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Relations between some analytic representations of one-loop scalar integrals

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Abstract

We compare several parametrized analytic expressions for an arbitrary off-shell one-loop n-point function in scalar field theory in D-dimensional space-time, and show their equivalence both directly and through path-integral methods.

1 Introduction

Inspired by the string-based work of Bern and Kosower [1, 2], Strassler [3] has given a set of rules for computing one-loop Green functions in field theory directly from path-integrals for relativistic point-particle models. Similar results in this direction were already obtained earlier by Polyakov [4]. The procedure has been extended by various authors [5]-[8] so as to allow for example the computation of processes with internal or external fermions, of higher-order terms in the derivative-expansion of the effective potential, or of higher-loops. In many of these applications, the emphasis is on the non-trivial structure due to spin-polarization and non-abelian charges, and in particular on the manifest gauge-invariance of the computational scheme, whilst the scalar loop-integrals are merely shown to reduce to a standard form. An exception is reference [7], where use of the method is made to construct the derivative expansion of the effective action of scalar field theory.

It is the purpose of this note to show how the expressions for the scalar loop-integrals, first obtained by the path-integral method, can also be derived directly from the standard Feynman-diagram representation. This provides an independent check of the correctness of the path-integral manipulations or, if one prefers, of the equivalence between the Feynman-diagram method and the functional-integral representation of the corresponding point-particle theory. Indeed, a step-by-step comparison between the path-integral and Feynman diagram calculation becomes possible. Basically it is another way of establishing the equivalence between 'first' and 'second' quantization.

2 The one-loop scalar n-point function

Consider the one-loop diagrams for $g\varphi^N$-theory with mass m in D-dimensional Minkowiski space, see fig. 1. Since the particles in the theory are structureless, the only effect of any vertex in the loop is a multiplicative factor $-g/(2\pi)^D$, an additional propagator, and the insertion of a momentum

$$Q^\mu = \sum_{r=1}^{N-2} k_r^\mu, \tag{1}$$

into the loop; here the k_r^μ are the (off-shell) momenta of the $N-2$ individual external lines entering the vertex. The full one-loop contribution to the n-point function is obtained by symmetrization over all vertices, or equivalently all external total momenta Q_i, $i = 1, ..., n$, and taking into account over-all momentum conservation:

$$\sum_{i=1}^{n} Q_i = 0. \tag{2}$$

The well-known result for the amplitude is

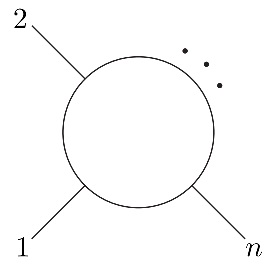

Figure 1: 1-loop scalar n-point diagram

$$\Gamma_n[Q_i] = \frac{1}{n}\left[\frac{-g}{(2\pi)^D}\right]^n \delta^{(D)}\left(\sum_{k=1}^n Q_k\right) \times \\ \times \sum_{\{i_k\}} \int_0^1 \prod_{i=1}^n dx_i \int d^D p \frac{\delta\left(1-\sum_{j=1}^n x_j\right)}{\left[\sum_{l=1}^n x_l p_l^2 \, + \, m^2 \, - \, i\epsilon\right]^n}, \tag{3}$$

where $\{i_k\}$ denotes a permutation of $(1, ..., n)$, whilst

$$p_1 = p, \qquad \text{and} \qquad p_l = p + \sum_{k=1}^{l-1} Q_{i_k}, \quad \text{for} \quad l = 2, ..., n. \tag{4}$$

Clearly the last expression depends on the particular permutation of the Q_i. The x_i with values in the interval $[0, 1]$ are the usual Feynman parameters.

The integrand in eq.(3) can be converted to exponential form using the Schwinger trick; introducing a factor $1/2m$ for later convenience, we write

$$\frac{1}{\left[\sum_{j=1}^n x_j p_j^2 \, + \, m^2 \, - \, i\epsilon\right]^n} = \frac{i^n}{(n-1)!(2m)^n} \int_0^\infty dT \, T^{n-1} e^{-\frac{iT}{2m} \sum_{j=1}^n x_j (p_j^2 + m^2 - i\epsilon)}. \tag{5}$$

The factor T can be removed from the exponent by absorbing it into the Feynman parameters; hence we define

$$\alpha_i \equiv T x_i, \tag{6}$$

taking values in $[0, T]$. Thus we obtain the following expression for the n-point amplitude in one loop:

$$\Gamma_n[Q_i] = \frac{1}{n!}\left[\frac{-ig}{2m(2\pi)^D}\right]^n \delta^{(D)}\left(\sum_{k=1}^n Q_k\right) \times$$
$$\times \sum_{\{i_k\}} \int_0^\infty dT \int_0^T \prod_{i=1}^n d\alpha_i\, \delta\left(T - \sum_{i=1}^n \alpha_i\right) \int d^D p e^{-\frac{i}{2m}\sum_{j=1}^n \alpha_j (p_j^2+m^2)}. \tag{7}$$

In the exponent we have tacitly absorbed the $i\epsilon$ term in the mass m^2, and we will no longer write it explicitly.

The integral over the loop-momentum p has now been reduced to a Gaussian and is therefore straightforward to perform. One finds (in Minkowski space):

$$\Gamma_n[Q_i] = \frac{i}{n!}\left[\frac{-ig}{2m(2\pi)^D}\right]^n \delta^D\left(\sum_{k=1}^n Q_k\right) \times$$
$$\times \sum_{\{i_k\}} \int_0^\infty dT \int_0^T \prod_{i=1}^n d\alpha_i\, \delta\left(T - \sum_{i=1}^n \alpha_i\right) \left(\frac{-2\pi i m}{T}\right)^{D/2} e^{-\frac{imT}{2} - \frac{i}{2mT} R_n[Q_i]}, \tag{8}$$

where $R_n\left[Q_i\right]$ is a quadratic polynomial in the external momenta; explicitly:

$$R_n\left[Q_i\right] = T\sum_{j=2}^n \left[\alpha_j \left(\sum_{k=1}^{j-1} Q_{i_k}\right)^2\right] - \sum_{j=2}^n \left[\alpha_j \sum_{k=1}^{j-1} Q_{i_k}\right]^2. \tag{9}$$

The sums on the right-hand side start only at $j = 2$, because α_1 is multiplied by $\sum_{k=1}^n Q_{i_k}$, which vanishes acording to eq.(2) in any permutation.

We proceed to show that in the symmetric integral over the parameters α_i the polynomial $R_n\left[Q_i\right]$ may be replaced by

$$R_n\left[Q_i\right] = 2T \sum_{k=2}^n \sum_{l=1}^{k-1} Q_{i_k} \cdot Q_{i_l}\, G\left[\Delta_{kl}\right], \tag{10}$$

where the function $G[\Delta_{kl}]$ of the argument $\Delta_{kl} = \sum_{j=l+1}^k \alpha_j$ is defined as

$$G[\Delta_{kl}] = -\frac{1}{2}\,|\Delta_{kl}| + \frac{\Delta_{kl}^2}{2T}. \tag{11}$$

This rearrangement of terms is possible, because of the momentum conservation condition (2) and the constraint

$$\sum_{i=1}^n \alpha_i = T, \tag{12}$$

which show the full set of variables and parameters in the polynomial $R_n\left[Q_i\right]$ to be redundant. Since the proof is independend of the ordering of the external momenta, we may take any particular ordering. For convenience of notation we take $\{Q_{i_k}\} = \{Q_k\}$. Using the resummation formula

$$\sum_{k=2}^{n}\left(\alpha_k \sum_{l=1}^{k-1} B_l\right) = \sum_{l=1}^{n-1}\left(B_l \sum_{k=l+1}^{n} \alpha_k\right), \tag{13}$$

and splitting off the terms which contain pure squares:

$$\sum_{k,l=1}^{n-1} c_{kl}\, Q_k \cdot Q_l = \sum_{k=1}^{n-1} c_{kk}\, Q_k^2 + 2\sum_{k=2}^{n-1}\sum_{l=1}^{k-1} c_{kl}\, Q_k \cdot Q_l, \tag{14}$$

we can rewrite the expression (9) for $R_n[Q_i]$ in the form

$$\begin{aligned} R_n[Q_i] &= \sum_{k=1}^{n-1} Q_k^2 \left[T \sum_{j=k+1}^{n} \alpha_j - \left(\sum_{j=k+1}^{n} \alpha_j\right)^2 \right] + \\ &\quad + 2\sum_{k=2}^{n-1}\sum_{l=1}^{k-1} Q_k \cdot Q_l \left[T \sum_{j=k+1}^{n} \alpha_j - \left(\sum_{j=k+1}^{n} \alpha_j\right)^2 \right]. \end{aligned} \tag{15}$$

Note that the mixed term has been arranged such that the sum runs only over $k > l$, giving rise to the factor of 2. Now we can eliminate the pure squares by using momentum conservation:

$$\begin{aligned} Q_1^2 &= -Q_1 \cdot \sum_{l=2}^{n} Q_l, \\ Q_k^2 &= -Q_k \cdot \left(\sum_{l=1}^{k-1} Q_l + \sum_{l=k+1}^{n} Q_l\right), \qquad k \geq 2, \end{aligned} \tag{16}$$

Using the constraint (12) to expand T in terms of the α_i, eq.(15) for $R_n[Q_i]$ then becomes

$$\begin{aligned} R_n[Q_i] &= \sum_{k=2}^{n-1}\sum_{l=1}^{k-1} Q_k \cdot Q_l \left[2\sum_{i=1}^{l} \alpha_i \sum_{j=k+1}^{n} \alpha_j - \sum_{i=1}^{k} \alpha_i \sum_{j=k+1}^{n} \alpha_j - \sum_{i=1}^{l} \alpha_i \sum_{j=l+1}^{n} \alpha_j \right] - \\ &\quad - Q_n \cdot \sum_{l=1}^{n-1} Q_l \left[\sum_{i=1}^{l} \alpha_i \sum_{j=l+1}^{n} \alpha_j \right]. \end{aligned} \tag{17}$$

Performing the subtractions inside the double sum gives

$$
\begin{aligned}
R_n\left[Q_i\right] &= -\sum_{k=2}^{n-1}\sum_{l=1}^{k-1} Q_k\cdot Q_l \sum_{j=l+1}^{k}\alpha_j\left[\sum_{i=1}^{l}\alpha_i+\sum_{i=k+1}^{n}\alpha_i\right]-Q_n\cdot\sum_{l=1}^{n-1}Q_l\left[\sum_{i=1}^{l}\alpha_i\sum_{j=l+1}^{n}\alpha_j\right] \\
&= -\sum_{k=2}^{n}\sum_{l=1}^{k-1} Q_k\cdot Q_l \sum_{j=l+1}^{k}\alpha_j\left(T-\sum_{i=l+1}^{k}\alpha_i\right).
\end{aligned} \tag{18}
$$

To obtain the last line we have again used relation (12). Note that the remaining sums over α_i are equal precisely the quantity Δ_{kl} introduced before. Therefore eq.(18) indeed reduces to eq.(10). Using this relation under the integral with the constraints on the external momenta and on the parameters α_i, we can then bring the n-particle amplitude into the form

$$
\begin{aligned}
\Gamma_n[Q_i] &= \frac{i}{n!}\left[\frac{-ig}{2m(2\pi)^D}\right]^n \delta^D\left(\sum_{k=1}^{n}Q_k\right)\sum_{\{i_k\}}\int_0^\infty dT\int_0^T\prod_{i=1}^{n}d\alpha_i\,\delta\left(T-\sum_{i=1}^{n}\alpha_i\right)\times \\
&\quad\times\left(\frac{-2\pi i m}{T}\right)^{D/2} e^{-\frac{imT}{2}-\frac{i}{m}\sum_{k=2}^{n}\sum_{l=1}^{k-1}Q_{i_k}\cdot Q_{i_l}\,G[\Delta_{kl}]}.
\end{aligned} \tag{19}
$$

Eqs. 3, 8 and 19 represent three equivalent expressions for the one-loop scalar n-point function, obtained from the standard Feynman rules for $g\varphi^N$-theory. In refs.[3]-[8] it was shown, that similar results are obtained from the path-integral for a relativistic point-particle of mass m calculated in a specific way. In the following we rederive some of these results, and show that each of our three expressions results directly from a single functional integral using different methods of evaluation. In the course of these derivations we will encounter yet another analytic parametrization of the one-loop scalar integral.

3 The path-integral representation

The starting point for the path-integral calculations is the expression [3]

$$
\Gamma_n[Q_i] = \frac{1}{n!}\left[\frac{-ig}{2m(2\pi)^D}\right]^n\int_0^\infty\frac{dT}{T}\oint D\xi[\tau]\int_0^T\prod_{i=1}^{n}d\tau_i\,e^{\frac{im}{2}\int_0^T d\tau\left(\dot{\xi}_\mu^2-1\right)+i\sum_{j=1}^{n}Q_j\cdot\xi_j}, \tag{20}
$$

which is claimed to be equal to the one-loop amplitude of eq.(3). The right-hand side of (20) can be interpreted as the sum over all closed world lines (satisfying periodic boundary conditions) of a scalar particle, of all possible proper-time lengths $T>0$, with n insertions of interactions with an external potential, giving the particle a kick of momentum Q_{i_k} at proper time τ_k when the particle co-ordinates are $\xi_k^\mu=\xi^\mu(\tau_k)$. Note that in pure $g\varphi^N$-theory the only point-like interaction is the one described by one single type of vertex; in particular, the values of τ_k for two different vertices never coincide, since this would correspond to the introduction of a new point-like vertex. This argument can of course be made rigorous only in a renormalizable field theory; nevertheless in the following we take for granted that all interactions are separated in proper time.

We can now order the interactions in proper time:

$$0 \leq \tau_1 < \tau_2 < ... < \tau_n \leq T, \tag{21}$$

In principle, there are $n!$ such orderings, but the various possibilities are all taken into account by symmetrizing over the permutations $\{i_k\}$ of the external momenta $\{Q_{i_k}\}$. Therefore eq.(20) is equivalent to

$$\begin{aligned} \Gamma_n[Q_i] &= \frac{1}{n!}\left[\frac{-ig}{2m(2\pi)^D}\right]^n \sum_{\{i_k\}} \int_0^\infty \frac{dT}{T} \int_0^T d\tau_n \int_0^{\tau_n} d\tau_{n-1} ... \int_0^{\tau_2} d\tau_1 \times \\ &\times \oint D\xi[\tau]\, e^{\frac{im}{2}\int_0^T d\tau(\dot{\xi}_\mu^2 - 1) + i\sum_{j=1}^n Q_j \cdot \xi_j}. \end{aligned} \tag{22}$$

The action integral from 0 to T in the exponent can of course be decomposed into a sum

$$\int_0^T d\tau = \sum_{k=1}^n \int_{\tau_{k-1}}^{\tau_k} d\tau, \tag{23}$$

with $\tau_0 = \tau_n$. In the following we use the standard functional integral results for free particles in D-dimensional Minkowski space, expressed by

$$\begin{aligned} \int_{\xi_1}^{\xi_2} D\xi[\tau]\, e^{\frac{im}{2}\int_{\tau_1}^{\tau_2} d\tau(\dot{\xi}_\mu^2 - 1)} &= i\left(\frac{m}{2\pi i\alpha}\right)^{D/2} e^{-\frac{im\alpha}{2} + \frac{im}{2\alpha}(\xi_2 - \xi_1)^2} \\ &= \int \frac{d^D p}{(2\pi)^D}\, e^{-\frac{i\alpha}{2m}(p^2 + m^2) + ip\cdot(\xi_2 - \xi_1)}, \end{aligned} \tag{24}$$

where $\alpha = \tau_2 - \tau_1$. In eq.(22) this result is to be used n times, once for each of the functional integrals over the various sections of the path, from τ_{k-1} to τ_k. This requires the introduction of n parameters

$$\alpha_1 = \tau_1 - \tau_n + T, \qquad \text{and} \qquad \alpha_k = \tau_k - \tau_{k-1}, \quad 2 \leq k \leq n, \tag{25}$$

Clearly, these parameters satisfy the constraint (12), hence they are not all independend; this is the result of translation invariance. The same invariance also makes the integral over τ_n trivial, since this amounts to an integration over all possible choices of the origin of proper time. Thus we can trade the multiple integral over the τ_k for a constrained integral over the α_i:

$$\int_0^T d\tau_n \int_0^{\tau_n} d\tau_{n-1} ... \int_0^{\tau_2} d\tau_1 = T\int_0^T \prod_{i=1}^n d\alpha_i\, \delta\left(T - \sum_{j=1}^n \alpha_j\right). \tag{26}$$

After performing the functional integrals between each pair of successive interaction points, and making the variable transformation (26), one is left only with an integral over the vertex co-ordinates ξ_k^μ and over each of the momenta p_k of the free particles propagating between the vertices. The result is

$$\begin{aligned}\Gamma_n[Q_i] &= \frac{1}{n!}\left[\frac{-ig}{2m(2\pi)^D}\right]^n \sum_{\{i_k\}} \int_0^\infty dT \int_0^T \prod_{i=1}^n d\alpha_i\, \delta\left(T - \sum_{j=1}^n \alpha_j\right) \times \\ &\quad \times \int \prod_{k=1}^n \left[\frac{d^D\xi_k d^D p_k}{(2\pi)^D}\right] e^{-\frac{i}{2m}\sum_{l=1}^n \alpha_l (p_l^2+m^2) + i\sum_{l=1}^n (Q_{i_l}+p_l-p_{l+1})\cdot\xi_l}. \end{aligned} \tag{27}$$

As expected, the integrations over the vertex positions ξ_k^μ produce δ-functions for momentum conservation at each vertex. The integrals over the momenta p_k eliminate all of these δ-functions except one: there remains the δ-function expressing conservation of the external momenta. This leaves one momentum integral, and the formula for the amplitude becomes

$$\begin{aligned}\Gamma_n[Q_i] &= \frac{1}{n!}\left[\frac{-ig}{2m(2\pi)^D}\right]^n \delta^{(D)}\left(\sum_{k=1}^n Q_k\right) \sum_{\{i_k\}} \int_0^\infty dT \int_0^T \prod_{i=1}^n d\alpha_i\, \delta\left(T - \sum_{j=1}^n \alpha_j\right) \times \\ &\quad \times \int d^D p\, e^{-\frac{i}{2m}\sum_{l=1}^n \alpha_l (p_l^2+m^2)}, \end{aligned} \tag{28}$$

where the momenta p_l are to be interpreted as in eq.(4). Eq.(28) is identical to eq.(7). Hence from this result we can directly obtain either eq.(3) by integration over the parameter T, or eq.(8) by integrating out the loop momentum p. Therefore both these equations are indeed seen to follow in a straightforward way from the functional integral (20).

We proceed to show, that eq.(19) can be derived from this same functional integral, but using the evaluation procedure of ref.[3], rather than the procedure sketched above. Introducing the notation

$$j^\mu(\tau) = \sum_{k=1}^n Q_{i_k}^\mu\, \delta(\tau - \tau_k), \tag{29}$$

eq.(22) can be written in the form

$$\begin{aligned}\Gamma_n[Q_i] &= \frac{1}{n!}\left[\frac{-ig}{2m(2\pi)^D}\right]^n \sum_{\{i_k\}} \int_0^\infty \frac{dT}{T} \int_0^T d\tau_n \int_0^{\tau_n} d\tau_{n-1} \dots \int_0^{\tau_2} d\tau_1 \times \\ &\quad \times \oint D\xi[\tau]\, e^{i\int_0^T d\tau\left[-\frac{m}{2}\xi_\mu\left(\frac{d^2}{d\tau^2}\right)\xi^\mu + j\cdot\xi - \frac{m}{2}\right]}. \end{aligned} \tag{30}$$

Clearly the external momenta act as external forces, and the partial integration in the action is allowed because of the periodic boundary conditions. Since the action integral in the exponent is a quadratic expression in the $\xi^\mu(\tau)$, we can perform the functional integration by completing the square. This requires the introduction of the inverse of the operator $-d^2/d\tau^2$ in the space of continuous functions on a closed line of length T. As discussed in ref.[3], such an inverse does not exist on the full function space, because of the existence of zero modes: the constant functions. When one restricts oneself to periodic functions $\bar{\xi}(\tau)$ which have no constant component:

$$\int_0^T d\tau \, \bar{\xi}(\tau) \, = \, 0, \tag{31}$$

the inverse does exist, and is in fact given by $G[\tau - \tau']$ defined in eq.(11):

$$-\frac{d^2}{d\tau^2} G[\tau - \tau'] \, = \, \delta\left(\tau - \tau'\right) \, - \, \frac{1}{T}. \tag{32}$$

This Green function is translation invariant, reflection symmetric and periodic with period T. From eq.(32) it follows that for functions restricted as in (31), one has

$$-\frac{d^2}{d\tau^2} \int_0^T d\tau' \, G[\tau - \tau'] \, \bar{\xi}(\tau') \, = \, \bar{\xi}(\tau). \tag{33}$$

The procedure to be followed is therefore, as explained in [7], to split off the constant mode from $\xi^\mu(\tau)$:

$$\xi^\mu(\tau) \, = \, \xi^\mu_0 \, + \, \bar{\xi}^\mu(\tau), \tag{34}$$

with

$$\xi^\mu_0 \, = \, \frac{1}{T} \int_0^T d\tau \, \xi^\mu(\tau). \tag{35}$$

Then the Fourier decomposition of $\bar{\xi}^\mu(\tau)$ has no constant component. Defining a new variable of integration

$$\tilde{\xi}^\mu(\tau) \, = \, \bar{\xi}^\mu(\tau) \, + \, \frac{1}{m} \int_0^T d\tau' \, G[\tau - \tau'] \, j^\mu(\tau'), \tag{36}$$

the action integral in the exponent in (30) can be brought into the form

$$\begin{aligned} \int_0^T d\tau \left[-\frac{m}{2} \, \xi_\mu \, \frac{d^2}{d\tau^2} \, \xi^\mu \, + \, j_\mu \xi^\mu \right] \, &= \, \int_0^T d\tau \left[-\frac{m}{2} \, \tilde{\xi}_\mu \, \frac{d^2}{d\tau^2} \, \tilde{\xi}^\mu \right] \, - \\ - \frac{1}{2m} \int_0^T d\tau \int_0^T d\tau' \, j_\mu(\tau) \, G[\tau - \tau'] \, j^\mu(\tau') \, &+ \, \xi^\mu_0 \int_0^T d\tau \, j_\mu(\tau). \end{aligned} \tag{37}$$

Note that ξ^μ_0 has become a Lagrange multiplier imposing the constraint

$$\int_0^T d\tau \, j^\mu(\tau) \, = \, \sum_{k=1}^{n} Q^\mu_{i_k} \, = \, 0 \tag{38}$$

In the functional integral we therefore integrate separately over ξ_0 and $\tilde{\xi}$; the first integration gives the δ-function for momentum conservation, the second is a pure Gaussian and can be evaluated using eq.(24) with $\xi_2 = \xi_1$, owing to the periodic boundary conditions. It is straightforward to see that

$$\int_0^T d\tau \int_0^T d\tau' \, j_\mu(\tau) \, G[\tau - \tau'] \, j^\mu(\tau') \; = \; \sum_{k,l=1}^{n} Q_{i_k} \cdot Q_{i_l} \, G\left[\tau_k - \tau_l\right]. \tag{39}$$

Therefore we obtain

$$\Gamma_n[Q_i] \;\; = \;\; \frac{i}{n!} \left[\frac{-ig}{2m(2\pi)^D}\right]^n \delta\left(\sum_{k=1}^{n} Q_k\right) \sum_{\{i_k\}} \int_0^\infty \frac{dT}{T} \left(-\frac{2\pi i m}{T}\right)^{D/2} \times$$
$$\times \int_0^T d\tau_n \int_0^{\tau_n} d\tau_{n-1} \ldots \int_0^{\tau_2} d\tau_1 \, e^{-\frac{im}{2}T - \frac{i}{m}\sum_{k=2}^{n}\sum_{l=1}^{k-1} Q_{i_k} \cdot Q_{i_l} \, G[\tau_k - \tau_l]}. \tag{40}$$

We have used the fact that $G[0] = 0$ and the reflection symmetry of $G[\tau_k - \tau_l]$ to resum the terms in the exponent in such a way that the argument of G is always positive: $\tau_k > \tau_l$. This accounts for a factor of 2 difference compared to eq.(39).

Finally, using the transformation (26), eq.(40) is seen to equal the right-hand side of eq.(19). In particular, we have

$$\tau_k \; - \; \tau_l \; = \; \sum_{j=l+1}^{k} \alpha_j \; = \; \Delta_{kl}. \tag{41}$$

Therefore the two methods of evaluating the path-integral, either directly or using the Green function technique, give fully equivalent results.

Eq.(40) may be used to derive yet another expression for the n-point scalar amplitude. Namely, if one rescales the variables τ_k like the α_k in (6):

$$u_k \; \equiv \; \frac{\tau_k}{T}, \tag{42}$$

taking values in the range $[0,1]$, with a simultaneous rescaling $G[\tau_k - \tau_l] \rightarrow T\tilde{G}[u_k - u_l]$, then one obtains an integral of Γ-function type, as in the right-hand side of (5):

$$\Gamma_n[Q_i] \; = \; \frac{i}{n!} \; (-2\pi i m)^{D/2} \left[\frac{-ig}{2m(2\pi)^D}\right]^n \delta\left(\sum_{k=1}^{n} Q_k\right) \sum_{\{i_k\}} \int_0^\infty dT \, T^{n-1-(D/2)} \times$$
$$\times \int_0^1 du_n \int_0^{u_n} du_{n-1} \ldots \int_0^{u_2} du_1 \, e^{-\frac{im}{2}T - \frac{iT}{m}\sum_{k=2}^{n}\sum_{l=1}^{k-1} Q_{i_k} \cdot Q_{i_l} \, \tilde{G}[u_k - u_l]}. \tag{43}$$

Remembering the $i\epsilon$-prescription, the integral over T can be carried out and gives

$$\Gamma_n[Q_i] = \frac{i\pi^{D/2}}{n!} \left[\frac{-g}{(2\pi)^D}\right]^n \Gamma\left(n - \frac{D}{2}\right) \delta\left(\sum_{k=1}^n Q_k\right) \times$$

$$\times \sum_{\{i_k\}} \int_0^1 du_n \int_0^{u_n} du_{n-1} \ldots \int_0^{u_2} du_1 \left[\sum_{k,l=1}^n Q_{i_k} \cdot Q_{i_l} \tilde{G}[u_k - u_l] + m^2\right]^{-n+D/2}. \tag{44}$$

4 Conclusions

From the computations presented here, as well as in the earlier papers [3]-[8], we can draw two main conclusions. First, different ways of evaluating the functional integral (20) give equivalent results. Second, these results agree fully with a direct computation of the corresponding Feynman diagrams, which are derived from the appropriate relativistic quantum field theory. In comparing the various approaches we have established four different parametrizations of the one-loop n-point function[1] given by eqs. 3, 8, 19 and 44.

From these observations, it is now possible to construct directly the effective action for the scalar field theory in the one-loop approximation. Let $U[\varphi]$ denote a potential constructed out of the scalar fields; its momentum-space components are defined by

$$U[\varphi(\xi)] = \int \frac{d^DQ}{(2\pi)^D} \tilde{U}(Q)\, e^{iQ\cdot\xi}. \tag{45}$$

Now introduce the generating functional for the one-loop scalar n-point functions:

$$\begin{aligned} W[U] &= \sum_{n=0}^{\infty} \int \prod_{i=1}^n d^DQ_i\, \Gamma_n[Q_i]\, \tilde{U}(Q_n) \ldots \tilde{U}(Q_1) \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \left[\frac{-ig}{2m}\right]^n \int_0^\infty \frac{dT}{T} \oint D\xi[\tau] \prod_{i=1}^n \left(\int_0^T d\tau_i\, U[\varphi(\xi_i)]\right) e^{\frac{im}{2}\int_0^T d\tau\, (\dot{\xi}_\mu^2 - 1)}. \end{aligned} \tag{46}$$

In this expression the sum over n can be carried out to give an exponential, leading to the result

$$W[U] = \int_0^\infty \frac{dT}{T} \oint D\xi[\tau]\, e^{i\int_0^T d\tau\, \left(\frac{m}{2}\dot{\xi}_\mu^2 - \frac{m}{2} - \frac{g}{2m} U[\varphi(\xi)]\right)}. \tag{47}$$

This is the path-integral for a relativistic scalar particle moving in an external scalar potential $U[\varphi]$. This path-integral is therefore seen to generate by its expansion in powers of g all one-loop n-point functions for the scalar field theory

[1] An intermediate fifth one is the highly symmetric integral expression (27).

for which $U[\varphi]$ is the second derivative of the classical potential. For example, the one-loop scalar integrals of $g\varphi^4$-theory in four dimensions are obtained by taking $U[\varphi] = \varphi^2/2$. $W[U]$ thus corresponds to the one-loop approximation to the logarithm of the generating functional for Green functions of the corresponding field theory (modulo a constant):

$$W[U] \;=\; -\log\det\Bigl(-\Box + m^2 + gU[\varphi]\Bigr)\,. \tag{48}$$

In this way we reproduce, here for pure scalar field theories, the starting point of ref.[3] and the subsequent papers cited below.

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