, A_\nu\}T^*_{\mu\nu} - S\{A_\mu, T^*_{\mu\nu}\}A_\nu + A_\nu\{A_\mu, T^*_{\mu\nu}\}S\ \rangle\end{aligned}$$

$$
\begin{aligned}
\mathcal{A}[PV^2T] &= i\langle\ \frac{1}{2}\{\{T_{\mu\nu},P\},[i\mathcal{D}_\mu,i\mathcal{D}_\nu]\}-[[i\mathcal{D}_\mu,T_{\mu\nu}],[i\mathcal{D}_\nu,P]]\ \rangle \\
\mathcal{A}[PA^2T] &= i\langle\ \frac{1}{3}T_{\mu\nu}A_\mu A_\nu P+\frac{1}{3}PA_\mu A_\nu T_{\mu\nu}-A_\mu\{P,T_{\mu\nu}\}A_\nu+T_{\mu\nu}[A_\mu,P]A_\nu \\
&\quad -A_\mu[A_\nu,P]T_{\mu\nu}-\frac{1}{3}P[A_\mu,T_{\mu\nu}]A_\nu+\frac{1}{3}A_\mu[A_\nu,T_{\mu\nu}]P\ \rangle \\
\mathcal{A}[SVAT] &= \langle -[i\mathcal{D}_\mu,S]A_\nu T_{\mu\nu}+T_{\mu\nu}A_\nu[i\mathcal{D}_\mu,S]-[i\mathcal{D}_\mu,A_\nu]ST_{\mu\nu}+T_{\mu\nu}S[i\mathcal{D}_\mu,A_\nu] \\
&\quad +\frac{1}{3}[i\mathcal{D}_\mu,S]T_{\mu\nu}A_\nu-\frac{1}{3}A_\nu T_{\mu\nu}[i\mathcal{D}_\mu,S]-\frac{1}{3}[i\mathcal{D}_\mu,A_\nu]T_{\mu\nu}S \\
&\quad +\frac{1}{3}ST_{\mu\nu}[i\mathcal{D}_\mu,A_\nu]+\frac{1}{3}[i\mathcal{D}_\mu,T_{\mu\nu}]SA_\nu-\frac{1}{3}A_\nu S[i\mathcal{D}_\mu,T_{\mu\nu}] \\
&\quad +\frac{1}{3}[i\mathcal{D}_\mu,T_{\mu\nu}]A_\nu S-\frac{1}{3}SA_\nu[i\mathcal{D}_\mu,T_{\mu\nu}]+\frac{2}{3}T_{\mu\nu}[i\mathcal{D}_\mu,S]A_\nu \\
&\quad -\frac{2}{3}A_\nu[i\mathcal{D}_\mu,S]T_{\mu\nu}+\frac{2}{3}A_\nu[i\mathcal{D}_\mu,T_{\mu\nu}]S-\frac{2}{3}S[i\mathcal{D}_\mu,T_{\mu\nu}]A_\nu \\
&\quad +\frac{2}{3}T_{\mu\nu}[i\mathcal{D}_\mu,A_\nu]S-\frac{2}{3}S[i\mathcal{D}_\mu,A_\nu]T_{\mu\nu}\ \rangle \\
\mathcal{A}[PVAT] &= i\langle\ [i\mathcal{D}_\mu,P]A_\nu T^*_{\mu\nu}-T^*_{\mu\nu}A_\nu[i\mathcal{D}_\mu,P]+[i\mathcal{D}_\mu,P]T^*_{\mu\nu}A_\nu \\
&\quad -A_\nu T^*_{\mu\nu}[i\mathcal{D}_\mu,P]-\frac{1}{3}[i\mathcal{D}_\mu,A_\nu]T^*_{\mu\nu}P+\frac{1}{3}PT^*_{\mu\nu}[i\mathcal{D}_\mu,A_\nu] \\
&\quad -[i\mathcal{D}_\mu,A_\nu]PT^*_{\mu\nu}+T^*_{\mu\nu}P[i\mathcal{D}_\mu,A_\nu]+[i\mathcal{D}_\mu,T^*_{\mu\nu}]PA_\nu \\
&\quad -A_\nu P[i\mathcal{D}_\mu,T^*_{\mu\nu}]-\frac{1}{3}[i\mathcal{D}_\mu,T^*_{\mu\nu}]A_\nu P+\frac{1}{3}PA_\nu[i\mathcal{D}_\mu,T^*_{\mu\nu}] \\
&\quad +2T^*_{\mu\nu}[i\mathcal{D}_\mu,P]A_\nu-2A_\nu[i\mathcal{D}_\mu,P]T^*_{\mu\nu}+\frac{2}{3}P[i\mathcal{D}_\mu,A_\nu]T^*_{\mu\nu} \\
&\quad -\frac{2}{3}T^*_{\mu\nu}[i\mathcal{D}_\mu,A_\nu]P+\frac{2}{3}P[i\mathcal{D}_\mu,T^*_{\mu\nu}]A_\nu-\frac{2}{3}A_\nu[i\mathcal{D}_\mu,T^*_{\mu\nu}]P\ \rangle
\end{aligned}
$$

Appendix D. Explicit form of the counterterms in four dimensions

For all the terms which do not contain $i\mathcal{D}_\mu$ nor A_μ, one can find a compact expression for their counterterms, namely

$$
W[S,P,T]=-\frac{1}{24}\langle\mathbf{M}^4+\mathbf{M}^2\gamma_\mu\mathbf{M}^2\gamma_\mu+2\mathbf{M}^3\gamma_\mu\mathbf{M}\gamma_\mu\rangle \tag{D.1}
$$

where here $\mathbf{M}(x)=S+i\gamma_5P+\frac{1}{2}i\sigma_{\mu\nu}T_{\mu\nu}$ and $\delta_A\mathbf{M}(x)=-\{i\alpha(x)\gamma_5,\mathbf{M}(x)\}$. The rest of the counterterms are given by

$$
\begin{aligned}
W[A^4+\cdots] &= -2\Big\langle [i\mathcal{D}_\mu, A_\mu][i\mathcal{D}_\nu, A_\nu] - [i\mathcal{D}_\mu, A_\nu][i\mathcal{D}_\nu, A_\mu] \\
&\quad + \frac{1}{3}[i\mathcal{D}_\mu, A_\nu][i\mathcal{D}_\mu, A_\nu] - \frac{4}{3}A_\mu A_\mu A_\nu A_\nu + \frac{2}{3}A_\mu A_\nu A_\mu A_\nu \Big\rangle \\
W[S^2V^2+\cdots] &= 4\Big\langle \frac{1}{3}[i\mathcal{D}_\mu, S][i\mathcal{D}_\mu, S] + \frac{1}{2}[i\mathcal{D}_\mu, P][i\mathcal{D}_\mu, P] \\
&\quad + S^2 A_\mu A_\mu - P A_\mu P A_\mu - i[i\mathcal{D}_\mu, S]\{P, A_\mu\} \Big\rangle \\
W[V^2T^2+\cdots] &= 4\Big\langle -\frac{1}{6}[A_\mu, T_{\mu\alpha}][A_\nu, T_{\nu\alpha}] + \frac{1}{6}A_\mu A_\mu T_{\alpha\beta} T_{\alpha\beta} \\
&\quad - \frac{1}{3}[i\mathcal{D}_\mu, T_{\mu\alpha}][i\mathcal{D}_\nu, T_{\nu\alpha}] + \frac{1}{6}[i\mathcal{D}_\mu, T_{\nu\alpha}][i\mathcal{D}_\nu, T_{\mu\alpha}] \\
&\quad - \frac{1}{6}[i\mathcal{D}_\mu, A_\mu] T_{\alpha\beta} T^*_{\alpha\beta} + \frac{1}{3}[i\mathcal{D}_\mu, T_{\mu\alpha}]\{A_\nu, T^*_{\nu\alpha}\} \Big\rangle \\
W[SV^2T+\cdots] &= -4i\Big\langle -\frac{1}{3}\{S, T_{\mu\nu}\}A_\mu A_\nu + \frac{4}{3}S A_\mu T_{\mu\nu} A_\nu + \frac{1}{2}i[i\mathcal{D}_\mu, i\mathcal{D}_\nu]\{P, T_{\mu\nu}\} \\
&\quad + \frac{1}{3}[i\mathcal{D}_\mu, S][A_\nu, T^*_{\mu\nu}] - \frac{2}{3}[i\mathcal{D}_\mu, A_\nu][S, T^*_{\mu\nu}] - i[i\mathcal{D}_\mu, T_{\mu\nu}][P, A_\nu] \Big\rangle
\end{aligned}
$$

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Table captions

Table 4.1 Several useful particular values for the coefficients involved in the inverse mass expansion (4.2) and defined in Eqs. (4.5) and (4.6). The coefficients for the inverse mass expanded effective action Eq. (4.14) and the effective current Eq. (4.18) and (4.19) are also given. In the odd-dimensional case, the coefficients ζ_K, α_K^W, β_K^W, α_K^J and β_K^J have to be multiplied with an extra $\sqrt{\pi}$ factor.

Table 4.2 Explicit formulas for the effective action inverse mass coefficients, α_K^W and β_K^W (see formula (4.14)) in terms of ζ_K as given by Eqs. (4.5) and (4.6).

K	even D ϵ_K	ζ_K	α_K^W	β_K^W	α_K^J	β_K^J	odd D ϵ_K	ζ_K	α_K^W	β_K^W	α_K^J	β_K^J
-5	-5	24	$\frac{137}{300}$	$-\frac{1}{5}$	$\frac{25}{12}$	-1	2	0	0	0	0	0
-4	2	12	$\frac{25}{24}$	$-\frac{1}{2}$	$\frac{11}{3}$	-2	-4	0	0	0	0	0
-3	-3	-4	$-\frac{11}{9}$	$\frac{2}{3}$	-3	2	2	0	0	0	0	0
-2	2	-2	$-\frac{3}{2}$	1	-2	2	-2	0	0	0	0	0
-1	-1	2	2	-2	0	-2	2	0	0	0	0	0
0	2	1	0	-1	-1	0	0	-1	0	1	1	0
1	1	1	-1	0	1	0	2	$-\frac{1}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	0
2	2	$\frac{1}{2}$	$\frac{1}{2}$	0	-1	0	2	$-\frac{1}{4}$	$-\frac{1}{4}$	0	$\frac{1}{2}$	0
3	3	$\frac{1}{6}$	$-\frac{1}{3}$	0	1	0	2	$-\frac{1}{8}$	$\frac{1}{4}$	0	$-\frac{3}{4}$	0
4	2	$\frac{1}{12}$	$\frac{1}{2}$	0	-2	0	4	$-\frac{1}{32}$	$-\frac{3}{16}$	0	$\frac{3}{4}$	0
5	5	$\frac{1}{60}$	$-\frac{2}{5}$	0	2	0	2	$-\frac{1}{64}$	$\frac{3}{8}$	0	$-\frac{15}{8}$	0

Table 4.1

	$K<0$	$K=0$	$K>0$
α_K^W	$\frac{1}{\Gamma(-K+1)}\zeta_K \sum_{n=1}^{-K} n^{-1}$	0	$(-1)^K\Gamma(K)\zeta_K$
β_K^W	$-\frac{1}{\Gamma(-K+1)}\zeta_K$	$-\zeta_0$	0

Table 4.2