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Local BRST cohomology in the antifield formalism: I. General theorems

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Abstract

We establish general theorems on the cohomology $H^*(s|d)$ of the BRST differential modulo the spacetime exterior derivative, acting in the algebra of local p-forms depending on the fields and the antifields (=sources for the BRST variations). It is shown that $H^{-k}(s|d)$ is isomorphic to $H_k(\delta|d)$ in negative ghost degree $-k$ ($k > 0$), where δ is the Koszul-Tate differential associated with the stationary surface. The cohomological group $H_1(\delta|d)$ in form degree n is proved to be isomorphic to the space of constants of the motion, thereby providing a cohomological reformulation of Noether theorem. More generally, the group $H_k(\delta|d)$ in form degree n is isomorphic to the space of $n-k$ forms that are closed when the equations of motion hold. The groups $H_k(\delta|d)$ ($k > 2$) are shown to vanish for standard irreducible gauge theories. The group $H_2(\delta|d)$ is then calculated explicitly for electromagnetism, Yang-Mills models and Einstein gravity. The invariance of the groups $H^k(s|d)$ under the introduction of non minimal variables and of auxiliary fields is also demonstrated. In a companion paper, the general formalism is applied to the calculation of $H^k(s|d)$ in Yang-Mills theory, which is carried out in detail for an arbitrary compact gauge group.

1 Introduction

A major development of field theory in the eighties has been the construction of the antifield-antibracket formalism [1]. This formalism finds its roots in earlier work on the renormalization of Yang-Mills models [2, 3, 4] and quantization of supergravity [5, 6], and enables one to formulate the quantum rules (path integral, Feynman diagrams) for an arbitrary gauge theory in a manner that maintains manifest spacetime covariance.

The algebraic structure of the antifield formalism has been elucidated in [7, 8], where it has been shown that the BRST complex contains two crucial ingredients:

(i) the Koszul-Tate resolution, generated by the antifields, which implements the equations of motion in (co)homology ; and

(ii) the longitudinal exterior complex, which implements gauge invariance. The BRST differential combines the Koszul-Tate differential with the longitudinal exterior derivative along the systematic lines of homological perturbation theory [9]. As a result of that analysis, a simple rationale for the BRST construction has been obtained and, in particular, the role of the antifields has been understood. A pedagogical exposition of these ideas may be found in [10].

As we have just mentioned, a key feature of the BRST differential is that it incorporates the equations of motion through the Koszul-Tate resolution. This is true both classically [7], where the relevant equations are the classical Euler-Lagrange equations, and quantum-mechanically [8], where the relevant equations are now the Schwinger-Dyson equations. [A different (non cohomological) relation between the antifields and the Schwinger-Dyson equations has been analyzed recently in [11] for theories with a closed gauge algebra].

It is somewhat unfortunate that this important conceptual property of the BRST differential s is often underplayed in the Yang-Mills context, where what one usually calls the BRST differential is only a piece of it, namely the off-shell extended longitudinal exterior derivative along the gauge orbits. Such a differential exists because the gauge algebra closes off-shell. This accident of the Yang-Mills theory (closure off-shell) hides the fundamental fact that it is the *full* BRST differential s, including the Koszul-Tate piece δ, that is of direct physical interest. Indeed, it is the only differential available for a generic gauge theory. Moreover, it is the cohomology of s that appears in renormalization theory [2, 12] (where the antifields are named sources for the BRST variations), in the study of anomalies [13] as well as in the question of consistently deforming the classical action [14].

Actually, what really appears in those problems is not just the cohomology of s but rather the cohomology of s *in the space of local functionals*. The purpose of this paper is to investigate some general properties of the cohomological groups $H^k(s)$ of s acting in the space of local functionals with ghost number k, or rather, of the related and more tractable cohomological groups $H^k(s|d)$ of s acting in the space of local p-forms. Here, d is the exterior derivative in spacetime. According to homological perturbation theory, these goups are isomorphic to $H_{-k}(\delta|d)$ for negative k's, where the subscript denotes the antighost number, and to $H^k(\gamma|d, H_0(\delta))$ for positive k's, where the differential γ is the exterior derivative along the gauge orbits (see below). Our main results can be summarized as follows:

(i) The group $H_1(\delta|d)$ in form degree n is isomorphic to the space of non trivial conserved currents. This is actually a cohomological reformulation of Noether theorem. More generally, the groups $H_k(\delta|d)$ in form degree n are isomorphic to the space of non trivial $n-k$ forms that are closed modulo the equations of motion ("characteristic cohomology").

(ii) The groups $H_q(\delta|d)$ vanish for $q > p$ for field theories of Cauchy order p. (The "Cauchy order" of a theory is defined below. Usual irreducible gauge theories are of Cauchy order 2).

(iii) The complete calculation of $H_2(\delta|d)$ is carried out for electromagnetism, Yang-Mills models and Einstein gravity. In the latter two cases, $H_2(\delta|d)$ vanishes. For Einstein gravity, the vanishing of $H_2(\delta|d)$ is a consequence of the absence of Killing vectors for a generic Einstein metric.

(iv) Non-minimal sectors, as well as the "ultralocal" shift symmetries of [11], do not contribute to $H(s|d)$.

(v) The invariance of $H(s|d)$ under the introduction of auxiliary fields is established.

These general results are applied in a companion paper [15] to the computation of $H^k(s|d)$ for Yang-Mills theory.

The next five sections (2 through 6) are mostly recalls of the BRST features needed for the subsequent analysis: how to handle locality [16, 17], examples, BRST construction and main theorem of homological perturbation theory. Section 7 is then devoted to the isomorphism between $H_1(\delta|d)$ and the space of constants of the motion. In section 8, we introduce the concept of Cauchy order and establish some theorems on $H_p(\delta|d)$ for theories of

Cauchy order q. The general analysis is pursued further in sections 9 and 10. In section 11, we prove some general results on $H_2(\delta|d)$ for irreducible gauge theories. These are then used in sections 12 and 13 to compute $H_2(\delta|d)$ for Yang-Mills models. We show that there is no $n-2$-form that is closed modulo the equations of motion for semi-simple gauge groups. This result holds also for Einstein gravity. Sections 14 and 15 respectively show that non minimal sectors or auxiliary fields do not modify the local BRST cohomology.

We assume throughout our analysis that the topology of spacetime is simply that of the n-th dimensional euclidean space $\mathbf{R}^n$.

2 Cohomological groups $\mathbf{H^k(s)}$ and $\mathbf{H^k(s|d)}$

The way to incorporate locality in the BRST formalism is quite standard and proceeds as follows. First one observes that local functions, i.e., (smooth) functions of the field components and a finite number of their derivatives, are functions defined over *finite dimensional spaces*. These spaces, familiar from the theory of partial differential equations, are called "jet spaces" and are denoted here by V^k ($k = 0, 1, 2, \ldots$). Local coordinates on V^k are given by x^μ (the spacetime coordinates), the field components ϕ^i, their derivatives $\partial_\mu \phi^i$ and their subsequent derivatives $\partial_{\mu_1 \ldots \mu_j} \phi^i$ up to order k ($j = 0, 1, 2, \ldots, k$). Since we assume that spacetime is $\mathbf{R}^n$, these local coordinates on V^k are also global coordinates. The Lagrangian involves usually only the fields and their first derivatives and so is a function on V^1. We refer the reader unfamiliar with this approach to [18, 19, 20, 21] for more information.

Local functionals are by definition integrals of local functions. More precisely, consider the exterior algebra of differential forms on $\mathbf{R}^n$ with coefficients that are local functions. These will be called "local q-forms". Local functionals are integrals of local n-forms. The second idea for dealing with locality is to reexpress all the equations involving local functionals in terms of their integrands. To achieve this goal, one needs to know how to remove the integral sign. This can be done by means of the following elementary results:

(i) Let α be an exact local n-form, $\alpha = d\beta$. Assume $\oint \beta = 0$, where the surface integral is evaluated over the boundary of the spacetime region under consideration. Then $\int \alpha = 0$ (Stokes theorem).

(ii) Conversely, if α is a local n-form such that $\int \alpha = 0$ for all allowed field configurations, then $\alpha = d\beta$ with $\oint \beta = 0$.

These results are well known and proved for instance in [10] chapter 12. The differential d is the exterior derivative in spacetime, defined in the algebra of local q-forms through

$$df(x^\mu, \phi^i, \partial_\mu \phi^i, \dots \partial_{\mu_1 \dots \mu_s} \phi^i) = (\frac{\partial^R f}{\partial x^\nu} + \frac{\partial^R f}{\partial \phi^i} \partial_\nu \phi^i$$
$$+ \frac{\partial^R f}{\partial(\partial_\mu \phi^i)} \partial_\nu \partial_\mu \phi^i + \dots + \frac{\partial^R f}{\partial(\partial_{\mu_1 \dots \mu_s} \phi^i)} \partial_\nu \partial_{\mu_1 \dots \mu_s} \phi^i) dx^\nu \tag{2.1}$$

and $d(dx^\mu) = 0$. As in [10], we shall take all differentials to act from the right.

On account of (i) and (ii), the correspondence between local n- forms and local functionals is not unique. If one allows only for $(n-1)$-forms such that $\oint \beta = 0$, then, local functionals may be viewed as equivalence classes of local n-forms (which are necessarily d-closed) modulo d-exact ones. They are thus the elements of the cohomological space $H^n(d)$ [22].

The BRST differential s is defined in the algebra of local functions. It may easily be extended to the algebra of local q-forms by setting $s(dx^\mu) = 0$. One has

$$sd + ds = 0 \tag{2.2}$$

since $s\partial_\mu = \partial_\mu s$. Let $A = \int \alpha$ be a BRST-closed local functional. From

$$sA = \int s\alpha = 0 \tag{2.3}$$

one gets

$$s\alpha + d\beta = 0 \tag{2.4}$$

with $\oint \beta = 0$. Thus, the integrand α is a local n- form that is BRST-closed modulo d. Furthermore, A is BRST-exact iff $\alpha = d\lambda + s\mu$ (with $\oint \lambda = 0$). Accordingly, the cohomological groups $H^k(s)$ of s acting in the space of local functionals is isomorphic to the cohomological groups $H^{k,n}(s|d)$ of s acting in the space of local n-forms (k = ghost number, n = form degree).

The condition that $\oint \beta$ should vanish is rather awkward to take into account, if only because it depends on the precise conditions imposed on the fields at the boundaries. For this reason, it is customary to drop it and to investigate $H^{k,n}(s|d)$ without restrictions on the $(n-1)$-forms at the boundary. This approach will be followed here. By doing so, one allows elements of $H^{k,n}(s|d)$ that *do not define* s-closed local functionals because of non-vanishing surface terms. We shall comment further on this point below (end of section 7).

3 Regularity conditions. Examples

Since one can reformulate questions involving local functionals in terms of local n-forms, we shall exclusively work from now on with the algebra of local q- forms. That is, any element $a, b, c, \alpha, \beta, \dots$ upon which the differential s acts will be a local q-form with no restrictions at the boundaries unless otherwise specified.

Let $\mathcal{L}_0 = \mathcal{L}_0(\phi^i, \partial_\mu \phi^i, \dots, \partial_{\mu_1 \dots \mu_s} \phi^i)$ be the original gauge invariant Lagrangian. The equations of motion are

$$\mathcal{L}_i \equiv \frac{\delta \mathcal{L}_0}{\delta \phi^i} = 0 \tag{3.1}$$

where $\frac{\delta \mathcal{L}_0}{\delta \phi^i}$ are the Euler-Lagrange derivatives of $\mathcal{L}_0$ with respect to ϕ^i,

$$\mathcal{L}_i \equiv \frac{\delta \mathcal{L}_0}{\delta \phi^i} \equiv \frac{\partial \mathcal{L}_0}{\partial \phi^i} - \partial_\mu \frac{\partial \mathcal{L}_0}{\partial(\partial_\mu \phi^i)} + \dots + (-1)^s \partial_{\mu_1 \dots \mu_s} \frac{\partial \mathcal{L}_0}{\partial(\partial_{\mu_1 \dots \mu_s} \phi^i)}. \tag{3.2}$$

Two Lagrangians $\mathcal{L}_0$ and $\mathcal{L}'_0$ are regarded as being equivalent if they yield identical equations of motion, i.e., if they have the same Euler- Lagrange derivatives, $\mathcal{L}_i \equiv \mathcal{L}'_i$. The corresponding n-forms $\alpha_0 = \mathcal{L}_0 dx^0 \dots dx^{n-1}$ and $\alpha'_0 = \mathcal{L}'_0 dx^0 \dots dx^{n-1}$ are then also called equivalent. One has $\alpha_0 = \alpha'_0 + d\beta$ for some local $(n-1)$-form β. Conversely, if $\alpha_0 = \alpha'_0 + d\beta$, then $\mathcal{L}_i \equiv \mathcal{L}'_i$ (see theorem 5.3).

We shall make the same regularity assumptions on $\mathcal{L}_0$ and on the gauge transformations as in [17]. These state that one can (locally in the jet bundle spaces) separate the field equations $\mathcal{L}_i = 0$ and their derivatives $\partial_{\mu_1 \dots \mu_k} \mathcal{L}_i = 0$ $(k = 1, 2, \dots)$ into two groups. The first group contains the "independent" equations $L_a = 0$. The second group contains the dependent equations $L_\Delta = 0$, which hold as consequences of the others. Furthermore, one may introduce new local coordinates in V^k (for each k) in such a way that the independent

L_a are some of the new coordinates in the vicinity of the surface defined by the equations of motion. So, one can also split the field components and their derivatives into two groups. The first group contains the independent field variables, denoted by x_A, which are not constrained by the equations of motion in V^k. The second group contains the dependent field variables, denoted by z_a, which can be expressed in terms of the x_A and the L_b, in such a way that $z_a = z_a(x_A, L_b)$ is smooth and invertible for the L_b's. In the case of reducible gauge theories, similar conditions are imposed on the reducibility functions.

These conditions are easily seen to hold for the usual gauge theories. This is explicitely verified in [23] for the Klein-Gordon field ϕ and the Yang- Mills A^a_ρ. We list here the corresponding L_a, L_Δ, x_A and z_a.
Klein-Gordon:

$$\{L_a\} \equiv \{\mathcal{L} \equiv \Box\,\phi, \partial_\mu \mathcal{L}, \partial_{\mu_1\mu_2}\mathcal{L}, \ldots\} \tag{3.3}$$

$$\{L_\Delta\}\ \textit{is empty} \tag{3.4}$$

$$\{x_A\} \equiv \{\phi, \partial_\rho\phi, \partial_{s_1}\partial_\rho\phi, \ldots, \partial_{s_1\ldots s_m}\partial_\rho\phi, \ldots\} \tag{3.5}$$

$$\{z_a\} \equiv \{\partial_{00}\phi, \partial_{\rho_1}\partial_{00}\phi, \ldots, \partial_{\rho_1\ldots\rho_m 00}\phi, \ldots\} \tag{3.6}$$

$$\tag{3.7}$$

Yang-Mills:

$$\{L_a\} \equiv \{\mathcal{L}^\mu_a \equiv D_\nu F^{\mu\nu}_a, \partial_\rho \mathcal{L}^m_a, \partial_{\rho_1\rho_2}\mathcal{L}^m_a, \ldots, \partial_{\rho_1\ldots\rho_s}\mathcal{L}^m_a, \ldots, \\ \partial_k\mathcal{L}^0_a, \partial_{k_1k_2}\mathcal{L}^0_a, \ldots, \partial_{k_1\ldots k_2}\mathcal{L}^0_a, \ldots\} \tag{3.8}$$

$$\{L_\Delta\} \equiv \{\partial_0\mathcal{L}^0_a, \partial_\rho\partial_0\mathcal{L}^0_a, \partial_{\rho_1\rho_2}\partial_0\mathcal{L}^0_a, \ldots, \partial_{\rho_1\ldots\rho_s}\partial_0\mathcal{L}^0_a, \ldots\} \tag{3.9}$$

$$\{x_A\} \equiv \{A^a_\mu, \partial_\rho A^a_\mu, \partial_{s_1}\partial_\rho A^a_m, \ldots, \partial_{s_1\ldots s_k}\partial_\rho A^a_m, \ldots, \partial_\lambda\partial_0 A^a_0, \ldots, \\ \partial_{\lambda_1\ldots\lambda_k}\partial_0 A^a_0, \ldots, \partial_{\bar{l}}\partial_m A^a_0, \ldots, \partial_{\bar{l}_1\ldots\bar{l}_k}\partial_m A^a_0, \ldots\}, \quad (\bar{l}, \bar{l}_i \neq 1) \tag{3.10}$$

$$\{z_a\} \equiv \{\partial_{00}A^a_m, \partial_{\rho_1}\partial_{00}A^a_m, \ldots, \partial_{\rho_1\ldots\rho_s}\partial_{00}A^a_m, \ldots, \partial_{11}A^a_0, \ldots, \\ \partial_{s_1\ldots s_n}\partial_{11}A^a_0 \ldots\} \tag{3.11}$$

In the Klein-Gordon case, for which $\{L_\Delta\}$ is empty, the set I_0 of independent variables x_A enjoys a useful property: it is stable under spatial differentiation, i.e., $\partial_k x_A \in I_0$ for all k's and A's ($\partial_k I_0 \subset I_0$). The equations of motion constrain only the temporal derivatives of the x_A's. This is not true for the Yang-Mills model, since $\partial_{11}A^a_0$ does not belong to I_0 even though $\partial_1 A^a_0$ does. However, it is true that I_0 is preserved under differentiation with respect to $x^{\bar{l}}$, $\partial_{\bar{l}} I_0 \subset I_0$ ($\bar{l} = 2, 3, \ldots, n-1$). Furthermore, the set of independent equations L_a is stable under spatial differentiation. We shall study the implications of these properties when we introduce the concept of Cauchy order below (section 8).

The regularity conditions also hold for p-form gauge theories, which are reducible. Namely, the reducibility identities on the equations of motion ("Noether identities")

$$R_\alpha \equiv R^i_\alpha \mathcal{L}_i + R^{i\mu}_\alpha \partial_\mu \mathcal{L}_i + \ldots + R^{i\mu_1 \ldots \mu_k}_\alpha \partial_{\mu_1 \ldots \mu_k} \mathcal{L}_i \equiv 0, \tag{3.12}$$

together with their derivatives $\partial_\mu R_\alpha, \partial_{\mu_1 \mu_2} R_\alpha, \ldots$ fall into two groups: the independent identities R_u ; and the dependent identities R_U, which hold as consequences of $R_u = 0$. [More precisely, when one says that $R_U = 0$ holds as a consequence of $R_u = 0$, one views the $\mathcal{L}_i$ in (3.12) as independent variables not related to the ϕ^i ; the statement would otherwise be meaningless]. Similar properties are verified for the higher order reducibility functions.

For a 2-form abelian gauge theory with gauge field $B_{\mu\nu} = -B_{\nu\mu}$ and equations of motion $\mathcal{L}^{\mu\nu} \equiv \partial_\rho H^{\rho\mu\nu} = 0$ ($H_{\rho\mu\nu} = \partial_\rho B_{\mu\nu} + \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu}$), one has $R^\nu \equiv \partial_\mu \mathcal{L}^{\mu\nu}$. The reducibility relations $R^\nu = 0$ are not independent in the sense that they are subject to algebraic identities $\partial_\nu R^\nu = 0$ holding no matter what $\mathcal{L}^{\mu\nu} = -\mathcal{L}^{\nu\mu}$ is. This is in contrast with the Yang-Mills case where the relations $D_\nu \mathcal{L}^\nu_a \equiv 0$ are independent. Thus, while the set $\{R_U\}$ is empty in the Yang-Mills case, one has for the 2-form gauge field

$$\begin{aligned} \{R_u\} \equiv \{R^\nu \equiv \partial_\mu \mathcal{L}^{\mu\nu}, \partial_\lambda R^n, \partial_{\lambda_1 \lambda_2} R^n, \ldots, \partial_{\lambda_1 \ldots \lambda_s} R^n, \ldots, \\ \partial_l R^0, \partial_{l_1 l_2} R^0, \ldots, \partial_{l_1 \ldots l_s} R^0, \ldots\} \end{aligned} \tag{3.13}$$

$$\{R_U\} \equiv \{\partial_0 R^0, \partial_\rho \partial_0 R^0, \partial_{\rho_1 \rho_2} \partial_0 R^0, \ldots, \partial_{\rho_1 \ldots \rho_s} \partial_0 R^0, \ldots\}. \tag{3.14}$$

For the equations and the field variables, the split reads

$$\begin{aligned} \{L_a\} \equiv \{\mathcal{L}^{\mu\nu}, \partial_\rho \mathcal{L}^{mn}, \partial_{\rho_1 \rho_2} \mathcal{L}^{mn}, \ldots, \partial_{\rho_1 \ldots \rho_s} \mathcal{L}^{mn}, \ldots, \partial_k \mathcal{L}^{0\bar{m}}, \\ \partial_{k_1 k_2} \mathcal{L}^{0\bar{m}}, \ldots, \partial_{k_1 \ldots k_s} \mathcal{L}^{0\bar{m}}, \ldots, \partial_{\bar{k}} \mathcal{L}^{01}, \partial_{\bar{k}_1 \bar{k}_2} \mathcal{L}^{01}, \ldots, \partial_{\bar{k}_1 \ldots \bar{k}_s} \mathcal{L}^{01}, \ldots\} \end{aligned} \tag{3.15}$$

$$\begin{aligned} \{L_\Delta\} \equiv \{\partial_0 \mathcal{L}^{0m}, \partial_\rho \partial_0 \mathcal{L}^{0m}, \partial_{\rho_1 \rho_2} \partial_0 \mathcal{L}^{0m}, \ldots, \partial_{\rho_1 \ldots \rho_s} \partial_0 \mathcal{L}^{0m}, \ldots, \\ \partial_1 \mathcal{L}^{01}, \partial_k \partial_1 \mathcal{L}^{01}, \partial_{k_1 k_2} \partial_1 \mathcal{L}^{01}, \ldots, \partial_{k_1 \ldots k_s} \partial_1 \mathcal{L}^{01}, \ldots\} \end{aligned} \tag{3.16}$$

$$
\begin{aligned}
\{x_A\} \equiv \{&B_{\mu\nu}, \partial_\rho B_{\mu\nu}, \partial_k \partial_\rho B_{mn}, \dots, \partial_{k_1 \dots k_s} \partial_\rho B_{mn}, \dots, \partial_\rho \partial_0 B_{0m}, \\
&\partial_{\rho_1 \rho_2} \partial_0 B_{0m}, \dots, \partial_{\rho_1 \dots \rho_s} \partial_0 B_{0m}, \dots, \partial_{\bar{k}} \partial_n B_{0\bar{m}}, \partial_{\bar{k}_1 \bar{k}_2} \partial_n B_{0\bar{m}}, \dots, \partial_{\bar{k}_1 \dots \bar{k}_s} \\
&\partial_n B_{0\bar{m}}, \dots, \partial_k \partial_1 B_{01}, \partial_{k_1 k_2} \partial_1 B_{01}, \dots, \partial_{k_1 \dots k_s} \partial_1 B_{01}, \dots, \partial_{\bar{\bar{l}}} \partial_{\bar{m}} B_{01}, \\
&\partial_{\bar{\bar{l}}_1 \bar{\bar{l}}_2} \partial_{\bar{m}} B_{01}, \dots, \partial_{\bar{\bar{l}}_1 \dots \bar{\bar{l}}_s} \partial_{\bar{m}} B_{01}, \dots\}, \quad (\bar{\bar{l}}, \bar{\bar{l}}_i = 3, 4, \dots, n-1).
\end{aligned} \tag{3.17}
$$

$$
\begin{aligned}
\{z_a\} \equiv \{&\partial_0 \partial_0 B_{mn}, \partial_\rho \partial_0 \partial_0 B_{mn}, \dots, \partial_{\rho_1 \dots \rho_s} \partial_0 \partial_0 B_{mn}, \dots, \partial_1 \partial_1 B_{0\bar{m}}, \\
&\partial_k \partial_1 \partial_1 B_{0\bar{m}}, \partial_{k_1 k_2} \partial_1 \partial_1 B_{0\bar{m}}, \dots, \partial_{k_1 \dots k_s} \partial_1 \partial_1 B_{0\bar{m}}, \dots, \partial_2 \partial_2 B_{01}, \\
&\partial_{\bar{s}} \partial_2 \partial_2 B_{01}, \partial_{\bar{s}_1 \bar{s}_2} \partial_2 \partial_2 B_{01}, \dots, \partial_{\bar{s}_1 \dots \bar{s}_n} \partial_2 \partial_2 B_{01}, \dots\}.
\end{aligned} \tag{3.18}
$$

This time, it is not true that $\partial_{\bar{l}} I_0 \subset I_0$ for $\bar{l} \geq 2$. But $\partial_{\bar{\bar{l}}} I_0 \subset I_0$ if $\bar{\bar{l}} \geq 3$, while $\partial_{\bar{l}} L_a \in \{L_a\}$ and $\partial_l R_u \in \{R_u\}$. We leave it to the reader to investigate the local structure of the equations of motion for higher order p-form gauge fields or gravity along similar lines. Note that in all cases treated here, the x_A and the independent equations L_a provide global coordinates in the jet spaces and not just local coordinates in the vicinity of the stationary surface. This property is clear for linear theories, but it also holds for Yang-Mills models or gravity because the terms with highest derivatives in L_a are the linear ones. Thus, modulo terms belonging to the previous space V^k, the relationship between the field derivatives of order $k+1$ and the variables x_A and L_a is the same as in the linear case. Furthermore, in the Yang-Mills case, the change of parametrization between the field variables and their derivatives on the one hand, and the (x_A, L_a) on the other hand, is polynomial. Its inverse also fulfills this property. Polynomiality in the original field variables and their derivatives is accordingly completely equivalent to polynomiality in the x_A, L_a. In the case of gravity, the same property is true for the transformation restricted to variables carrying derivatives, since the quantities $\sqrt{g}$ or $g^{\lambda\mu}$, which are non polynomial in the undifferentiated fields, occur in the field equations.

There is clearly a lot of freedom in the explicit choice of what is meant by the "independent variables" x_A, since any other choice $x_A \rightarrow x'_A = x'_A(x_B, L_a)$ with $\partial x'_A / \partial x_B$ invertible is also acceptable. The subsequent results do not depend on the precise choice that is being made. All that matters is that the split of the field variables and equations of motion with the above properties can indeed be performed if desired.

A different split adapted to the Lorentz symmetry - or to the $SO(n)$ symmetry in the Euclidean case - could have been actually achieved. This is because the Lorentz group is semi-simple. Hence, for each k, the representation to which the derivatives of order k belong, is completely reducible. The equations of motion restrict an invariant subspace of that representation. A covariant split is achieved by working with a basis adopted to the irreducible subspaces of the representation of order k (for each k). Such a covariant split is useful in maintaining manifest covariance. However, the non covariant splits given here, such that $\partial_{\bar{l}} I_0 \subset I_0$ or $\partial_{\bar{\bar{l}}} I_0 \subset I_0$, are useful in establishing the vanishing theorems on $H(\delta|d)$ derived below. Covariance can be controled differently, as we shall mention in section 5.

4 Local p-forms and antifields

To fix the ideas, we shall assume from now on that the theory is at most a reducible gauge theory of order one and that the fields ϕ^i are bosonic. The gauge transformations

$$\delta_\varepsilon \phi^i = \int R^i_\alpha(x,x')\varepsilon^\alpha(x')dx' \quad (4.1)$$

$$R^i_\alpha(x,x') = R^i_\alpha \delta(x,x') + R^{i\mu}_\alpha \partial_\mu \delta(x,x') + \ldots + R^{i\mu_1\ldots\mu_k}_\alpha \partial_{\mu_1\ldots\mu_k}\delta(x,x') \quad (4.2)$$

$$\Longleftrightarrow \delta_\varepsilon \phi^i = R^i_\alpha \varepsilon^\alpha + R^{i\mu}_\alpha \partial_\mu \varepsilon^\alpha + \ldots + R^{i\mu_1\ldots\mu_k}_\alpha \partial_{\mu_1\ldots\mu_k}\varepsilon^\alpha \quad (4.3)$$

are not independent

$$\int Z^\alpha_\Delta(x,x')R^i_\alpha(x',x'')dx' = 0 \quad (4.4)$$

but there are no non trivial relations among the Z^α_Δ. The ghost and antifield spectrum relevant to that case is given by

$$\phi^A \equiv (\phi^i, C^\alpha, C^\Delta) \quad (4.5)$$

$$\phi^*_A \equiv (\phi^*_i, C^*_\alpha, C^*_\Delta) \quad (4.6)$$

with

$$antigh\ \phi^i = antigh\ C^\alpha = antigh\ C^\Delta = 0 \quad (4.7)$$

$$antigh\ \phi^*_i = 1,\ antigh\ C^*_\alpha = 2,\ antigh\ C^*_\Delta = 3 \quad (4.8)$$

$$puregh\ \phi^i = 0,\ puregh\ C^\alpha = 1,\ puregh\ C^\Delta = 2 \tag{4.9}$$

$$puregh\ \phi^*_i = puregh\ C^*_\alpha = puregh\ C^*_\Delta = 0 \tag{4.10}$$

$$gh\ \phi^i = 0,\ gh\ C^\alpha = 1,\ gh\ C^\Delta = 2 \tag{4.11}$$

$$gh\ \phi^*_i = -1,\ gh\ C^*_\alpha = -2,\ gh\ C^*_\Delta = -3 \tag{4.12}$$

Irreducible gauge theories have no Z^α_Δ, and thus no ghosts of ghosts C^Δ and their antifields C^*_Δ. Theories without gauge freedom have only ϕ^i and ϕ^*_i ($R^i_\alpha \equiv 0,\ Z^\alpha_\Delta \equiv 0$). The assumption that the gauge theory is at most reducible of order one or that the fields are bosonic is by no means important. It is only made to keep the formulas and the field spectrum simple. The general theorems of sections 6,7 or 8 below hold for reducible gauge theories of higher order as well.

The local p-forms introduced previously depended only on the original fields and their derivatives. From now on, local p-forms will also involve the ghosts, the antifields and their derivatives. Because the objects under consideration may have terms of arbitrarily high antighost number, we shall actually define two different types of local p-forms and local functionals. The p-forms of the first type are given by unrestricted formal sums

$$a = \sum_{k\geq 0} \overset{(k)}{a}\ , \tag{4.13}$$

where k is the antighost number and where $\overset{(k)}{a}$ are p-forms of antighost number k involving the fields, the ghosts, the antifields and a finite number of their derivatives. Because a is assumed to have given total ghost number, and because $antigh\ \overset{(k)}{a} = k$, the $\overset{(k)}{a}$ are actually polynomials in the ghosts and their derivatives. The sum (4.13) may not terminate, i.e., a may be an infinite formal series in the antifields. So, while $\overset{(k)}{a}$ are local p-forms in the usual sense, the formal sum (4.13) may in principle involve derivatives of arbitrarily high order since the order of the derivatives present in $\overset{(k)}{a}$ may increase with k. In the same way, local functionals of the first type are given by unrestricted sums of integrated terms

$$A = \int \sum_{k\geq 0} \overset{(k)}{a}\ , \tag{4.14}$$

where $\overset{(k)}{a}$ are usual n-forms of antighost number k.

For a generic gauge theory with open algebra, there is a priori no control as to whether the formal sums (4.13) or (4.14) stop after a finite number of steps. The local p-forms and functionals (4.13) and (4.14) are accordingly the natural objects to be considered. It is to those objects that the general theorems of homological perturbation theory apply.

For the usual theories like Yang-Mills models or gravity, however, it is possible to control the expansion (4.13) and (4.14). For such theories, we shall consider a second type of local p-forms and local functionals, namely, those for which the expansions (4.13) or (4.14) stop after a finite number of steps,

$$a = \sum_{0 \leq k \leq L} \overset{(k)}{a} \tag{4.15}$$

$$A = \int \sum_{0 \leq k \leq L} \overset{(k)}{a} \,. \tag{4.16}$$

There is no difference between the individual terms appearing in the expansion (4.13) or (4.15). When projected to a definite value of the antighost number, the local forms (4.13) or (4.15) are identical. The difference lies only in the fact that (4.13) (or (4.14)) may involve terms of arbitrarily high antighost number.

In the case of Yang-Mills and gravity, we shall restrict even more the functional spaces to which the local functions and functionals belong, by demanding that they be polynomials in the derivatives of the fields (and also in the undifferentiated fields in the Yang-Mills case). That is, we exclude local functions like $exp(\partial_0 A_1)$. This is quite natural from the point of view of perturbative quantum field theory.

Thus, we require that the local q-forms be polynomials in all the variables ϕ^i, C^α, ϕ^*_i, C^*_α and their derivatives for Yang-Mills models ; and for Einstein gravity, that they be polynomials in C^α, ϕ^*_i, C^*_α and their derivatives, as well as in the derivatives of the fields ϕ^i, with coefficients that may be infinite series in the undifferentiated fields (to allow the inverse metric $g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} + \dots$).

5 BRST differential

The Koszul-Tate differential δ is defined in the algebra of local p-forms by:

$$\delta\phi^A(x) = 0,\ \delta\phi^*_i(x) = -\frac{\delta\mathcal{L}_0}{\delta\phi^i}(x)$$
$$\delta C^*_\alpha(x) = \int \phi^*_i(x')R^i_\alpha(x,x')dx'$$
$$\delta C^*_\Delta(x) = \int -C^*_\alpha(x')Z^\alpha_\Delta(x,x')dx' + \frac{1}{2}\int \phi^*_i(x')\phi^*_j(x'')M^{ij}_\Delta(x;x'x'')dx'dx''$$
$$\delta dx^\mu = 0 \tag{5.1}$$

where ϕ^A denotes collectively ϕ^i, C^α and C^Δ. [Like s and γ below, the differential δ is extended to the derivatives of the fields by requiring $\delta\partial_\mu = \partial_\mu\delta$, and to arbitrary functions of the generators by means of the Leibnitz rule]. It has antighost number -1 and is such that

$$H_0(\delta) = C^\infty(\Sigma)\otimes\mathbf{C}[C^\alpha,\partial_\mu C^\alpha,\ldots,C^\Delta,\partial_\mu C^\Delta,\ldots,dx^\mu]$$
$$H_k(\delta) = 0,\ k>0 \tag{5.2}$$

where $C^\infty(\Sigma)$ denotes the quotient algebra of the smooth functions of the fields ϕ^i and their derivatives modulo the ideal of functions that vanish when the field equations hold. One says that δ provides a "resolution" of the algebra $C^\infty(\Sigma)\otimes\mathbf{C}[C^\alpha,\partial_\mu C^\alpha,\ldots,C^\Delta,\partial_\mu C^\Delta,\ldots,dx^\mu]$, the antighost number being the resolution degree. The same result holds if instead of arbitrary smooth functions of the fields and their derivatives, one considers polynomial functions, provided that the change of variables $(\phi^i,\partial\phi^i,\partial^2\phi^i,\ldots)\leftrightarrow(x_A,L_a)$ and its inverse are both polynomial. Any polynomial cycle a of antighost number $k>0$ can be written as $a=\delta b$ where b is also polynomial ([24] footnote 1).

The acyclicity of δ in strictly positive antighost number is most easily proved by introducing a homotopy for the operator that counts the antifields and the equations of motion [10], for instance in the basis of section 3. Since that basis is not manifestly covariant, one may wonder whether acyclicity also holds in the space of Lorentz invariant local forms. More generally, if the theory is invariant under a global symmetry group G, one may wonder whether acyclicity also holds in the space of local forms belonging to a definite representation of G. That this is so if the group is semi-simple can be seen either by redefining the basis of section 3 in a manner compatible with the symmetry, or by using the fact that δ commutes with the action of G and hence maps any irreducible representation occuring in the decomposition of the completely reducible representation of G given by the local forms on an equivalent representation or on zero [24]. [The argument uses the fact that the representations of G are completely reducible ; this is guaranteed to hold if G is semi-simple, but holds also if G is compact provided (in both cases) that the representations under consideration are effectively finite-dimensional. This is for instance so if the local forms are polynomials in the fields and their derivatives].

The differential γ has antighost number equal to zero. It is defined on the fields through

$$\gamma\phi^i(x) = \int R^i_\alpha(x,x')C^\alpha(x')dx'$$
$$\gamma C^\alpha(x) = \int Z^\alpha_\Delta(x,x')C^\Delta(x')dx' + \frac{1}{2}\int C^\alpha_{\beta\gamma}(x;x'x'')C^\beta(x')C^\gamma(x'')dx'dx''$$
$$\gamma C^\Delta(x) = \int C^\Delta_{\alpha\Gamma}(x;x'x'')C^\alpha(x')C^\Gamma(x'')dx'dx''$$
$$+\frac{1}{3}\int C^\Delta_{\alpha\beta\gamma}(x;x'x''x''')C^\alpha(x')C^\beta(x'')C^\gamma(x''')dx'dx''dx''' \tag{5.3}$$

so that $H^*(\gamma)$ in $C^\infty(\Sigma)\otimes\mathbf{C}[C^\alpha,\partial_\mu C^\alpha,\ldots,C^\Delta,\partial_\mu C^\Delta,\ldots]$ is isomorphic to the cohomology of the exterior derivative along the gauge orbits. Furthermore, it is extended to the antifields in such a way that it is a differential modulo δ, i.e.,

$$\gamma\delta + \delta\gamma = 0$$
$$\gamma^2 = -(\delta s_1 + s_1\delta) \tag{5.4}$$

for some derivation s_1 of antighost number +1 [10].

The following theorems are standard results of the BRST formalism.

Theorem 5.1 : *There exists a derivation s of total ghost number equal to* 1 *such that*

(i) $s = \delta + \gamma + s_1 +$ *"higher orders", antigh ("higher orders")*≥ 2 *;*

(ii) $s^2 = 0$ *(s is a differential).*

Furthermore, one may choose s so that it is canonically generated in the antibracket, $sa = (a, S)$, where S is a solution of the classical master equation

$(S,S)=0$ and starts like $S=\int \mathcal{L}dx+\int \ldots$. The differential s is the BRST differential.

Theorem 5.2 (on the cohomology of s) : *The cohomology of the BRST differential s in the algebra of local q-forms is given by*

(i) $H^k(s)=H_{-k}(\delta)=0$ for $k<0$

(ii) $H^k(s)\simeq H^k(\gamma,H_0(\delta))$ for $k\geq 0$

where $H^k(\gamma,H_0(\delta))$ is the cohomology of γ in the cohomology of δ.

Furthermore, the correspondence between $H^k(s)$ and $H^k(\gamma,H_0(\delta))$ is given by

$$[a]\in H^k(s)\longleftrightarrow [a_0]\in H^k(\gamma,H_0(\delta)) \tag{5.5}$$

where a_0 is the component of a of antighost number zero. *That is, for non negative ghost number, any cohomological class of the BRST cohomology is completely determined by its antifield independent component, which is a solution of $\gamma a_0+\delta a_1=0$ or, what is the same $\gamma a_0\approx 0$.* Here, $\approx$ means "equal modulo the equations of motion".

Proof. The proofs of theorem 5.1 and theorem 5.2 may be found in [10]. □.

We stress again that here, s and a are a priori infinite formal power series with terms of arbitrarily high antighost numbers.

To analyze the cohomology of s modulo d, we shall also need the following two results

Theorem 5.3 (on the cohomology of d) : *The cohomology of d in the algebra of local p-forms is given by*

$$H^0(d)\simeq \mathbf{R},$$
$$H^k(d)=0 \ for\ k\neq 0,\ k\neq n,$$
$$H^n(d)\simeq space\ of\ equivalence\ classes\ of\ local\ n-forms, \tag{5.6}$$

where two local n-forms $\alpha=fdx^0\ldots dx^{n-1}$ and $\alpha'=f'dx^0\ldots dx^{n-1}$ are equivalent if and only if f and f' have identical Euler- Lagrange derivatives with respect to all the fields and antifields,

$$\frac{\delta(f-f')}{\delta\phi^A}=0=\frac{\delta(f-f')}{\delta\phi^*_A}\Longleftrightarrow \alpha\ and\ \alpha'\ are\ equivalent. \tag{5.7}$$

Proof. The proof of this theorem may be found in various different places [25, 26, 27, 18, 28, 29]. □.

Theorem 5.4 (on the cohomology of δ modulo d) : *In the algebra of local forms,*

$$H_k(\delta|d) = 0 \tag{5.8}$$

for $k > 0$ <u>and</u> pureghost number > 0.

Proof. see [17] or [10], chapter 12. □.

Theorem 5.3 is sometimes referred to as the "algebraic Poincaré lemma" because it reminds one of the usual Poincaré lemma. However, it is not the standard Poincaré lemma, which states that $d\Psi = 0 \Rightarrow \Psi = d\chi$ locally in spacetime but without guarantee that χ involves the fields and a finite number of their derivatives if Ψ is a local p-form. Note also that $H^k(d)$ vanishes for $k \neq 0$ and $k \neq n$ only if one allows for an explicit coordinate dependence of the local forms. Otherwise, $H^k(d)$ is isomorphic to the set of constant forms for $k \neq 0$ and $k \neq n$.

6 Homological Perturbation Theory and $H^*(s|d)$

Theorem 5.2 relates the cohomology of s to the cohomology of δ and γ. This is done through the methods of homological perturbation theory. A straightforward application of the same techniques enables one to characterize the cohomology of s modulo d.

Theorem 6.1 (on the cohomology of s modulo d) :

$$(i) H^k(s|d) \simeq H_{-k}(\delta|d) \;\; for \;\; k < 0 \tag{6.1}$$

$$(ii) H^k(s|d) \simeq H^k(\gamma|d, H_0(\delta)) \;\; for \;\; k \geq 0 \tag{6.2}$$

Proof. the proof proceeds as the proof of 5.2 (see [10], chapter 8, section 8.4.3). We shall thus only sketch it here. Let a be a representative of a cohomological class of s modulo d, $sa + db = 0$. Assume $gh\ a = k$. Expand a according to the antighost number,

$$a = a_i + a_{i+1} + \ldots \tag{6.3}$$

$$antigh\ a_j = j \geq 0 \tag{6.4}$$
$$pure\ gh\ a_j = k + j \tag{6.5}$$
$$gh\ a_j = k \tag{6.6}$$

The first term a_i in (6.3) has antighost number equal to max $(0, -k)$ (i.e., $i = 0$ if $k \geq 0$ and $i = -k$ if $k < 0$). If $k \geq 0$, a_0 fulfills $\gamma a_0 + \delta a_1 + db_0 = 0$ and thus, defines an element of $H^k(\gamma|d, H_0(\delta))$ (both a_0 and b_0 fulfill $\delta a_0 = \delta b_0 = 0$; furthermore, *pure gh* $a_0 = k$). It is easy to verify that the map $H^k(s|d) \longrightarrow H^k(\gamma|d, H_0(\delta)) : [a] \longmapsto [a_0]$ is well defined, i.e., does not depend on the choice of representatives. One proves that it is injective and surjective as in [10] using the crucial property that $H_j(\delta|d)$ vanishes for both $j > 0$ *and* strictly positive pure ghost number (the pure ghost number of the higher order terms in the expansion (6.3) is > 0). Turn now to the case $k = -k' < 0$. Then the expansion of a reads

$$a = a_{k'} + a_{k'+1} + \dots \tag{6.7}$$

The term $a_{k'}$ fulfills $\delta a_{k'} + db_{k'-1} = 0$, i.e., defines an element of $H_{k'}(\delta|d)$. As for $k \geq 0$, the map $H^k(s|d) \longrightarrow H_{-k}(\delta|d) : [a] \longmapsto [a_{-k}]$ is well defined and is both injective and surjective thanks to the triviality of $H_j(\delta|d)$ in positive antighost and pure ghost numbers. This proves the theorem. Note again that the series (6.3) or (6.7) under consideration may be infinite formal series in the antifields, just as in theorem 5.2 ; there is at this stage no guarantee that they stop. $\square$.

Comments:

(i) For $k \geq 0$, the cohomological classes of $H^k(s|d)$ are completely determined by their antifield independent components. In particular, to determine whether there exist non trivial elements of $H^k(s|d)$, it is enough to determine whether there exist non trivial solutions of $\gamma a_0 + db_0 + \delta a_1 = 0$, or, what is the same, $\gamma a_0 + db_0 \approx 0$. This is true for any value of the (positive) ghost number, in particular for $k = 1$ (anomalies). It is just the transcription, in terms of local functionals, of standard and well-established properties of $H^k(s)$. Theorem 6.1 is discussed along the same lines in exercise 12.9 of [10].

(ii) For $k < 0$, the cohomological classes of $H^k(s|d)$ are also determined by their components of lowest antighost number. In this case, these components do involve the antifields but do not involve the ghosts.

(iii) The surjectivity of the map $[a] \mapsto [a_{-k}]$ for $k < 0$ shows that any solution $a_{k'}$ of $\delta a_{k'} + db_{k'-1} = 0$ is automatically annihilated by γ up to

δ- and d- exact terms ($\gamma a_{k'} = -\delta a_{k'+1} - db_{k'}$). That is, any solution of $\delta a_{k'} + db_{k'-1} = 0$ is "weakly gauge invariant" up to d-exact terms.

7 Constants of the motion and $\mathbf{H_1^n}(\delta|\mathbf{d})$.

Although $H_j(\delta)$ vanishes for antighost number $j > 0$, this is not true for $H_j(\delta|d)$. A counterexample was provided in [17]. In this section, we characterize more completely $H_1^n(\delta)$ (where n is the form degree). We show that there is a bijective correspondence between $H_1^n(\delta|d)$ ($\equiv H^{-1,n}(s|d)$) and the space of non trivial conserved currents. That the BRST cohomology involves the constants of the motion is not surprising, in view of the fact that the BRST differential incorporates explicitly the equations of motion.

Elements of $H_1^n(\delta|d)$ are determined by n-forms of antighost number one solving

$$\delta a + dj = 0 \tag{7.1}$$

where j is a $(n-1)$-form of antighost number zero. Both a and j may be assumed not to depend on the ghosts (ghost dependent contributions are trivial, see theorem 5.4). If one substitutes

$$a = a^i \phi_i^* + a^{i\mu}\partial_\mu \phi_i^* + \ldots + a^{i\mu_1\ldots\mu_t}\partial_{\mu_1\ldots\mu_t}\phi_i^* \tag{7.2}$$

in (7.1), one gets using (5.1),

$$a^i \frac{\delta \mathcal{L}_0}{\delta \phi^i} + a^{i\mu}\partial_\mu \frac{\delta \mathcal{L}_0}{\delta \phi^i} + \ldots + a^{i\mu_1\ldots\mu_t}\partial_{\mu_1\ldots\mu_t}\frac{\delta \mathcal{L}_0}{\delta \phi^i} = dj, \tag{7.3}$$

or, in dual notations ($a \equiv X dx^0 \ldots dx^{n-1}$),

$$X^i \frac{\delta \mathcal{L}_0}{\delta \phi^i} + X^{i\mu}\partial_\mu \frac{\delta \mathcal{L}_0}{\delta \phi^i} + \ldots + X^{i\mu_1\ldots\mu_t}\partial_{\mu_1\ldots\mu_t}\frac{\delta \mathcal{L}_0}{\delta \phi^i} = \partial_\mu j^\mu. \tag{7.4}$$

Thus, j^μ is a current that is conserved by virtue of the equations of motion.

The current j is not completely determined by (7.1). One may add to it an arbitrary solution $\bar{j}$ of $d\bar{j} = 0$ without changing a. Since $H^{n-1}(d) = 0$ (theorem 5.3), $\bar{j}$ is of the form dk. Thus given a, j is determined up to $j \to j + dk$. But a is not even given completely ; what is fixed is the cohomological class of a in $H^n_1(\delta|d)$, i.e., a up to $\delta m + dn$. The modification $a \rightarrow a + \delta m + dn$ yields $j \rightarrow j + \delta n + dk$. Since j is d closed modulo δ, there is accordingly a well defined map from $H^n_1(\delta|d)$ to $H^{n-1}_0(d|\delta)$,

$$H^n_1(\delta|d) \longrightarrow H^{n-1}_0(d|\delta), [a] \longmapsto [j] \tag{7.5}$$

The map is injective because $H_1(\delta) = 0$ (if $[j] = 0$ in $H(d|\delta)$, i.e., $j = \delta n + dk$, then $\delta(a - dn) = 0$, i.e., $a = dn + \delta b$, i.e., $[a] = 0$ in $H(\delta|d)$). It is also clearly surjective. Thus there is the isomorphism

$$H^n_1(\delta|d) \simeq H^{n-1}_0(d|\delta). \tag{7.6}$$

This result is a particular case of a proposition (Eq.(16)) of [27].

To fully appreciate the physical content of Equation (7.6), one needs to introduce the concept of non trivial conserved currents and non trivial global symmetries.

(i) A current is said to be identically conserved if it is conserved independently of the dynamics, i.e., if $dj = 0$ or $j = dk$. A conserved current is said to be non trivial if it does not coincide on-shell with an identically conserved current, $j \not\approx dk$. The space $H^{n-1}_0(d|\delta)$ is just the space of inequivalent non trivial conserved currents.

(ii) By making integrations by parts if necessary, one may assume that $a \equiv X dx^0 \dots dx^{n-1}$ does not involve the derivatives of the antifields. With $X = X^i \phi^*_i$, Eq.(7.4) reduces to

$$X^i \frac{\delta \mathcal{L}_0}{\delta \phi^i} = \partial_\mu j^\mu \tag{7.7}$$

and shows that a defines the symmetry $\delta_X \phi^i = X^i$ of the action (any linear function of the ϕ^*_i is naturally viewed as a tangent vector to field space [30]). Gauge symmetries (including on-shell trivial symmetries [10]) are physically irrelevant since two configurations differing by a gauge symmetry must be identified. They correspond to X^i of the form

$$X^i(x) = \int R^i_\alpha(x, x') t^\alpha(x') dx' + \int \mu^{ij}(x, x') \frac{\delta S_0}{\delta \phi^j(x')} dx' \tag{7.8}$$

with $\mu^{ij}(x, x') = -\mu^{ji}(x, x')$, which is equivalent to

$$X^i \phi^*_i = \delta \mu + \partial_\mu b^\mu \tag{7.9}$$

[e.g., if the gauge transformations are $\delta_\varepsilon \phi^i = R^i_\alpha \varepsilon^\alpha + R^{i\mu}_\alpha \partial_\mu \varepsilon^\alpha$, then $\delta C^*_\alpha = R^i_\alpha \phi^*_i - \partial_\mu (R^{i\mu}_\alpha \phi^*_i)$. If $X^i = R^i_\alpha \lambda^\alpha + R^{i\mu}_\alpha \partial_\mu \lambda^\alpha + \mu^{ij} \frac{\delta \mathcal{L}_0}{\delta \phi^j}$ for some $\lambda^\alpha(\phi, \partial\phi, \dots)$ and $\mu^{ij}(\phi, \partial\phi, \dots) = -\mu^{ji}(\phi, \partial\phi, \dots)$, then, $X^i \phi^*_i = \delta(\lambda^\alpha C^*_\alpha - \frac{1}{2}\mu^{ij}\phi^*_i\phi^*_j) + \partial_\mu b^\mu$ with $b^\mu = \lambda^\alpha R^{i\mu}_\alpha \phi^*_i$]. A symmetry of the action is said to be a nontrivial global symmetry if it does not coincide with a gauge symmetry modulo on-shell trivial symmetries, i.e., if it is not of the form $\delta\mu + \partial_\mu b^\mu$. Thus, one can identify $H^n_1(\delta|d)$ with the space of inequivalent global symmetries.

We have therefore established

Theorem 7.1 : *The space of inequivalent non trivial conserved currents is isomorphic with the space of non trivial inequivalent global symmetries,*

$$H^{n-1}_0(d|\delta) \simeq H^n_1(\delta|d) \tag{7.10}$$

This theorem provides a cohomological reformulation of the physical version of Noether theorem. It shows that each non trivial symmetry defines a non trivial conserved charge and vice-versa.

Comments:

(i) Given a conserved current, one may define a conserved charge through $\int j^0 d^{n-1}x$. Now, the antibracket induces a map $H^n_1(\delta|d) \times H^n_1(\delta|d) \longrightarrow H^n_1(\delta|d)$, which corresponds to the Lie bracket of the global symmetries (the antibracket coincides with the Schouten bracket [30, 10], which reduces to the Lie bracket for vector fields). It is easy to verify that the isomorphism (7.10) associates with the Lie bracket of two global symmetries the Poisson bracket of the corresponding conserved charges. This will be made more precise in [31].

(ii) The fact that gauge symmetries lead to "trivial" conserved currents is well known, see e.g. [32] in that context. Our analysis reformulates the question as a cohomological problem.

(iii) One may wonder how the non triviality of $H^n_1(\delta|d)$ in the space of local functions is compatible with the triviality of $H_1(\delta, \mathcal{A})$ in the space $\mathcal{A}$ of all (local and non-local) functionals. It turns out that this follows from a combination of two features: (α) non trivial elements of $H^n_1(\delta|d)$ do not necessarily define elements of $H_1(\delta)$ upon integration because of non vanishing surface terms ; (β) those that do, actually turn out to define trivial elements that can be written as the δ-variation of non local functionals. [If this is not the case, then there are missing global antifields for antifields, since the triviality of $H_1(\delta)$ is a central property of the BRST formalism that fixes the antifield spectrum].

To see this, let us assume that the spacetime volume under consideration is limited by two spacelike hypersurfaces "at t_1 and t_2". Let us also assume that the field configurations are not restricted at t_1 and t_2, so that the allowed histories include all the solutions of the equations of motion and not just one. Then a solution a of the equation $\delta a + dj = 0$ defines a solution $A = \int a$ of $\delta A = 0$ iff $\oint j = Q(t_2) - Q(t_1) = 0$ with $Q(t) = \int_{\Sigma(t)} j^0 d^{n-1}x$. This is a strong condition on j. Indeed, the requirement $Q(t_2) - Q(t_1) = 0$ for all allowed field configurations at t_2 and t_1 generically implies $Q(t) = constant$ and thus $j^0 = \partial_k S^{0k}$. Thus, $\tilde{j}^\mu = j^\mu - \partial_\rho S^{\mu\rho}$ with $S^{mn} = 0$ and $S^{k0} = -S^{0k}$, is a current such that (i) $\tilde{j}^0 = 0$; and (ii) $\delta a + d\tilde{j} = 0$. The corresponding charge $Q(t) = \int \tilde{j}^0 d^{n-1}x$ identically vanishes. Thus the transformations of the fields associated with it is a gauge symmetry (exercise 3.3 (b) of [10]). If all gauge symmetries have been properly taken into account, locally and non locally, then, $\int a = \delta X$. As an example, one may consider electromagnetism with $a = A^{*0} dx^0 \dots dx^{n-1}$. One has $\delta A^{*0} + \partial_\mu j^\mu = 0$ with $j^\mu = F^{\mu 0}$, i.e., $j^0 = 0$, $j^k = F^{k0}$. Thus $A = \int A^{*0} dx^0 \dots dx^{n-1}$ solves $\delta A = 0$. Even though a is a non trivial element of $H(\delta|d)$ in the space of local functions, one has $A^{*0}(t, \vec{x}) = \int^t [-\delta C^*(u, \vec{x}) + \partial_k A^{*k}(u, \vec{x})] du$ and thus $A = \delta \int dt \dots dx^{n-1} \int^t -C^*(u, \vec{x}) du$: A is a trivial element of $H_1(\delta)$ if one includes non local functionals.

(iv) There exist interesting theories for which there is no non trivial, local conserved current. For example, pure Einstein gravity is such a theory [33]. In that case, the cohomological groups $H^n_1(\delta|d)$ and thus also $H^{-1,n}(s|d)$ are empty.

8 Results on $\mathbf{H^p_k}(\delta|\mathbf{d})$ (p, k arbitrary)

The above theorem characterizes $H^n_1(\delta|d)$ in terms of conserved currents. What is the cohomology of δ modulo d for the other values of the antifield number and form degree? As a first step in characterizing $H^p_k(\delta|d)$ for arbitrary k's and p's we establish

Theorem 8.1 (descent equations for δ and d) : *if $p \geq 1$ and $k > 1$, then*

$$H^p_k(\delta|d) \simeq H^{p-1}_{k-1}(\delta|d). \tag{8.1}$$

Proof. from $\delta a_k^p + da_{k-1}^{p-1} = 0$, one gets $d\delta a_{k-1}^{p-1} = 0$ and thus using the triviality of d in degree $p-1$,

$$\delta a_{k-1}^{p-1} + da_{k-2}^{p-2} = 0 \tag{8.2}$$

(If $p-1=0$, $H^0(d)$ is trivial because a_{k-1}^0 has non vanishing antifield number and cannot be constant). This shows that a_{k-1}^{p-1} defines an element of $H_{k-1}^{p-1}(\delta|d)$. It is easy to check that this element does not depend on the choices of representatives and thus, there is a well defined map from $H_k^p(\delta|d)$ to $H_{k-1}^{p-1}(\delta|d)$. This map is injective because $H_k(\delta)=0$ and surjective because $H_{k-1}(\delta)=0$. This proves the theorem. □.

Of course one has also, by the same techniques as in the previous section

Theorem 8.2 : *if $p \geq 1$ and $k \geq 1$ with $(p,k) \neq (1,1)$, then*

$$H_k^p(\delta|d) \simeq H_{k-1}^{p-1}(d|\delta) \tag{8.3}$$

Furthermore,

$$H_1^1(\delta|d) \simeq H_0^0(d|\delta)/\mathbf{R}. \tag{8.4}$$

[If one does not allow for an explicit x dependence in the local forms, then, (8.3) must be replaced by $H_1^p(\delta|d) \simeq H_0^{p-1}(d|\delta)/\{constant\ forms\}$ for $k=1$].

In particular, $H_k^n(\delta|d) \simeq H_0^{n-k}(d|\delta)$: the equivalence classes of n-forms that are δ-closed modulo d at antighost number k are in bijective correspondence with the equivalence classes of antifield independent $(n-k)$-forms that are d-closed modulo the equations of motion.

The calculation of the general solution of $da \approx 0$, $antigh\ a = 0$, is a question that is of interest independently of the BRST symmetry. It can be analyzed without ever introducing the antifields or the Koszul-Tate resolution and carries the name of "characteristic cohomology" [34]. However, as we shall see in the explicit case of the Yang-Mills theory, the direct calculation of $H_k^n(\delta|d)$ may be simpler than that of $H_0^{n-k}(d|\delta)$ for $k=2$. Thus, it appears to be useful to bring in the tools of the antifield formalism even in the analysis of questions that are a priori unrelated to the BRST symmetry, like that of calculating $H_0^p(d|\delta)$.

A direct consequence of theorem 8.1 is that $H_k^p(\delta|d)$ vanishes whenever $k > p$. Indeed, by using repeatedly (8.1), one gets $H_k^p(\delta|d) \simeq H_{k-p}^0(\delta|d) \simeq H_{k-p}^0(\delta) \simeq 0$ $(k > p)$.

To determine the cohomological groups $H^p_k(\delta|d)$, it is enough to compute $H^n_k(\delta|d)$ for $k = 1, 2, \ldots, n$ or $H^p_1(\delta|d)$ for $p = 1, 2, \ldots, n$. In general, this is a difficult task. For theories of Cauchy order q, however, one can locate more precisely the values of the degrees where the non trivial cohomology may lie.

To define the Cauchy order of a theory, we come back to the split of the field components, the field equations and their derivatives performed in section 3. We recall that the set of independent field variables x_A was denoted by I_0. We shall say that the split has Cauchy order q if $\partial_\alpha I_0 \subset I_0$ for $\alpha = q, q+1, \ldots, n-1$. This terminology is motivated by the fact that the split of the derivatives is somewhat adapted to the Cauchy problem. Thus, the above split for the Klein-Gordon theory has Cauchy order 1 ; the one for electromagnetism and Yang-Mills theories, has Cauchy order 2 ; and the one for p-form gauge fields, has Cauchy order $p+1$.

As such the Cauchy order depends on the choice of I_0 but also on the coordinate system. For instance, the two-dimensional Klein-Gordon equation $\partial^\mu \partial_\mu \phi = 0$ reads in light-like coordinates $\partial_+ \partial_- \phi = 0$. One may take as independent field variables ϕ, $\partial_-^{(k)} \phi$ and $\partial_+^{(k)} \phi$ ($k = 1, 2, 3, \ldots$). These are, however, preserved neither by ∂_+ nor by ∂_-, so that the value of q associated with this choice is 2. We shall define the Cauchy order of a theory as the minimum value of q for which $\partial_\alpha I_0 \subset I_0$ ($\alpha = q, q+1, \ldots$). The minimum is taken over all sets of spacetime coordinates and all choices of I_0.

So, the Cauchy orders of the Klein-Gordon theory and electromagnetism are respectively ≤ 1 and ≤ 2. The fact that $H_1(\delta|d)$ (respectively $H_2(\delta|d)$) does not vanish for those models implies, however, that $q = 1$ (respectively $q = 2$). Indeed, one has

Theorem 8.3 : *for theories of Cauchy order q,*

$$H^i_1(\delta|d) = 0 \ if \ i \leq n - q \tag{8.5}$$

Thus, for the Klein-Gordon model, only $H^n_1(\delta|d)$ is non vanishing. For electromagnetism and Yang-Mills, only $H^n_1(\delta|d)$ and $H^{n-1}_1(\delta|d) \simeq H^n_2(\delta|d)$ may be non vanishing. And for p-form gauge fields, only $H^n_1(\delta|d)$, $H^{n-1}_1(\delta|d) \simeq H^n_2(\delta|d)$, ... up to $H^{n-p}_1(\delta|d) \simeq H^{n-p+1}_2(\delta|d) \simeq \ldots \simeq H^n_{p+1}(\delta|d)$ may differ from zero.

Proof. We set $d = \bar{d} + \bar{\bar{d}}$ where $\bar{d} = \partial_0 dx^0 + \partial_1 dx^1 + \ldots + \partial_{q-1} dx^{q-1}$ and $\bar{\bar{d}} = \partial_q dx^q + \partial_{q+1} dx^{q+1} + \ldots + \partial_{n-1} dx^{n-1}$. Let a be a solution of $\delta a + db = 0$

with *antigh* $a = 1$ and *deg* $a = i \leq n - q$. One has *antigh* $b = 0$ and *deg* $b = i - 1$. Let $a = a_1 + a_2$ and $b = b_1 + b_2$, where a_1 (respectively b_1) involves at least one dx^β with $\beta \leq q - 1$, while a_2 (respectively b_2) involve only the dx^α with $\alpha \geq q$. One may assume without loss of generality that b_2 involves only the independent variables $x_A \in I_0$. This can always be achieved by adding to b a δ-exact term if necessary.This modifies a by a d-exact term. The equations $\delta a + db = 0$ splits as

$$\begin{aligned} \delta a_1 + db_1 + \bar{d}b_2 = 0 \\ \delta a_2 + \bar{\bar{d}}b_2 = 0. \end{aligned} \tag{8.6}$$

Now, $\bar{\bar{d}}b_2$ contains only the variables not constrained by the equations of motion since $\partial_\alpha I^0 \in I^0$ for $\alpha = q, q+1, \ldots, n-1$, while δa_2 vanishes by the equations of motion. Hence $\bar{\bar{d}}b_2$ and δa_2 must be zero separately. This implies $a_2 = \delta m_2$ and $b_2 = \bar{\bar{d}}c_2$ because $H^{i-1}(\bar{\bar{d}}) = 0$ (b_2 is a $(i-1)$-form in the $(n-q)$-dimensional space of the $x^q, x^{q+1}, \ldots, x^{n-1}$; and $i \leq n - q$ by assumption). Thus, by making the redefinitions $a \rightarrow a - \delta m$ and $b \rightarrow b - dc$, we may assume $a = a_1$ and $b = b_1$.

To pursue the analysis, we split further a_1 and b_1 into components a_{11} (b_{11}) involving at least two dx^β with $\beta \leq q-1$ and a_{12} (b_{12}) involving only one dx^β ($\beta \leq q-1$). We further redefine b_{12} in such a way that it involves only x_A, $b_{12} \rightarrow b_{12} + \delta t$. Because t involves one dx^β, the corresponding redefinition of a ($a \rightarrow a - dt$) leaves a_2 equal to zero. The equation $\delta a + db = 0$ yields

$$\delta a_{12} + \bar{\bar{d}}b_{12} = 0. \tag{8.7}$$

from which one infers as above that $\delta a_{12} = 0$ and $\bar{\bar{d}}b_{12} = 0$. It is easy to see that this implies not only $a_{12} = \delta m_{12}$ but also $b_{12} = \bar{\bar{d}}c_{12}$ (write b_{12} as $\sum_{\beta=0}^{q-1} b_{12\beta} dx^\beta$ where $b_{12\beta}$ are $(i-2)$-forms which must separately fulfill $\bar{\bar{d}}b_{12\beta} = 0$. Thus one can remove a_{12} from a_1 and b_{12} from b_1. By going on in the same fashion, one arrives in maximal form degree for $\bar{d}$ at $\delta a_{1q} + \bar{\bar{d}}b_{1q} = 0$. Again both terms have to vanish separately and can be absorbed by redefinitions, which proves the theorem. Note that (8.5) holds also in the space of polynomial q-forms if the change of parametrization $(\phi^i, \partial\phi^i, \ldots) \leftrightarrow (x_A, L_a)$ and its inverse are polynomial. □.

An analogous vanishing theorem for the characteristic cohomology of an exterior differential system, which probably encompasses theorem 8.3, has been derived in [34].

Comment:

As a side comment, we note that the result (8.5) extends to the other rows of the variational bicomplex[1]: by theorem 8.2, theorem8.3 is equivalent to the statement that $H^i_0(d|\delta) = 0$ for $i \leq n - q - 1$ for theories of Cauchy order q. This corresponds, in the terminology of the variational bicomplex for differential equations [18], to the exactness of the bottom row of this complex up to horizontal degree $n - q - 1$.

Now, it is straightforward to check that the same result remains true for any other row with vertical degree s different from zero. Indeed, in the proof of theorem 8.3, one has to replace b by a linear combination of b's multiplied by s of the generators $d_V\phi^i, d_V\phi^i_{,\mu}, d_V\phi^i_{,\mu_1\mu_2}, \dots$, whereas a becomes a linear combination of a's linear in the antifields and their derivatives multiplied by s of the above generators plus a linear combination of antifield independent a's multiplied by $s - 1$ of the above generators and one of the generators $d_V\phi^*_i, d_V\phi^*_{i,\mu}, d_V\phi^*_{i,\mu_1\mu_2}, \dots$. Using the split of the fields and their derivatives into L_a and x_A, one can choose b to depend only on x_A and $d_V x_A$. Indeed, both L_a and its vertical derivative $d_V L_a$ can be removed from b since they are δ-exact ($d_V\delta + \delta d_V = 0$). Since d_V and $d \equiv d_H$ (respectively $\bar{\bar{d}}$) anticommute, $\bar{\bar{d}}b$ will also only depend on x_A and $d_V x_A$. Because $H^i(\bar{\bar{d}}) = 0$ in vertical degree s, the proof of theorem 8.3 goes through exactly in the same way.

This means that the variational bicomplex for differential equations of Cauchy order q is exact up to order $n - q - 1$ (all the columns for this bicomplex are exact like in the case of the free complex [18]). This question will be developed further in [31].

9 Linear gauge theories

The vanishing theorem 8.3 for the δ-cohomology modulo d can be derived, in the case of linear gauge theories and perturbations of them (in a sense to be made precise), under different conditions. The techniques necessary for deriving this alternative vanishing theorem are quite useful and based on the well known fact that the Euler-Lagrange derivatives of a divergence $\partial_\mu j^\mu$ identically vanish.

[1]We thank Niky Kamran for asking us this question.

Theorem 9.1 : *for a linear gauge theory of reducibility order r, one has,*

$$H^n_j(\delta|d) = 0, \qquad j > r+2 \tag{9.1}$$

whenever j is strictly greater than $r+2$ (we set $r=-1$ for a theory without gauge freedom).

Proof. assume for definiteness $r = 0$ (irreducible gauge theory). The case of arbitrary r is treated along identical lines. Since the equations are linear, one has

$$\frac{\delta^L \mathcal{L}}{\delta \phi^i} = D_{ij}\phi^j \tag{9.2}$$

where D_{ij} is a linear differential operator with field independent coefficients,

$$D_{ij} = \sum_{l\geq 0} d^{\mu_1\ldots\mu_l}_{ij}\partial_{\mu_1\ldots\mu_l}. \tag{9.3}$$

Similarily, the Noether identity reads

$$U^i_\alpha \frac{\delta^L \mathcal{L}}{\delta \phi^i} = 0 \tag{9.4}$$

with

$$U^i_\alpha = \sum_{k\geq 0} u^{i\mu_1\ldots\mu_k}_\alpha \partial_{\mu_1\ldots\mu_k}. \tag{9.5}$$

Let a be a n-form solution of $\delta a + \partial_\mu b^\mu = 0$ (in dual notations), with *antighost* $a \geq 3$. By taking the Euler-Lagrange derivatives of this cycle condition with respect to C^*_α, ϕ^*_i and ϕ^i, one gets $(\delta(\partial_\mu b^\mu)/\delta(anything) = 0)$,

$$\delta\frac{\delta^R a}{\delta C^*_\alpha} = 0, \qquad \delta\frac{\delta a^R}{\delta \phi^*_i} - U^{i+}_\alpha \frac{\delta^R a}{\delta C^*_\alpha} = 0$$
$$\delta\frac{\delta^R a}{\delta \phi^i} - D^+_{ji}\frac{\delta^R a}{\delta \phi^*_j} = 0 \tag{9.6}$$

where

$$U^{i+}_\alpha = \sum_{k\geq 0}(-)^k u^{i\mu_1\ldots\mu_k}_\alpha \partial_{\mu_1\ldots\mu_k} \tag{9.7}$$

$$D^+_{ij} = \sum_{l\geq 0}(-)^l d^{\mu_1\ldots\mu_l}_{ij}\partial_{\mu_1\ldots\mu_l}. \tag{9.8}$$

Since the variational derivatives of a have non-vanishing antighost number ($antigh\ a \geq 3$), the relation $H_k(\delta) = 0$ ($k > 0$) implies, using the operator identity $U^i_\alpha D_{ij} = 0$ that follows from (9.4),

$$\frac{\delta^R a}{\delta C^*_\alpha} = \delta f^\alpha \tag{9.9}$$

$$\frac{\delta^R a}{\delta \phi^*_i} = U^{i+}_\alpha f^\alpha + \delta f^i \tag{9.10}$$

$$\frac{\delta^R a}{\delta \phi^i} = D^+_{ji} f^j + \delta f_i \tag{9.11}$$

for some f^α, f^j and f_i. The equations (9.9)-(9.11) are valid for any field configuration. Thus, we may replace in them the fields, the antifields and their derivatives by t times themselves, where t is a real parameter. For instance, $(\frac{\delta a}{\delta C^*_\alpha})(t) = (\delta f^\alpha)(t)$ with $F(t) \equiv F(t\phi^i, t\phi^*_i, tC^*_\alpha)$.

Now, one can reconstruct a from its Euler-Lagrange derivatives through the formula

$$a = \int_0^1 [\frac{\delta^R a}{\delta C^*_\alpha}(t) C^*_\alpha + \frac{\delta^R a}{\delta \phi^*_i}(t)\phi^*_i + \frac{\delta^R a}{\delta \phi^j}(t)\phi^j] dt + \partial_\mu k^\mu. \tag{9.12}$$

If one inserts (9.9)-(9.11) in (9.12) one gets, using the fact that δ does not depend on t because the equations of motion are linear ($(\delta x)(t) = \delta(x(t))$), that the cycle a is given by

$$a = \delta[(\int_0^1 f^\alpha(t)dt)C^*_\alpha - (\int_0^1 f^i(t)dt)\phi^*_i + (\int_0^1 f_i(t)dt)\phi^i] + \partial_\mu k'^\mu. \tag{9.13}$$

That is, a is δ-trivial modulo d as claimed above. $\Box$.

10 Normal theories

Theorem 9.1 can be extended to non linear theories under the condition that the linear part of the theory contains the maximum number of derivatives. We shall call such theories "normal theories". We shall first illustrate the concept in the case of the Yang-Mills field coupled to coloured multiplets, and we shall then define it in general.

The Lagrangian for the Yang-Mills field coupled to matter reads

$$\mathcal{L} = \frac{1}{8} tr F^{\mu\nu} F_{\mu\nu} + \mathcal{L}^y(y^i, D^y_\mu y^i) \tag{10.1}$$

with

$$D^y_\mu y^i = \partial_\mu y^i - g A^a_\mu T^i_{aj} y^j, \tag{10.2}$$
$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g C^a_{bc} A^b_\mu A^c_\nu. \tag{10.3}$$

Here, the T_a's are the generators of the representation to which the matter fields y^i belong. We shall consider for definiteness the case of Dirac fermions, $y^i \equiv (\psi^i, \bar{\psi}_i)$,

$$\mathcal{L}^y = \bar{\psi}_i \gamma^\mu D_\mu \psi^i + m^i_j \bar{\psi}_i \psi^j, \tag{10.4}$$

where m^i_j is the mass matrix, which commutes with T_a, $[m, T_a] = 0$, and we have absorbed a factor i in the definition of γ^μ.

The Koszul-Tate differential is given by

$$\begin{aligned} &\delta A^a_\mu = 0,\ \delta C^a = 0,\ \delta y^i = 0, \\ &\delta A^{*\mu}_a = -D_\nu F^{\nu\mu}_a + g j^\nu_a,\ \delta C^*_a = -D_\mu A^{*\mu}_a + g T^j_{ai} y^*_j y^i, \\ &\delta \bar{\psi}^{*i} = -\gamma^\mu D_\mu \psi^i - m^i_j \psi^j,\ \delta \psi^*_i = -D_\mu \bar{\psi}_i \gamma^\mu + \bar{\psi}_j m^j_i. \end{aligned} \tag{10.5}$$

One can split both the Lagrangian and the Koszul-Tate differential into free and interacting pieces,

$$\mathcal{L} = \mathcal{L}^{free} + \mathcal{L}^{int}, \tag{10.6}$$
$$\delta = \delta^{free} + \delta^{int}. \tag{10.7}$$

A crucial feature of the free Lagrangian is that it contains the maximum number of derivatives, namely, two derivatives of A^a_μ and one derivative of ψ^i. The interaction vertices have at most one derivative of A^a_μ and no derivative of ψ^i.

To formalize this property, we introduce a degree K defined as

$$K = N_\partial + A \tag{10.8}$$

where N_∂ is the derivation counting the number of derivatives of the fields and of the antifields,

$$N_\partial = \sum_{A,(k)} \frac{\partial^R}{\partial \phi^{A(k)}} |k| \phi^{A(k)}, \tag{10.9}$$

and where A is defined by

$$A = \sum_{(k)} \frac{\partial^R}{\partial(\partial^{(k)} A^{*\mu}_a)} 2\partial^{(k)} A^{*\mu}_a + \sum_{(k)} \frac{\partial^R}{\partial(\partial^{(k)} C^*_a)} 3\partial^{(k)} C^*_a + \sum_{(k)} \frac{\partial^R}{\partial(\partial^{(k)} y^*_i)} \partial^{(k)} y^*_i. \tag{10.10}$$

The differential δ splits into components of definite K-degree,

$$[K, \delta^j] = j\delta^j. \tag{10.11}$$

Since we have assigned A-weight 2 to the antifields associated with second order equations of motion and A-weight one to the antifields associated with first order equations of motion, the differential δ contains only components of non positive K-degree. Explicitly, one has

$$\delta = \delta^0 + \delta^{-1} + \delta^{-2} \tag{10.12}$$

Similarly, one gets

$$\delta^{free} = \delta^{free,0} + \delta^{free,-1}. \tag{10.13}$$

The derivation $\delta^{free,0}$ is simply the mass-independent piece of δ^{free}, $(\delta^{free})_{m=0} = \delta^{free,0}$. In addition, the zeroth component of δ coincides with the zeroth component of δ^{free} since the free part of the equations of motion contains the maximum number of derivatives.

The differential $\delta^{free,0} \equiv \delta^0$, like δ^{free} and δ, is acyclic at positive antighost number. Thus, if the local q-form a (i) has positive antighost number; (ii) is δ-closed (respectively δ^{free}-closed); and (iii) has no component of K-degree higher than k, then $a = \delta b$ (respectively, $a = \delta^{free} b$), where b has also no component of K-degree higher than k.

One easily verifies that

$$[K, \partial_\mu] = \partial_\mu. \tag{10.14}$$

Furthermore, if a has K-degree k, then, $\delta a/\delta A^a_\mu$ and $\delta a/\delta y^i$ have K-degree k, $\delta a/\delta y^*_i$ has K-degree $k-1$, $\delta a/\delta A^{*\mu}_a$ has K-degree $k-2$, while $\delta a/\delta C^*_a$ has K-degree $k-3$. Finally, any q-form with bounded K-degree is necessarily a polynomial in the derivatives of the fields, the antifields and their derivatives, since these variables have all strictly positive K-degree. It may, however, be an infinite series in the undifferentiated fields, which carry zero K-degree.

The existence of a K-degree with the above properties is the characteristic feature of the so-called "normal theories". This concept applies to reducible or irreducible gauge theories and is captured as follows. Let $\mathcal{L}(\phi, \partial\phi, \partial^2\phi, \dots, \partial^s\phi)$ be the Lagrangian of a theory,

$$\mathcal{L} = \mathcal{L}^{free} + \mathcal{L}^{int}. \tag{10.15}$$

The free term is quadratic in the fields and their derivatives. We shall say that (10.15) describes a normal theory iff

(i) the free theory and the full theory have the same number of gauge invariances, with the same reducibility properties, so that δ^{free} is acyclic at positive antighost number (with the antifield spectrum of the full theory);

(ii) it is possible to define an even derivation K along the lines of the Yang-Mills case, which is the sum of the operator counting the derivatives and an operator A commuting with ∂_μ, $K = N_\partial + A$. The even derivation A should assign strictly positive degree to the antifields, and non negative degree to the fields ϕ^i. The even derivation K should be such that the differential δ^{free} has only components of non positive K-degree

$$\delta^{free} = \sum_t \delta^{free,t}, \qquad [K, \delta^{free,t}] = t\delta^{free,t}, \qquad t \leq 0, \tag{10.16}$$

and

$$[K, \partial_\mu] = \partial_\mu. \tag{10.17}$$

Furthermore, the zeroth order differential $\delta^{free,0}$ should be acyclic at positive antighost number ; as in the Yang-Mills case, the A-weight of the antifields ϕ^*_i is determined by the differential order of the corresponding free equations of motion ; the A-weight of the antifields C^*_α (and C^*_Δ if any) is determined by the differential order of the corresponding reducibility identities and the A-weight of the previous antifields ϕ^*_i (or C^*_α) ;

(iii) finally, the interacting part of δ must contain only terms of non positive K-degree,

$$\delta^{int} = \sum_t \delta^{int,t}, \qquad [K, \delta^{int,t}] = t\delta^{int,t}, \qquad t \leq 0. \tag{10.18}$$

This condition expresses that there are at most as many derivatives in δ^{int} as there are in δ^{free}. Note that we do not require $\delta^{int,0}$ to vanish, but that the sum $\delta^0 = \delta^{free,0} + \delta^{int,0}$ is always acyclic because of (ii).

It may happen that condition (ii) is fulfilled only after one has redefined the fields. Einstein gravity (with or without cosmological constant) is a normal theory, characterized by a non-vanishing $\delta^{int,0}$. Even though one can split the derivatives as in section 3, a theory that is not a normal theory is

$$\mathcal{L} = \phi \Box^2 \phi + (\partial_\mu \phi \partial^\mu \phi)^{10} \tag{10.19}$$

since the interaction vertices contain 20 derivatives while the free part contains only 4 derivatives.

Let a be a solution of $\delta a + db = 0$, with antighost number ≥ 3 and bounded K-degree. Then, a is a polynomial in the antifields, their derivatives and the derivatives of the fields, with coefficients that may be infinite series in the undifferentiated fields. Let us expand a according to its polynomial degree, $a = a_2 + a_3 + a_4 + ...$ The lower index denotes the polynomial degree of a (not the K-degree) and the series terminates if a is polynomial in the undifferentiated fields. The first term is at least quadratic because we assume the antighost number of a to be ≥ 3. The term of degree 2 in the cocyle condition reads

$$\delta^{free} a_2 + db_2 = 0. \tag{10.20}$$

By theorem 9.1, this implies

$$a_2 = \delta^{free} c_2 + de_2. \tag{10.21}$$

Thus, $a - \delta c_2 - de_2$ has no quadratic piece and reads $a - \delta c_2 - de_2 = a'_3 + a'_4 + \ldots$. That is, one can remove a_2 through the addition of δ-exact modulo d terms.

One can repeat the argument to remove successively a_3, $a_4, \ldots$. This shows that a is δ-trivial modulo d, $a = \delta c + de$. The conclusion is correct, however, only if one can prove that the procedure does not introduce arbitrarily high derivatives of the variables. The question is not entirely straightforward because when one removes a_i, one generically modifies the next terms a_{i+1} and a_{i+2}. Thus, even if $a_{i+1} = a_{i+2} = 0$ originally, one may have $a_{i+1} \neq 0$ and $a_{i+2} \neq 0$ after a_i has been set equal to zero by the addition of a δ-exact modulo d term.

It is here that the fact that $\mathcal{L}^{free}$ contains the maximum number of derivatives, or more precisely, that the theory is a normal theory, plays a crucial role. Indeed, the components of a are bounded in K-degree, let us say by k. Then the reconstruction formula (9.13) and (9.9)-(9.11) show that the K-degree of c_2, given by the first terms in the right-hand side of (9.13), cannot exceed k. It then follows that the K-degree of e_2 cannot exceed $k-1$ since $[K, \partial_\mu] = \partial_\mu$. Therefore, the term $\delta^{int} c_2$, which modifies a_3, a_4, etc has K-degree smaller than (or equal to if $\delta^{int,0}$ does not vanish) k. The same reasoning applies next to c_3, e_3, c_4, $e_4, \ldots$. We can thus conclude that the K-degree of c, respectively e, does not exceed k, respectively $k-1$. Thus, c and e are polynomial in the derivatives of the fields, the antifields, and their derivatives. Moreover, if $\delta^{int,0}$ is absent and if the initial a is a polynomial of order L in all the variables and their derivatives, $a = a_2 + a_3 + \ldots + a_L$, then the process of successively eliminating a_2, $a_3, \ldots$ stops after at most $L + k$ steps. This is because the K-degree strictly decreases at each step (it cannot remain equal to k). Accordingly, c and e are polynomial not just in the derivatives, but also in the undifferentiated variables.

We have thus established:

Theorem 10.1 : *let $\mathcal{L}$ be the Lagrangian of a normal, reducible gauge theory of order r. Then*

$$H_k(\delta|d) = 0 \tag{10.22}$$

for $k > r + 2$ in the space of forms with coefficients that are polynomials in the differentiated variables and the antifields, and formal series in the undifferentiated fields.

Theorem 10.2 : *If, in addition, $\delta^{int,0} = 0$, then*

$$H_k(\delta|d) = 0 \qquad (10.23)$$

for $k > r + 2$ in the space of forms with coefficients that are polynomials in all the variables and their derivatives.

Theorem 10.1 applies to gravity, where the infinite series $g^{\mu\nu} = \eta^{\mu\nu} - h^{\mu\nu} + \dots$ are allowed; while Theorem 10.2 applies to Yang-Mills theory.

Since reducible gauge theories of order r are usually not only normal theories, but also theories of Cauchy order $r + 2$, Theorems 10.1 (or 10.2) and 8.3 are equivalent in practice. However, the conditions under which they apply may in principle be different (see (10.19) and section 14 below on shift symmetry) and so, Theorems 10.1 (or 10.2) and 8.3 are in those cases inequivalent.

Finally, we point out that it would be of interest to extend the conditions under which the perturbative argument behind Theorems 10.1 and 10.2 applies.

11 Results on $H_2^n(\delta|d)$

It follows from the above analysis that $H_j(\delta|d)$ vanishes whenever $j > 2$ for the usual irreducible gauge theories. We shall now establish some general theorems on $H_2^n(\delta|d)$.

Let a be a representative of $H_2^n(\delta|d)$. By adding a δ-exact term and a total derivative if necessary, one has

$$a = f^\alpha C^*_\alpha + \mu \qquad (11.1)$$

where f^α may be assumed to depend on x_A only and where μ is quadratic in the antifields ϕ^*_i and their derivatives.

Theorem 11.1 : *a necessary condition for a to be a δ-cycle modulo d is that f^α be the parameter of a gauge transformation that leaves the fields invariant on-shell,*

$$f^\alpha R^i_\alpha + \partial_\mu f^\alpha R^{i\mu}_\alpha + \dots + \partial_{\mu_1 \dots \mu_k} f^\alpha R^{i\mu_1 \dots \mu_k}_\alpha \approx 0 \qquad (11.2)$$

("global reducibility identity").

Proof. the proof is direct. One has

$$\delta a = (f^\alpha R^i_\alpha + \partial_\mu f^\alpha R^{i\mu}_\alpha + \ldots + \partial_{\mu_1 \ldots \mu_k} f^\alpha R^{i\mu_1 \ldots \mu_k}_\alpha)\phi^*_i + \delta\mu + \partial_\mu k^\mu \tag{11.3}$$

where $\delta\mu$ vanishes on-shell. The Euler-Lagrange derivative of $\delta a + db = 0$ with respect to ϕ^*_i yields then (11.2), as desired. □.

Thus, if there is no solution f^α to (11.2), one may assume that a is quadratic in the antifields ϕ^*_i and their derivatives. This occurs in electromagnetism with charged matter fields since then (11.2) reads

$$\partial_\mu f \approx 0, \qquad ief\psi \approx 0 \tag{11.4}$$

from which it follows that $f \approx 0$ ($e \neq 0$) and thus $f = 0$ (f depends only on x_A). This also occurs in (i) Yang-Mills theory with a semi-simple gauge group, for which (11.2) becomes

$$D_\mu f^a \equiv \partial_\mu f^a - gC^a_{bc} A^b_\mu f^c \approx 0 \tag{11.5}$$

which has no solution $f^a(A^a_\mu, \partial_\rho A^a_\mu, \ldots)$ besides $f^a = 0$; and (ii) Einstein gravity, for which (11.2) reads

$$\xi_{\alpha;\beta} + \xi_{\beta;\alpha} \approx 0 \tag{11.6}$$

which again has no solution $\xi_\alpha(g_{\rho\sigma}, \partial_\lambda g_{\rho\sigma}, \ldots)$ besides $\xi_\alpha = 0$ (a generic metric has no Killing vectors ; thus $\xi_\alpha(g_{\rho\sigma}, \partial_\lambda g_{\rho\sigma}, \partial_{\lambda\mu} g_{\rho\sigma}, \ldots) = 0$ for generic $g_{\rho\sigma}$'s and by continuity, $\xi_\alpha = 0$).

Consider now a solution μ of $\delta\mu + \partial_\mu b^\mu = 0$ which is purely quadratic in the antifields and their derivatives.

Theorem 11.2 : *for linear gauge theories, there is no nontrivial element of $H^n_2(\delta|d)$ that is purely quadratic in the antifields ϕ^*_i and their derivatives. That is, if μ is quadratic in the antifields ϕ^*_i and their derivatives and if $\delta\mu + \partial_\mu b^\mu = 0$ then $\mu = \delta C + \partial_\mu V^\mu$.*

Proof. the proof proceeds as the proof of theorem 9.1. One computes first the variational derivatives of μ with respect to ϕ^*_i and ϕ^i from the cycle condition $\delta\mu + \partial_\mu b^\mu = 0$. One then reconstructs μ by a formula analogous to (9.13) recalling that $\delta\mu/\delta C^*_\alpha = 0$ since μ is purely quadratic in the antifields ϕ^*_i and their derivatives. This yields immediately the desired result. □.

Again, theorem 11.2 can be extended to non linear, normal theories, as the perturbative argument of the previous section indicates.

12 Calculation of $H_2^n(\delta|d)$ for electromagnetism

As we have seen, electromagnetism and Yang-Mills models have Cauchy order 2. Hence, only $H_2^n(\delta|d) \simeq H_1^{n-1}(\delta|d)$ can be different from zero besides $H_1^n(\delta|d)$. The explicit calculation of $H_1^n(\delta|d)$ is a difficult question that depends explicitly on the model under consideration and its rigid symmetries. It turns out that, by contrast, the calculation of $H_2^n(\delta|d)$ can be carried out completely. We first compute $H_2^n(\delta|d)$ for free electromagnetism. The Koszul-Tate differential acting on the undifferentiated generators reads explicitly

$$\delta A_\mu = \delta C = 0,\ \delta A^{*\mu} = -\partial_\rho F^{\rho\mu},\ \delta C^* = -\partial_\mu A^{*\mu}. \tag{12.1}$$

Theorem 12.1 : *For a free abelian gauge field A_μ, the groups $H_2^n(\delta|d)$ and $H_1^{n-1}(\delta|d)$ are one-dimensional. One can take as representatives:*

$$for\ H_2^n(\delta|d)\ : C^* dx^0 \wedge \ldots \wedge dx^{n-1} \tag{12.2}$$

$$for\ H_1^{n-1}(\delta|d)\ : \frac{1}{(n-1)!} A^{*\mu} \varepsilon_{\mu\alpha_1\ldots\alpha_{n-1}} dx^{\alpha_1} \wedge \ldots \wedge dx^{\alpha_{n-1}} \tag{12.3}$$

(or, in dual notations, C^* and $A^{*\mu}$, respectively).

Proof. we consider explicitly $H_2^n(\delta|d)$ and work in dual notations. By adding a divergence if necessary, any representative a of an element of $H_2^n(\delta|d)$ can be chosen to be of the form $a = C^* f(x_A) + \mu$ where μ depends on the fields and is quadratic in the antifields $A^{*\mu}$. One has $\delta a + \partial_\mu b^\mu = 0$ for some b^μ. By theorem 11.1, the function f must fulfill $\partial_\mu f \approx 0$, i.e., $df + \delta k = 0$ for some k of antighost number 1 and form degree 1. But $H_0^0(d|\delta)/\mathbf{R} \simeq H_1^1(\delta|d)$ vanishes (we assume the spacetime dimension to be strictly greater than 2). Accordingly, f is trivial, $f \approx const.$ Since f does not involve the equations of motion, this forces $f = const.$ (strongly).

The cycle fC^* with $f = const.$ is a solution of $\delta(fC^*) + \partial_\mu(fA^{*\mu}) = 0$ by itself. By substracting it from a, one may assume $a = \mu$ where μ is quadratic in the antifields $A^{*\mu}$. Using theorem 11.2, one then finds that μ is δ-trivial modulo d and does not contribute to the cohomology. This completes the demonstration of the theorem. □.

Remarks: (i) The solutions (12.2) and (12.3) are non trivial because they do not contain derivatives of the fields (while a trivial term $\delta m + dn$ contains necessarily derivatives).

(ii) The condition $n > 2$ is essential. For $n = 2$ there are other non- trivial cocycles. For instance, $a = \frac{1}{2}\varepsilon_{\mu\nu}F^{\mu\nu}C^* + \frac{1}{2}\varepsilon_{\mu\nu}A^{*\mu}A^{*\nu}$ fulfills $\delta a + db = 0$ and is non trivial because it contains $F^{\mu\nu}$, C^* and $A^{*\mu}$ undifferentiated. To analyze completely the 2- dimensional case, we use the chain of isomorphisms $H_2^2(\delta|d) \simeq H_1^1(\delta|d) \simeq H_0^0(d|\delta)/\mathbf{R}$. Thus it is enough to find all non-trivial 0-forms f that are closed (constant) modulo the equations of motion. These 0-forms may be assumed to depend on the x_A only. It is convenient at this stage to adopt a different parametrisation of the field variables: one may express any function of the A_μ and their derivatives in terms of A_μ, its symmetrized derivatives $A_{\mu_1\ldots\mu_k} = \partial_{(\mu_1}\ldots\partial_{\mu_{k-1}}A_{\mu_k)}$ (which are independent), $F_{01} \equiv \frac{1}{2}\varepsilon_{\alpha\beta}F^{\alpha\beta} \equiv^* F$ and its derivatives (which are also independent, the Bianchi identities are empty in two dimensions). The equations of motion set the derivatives of $^*F$ equal to zero and leave A_μ and its symmetrized derivatives free. We may assume $f = f(A_\mu, A_{\mu\nu}, \ldots, A_{\mu_1\ldots\mu_k}, F_{01})$ since $\partial_\rho F_{01} \approx 0$. Impose now $\partial_\rho f \approx 0$. One has $\partial_\rho f = (\partial f/\partial A_\mu)\partial_\rho A_\mu + \ldots + (\partial f/\partial A_{\mu_1\ldots\mu_k})\partial_\rho A_{\mu_1\ldots\mu_k} + (\partial f/\partial F_{01})\partial_\rho F_{01}$. The last term is weakly zero. Since $\partial_\rho A_{\mu_1\ldots\mu_k} \approx \frac{1}{k+1}A_{\rho\mu_1\ldots\mu_k}$ and since $A_{\rho\mu_1\ldots\mu_k}$ is unconstrained by the equations of motion, the requirement $\partial_\rho f \approx 0$ imposes $\partial f/\partial A_{\mu_1\ldots\mu_k} \approx 0$, i.e., $\partial f/\partial A_{\mu_1\ldots\mu_k} = 0$ (f does not involve the derivatives of F_{01}). Thus, f cannot depend on the symmetrized derivatives of order k. Similiarily, it cannot depend on the symmetrized derivatives of order $k-1$, etc ..., i.e., $f = f(F_{01})$. Any function of F_{01} is a solution of the problem and is non trivial. Accordingly, $H_0^0(d|\delta) \simeq C^\infty(F_{01})$. Note that $H_0^0(d|\delta) \simeq H^0(d, H_0(\delta))$ and is an algebra.

13 Calculation of $H_2^n(\delta|d)$ for Yang-Mills models and gravity

The previous section shows that $H_2^n(\delta|d)$ is non empty for free electromagnetism, because there is then a global reducibility identity on the gauge transformations[2]. Now, there is no global reducibility identity when self-couplings or couplings to matter are included. Indeed, gauge transformations leaving the Yang-Mills field A^a_μ and the matter fields y^i invariant should fulfill

$$D_\mu f^a \approx 0, \; f^a (T_a)^i_j y^j \approx 0, \tag{13.1}$$

whose only solution $f^a(A^a_\mu, \partial_\rho A^a_\mu, \dots, y^i, \partial_\mu y^i, \dots)$ is $f^a \approx 0$, and thus $f^a = 0$ if one assumes - as one can - that f^a involves only the x_A. Accordingly, by theorem 11.1, an element of $H^n_2(\delta|d)$ should be quadratic in the antifields of antighost number 1. Since the Yang-Mills theory is a normal theory, theorem 11.2 implies then that $H^n_2(\delta|d)$ is empty. Non-vanishing cohomology arises only when there are uncoupled abelian factors since equation (13.1) has then non trivial solutions.

We have thus proved

Theorem 13.1 : *In spacetime dimensions ≥ 3, the group $H^n_2(\delta|d)$ vanishes unless there are free abelian gauge fields A^α_μ. In that case, a basis of δ-cycles modulo d is given by*

$$f^\alpha C^*_\alpha, \; f^\alpha = const. \tag{13.2}$$

where the C^*_α are the antifields of antighost number 2 associated with the uncoupled abelian factors.

The same reasoning applies also to Einstein gravity: linearized gravity has ten global reducibility identities corresponding to the ten Killing vectors of Minkowski space. These define cohomological classes of $H^n_2(\delta|d)$. The full theory, however, has no global reducibility identity (a generic solution of Einstein equations has no Killing vector). Thus $H^n_2(\delta|d)$ is empty in Einstein gravity.

Comments: (i) It follows from $H^n_2(\delta|d) = 0$ and the isomorphism $H^n_2(\delta|d) \simeq H^{n-2}_0(d|\delta)$ that there is no local $(n-2)$-form that is closed modulo the equations of motion for generic Yang-Mills models, or Einstein gravity.

(ii) In two dimensions, one may compute $H^2_2(\delta|d) \simeq H^0_0(d|\delta)/\mathbf{R}$ directly. The analysis proceeds as in the Abelian case. The equations of motion are

[2] There is no contradiction between the fact that electromagnetism is a irreducible gauge theory because gauge transformations should vanish at infinity. This kills constant gauge parameters. However, in analyzing $H(\delta|d)$, no boundary condition is imposed.

equivalent to $D_\mu F^a_{01} = 0$ (in the absence of coupling to matter) and thus, eliminate the derivatives of F^a_{01}. Any function $f(A^a_\mu, A^a_{\mu\nu}, \dots, A^a_{\mu_1\dots\mu_k}, F^a_{01})$ solution of $df + \delta m = 0$ must be gauge invariant since $\gamma df + \gamma\delta m = 0$ implies $\gamma m = \delta u + dv$ and hence $\gamma f - \delta v = 0$, i.e., $\gamma f = 0$ (f does not contain the derivatives of F^a_{01}). This means that f must be an invariant function of F^a_{01}. Any such function fulfills $df \approx 0$ and is thus a solution. That is, $H^0_0(d|\delta)$ is isomorphic to the set of invariant functions of F^a_{01} in the absence of matter. When couplings to matter are included, however, $H^0_0(d|\delta)/\mathbf{R}$ is generically zero dimensional.

14 Non minimal sector - Shift symmetry

In order to fix the gauge, it is often useful to add further variables known as the "variables of the non-minimal sector". This procedure is physically acceptable because these variables do not modify the BRST cohomology $H^*(s)$ [10]. We show that they do not modify the local BRST cohomology $H^*(s|d)$ either.

The standard non minimal sector contains the "antighosts" $\bar{C}_a$, the auxiliary fields b_a and the corresponding antifields $\bar{C}^{*a}$ and b^{*a}. The action of the BRST differential on those variables is

$$s\bar{C}_a = b_a,\ sb_a = 0,\ s\bar{C}^{*a} = 0,\ sb^{*a} = -\bar{C}^{*a}. \tag{14.1}$$

This is the caracteristic form of a contractible differential algebra. As usual, one extends s to the derivatives of $\bar{C}_a$, b_a, $\bar{C}^{*a}$ and b^{*a} so that $s\partial_\mu = \partial_\mu s$.

The triviality of s in the non minimal sector is proved by introducing the contracting homotopy ρ,

$$\rho = \sum_{k\geq 0}[\frac{\partial^R}{\partial(\partial_{\mu_1\dots\mu_k} b_a)}\partial_{\mu_1\dots\mu_k}\bar{C}_a - \frac{\partial^R}{\partial(\partial_{\mu_1\dots\mu_k}\bar{C}^{*a})}\partial_{\mu_1\dots\mu_k} b^{*a}] \tag{14.2}$$

such that $\rho s + s\rho = \tilde{N}$, where $\tilde{N}$ counts the number of variables of the non minimal sector and their derivatives. The crucial feature of the contracting homotopy ρ is that it commutes with ∂_μ, $\rho\partial_\mu = \partial_\mu\rho$. Thus, it anticommutes with d,

$$\rho d + d\rho = 0. \tag{14.3}$$

The existence of a contracting homotopy that commutes with ∂_μ follows from the fact that the action of s on the fields of the non minimal sector does not increase the order of the derivatives. Such a homotopy does not exist for δ in the minimal sector whenever $H_k(\delta|d)$ is non trivial [17].

Because of (14.3), one may easily establish that the non minimal sector does not contribute to $H(s|d)$. Let a be a solution of $sa+db=0$. Decompose a according to the its $\tilde{N}$-degree, $a = \Sigma_{k\geq 0} a_k$. The term a_0 does not contain the variables of the non-minimal sector. From $sa+db=0$, one infers

$$a - a_0 = \Sigma_{k\geq 1} \frac{\tilde{N} a_k}{k} = \Sigma_{k\geq 1} (\rho s + s\rho) \frac{a_k}{k} = s(\Sigma_{k\geq 1} \rho \frac{a_k}{k}) + d(\Sigma_{k\geq 1} \rho \frac{b_k}{k}). \quad (14.4)$$

Hence, $a - a_0$ is s-exact modulo d and can be removed from $H(s|d)$.

We have thus proved

Theorem 14.1 : *The variables of the non minimal sector do not contribute to $H(s|d)$: one can remove the variables of the non minimal sector from any s-cocycle modulo d by adding to it a s-boundary modulo d.*

In particular, in Yang-Mills theory, one may analyze the local BRST cohomology in terms of the original variables of the "minimal sector" A^a_μ, C^a, $A^{*\mu}_a$, C^*_a, y^i, y^*_i introduced above.

A similar analysis applies to gauge symmetries not involving the derivatives of the fields, such as the "shift symmetry" of [11]. Consider a gauge theory such that (i) the fields φ^i split into two groups $\varphi^i \equiv (e^a, \omega^\alpha)$; and (ii) the gauge transformations are mere translations in the ω^α,

$$\delta_\varepsilon e^a = 0 \quad (14.5)$$

$$\delta_\varepsilon \omega^\alpha = \varepsilon^\alpha. \quad (14.6)$$

The Lagrangian depends only on the e^a and their derivatives. We shall call (14.5), (14.6) "shift symmetry" because (14.6) is just a shift of the ω's. An example (but not the only one) of such a gauge theory is obtained by replacing the fields ϕ^i by $\phi^i - \psi^i$ in a theory without gauge invariance. The Lagrangian is then invariant under the shifts $\delta_\varepsilon \phi^i = \varepsilon^i$, $\delta_\varepsilon \psi^i = \varepsilon^i$. The redefinition $e^i = \frac{1}{2}(\phi^i - \psi^i)$, $\omega^i = (\phi^i + \psi^i)$ brings the theory to the desired form (14.5), (14.6). Since this redefinition is invertible and local, it does not modify $H^*(s|d)$. For instance, if one starts with the Klein-Gordon theory, one gets the Lagrangian $\mathcal{L}(e,\omega,\partial e,\partial\omega) \equiv -\frac{1}{2}\partial_\mu e \partial^\mu e$. The theory has Cauchy order one (the x_A are e, $\partial_\rho e$ and their spatial derivatives together with ω and its derivatives). It is also a linear irreducible gauge theory. Thus, by theorem 9.1, $H_j(\delta|d) = 0$ for $j > 2$. This is strengthned by theorem 8.3 to $H_j(\delta|d) = 0$ for $j > 1$ (Cauchy order 1).

The BRST cohomology of the shift symmetry can be completely computed. Indeed, the BRST transformation for (14.5), (14.6) reads

$$se^a = 0, \ se^*_a = -\frac{\delta \mathcal{L}_0}{\delta e^a} \tag{14.7}$$

$$s\omega^\alpha = C^\alpha, \ sC^\alpha = 0, \ s\omega^*_\alpha = 0, \ sC^*_\alpha = \omega^*_\alpha. \tag{14.8}$$

The transformation (14.8) takes the same form as (14.1). Thus, the same argument shows that ω^α, C^α, ω^*_α and C^*_α do not contribute to the cohomology $H^*(s|d)$. Only the gauge invariant degrees of freedom e^a and e^*_a contribute to $H^*(s|d)$. In particular, one has

Theorem 14.2 : *The shift symmetry cannot be anomalous, $H^{1,n}(s|d) = 0$.*

These results can be straightforwardly extended to the case of a gauge group that is the direct product of a shift symmetry group by another group. One may always reshuffle terms in $H^{1,n}(s|d)$ so that the shift symmetry remains anomaly-free [35].

15 Auxiliary fields

The cohomological groups $H(s|d)$ and $H(\delta|d)$ are manifestly invariant under invertible, local change of variables. We shall now show that they are also invariant under the introduction of so called "auxiliary fields".

If the fields ϕ^i split as $(\phi^i) = (y^{\bar{\alpha}}, z^\alpha)$ where the z^α are such that the equations of motion $\delta S_0/\delta z^\alpha = 0$ can be solved for z,

$$\frac{\delta S_0}{\delta z^\alpha} = 0 \Longleftrightarrow z^\alpha = Z^\alpha(y^{\bar{\alpha}}, \partial_\mu y^{\bar{\alpha}}, \ldots, \partial_{\mu_1\ldots\mu_k} y^{\bar{\alpha}}), \tag{15.1}$$

where Z^α are local functions, one says that the z^α are "auxiliary fields". Given a theory with auxiliary fields, one defines the reduced action $\bar{S}_0[y]$

by eliminating the auxiliary fields from $S_0[y,z]$ using their own equations of motion

$$\bar{S}_0[y] = S_0[y, z = Z(y)]. \tag{15.2}$$

The theories based on $S_0[y,z]$ and $\bar{S}_0[y]$ are classically equivalent. They are also quantum-mechanically equivalent, at least formally [36].

Auxiliary fields can be useful for closing gauge algebras off-shell and occur at various places in physics. The conjugate momenta of the Hamiltonian formalism can be viewed as auxiliary fields. Other examples of auxiliary fields are the field strengths in the first order formulation of Yang-Mills theory,

$$S_0[A^a_\mu, H^a_{\mu\nu}] = \int d^n x\ H^a_{\mu\nu} H^{\mu\nu}_a + H^a_{\mu\nu}(\partial^\mu A^\nu_a - \partial^\nu A^\mu_a + f_{abc} A^{\mu b} A^{\nu c}). \tag{15.3}$$

In gravity, the Christoffel coefficients $\Gamma^\alpha_{\beta\gamma}$ in the Palatini formulation of the Hilbert action are auxiliary fields, as are the connexion components $\omega_{ab\mu}$ in the first order tetrad formalism.

For the subsequent discussion, it is useful to define $T[y,z]$ through

$$S_0[y,z] = \bar{S}_0[y] + T[y,z]. \tag{15.4}$$

The equations $\delta S_0/\delta z = 0$ coincide with $\delta T/\delta z = 0$ and one has [36]

$$\frac{\delta T}{\delta y^i(x)} = \int \mu^A_i(x,x') \frac{\delta T}{z^A(x')} dx' \tag{15.5}$$

where $\mu^A_i(x,x')$ is a combination of $\delta(x,x')$ and its derivatives.

The relationship between the BRST cohomologies of two formulations of the same theory differing in auxiliary field content is easily derived by following the approach of [36]. As shown in [36], one may redefine the gauge transformations in such a way that the gauge transformations $\delta_\varepsilon y^i$ of the theory with auxiliary fields coincide with the gauge transformations of the reduced theory and

$$\delta_\varepsilon z^A(x) = -\int \mu^A_i(x,x') \delta_\varepsilon y^i(x') dx'. \tag{15.6}$$

With that choice, a solution S of the master equation of the full theory is given by

$$S = \overset{(0)}{S} + \overset{(1)}{S} + \sum_{k\geq 2} \overset{(k)}{S} \tag{15.7}$$

where $\overset{(0)}{S}$ is a solution $\bar{S}$ of the master equation of the reduced theory and where $\overset{(1)}{S}$ is given by

$$\overset{(1)}{S} = -\int z^*_A(x')\mu^A_i(x',x)\frac{\delta \overset{(0)}{S}}{\delta y^*_i(x)}dxdx'. \tag{15.8}$$

The index k denotes the number of antifields z^*_A in $\overset{(k)}{S}$ (not the antighost number) and the terms $\overset{(k)}{S}$ ($k \geq 2$) are successively determined by the method of homological perturbation theory by equations of the form

$$\delta' \overset{(k+1)}{S} = \overset{(k)}{D} (\overset{(0)}{S}, \ldots, \overset{(k)}{S}), \qquad k = 1, 2, 3, \ldots \tag{15.9}$$

where (i) $\overset{(k)}{D}$ involves the antibracket of the S's of lower order ; and (ii) δ' (acting from the left like in [36]) is the Koszul-Tate resolution of the surface where the auxiliary fields are on-shell,

$$\delta' z^*_A = \frac{\delta S_0}{\delta z^A}, \qquad \delta'(everything\ else) = 0. \tag{15.10}$$

Because the equations of motion (15.10) for z^A are equivalent to algebraic equations $z^A = Z^A(y, \partial y, \ldots)$, the theory based on the equations (15.10), viewed as equations for z^A with fixed y's, is a normal theory of order 0. Indeed, there is no independent derivatives of z^A since they are all determined by the equations (15.10) and their derivatives. Thus the set I_0 (for the z's) is empty and $\partial_\alpha I_0$ is clearly contained in I_0. By theorem 8.3, one concludes that $H_k(\delta'|d) = 0$ for $k > 0$ (besides $H_k(\delta') = 0$).

It follows from the standard method of homological perturbation that the terms $\overset{(k+1)}{S}$ subject to (15.9) exist and can be taken to be local functionals. Similarily, let $\bar{A}[y, y^*]$ be a local functional solution of $(\bar{A}, \bar{S}) = 0$. Then there exist a functional A,

$$A = \bar{A} + \overset{(1)}{A} + \Sigma_{k\geq 2} \overset{(k)}{A} \tag{15.11}$$

$$\overset{(1)}{A} = -\int z^*_A(x')\mu^A_i(x',x)\frac{\delta \bar{A}}{\delta y^*_i(x)}dxdx'. \tag{15.12}$$

which solves $(A, S) = 0$. This functional is determined recursively by equations of the same form as (15.9),

$$\delta' \overset{(k+1)}{A} = \overset{(k)}{F} (\overset{(0)}{A}, \ldots, \overset{(k)}{A}) \tag{15.13}$$

where $\overset{(k)}{F}$ involves the brackets of the lower order $\overset{(i)}{A}$'s with the $\overset{(j)}{S}$. Again, each term in the expansion (15.12) is a local functional because $H_k(\delta'|d) = 0$. We have thus proved

Theorem 15.1 : *The BRST cohomology groups $H(s|d)$ and $H(\bar{s}|d)$ respectively associated with two different formulations of the same theory differing only in the auxiliary field content are isomorphic.*

This theorem is the analog for local functionals of the isomorphism theorem $H(s) \simeq H(\bar{s})$ that holds for local p-forms or arbitrary functionals. It can easily be extended to the generalized auxiliary fields introduced in [37] (see also [38]) as we now show.

Assume that the solution of the master equation $S[y, y^*, z, z^*]$ is such that the equations $\delta S/\delta z^A = 0$ can be solved at $z^*_A = 0$ for the z^A as functions of the y^i *and* y^*_i,

$$\frac{\delta S}{\delta z^A} |_{z^*=0} = 0 \Longleftrightarrow z^A = Z^A(y, \partial y, \ldots, y^*, \partial y^*, \ldots) \tag{15.14}$$

If this is the case, one says that the z^A are generalized auxiliary fields. Ordinary auxiliary fields are a particular case of (15.14) ; they do not depend on y^* because the equations $\delta S/\delta z^A |_{z^*=0} \equiv \delta T/\delta z^A = 0$ do not involve the antifields y^*_i or their derivatives. Generalized auxiliary fields occur in the transition from the total action to the extended action of the Hamiltonian formalism [37, 10]. They have properties quite similar to ordinary auxiliary fields. In particular, the relations that replace (15.4) and (15.5) are respectively

$$S[y, y^*, z, z^* = 0] = \bar{S}[y, y^*] + T'[y, y^*, z] \tag{15.15}$$

$$\frac{\delta T}{\delta y^i(x)} = \int \mu^A_i(x, x') \frac{\delta T}{z^A(x')} dx'$$

$$\frac{\delta T}{\delta y^*_i(x)} = \int \nu^{Ai}(x, x') \frac{\delta T}{z^A(x')} dx' \tag{15.16}$$

where $\bar{S}[y, y^*]$ is the solution of the master equation for the reduced theory obtained by setting $z^*_A = 0$ and eliminating z^A through (15.14).

Because the equations (15.14) are algebraic in z^A, one finds again that $H_k(\delta'|d) = 0$ for $k > 0$, where δ' is now defined through

$$\delta' z^*_A = \frac{\delta S}{\delta z^A} \mid_{z^*=0}, \qquad \delta'(\textit{everything else}) = 0. \tag{15.17}$$

The standard methods of homological perturbation then enable one to establish

Theorem 15.2 : *the generalized auxiliary fields do not modify $H_k(s|d)$. Namely, $H_k(\bar{s}|d) \simeq H_k(s|d)$, where (i) s is the BRST differential for the formulation with z and z^* present ; and (ii) $\bar{s}$ is the BRST differential for the formulation in which the fields z and z^* are eliminated through $\delta S/\delta z^A \mid_{z^*=0} = 0$, $z^*_A = 0$.*

Theorems 15.1 and 15.2 imply in particular that $H(s|d)$ is invariant under transition to the Hamiltonian formalism, provided that the inverse transformation that expresses the momenta and the Lagrange multipliers in terms of the velocities is local [10] (in space - it is of course always local in time).

Both theorems are valid in the space of infinite formal series in the antighost number. For the auxiliary fields that usually occur in practise, one may improve the results as follows. Standard auxiliary fields appear quadratically in the action, with coefficients that depend only on the fields but not on their derivatives, and which may be assumed to be constant under redefinition,

$$T = \frac{1}{2}(z^A - Z^A)(z^B - Z^B) g_{AB}. \tag{15.18}$$

The redefinition $z^A \to z'^A = z^A - Z^A(y, \partial y, \ldots)$ - which may be completed to a canonical transformation - enables one to write T as

$$T = \frac{1}{2} z'^A z'^B g_{AB}. \tag{15.19}$$

It is then straightforward to verify that the solutions of the master equations of the reduced and the unreduced theories are related as

$$S = \bar{S} + T \tag{15.20}$$

and that the BRST invariant function(al)s may be taken to coincide ;

$$A = \bar{A}. \tag{15.21}$$

Hence if $\bar{S}$ (respectively $\bar{A}$) is polynomial in the antighost number, then so is S (respectively A) and vice versa.

16 Conclusion

In this paper, we have derived some general theorems on the local BRST cohomological groups $H^k(s|d)$. We have established their link with the groups $H_k(\delta|d)$, which are in turn connected to the groups $H_0^k(d|\delta)$ of k-forms that are closed when the equations of motion hold. These groups are of interest in the study of the dynamics of the theory and have already been discussed from that point of view in the mathematical literature ("characteristic cohomology"). Our work makes thus a bridge between the local BRST cohomology and the characteristic cohomology.

We have also developed tools for calculating explicitely the groups $H_k(\delta|d)$ for $k > 1$. These tools include a vanishing theorem for $H_k(\delta|d)$ whenever k is strictly greater than the Cauchy order of the theory. By a perturbative argument, we have then proved that $H_2^n(\delta|d)$ vanishes for Yang-Mills theory and Einstein gravity. This theorem is equivalent to the absence of a non trivial 2-form that is closed modulo the equations of motion.

In a companion paper, we shall illustrate the usefulness of the theorems demonstrated here by computing explicitely all the cohomological groups $H^k(s|d)$ in Yang-Mills theory, with sources included.

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References

[1] I.A. Batalin and G.A. Vilkovisky, *Phys. Lett.* **B102** (1981) 27 ; *Phys. Rev.* **D28** (1983) 2567 ; *Phys. Rev.* **D30** (1984) 508.

[2] C. Becchi, A. Rouet and R. Stora, *Commun. Math. Phys.* **42** (1975) 127 ; *Ann. Phys.* (NY) **98** (1976) 287.

[3] I.V. Tyutin, *Gauge invariance in field theory and statistical mechanics*, Lebedev preprint FIAN, $n^0$39 (1975).

[4] J. Zinn-Justin, *Renormalisation of gauge theories* in *Trends in elementary particle theory*, Lecture notes in Physics $n^0$37 (Springer, Berlin 1975).

[5] R.E. Kallosh, *Nucl. Phys.* **B141** (1978) 141.

[6] B. de Wit and J.W. van Holten, *Phys. Lett.* **B79** (1978) 389.

[7] J.M.L. Fisch and M. Henneaux, *Commun. Math. Phys.* **128** (1990) 627.

[8] M. Henneaux, *Nucl. Phys. B (Proc. Suppl.)* **18A** (1990) 47.

[9] G. Hirsch, *Bull. Soc. Math. Belg.* **6** (1953) 79 ; J.D. Stasheff, *Trans. Am. Math. Soc.* **108** (1963) 215,293 ; V.K.A.M. Gugenheim, *J. Pure Appl. Alg.* **25** (1982) 197 ; V.K.A.M. Gugenheim and J.D. Stasheff, *Bull. Soc. Math. Belg.* **38** (1986) 237. The ideas of Homological Perturbation Theory have been applied to physics first in the Hamiltonian formulation of the BRST symmetry in J. Fisch, M. Henneaux, J. Stasheff and C. Teitelboim, *Commun. Math. Phys.* **127** (1989) 379.

[10] M. Henneaux and C. Teitelboim, *Quantization of Gauge Systems*, Princeton University Press (Princeton: 1992).

[11] J. Alfaro and P.H. Damgaard, *Nucl. Phys.* **B404** (1993) 751 ; see also F. De Jonghe, Ph. D. Thesis, K.U.L. (1994).

[12] J. Zinn-Justin, *Quantum Field Theory and Critical Phenomena*, 2^{nd} Edition, Clarendon Press (Oxford: 1993).

[13] P.S. Howe, U. Lindström and P. White, *Phys. Lett.* **B246** (1990) 430 ; W. Troost, P. van Nieuwenhuizen and A. van Proeyen, *Nucl. Phys.* **B333** (1990) 727.

[14] G. Barnich and M. Henneaux, *Phys. Lett.* **B311** (1993) 123.

[15] G. Barnich, F. Brandt and M. Henneaux, *Local BRST cohomology in the antifield formalism: II. Application to Yang-Mills theory*, preprint ULB-TH-94/07.

[16] B.L. Voronov and I.V. Tyutin, *Theor. Math. Phys.* **50** (1982) 218.

[17] M. Henneaux, *Commun. Math. Phys.* **140** (1991) 1.

[18] I.M. Anderson, *The variational bicomplex*, Academic Press (Boston: 1994) ; *Contemp. Math.* **132** (1992) 51.

[19] P.J. Olver, *Applications of Lie Groups to Differential Equations*, Graduate Texts in Mathematics, volume 107, Springer Verlag (New York: 1986).

[20] D.J. Saunders, *The Geometry of Jet Bundles*, London Mathematical Society Lecture Note Series 142, Cambridge University Press (Cambridge 1989).

[21] P. Mc Cloud, Class. Quant. Grav. **11** (1994) 567.

[22] I.M. Gel'fand and I. Ya. Dorfman, *Funct. Anal. Appl.* **13** (1979) 174.

[23] M. Henneaux, *Spacetime locality of the antifield formalism: a "mise au point"* preprint hep-th 9402068.

[24] M. Henneaux, *Phys. Lett.* **B 313** (1993) 35.

[25] A.M. Vinogradov, *Sov. Math. Dokl.* **18** (1977) 1200, **19** (1978) 144, **19** (1978) 1220 ; F. Takens, *J. Diff. Geom* **14** (1979) 543 ; M. De Wilde, *Lett. Math. Phys.* **5** (1981) 351 ; W.M. Tulczyjew, *Lecture Notes in Math.* **836** (1980) 22 ; P. Dedecker and W.M. Tulczyjew, *Lecture Notes in Math.* **836** (1980) 498 ; T. Tsujishita, *Osaka J.of Math.* **19** (1982) 311.

[26] F. Brandt, N. Dragon and M. Kreuzer, *Nucl. Phys.* **B332** (1990) 224.

[27] M. Dubois-Violette, M. Henneaux, M. Talon and C.M. Viallet *Phys. Lett.* **B267** (1991) 81.

[28] R.M. Wald, *J. Math. Phys.* **31** (1990) 2378.

[29] L.A. Dickey, *Contemp. Math.* **132** (1992) 307.

[30] E. Witten, *Mod. Phys. Lett.* **A 5** (1990) 487.

[31] G. Barnich, M. Henneaux and F. Brandt, *in preparation.*

[32] D. Bak, D. Cangemi and R. Jackiw, *Energy- momentum conservation in general relativity*, preprint CTP#2245, hep-th/9310025.

[33] C.G. Torre and I.M. Anderson, *Phys. Rev. Lett.* **70** (1993) 3525 ; *Classification of generalized symmetries for the vacuum Einstein equations I*, preprint gr-qc 9404030.

[34] R.L. Bryant and P.A. Griffiths, *Characteristic Cohomology of Differential Systems (I): General Theory*, Duke University Mathematics Preprint Series, volume 1993 n^{0}1 (January 1993).

[35] F. De Jonghe, R. Siebelink and W. Troost, *Phys. Lett.* **B306** (1993) 295 ; see also J. Gomis and J. París *Nucl. Phys.* **B395** (1993) 288.

[36] M. Henneaux, *Phys. Lett.* **B238** (1990) 299.

[37] A. Dresse, P. Grégoire and M. Henneaux, *Phys. Lett.* **B245** (1990) 192.

[38] F. Brandt, *Phys. Lett.* **B320** (1994) 57.