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ANOTHER LOOK AT THE ELECTROWEAK VORTEX SOLUTION *

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Abstract

We discuss the position in configuration space of the electroweak vortex solution.

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Many, if not most, of the topics discussed at this workshop are non-perturbative in origin. For this reason it may be of interest to report here on one minor, but rigorous, non-perturbative result obtained in collaboration with P. Olesen [1]. Similar ideas have been presented at this meeting by M. James. The goal of this talk then is to suggest a new point of view for an old classical solution, namely the well-known vortex solution [2] as embedded in the electroweak standard model [3]. Specifically, we would like to know where in configuration space this solution fits in.

Let us start by reviewing what limited knowledge we have of configuration space, i. e. the abstract space of static, finite energy field configurations, with the gauge freedom eliminated. For simplicity, we consider only the bosonic fields of the electroweak standard model (the response of the fermionic fields is certainly important for the physics applications, but the crucial dynamics is beleived to be carried by the bosonic fields). The classical vacuum, with vanishing gauge fields and a constant Higgs field (energy $E_{\rm V} = 0$), corresponds to a single point V in configuration space. The energy surface at this point V is stationary, in other words the vacuum configuration solves the classical field equations. It turns out that the topology of configuration space is highly non-trivial and this leads to the existence of other stationary points, i. e. new classical solutions. We mention two of them.

First, there exists a non-contractible loop (NCL) in configuration space, parametrized by $\mu \in [-\pi, +\pi]$, which for $\mu = \pm\pi$ passes through the vacuum V and for $\mu = 0$ through a new classical solution, the sphaleron S [4]. This NCL captures some of the topology of configuration space, because it is based on a non-trivial map $\mathrm{S}_1 \times \mathrm{S}_2 \to \mathrm{SU}(2) \sim \mathrm{S}_3$, where S_1 refers to the loop parameter μ, S_2 to the angles on the sphere at spatial infinity and SU(2) to the non-abelian gauge group of the electroweak standard model. In short, the NCL wraps around a "hole" in configuration space. The energy of the corresponding sphaleron S is a function of the mass ratios M_H/M_W and M_Z/M_W, its order of magnitude being $E_{\rm S} = \mathrm{O}(M_W/\alpha) \sim 10$ TeV. Second, there exists a non-contractible sphere (NCS) in configuration space, parametrized by $\mu, \nu \in [-\pi, +\pi]$, which for $\mu = \pm\pi$ or $\nu = \pm\pi$ passes through V and for $\mu = \nu = 0$ through another sphaleron S* [5]. The essential non-trivial map is now $S_2 \times S_2 \rightarrow SU(2)$, where the first S_2 refers to the sphere with parameters μ and ν and the second to sphere at spatial infinity. The energy of the sphaleron S* is a little less than twice that of the sphaleron S. As to the physics applications, the crucial observation is that both the NCL and the NCS are related to anomalies, respectively the chiral U(1) anomaly and the global SU(2) anomaly. More concretely, the sphaleron S seems to play a role in B+L violating processes at high temperatures [4, 6, 7], whereas the sphaleron S*, or rather its related constrained instanton I*, may have to do with the asymptotics of perturbation theory [5, 8, 9].

These considerations were for 3-dimensional configurations of finite energy E, but we can also restrict ourselves to field configurations which are constant in, say, the x_3 direction. [Cylindrical coordinates may be defined in terms of the cartesian coordinates by $(\rho\cos\phi\,, \rho\sin\phi\,, z) \equiv (x_1, x_2, x_3)$.] The dimensionless "energy" for these 2-dimensional configurations is then given by

$$\begin{aligned} \epsilon &\equiv \frac{1}{\pi v^2}\int dx_1 dx_2\, e \\ &= \frac{1}{\pi v^2}\int dx_1 dx_2 \left[\frac{1}{4\,g^2}\,\left(W^a_{mn}\right)^2 + \frac{1}{4\,g'^2}\,\left(B_{mn}\right)^2 + |D_m\Phi|^2 + \lambda\left(|\Phi|^2 - v^2/2\right)^2\right], \end{aligned} \tag{1}$$

with field strengths and covariant derivatives

$$\begin{aligned} W_{mn} &\equiv W^a_{mn}\,\tau^a \equiv \partial_m W_n - \partial_n W_m + [W_m, W_n] \\ B_{mn} &\equiv \partial_m B_n - \partial_n B_m \\ D_m\Phi &\equiv \left(\partial_m + B_m/(2\,i) + W_m\right)\Phi \\ W_m &\equiv W^a_m\,\tau^a \\ \tau^a &\equiv \sigma^a/(2\,i)\,. \end{aligned}$$

Here W and B are the SU(2) and U(1) gauge fields and Φ is the complex doublet Higgs field. The semiclassical masses of the $W^\pm$ and Z^0 vector bosons are $M_W = \frac{1}{2}\,g\,v$ and $M_Z = M_W/\cos\theta_w$, with the weak mixing angle θ_w defined as $\tan\theta_w \equiv g'/g$, and the mass of the physical Higgs scalar is $M_H = \sqrt{8\,\lambda/g^2}\,M_W$. Our goal now is to investigate the topology of this different configuration space, i. e. the space of finite ϵ field configurations. Again, we will do this by constructing a non-contractible sphere.

Introducing for the parameters $\mu, \nu \in [-\pi, +\pi]$ the notation $[\mu\nu] \equiv \max(|\mu|, |\nu|)$, our NCS of 2-dimensional configurations is given by

$$\begin{aligned} \pi/2 \leq [\mu\nu] \leq \pi \quad &: \quad W = 0 \\ & \quad\;\; B = 0 \\ & \quad\;\; \Phi = \left(1 - (1-h)\sin[\mu\nu]\,\right)\,\frac{v}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \\[1ex] 0 \leq [\mu\nu] < \pi/2 \quad &: \quad W = -f\,G^a\,\tau^a \\ & \quad\;\; B = f\,\sin^2\theta_w\,F^3 \\ & \quad\;\; \Phi = h\,\frac{v}{\sqrt{2}}\,U\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \end{aligned} \tag{2}$$

with the following Lie algebra valued 1-forms

$$\begin{aligned} F^a\,\tau^a &\equiv U^{-1}\,\mathrm{d}U \\ G^a\,\tau^a &\equiv U\left[F^1\,\tau^1 + F^2\,\tau^2 + \cos^2\theta_w\,F^3\,\tau^3\right]U^{-1}\,, \end{aligned}$$

$SU(2)$ matrices

$$
\begin{aligned}
M(\mu,\nu,\phi) &= \begin{pmatrix} \sin\mu \\ \cos\mu\ \sin\nu \\ \cos\mu\ \cos\nu\ \sin\phi \\ \cos\mu\ \cos\nu\ \cos\phi \end{pmatrix} \cdot \begin{pmatrix} -i\sigma_1 \\ -i\sigma_2 \\ -i\sigma_3 \\ 1 \end{pmatrix} \\
U(\mu,\nu,\phi) &= M(\mu,\nu,0)^{-1}\ M(\mu,\nu,\phi)
\end{aligned}
$$

and boundary conditions for the axial functions $f(\rho)$ and $h(\rho)$

$$
\begin{aligned}
f(0) = h(0) &= 0 \\
\lim_{\rho\to\infty} f, h &= 1\,.
\end{aligned}
$$

The topology of configuration space is captured by the non-trivial map $M:\, \mathrm{S}_2 \times \mathrm{S}_1 \to \mathrm{SU}(2)$, where S_2 refers to the sphere with parameters μ,ν and S_1 to the circle at spatial infinity. This map with winding number $m = 1$ can be generalized by replacing $\sin\phi$ and $\cos\phi$ by $\sin m\phi$ and $\cos m\phi$. Also note that there is no longer the possibility of having a non-contractible loop, since the map $\mathrm{S}_1 \times \mathrm{S}_1 \to \mathrm{SU}(2)$ is always trivial.

With these configurations the resulting dimensionless energy density is for $[\mu\nu] < \pi/2$

$$
\begin{aligned}
e \;=\;& \cos^2\mu\ \cos^2\nu\ \left(1-\cos^2\mu\ \cos^2\nu\ \sin^2\theta_w\right)\ \cos^{-2}\theta_w\ \left[\rho^{-2}\,(\partial_\rho\, f)^2\right] + \\
& \cos^2\mu\ \cos^2\nu\ \left[\rho^{-2}\,h^2\ (1-f)^2\right] + (\partial_\rho\, h)^2 + \frac{1}{4}\left(\frac{M_H}{M_Z}\right)^2 \left(h^2-1\right)^2
\end{aligned}
\tag{3}
$$

and for $\pi/2 \le [\mu\nu] \le \pi$

$$
e \;=\; (\partial_\rho\, k)^2 + \frac{1}{4}\left(\frac{M_H}{M_Z}\right)^2 \left(k^2-1\right)^2\ , \tag{4}
$$

with $k \equiv 1-(1-h)\sin[\mu\nu]$. Perhaps the main subtlety in the ansatz (2) is the way the SU(2) and U(1) gauge fields are distributed, in order to keep their total kinetic energy density (3) down by partially cancelling the $\cos^{-2}\theta_w$ factor.

We now observe that the NCS configuration at $\mu = \nu = 0$ gives precisely the electroweak vortex solution (Z-string). Just recall that the fields of the Z-string can be written in the following form [2, 3]

$$
\begin{aligned}
W &= -\cos^2\theta_w\ f_Z\, \mathrm{d}V\ V^{-1} \\
B &= -\tan^2\theta_w\ W^3 \\
\Phi &= h_Z\,\frac{v}{\sqrt{2}}\,V \begin{pmatrix} 0 \\ 1 \end{pmatrix},
\end{aligned}
\tag{5}
$$

where the $SU(2)$ matrix V is simply

$$
V = \cos\phi\, 1 - \sin\phi\ i\sigma_3 \equiv U(0,0,\phi)
$$

and the functions f_Z and h_Z are the solutions of the field equations with the standard boundary conditions. The corresponding energy density of the Z-string

$$
e_Z = \rho^{-2}\,(\partial_\rho\, f_Z)^2 + \rho^{-2}\,h_Z^2\ (1-f_Z)^2 + (\partial_\rho\, h_Z)^2 + \frac{1}{4}\left(\frac{M_H}{M_Z}\right)^2 \left(h_Z^2-1\right)^2 \tag{6}
$$

	M_H/M_Z						
	1/8	1/4	1/2	1	2	4	8
ϵ_Z	0.47	0.59	0.76	1.00	1.34	1.79	2.35
a	0.16	0.18	0.20	0.21	0.21	0.20	0.18
b	0.19	0.22	0.25	0.29	0.35	0.43	0.52
c	0.50	0.43	0.36	0.29	0.22	0.16	0.12
d	0.15	0.17	0.19	0.21	0.22	0.21	0.18
$\sin^2\theta_w^{\rm NCS}$	0.69	0.69	0.69	0.70	0.73	0.76	0.79

Table 1: Numerical results for the Z-string solution : the dimensionless energy ϵ_Z, the integrals a–d and the critical mixing angle $\theta_w^{\rm NCS}$.

is independent of the value of the weak mixing angle θ_w.

With the axial functions of the Z-string $f = f_Z$ and $h = h_Z$, the energy profile over the NCS (2) is for $0 \leq [\mu\nu] < \pi/2$

$$\epsilon(\mu,\nu) = \epsilon_Z \left[\cos^2\mu\,\cos^2\nu \left(\frac{1-\cos^2\mu\,\cos^2\nu\,\sin^2\theta_w}{\cos^2\theta_w}\, a + b \right) + c + d \right] \qquad (7)$$

$$= \epsilon_Z \left[1 + (\mu^2+\nu^2) \left(\tan^2\theta_w - 1 - b/a \right) a + {\rm O}(\mu^4,\, \nu^4,\, \mu^2\nu^2) \right] , \qquad (8)$$

in terms of the following integrals

$$\begin{aligned}
a &\equiv \epsilon_Z^{-1} \int_0^\infty d\rho\, \rho^{-1}\, (\partial_\rho f_Z)^2 \\
b &\equiv \epsilon_Z^{-1} \int_0^\infty d\rho\, \rho^{-1}\, h_Z^2\, (1-f_Z)^2 \\
c &\equiv \epsilon_Z^{-1} \int_0^\infty d\rho\, \rho\, (\partial_\rho h_Z)^2 \\
d &\equiv \epsilon_Z^{-1} \int_0^\infty d\rho\, \rho\, 1/4\, (M_H/M_Z)^2 \left(h_Z^2 - 1 \right)^2 ,
\end{aligned}$$

which add up to 1. From (8) there is manifest instability of the Z-string for $\theta_w \leq \pi/4$ and arbitrary Higgs mass. In short, the Z-string of the electroweak interactions (with the experimental value $\theta_w \sim \pi/6$) is the 2-dimensional sphaleron of a non-contractible sphere.

Numerical results allow us to extend the range of instability beyond $\sin^2\theta_w = 1/2$. In Table 1 we give the maximum allowed value θ_w for instability according to (8). The Z-string solution has for $\theta_w < \theta_w^{\rm NCS}$ two negative modes, made explicit by the fields (2) of the NCS. This range of instability is consistent with the results of [10].

These numerical results give also the complete energy profile (7) of the non-contractible sphere through the Z-string, as a function of both θ_w and M_H/M_Z. For the case of perturbative stability [10] of the Z-string, we find an energy barrier for decay towards the vacuum. Of course, this is not yet the minimal energy barrier, but it may be indicative. Taking the parameter values $M_H/M_Z = 1/8$ and $\sin^2\theta_w = 63/64$, for example, we easily find the $\mu = 0$ section of the energy profile $\epsilon(\mu,\nu)$ over the NCS : starting at $\nu = 0$, with $\epsilon = \epsilon_Z = 0.47$, the energy rises to a maximum value $\epsilon = 1.57$ at $\nu \sim \pi/4$ and then drops to zero as $\nu \to \pi$.

To conclude, we have obtained a topological understanding for the existence and generic instability of the electroweak vortex solution.

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