

Killing-Yano tensors and generalized supersymmetries in black-hole and monopole geometries¹

J.W. van Holten

NIKHEF-H

Amsterdam NL

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Abstract

New kinds of supersymmetry arise in supersymmetric σ -models describing the motion of spinning point-particles in classical backgrounds, for example black-holes, or the dynamics of monopoles. Their geometric origin is the existence of Killing-Yano tensors. The relation between these concepts is explained and examples are given.

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1 Spinning particles and supersymmetric σ -models

At low energies, when back reaction effects may be neglected, the dynamics of spinning point-particles in a D -dimensional curved space-time is described by the one-dimensional supersymmetric σ -model [1]-[6] with Lagrangian

$$L = \frac{1}{2}g_{\mu\nu}(x)\dot{x}^\mu\dot{x}^\nu + \frac{i}{2}\eta_{ab}\psi^a\frac{D\psi^b}{D\tau}. \quad (1)$$

The position co-ordinates x^μ are Grassmann-even, whilst the spin co-ordinates ψ^a are Grassmann-odd and transform as a tangent-space Lorentz vector. The standard supersymmetry variations, under which L transforms into a total derivative, are

$$\delta x^\mu = -i\epsilon\psi^\mu, \quad \delta\psi^\mu = \dot{x}^\mu\epsilon, \quad (2)$$

where we have used the notation $\psi^\mu = e^\mu_a\psi^a$ with e^μ_a the inverse vielbein. The anti-symmetric spin tensor containing the electric and magnetic dipole moments [7] is

$$S^{ab} = -i\psi^a\psi^b. \quad (3)$$

In a covariant canonical formulation [8, 9] the dynamics follows from the Hamiltonian

$$H = \frac{1}{2}g^{\mu\nu}\Pi_\mu\Pi_\nu. \quad (4)$$

Here Π_μ is the covariant momentum, which can be expressed in terms of the canonical momentum p_μ and the spin-connection $\omega_{\mu ab}$ as

$$\Pi_\mu = p_\mu - \frac{1}{2}\omega_{\mu ab}S^{ab} = g_{\mu\nu}\dot{x}^\nu. \quad (5)$$

In this formulation the time evolution of any dynamical quantity $F(x, \Pi, \psi)$ can be computed in terms of a Poisson-Dirac bracket

$$\dot{F} = \{F, H\}, \quad (6)$$

with the non-vanishing elementary Poisson-Dirac brackets given by

$$\begin{aligned} \{x^\mu, \Pi_\nu\} &= \delta_\nu^\mu, & \{\Pi_\mu, \Pi_\nu\} &= R_{\mu\nu} \equiv \frac{1}{2}S^{ab}R_{ab\mu\nu}, \\ \{\psi^a, \psi^b\} &= \eta^{ab}, & \{\Pi_\mu, \psi^a\} &= -i\omega_\mu^a{}_b\psi^b. \end{aligned} \quad (7)$$

Note that $R_{\mu\nu}$ are the components of the spin-valued Riemann tensor. It follows, that for general scalar phase-space functions F and G the brackets read

$$\{F, G\} = \mathcal{D}_\mu F \frac{\partial G}{\partial \Pi_\mu} - \frac{\partial F}{\partial \Pi_\mu} \mathcal{D}_\mu G + R_{\mu\nu} \frac{\partial F}{\partial \Pi_\mu} \frac{\partial G}{\partial \Pi_\nu} + i(-1)^{a_F} \frac{\partial F}{\partial \psi^a} \frac{\partial G}{\partial \psi_a}, \quad (8)$$

with the covariant derivatives defined by

$$\mathcal{D}_\mu F = \partial_\mu F + \Gamma_{\mu\nu}{}^\lambda \Pi_\lambda \frac{\partial F}{\partial \Pi_\nu} + \omega_{\mu ab} \psi^b \frac{\partial F}{\partial \psi_a}. \quad (9)$$

Of course, any tensor-valued quantity can be converted into a scalar by decomposing its tangent-space components into irreducible representations of the D -dimensional Lorentz group and saturating the symmetric index sets with $\Pi_a = e^\mu_a \Pi_\mu$, and the anti-symmetric index sets with ψ^a ; then F takes the form

$$F(x, \Pi, \psi) = \sum_{m, n \geq 0} \frac{i^{\lfloor \frac{m}{2} \rfloor}}{m!n!} \psi^{a_1} \dots \psi^{a_m} f_{a_1 \dots a_m}^{\mu_1 \dots \mu_n}(x) \Pi_{\mu_1} \dots \Pi_{\mu_n}. \quad (10)$$

In this way the phase-space functions become generalized differential forms on a graded space.

As a simple application of the algebra (7) we compute the brackets of the spin tensor; we verify that it generates a realization of the Lorentz algebra

$$\{S^{ab}, S^{cd}\} = \eta^{ad} S^{bc} + \eta^{bc} S^{ad} - \eta^{ac} S^{bd} - \eta^{bd} S^{ac}. \quad (11)$$

This confirms the interpretation of S^{ab} as representing the spin of the particle.

2 Symmetries

In the hamiltonian formulation, symmetry transformations are generated by the constants of motion through the Poisson-Dirac brackets. In particular, the supersymmetry transformations (2) are obtained from the conserved supercharge

$$Q = \Pi \cdot \psi = e^\mu_a \Pi_\mu \psi^a, \quad (12)$$

by taking the bracket

$$\delta F = i \{F, Q\} \epsilon. \quad (13)$$

That Q is conserved and the super-transformations (13) represent a symmetry is a consequence of the bracket relations

$$\{Q, Q\} = -2iH, \quad \{Q, H\} = 0. \quad (14)$$

The second relation, which follows from the first by the Jacobi identity, implies at the same time the conservation of Q and the invariance of H under the transformations (13).

After the pattern established for the supercharge Q , we can now investigate the full set of symmetries for a given space-time by solving the equation

$$\{J, H\} = \Pi^\mu \left(\mathcal{D}_\mu J + R_{\mu\nu} \frac{\partial J}{\partial \Pi_\nu} \right) = 0, \quad (15)$$

which give all constants of motion $J(x, \Pi, \psi)$. This equation is the generalization of the usual Killing equation to spinning space [10]. However, unlike the usual case, in which the solutions of the Killing equation are single, completely symmetric tensors, here the solutions consist of linear combinations of symmetric tensors of different rank:

$$J(x, \Pi, \psi) = \sum_{n \geq 0} \frac{1}{n!} J^{\mu_1 \dots \mu_n}(x, \psi) \Pi_{\mu_1} \dots \Pi_{\mu_n}, \quad (16)$$

subject to the conditions

$$J_{(\mu_1 \dots \mu_n; \mu_{n+1})} + \omega_{(\mu_{n+1}}^{ab} \psi_b \frac{\partial J_{\mu_1 \dots \mu_n)}}{\partial \psi^a} = -\frac{i}{2} \psi^a \psi^b R_{ab}{}^\nu{}_{(\mu_{n+1}} J_{\mu_1 \dots \mu_n)\nu}. \quad (17)$$

A sufficient, though not necessary, condition for a solution of the generalized Killing equations is superinvariance of a dynamical variable:

$$\{J, Q\} = \psi \cdot \mathcal{D}J + i\Pi \cdot \frac{\partial J}{\partial \psi} = 0. \quad (18)$$

This equation may be considered as a kind of square root of the generalized Killing equation. The new supersymmetries we present later satisfy this superinvariance condition. An important exception is however the supercharge Q itself; according to (14) its bracket gives the hamiltonian, which generates proper-time translations.

The solutions of the generalized Killing equation (15) are of two distinct types: *generic* ones, which exist for any spinning particle model (1), and *non-generic* ones, which depend on the specific background space-time considered. To the first class belong supersymmetry and proper-time translations, generated by the supercharge and hamiltonian, respectively. In addition there also is a ‘chiral’ symmetry, generated by the conserved charge

$$\Gamma_* = -\frac{i^{\lfloor \frac{d}{2} \rfloor}}{d!} \varepsilon_{a_1 \dots a_d} \psi^{a_1} \dots \psi^{a_d}, \quad (19)$$

and a dual supersymmetry generated by

$$Q^* = i \{Q, \Gamma_*\} = -\frac{i^{\lfloor \frac{d}{2} \rfloor}}{(d-1)!} e^{\mu a_1} \Pi_\mu \varepsilon_{a_1 \dots a_d} \psi^{a_2} \dots \psi^{a_d}. \quad (20)$$

Note that Q^* is Grassmann odd in even-dimensional space-times and Grassmann even in odd-dimensional space-times. In the special case $d = 2$ dual supersymmetry is a real supersymmetry, in the sense that the bracket of Q^* with itself closes on the hamiltonian. For all $d > 2$ this bracket vanishes identically.

3 New supersymmetries

The existence of non-generic symmetries depends by definition on the background space-time considered. We now ask, what are the necessary conditions for the existence of new supersymmetries such that

$$\delta x^\mu = -i\epsilon f_a^\mu(x)\psi^a, \quad (21)$$

with f_a^μ some vector not equal to the vierbein e_a^μ . It is straightforward to establish that the solution to this problem is the existence of a constant of motion

$$Q_f = f_a^\mu \Pi_\mu \psi^a + \frac{i}{3!} c_{abc} \psi^a \psi^b \psi^c, \quad (22)$$

with the tensorial quantities f_a^μ and c_{abc} subject to

$$D_\mu f_\nu^a + D_\nu f_\mu^a = 0, \quad (23)$$

$$D_\mu c_{abc} = -R_{\mu\nu ab} f_c^\nu - R_{\mu\nu bc} f_a^\nu - R_{\mu\nu ca} f_b^\nu.$$

These conditions express the contents of the generalized Killing equation (15) for Q_f . The existence of new supersymmetries of this kind then implies automatically the existence of new Grassmann-even constants of motion Z , obtained by taking the brackets of the Q_f with themselves. Let us consider the case of r new supersymmetries with generators Q_i , $i = 1, \dots, r$, each of the general form (22); then the bracket of two supercharges reads

$$\{Q_i, Q_j\} = -2iZ_{ij}, \quad (24)$$

with Z_{ij} a Grassmann-even quadratic expression in the momenta Π_μ :

$$Z_{ij} = \frac{1}{2} K_{ij}^{\mu\nu} \Pi_\mu \Pi_\nu + I_{ij}^\mu \Pi_\mu + G_{ij}. \quad (25)$$

The explicit expressions for co-efficients of the three terms are

$$\begin{aligned} K_{ij}^{\mu\nu} &= K_{ij}^{\nu\mu} = \frac{1}{2} \left(f_{i a}^\mu f_j^{\nu a} + f_{j a}^\mu f_i^{\nu a} \right), \\ I_{ij}^\mu &= \frac{i}{2} \psi^a \psi^b \left(f_{i b}^\nu D_\nu f_{j a}^\mu + f_{j b}^\nu D_\nu f_{i a}^\mu + \frac{1}{2} f_i^{\mu c} c_{j abc} + \frac{1}{2} f_j^{\mu c} c_{i abc} \right), \\ G_{ij} &= -\frac{1}{4} \psi^a \psi^b \psi^c \psi^d \left(R_{\mu\nu ab} f_{i c}^\mu f_{j d}^\nu + \frac{1}{2} c_{i ab}^e c_{j cde} \right). \end{aligned} \quad (26)$$

Since Q_i are conserved, the Jacobi identities for the brackets (24) imply the conservation of the bosonic charges Z_{ij} :

$$\frac{dZ_{ij}}{d\tau} = \{Z_{ij}, H\} = 0. \quad (27)$$

Since this is a linear relation, we may regard the Z_{ij} as the components of a matrix Z of constants of motion, each of which is a solution of the generalized Killing equations (15). The matrix-valued co-efficients of the various terms in the momentum expansion of Z then satisfy the linear relations (17):

$$\begin{aligned} K_{(\mu\nu;\lambda)} &= 0, \\ D_{(\mu} I_{\nu)ab} &= R_{ab\lambda(\mu} K_{\nu)}^\lambda, \\ D_\mu G_{abcd} &= R_{\lambda\mu[ab} I_{cd]}^\lambda. \end{aligned} \quad (28)$$

In these equations parentheses denote symmetrization and the square brackets in the last expression denote anti-symmetrization over the (latin) indices enclosed.

4 Killing-Yano tensors

According to eqs.(22), a generalized supersymmetry Q_f exists if and only if we can find a pair of tensors (f_a^μ, c_{abc}) satisfying the differential relations (23). We now show that if Q_f is superinvariant:

$$\{Q, Q_f\} = 0, \quad (29)$$

then the conditions for the presence of a new supersymmetry in the σ -model (1) reduce to the existence of an anti-symmetric tensor $f_{\mu\nu}$ such that

$$f_{\mu\nu;\lambda} + f_{\lambda\nu;\mu} = 0. \quad (30)$$

This equation itself is equivalent to the first eq.(23), rewritten in terms of

$$f_{\mu\nu} = f_\mu^a e_{\nu a}. \quad (31)$$

The anti-symmetry of the tensor $f_{\mu\nu}$ is a consequence of condition (29). An anti-symmetric tensor of this type is called a Killing-Yano tensor.

Having a Killing-Yano tensor guarantees the existence of an anti-symmetric 3-index Lorentz tensor c_{abc} satisfying the second equation (23). Indeed, we observe that the anti-symmetry of $f_{\mu\nu}$ in combination with the constraint (30) implies that its covariant derivative is completely anti-symmetric:

$$\begin{aligned}
H_{\mu\nu\lambda} &= \frac{1}{3} (f_{\mu\nu;\lambda} + f_{\nu\lambda;\mu} + f_{\lambda\mu;\nu}) \\
&= f_{\mu\nu;\lambda}.
\end{aligned}
\tag{32}$$

Therefore the field strength of $f_{\mu\nu}$ is a pure gradient. Further differentiation of $H_{\mu\nu\lambda}$ and use of the Ricci identity then leads to the result

$$H_{\mu\nu\lambda;\kappa} = \frac{1}{2} \left(R_{\mu\nu\kappa}{}^{\sigma} f_{\sigma\lambda} + R_{\nu\lambda\kappa}{}^{\sigma} f_{\sigma\mu} + R_{\lambda\mu\kappa}{}^{\sigma} f_{\sigma\nu} \right).
\tag{33}$$

Comparing with the second Killing equation (23) we conclude, that it is solved by taking

$$c_{abc} = -2H_{abc}
\tag{34}$$

for the local Lorentz 3-form corresponding to the field strength tensor. Thus, given a Killing-Yano tensor $f_{\mu\nu}$, an anti-symmetric 3rd rank tensor of the desired type always exists.

We conclude, that the existence of a Killing-Yano tensor is both a necessary and a sufficient condition for the existence of a new supersymmetry of the type (22) obeying the superinvariance condition (29).

5 N-extended supersymmetry

A special case of a new supersymmetry Q_f satisfying eq.(29) is that of an additional conventional supersymmetry, for which the bracket closes on the Hamiltonian. Consider the simplest case, in which there is one independent new supersymmetry, generated by a charge \tilde{Q} :

$$\{Q, \tilde{Q}\} = 0, \quad \{\tilde{Q}, \tilde{Q}\} = -2iH.
\tag{35}$$

Since the bosonic constant of motion Z now coincides with H , we have

$$K^{\mu\nu} = g^{\mu\nu},
\tag{36}$$

whilst the other components of Z vanish. As a result we have

$$f_a{}^\mu f_\nu{}^a = \delta_\nu^\mu.
\tag{37}$$

The anti-commutativity of the two independent supercharges requires the anti-symmetry of $f_{\mu\nu}$ as before. Therefore we can rewrite eq.(37) in the form

$$f_\lambda{}^\mu f_\nu{}^\lambda = -\delta_\nu^\mu.
\tag{38}$$

Moreover, since the covariant hamiltonian contains no explicit ψ^4 -terms, there is no ψ^3 -term in \tilde{Q} :

$$c_{abc} = 0, \quad \Rightarrow \quad f^{\mu b} f_{\nu ; \mu}^a = f^{\mu b} f_{\nu ; \mu}^a. \quad (39)$$

We conclude, that the existence of a second conventional supersymmetry requires a complex structure $f_{\mu\nu}$, and this restricts the manifolds on which the models are defined to be of Kähler type. Hence in this case our general conditions reduce to the well-known standard requirements for $N = 2$ supersymmetry [11].

For higher N -extended supersymmetry these arguments are easily generalized. In particular, it requires the existence of N independent and mutually anti-commuting complex structures $f_{i\nu}^\mu$ ($i = 1, \dots, N$):

$$f_{i\lambda}^\mu f_{j\nu}^\lambda + f_{j\lambda}^\mu f_{i\nu}^\lambda = -2\delta_{ij}\delta_\nu^\mu. \quad (40)$$

This is an N -dimensional pseudo-Clifford algebra. For example, for $N = 4$ it becomes the quaternion algebra. Therefore theories with $N = 4$ supersymmetry can be realized only on target manifolds of hyper-Kähler type. In this way we reobtain the well-known connection between supersymmetry and the division algebras.

We can now also understand why an $N = 2$ supersymmetry is always present in 2-dimensional target space-time. For $D = 2$ the invariant anti-symmetric Lorentz tensor ϵ_{ab} provides a *generic* Killing-Yano tensor

$$f_a^\mu = e^{\mu b} \epsilon_{ba}, \quad H_{\mu\nu\lambda} = D_\lambda (e^{-1} \epsilon_{\mu\nu}) = 0. \quad (41)$$

Since the ϵ -symbol is covariantly constant, the corresponding 3-index tensor c_{abc} vanishes. The supercharge constructed with this special Killing-Yano tensor in $D = 2$ is then precisely the dual supercharge Q^* .

6 Examples

Except for the application to N -extended supersymmetry, there are also interesting examples of genuine generalized supersymmetries, which do not close on the Hamiltonian but on charges constructed out of other symmetric Stackel-Killing tensors. Here we give two examples, one pertaining to black-hole geometry, and one relevant to the theory of monopoles.

The first example is provided by the D=4 Kerr-Newman black holes. These are solutions of the combined Einstein-Maxwell equations; the line-element reads

$$ds^2 = -\frac{\Delta}{\rho^2} (dt - a \sin^2 \theta d\phi)^2 + \frac{\sin^2 \theta}{\rho^2} ((r^2 + a^2)d\phi - a dt)^2 + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2. \quad (42)$$

We have used the abbreviations

$$\rho^2 = r^2 + a^2 \cos^2 \theta, \quad \Delta = r^2 + a^2 - 2Mr + e^2 > 0. \quad (43)$$

In case of non-vanishing charge e the corresponding electro-magnetic field is described by the Maxwell 2-form

$$\begin{aligned} F &= \frac{e}{\rho^4} (r^2 - a^2 \cos^2 \theta) dr \wedge (dt - a \sin^2 \theta d\phi) \\ &+ 2 \frac{ear \cos \theta \sin \theta}{\rho^4} d\theta \wedge ((r^2 + a^2)d\phi - adt). \end{aligned} \quad (44)$$

For this geometry one can find a Killing-Yano tensor f_μ^a [12, 13] with components

$$\begin{aligned} f_\mu^0 dx^\mu &= \frac{\rho}{\Delta} \cos \theta dr, \\ f_\mu^1 dx^\mu &= -\frac{\sqrt{\Delta}}{\rho} \cos \theta (dt - a \sin^2 \theta d\phi), \\ f_\mu^2 dx^\mu &= -\frac{r \sin \theta}{a\rho} ((r^2 + a^2)d\phi - adt), \\ f_\mu^3 dx^\mu &= -\frac{r\rho}{a} d\theta. \end{aligned} \quad (45)$$

The corresponding components of the Lorentz 3-form c_{abc} obtained from the field strength are:

$$\begin{aligned} c_{012} &= \frac{2 \sin \theta}{\rho}, \\ c_{123} &= -\frac{2\sqrt{\Delta}}{a\rho}, \end{aligned} \quad (46)$$

$$c_{013} = c_{023} = 0.$$

The bosonic constant of motion Z then contains the symmetric Stackel-Killing tensor [14]

$$\begin{aligned} \frac{1}{2} K_{\mu\nu} \dot{x}^\mu \dot{x}^\nu &= \frac{\Delta \cos^2 \theta}{\rho^2} (\dot{t} - a \sin^2 \theta \dot{\phi})^2 + \frac{r^2 \sin^2 \theta}{\rho^2 a^2} ((r^2 + a^2) \dot{\phi} - a \dot{t})^2 \\ &- \frac{\rho^2 \cos^2 \theta}{\Delta} \dot{r}^2 + \frac{\rho^2}{r^2} a^2 \dot{\theta}^2, \end{aligned} \quad (47)$$

which is the square of the Killing-Yano tensor.

Our second example concerns the dynamics of magnetic monopoles. As shown in [15, 16] the effective action for the scattering of slowly-moving monopoles is given by the self-dual Taub-NUT solution of the D=4 Euclidean Einstein equations (with negative mass). The relevant Lagrangian for large distances² reads

$$L = \pi \left(\frac{ds}{d\tau} \right)^2, \quad (48)$$

where the line element in spherical co-ordinates is given by

$$ds^2 = \left(1 - \frac{2}{r} \right) \left[dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2) \right] + \frac{1}{\left(1 - \frac{2}{r} \right)} [d\psi + \cos \theta d\phi]^2. \quad (49)$$

The supersymmetric extension of this Lagrangian was investigated in [17, 18].

The metric (49) admits four Killing vectors, which transform as a scalar and a vector under rotations; they represent the relative charge q and the total angular momentum \vec{J} (which includes a contribution from the relative electric charge).

As observed in [19], the Taub-NUT geometry also possesses four Killing-Yano tensors. Three of these are rather special: they are covariantly constant, mutually anti-commuting and square to minus unity:

$$f_i f_j + f_j f_i = -2\delta_{ij}, \quad D_\mu f_{i\lambda}^\nu = 0. \quad (50)$$

Thus they are complex structures realizing the quaternion algebra. Indeed, the Taub-NUT manifold defined by (49) is hyper-Kähler and as a consequence the corresponding supersymmetric σ -model has an $N = 4$ supersymmetry. We also observe, that these three complex structures transform as a vector under rotations generated by \vec{J} .

Since the Killing-Yano tensors f_i are covariantly constant, their field strengths vanish and so do the corresponding 3-index tensors c_{iabc} . Therefore we find three vector-like supercharges of the form

$$Q_i = f_{ia}^\mu \Pi_\mu \psi^a. \quad (51)$$

Denoting the original supersymmetry by Q_0 , the complete set of brackets of the Q_A , $A = 0, \dots, 3$ realizes the $N = 4$ supersymmetry algebra:

$$\{Q_A, Q_B\} = -2i\delta_{AB} H. \quad (52)$$

There are no fermionic components on the right-hand side. This is consistent because of the properties (50), which imply by the Ricci identity and the self-duality of the Taub-NUT geometry that

²In dimensionless reduced co-ordinates: $r \gg 2$.

$$\epsilon^{abcd} R_{cd\mu\nu} f_{ia}^\mu f_{jb}^\nu = 0. \quad (53)$$

Eqs.(50,53) are sufficient to show the vanishing of the components I_{AB}^μ and G_{AB} on the right-hand side of the bracket (52)

In addition to the vector-like Killing-Yano tensors there also is a scalar one, called $Y_{\mu\nu}$, which has a non-vanishing field strength and, consequently, 3-index tensor c_{abc} . The supercharge Q_Y constructed out of Y is a scalar under rotations generated by the total angular momentum. It now turns out [19] that the only bosonic constants of motion which contain new dynamical information are those obtained from the brackets

$$\{Q_i, Q_Y\} = -2iZ_i. \quad (54)$$

The Stackel-Killing tensors $K_i^{\mu\nu}$ appearing on the right-hand side are those found in [20], forming a Runge-Lenz-like vector \vec{K} . We note in passing, that although the Z_i have a non-trivial contribution from spin [21], the scalar part G_i still vanishes due to the identity (53) and the vanishing of c_{iabc} .

7 Conclusion

From the examples given it is clear, that the concepts of Killing-Yano tensors and the generalized supersymmetries play an important role in clarifying the motion of fermions in the presence of black holes and monopoles. In this paper we have concentrated on the pseudo-classical aspects, but they have their direct translation in the quantum theory, where the supercharges are represented by Dirac operators

$$Q_f \rightarrow -i\gamma^a \left(f_a^\mu D_\mu - \frac{1}{3!} c_{abc} \sigma^{bc} \right). \quad (55)$$

The corresponding Laplacians which are appropriate extensions of

$$\square_K = D_\mu K^{\mu\nu} D_\nu, \quad (56)$$

represent the constants of motion Z . Some of these operators and their quantum mechanical interpretations have been studied in the literature [14, 13, 12, 15, 16]. The pseudo-classical treatment here provides an alternative road to quantization through the path-integral formulation. In both cases, the correspondence principle leads to equivalent algebraic structures.

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