

ON THE CONFIGURATION SPACE OF GAUGE THEORIES

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Abstract. We investigate the structure of the configuration space of gauge theories and its description in terms of the set of absolute minima of certain Morse functions on the gauge orbits. The set of absolute minima that is obtained when the background connection is a pure gauge is shown to be isomorphic to the orbit space of the pointed gauge group. We also show that the stratum of irreducible orbits is geodesically convex, i.e. there are no geometrical obstructions to the classical motion within the main stratum. An explicit description of the singularities of the configuration space of SU(2) theories on a topologically simple space-time and on the lattice is obtained; in the continuum case we find that the singularities are con-

cal and that the singular stratum is isomorphic to a \mathbf{Z}_2 -orbifold of the configuration space of electrodynamics.

^ℓ Heisenberg fellow

1 The space of gauge orbits

For both phenomenological and conceptual reasons, gauge theories play a prominent rôle in physics. A basic object of interest is the full configuration space of a gauge theory; in the context of quantization by path integrals this space is relevant simply because it is the path integration domain, while in the Hamiltonian formalism one has to deal with wave functionals that are defined as functionals over the whole configuration space. The most impressive successes of gauge theories emerged in the context of perturbation theory. In order to analyze with similar success various nonperturbative aspects of gauge theories, such as the confinement problem, a detailed understanding of the geometry of the configuration space, including its global features, will presumably be necessary.

In this note we comment on the configuration space of gauge theories and on its description in terms the absolute minima of certain Morse functions. More specifically, we consider pure Yang–Mills theories. Such theories are already interesting in themselves; moreover, for more complicated theories, such as Yang–Mills coupled to matter without or even with spontaneous symmetry breaking, the configuration space contains the configuration space of pure Yang–Mills theory as a subspace, so that a detailed knowledge of the latter is again compulsory. The action functional of pure Yang–Mills theory is given by

$$4g^2 S_{\text{YM}}[A] = |F|^2 \equiv \int_M d^4x \operatorname{tr} (F_{\mu\nu} F^{\mu\nu}). \quad (1.1)$$

Here g is the coupling constant, $A = A_\mu dx^\mu$ is a connection 1-form which takes values in the Lie algebra $\mathfrak{g} = \operatorname{Lie}(G)$ of a finite-dimensional compact nonabelian Lie group G (called the *structure group* of the theory), and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$ (the Yang–Mills field strengths) are the components of the curvature 2-form $F = dA + A \wedge A$. In (1.1) we also introduced a norm $|\cdot|$ on the space of equivariant p -forms over space-time M with values in \mathfrak{g} ; this norm is defined as $|B|^2 = (B, B)$, where (\cdot, \cdot) is a scalar product on that space which is defined by contracting group indices with the Killing form of \mathfrak{g} and integrating over space-time,

$$(B, C) := \int_M \operatorname{tr} (B \wedge *C). \quad (1.2)$$

To be able to make the path integral quantization well-defined, we take the space-time manifold to have euclidean signature. In the present article we take M to be the four-sphere

$M = S^4$ which is the conformal one point compactification $S^4 = \mathbf{R}^4 \cup \{\infty\}$ of \mathbf{R}^4 . This compactification corresponds to imposing suitable asymptotic conditions at infinity; it has the additional benefit of giving space-time a finite volume. Also note that the choice $M = S^4$ is familiar from the discussion of instanton effects [1]. In particular, the choice of asymptotic conditions for the connections A corresponds to the choice of a specific (isomorphism class of) principal bundle P with fiber G over M . For $M = S^4$ and G a simple group, P is labelled by the instanton number $k \in \mathbf{Z}$; in the sequel we consider a fixed principal bundle P , and hence the instanton number has a fixed value.

In a Hamiltonian formulation, the action (1.1) leads to first class constraints. These imply that in order to obtain a consistent system with unique time evolution, we have to describe the system not on the space \mathcal{A} of all connections, but rather to factor out the *gauge group* \mathcal{G} . (By definition, \mathcal{G} is the group of automorphisms of the principal bundle P which get projected to the identity on M . Equivalently, \mathcal{G} can be described by sections of the ‘gauge bundle’ $P \times_G G$ over M , where the fiber G is endowed with the adjoint action of G on itself. In the latter description, locally any $g \in \mathcal{G}$ is just a map from space-time to the finite-dimensional structure group G .) Thus the true configuration space is the space

$$\mathcal{M} := \mathcal{A} / \mathcal{G} = \{\mathcal{O}_A \mid A \in \mathcal{A}\} \quad (1.3)$$

of gauge orbits

$$\mathcal{O}_A := \{B \in \mathcal{A} \mid B = A^g \text{ for some } g \in \mathcal{G}\}. \quad (1.4)$$

Here for any $g \in \mathcal{G}$

$$A^g = g^{-1} A g + g^{-1} dg \quad (1.5)$$

is the gauge transform of the connection A . The norm introduced in (1.1) is invariant under the action (1.5) of the gauge group, and hence in particular $S_{\text{YM}}[A] = S_{\text{YM}}[A^g]$ so that S_{YM} is well-defined on \mathcal{M} .

While \mathcal{A} is an affine space, the space \mathcal{M} has a rather complicated structure; in particular for generic choices of the underlying space-time manifold the topology of \mathcal{M} is non-trivial [2]. For various purposes (such as the construction of a well-defined Feynman–Kac path integral [3]) one needs on \mathcal{G} not only the group structure, but also a topological structure, so that it becomes an infinite-dimensional Lie group; technically, this can be accomplished by completion with respect to a suitable Sobolev norm on \mathcal{G} [4, 5, 6, 7]. It follows that when considering a non-compact space-time, in particular all constant gauge transformations except for the unit element $\mathbf{1}$ (defined by $\mathbf{1}(x) = \mathbb{1} \in G$ for all $x \in M$) have to be excluded. In our approach this is avoided by compactifying space-time to S^4 .

When coupling the Yang–Mills theory to chiral fermions, the non-trivial topology of the orbit space \mathcal{M} is the source of the chiral anomaly. Namely, it leads to a gauge dependence of the phase of the fermion determinant so that the determinant is well-defined only over \mathcal{A} and not over \mathcal{M} . In other words, in the case of anomaly free theories the relevant bundle, the

determinant bundle over \mathcal{M} , is trivial, while for a theory with a chiral anomaly it is non-trivial and does not admit global sections like the fermion determinant [8, 9].

The rest of this paper is organized as follows. The next three sections deal with various aspects of gauge fixing. We work with background gauges in which these aspects are particularly transparent and which in the geometric setting serve as a natural starting point. In section 2 we introduce the Gribov region and recall its description in terms of Morse functionals. If the background connection is reducible, then some aspects of the gauge fixing have to be treated with special care; this is described in section 3. In particular, it is explained how \mathcal{M} acquires the structure of a stratified variety. In section 4 we analyze the structure of the fundamental modular domains, which have the property that they are isomorphic to the orbit space up to boundary identifications. When the background connection is a pure gauge, then the set of absolute minima of the Morse functionals is isomorphic to the orbit space of the pointed gauge group; this is proven in section 5, while in section 6 we show that the main stratum is geodesically convex. The final section 7 contains a detailed description of the configuration space of $SU(2)$ Yang–Mills theory on S^4 ; we show that in this case \mathcal{M} decomposes in three strata, and that the strata corresponding to reducible connections form conical singularities and can be described as an orbifold of the configuration space of electrodynamics; we also point out the existence of reducible connections in the lattice version of the theory.

Among the central objects of our interest are the reducible connections. The presence of reducible connections implies that the configuration space \mathcal{M} of Yang–Mills theories is *not* a manifold, but contains singularities; the singular strata of \mathcal{M} are formed by the orbits of reducible connections. While the physical implications of these singularities are still unclear, it is worth emphasizing that they provide an intriguingly rich structure that certainly deserves further investigation. In this paper we obtain several new results which allow for a more detailed description of the singular strata. We are confident that these will ultimately be helpful in relating the singularities of the configuration space to physical effects.

2 Gauge fixing

The non-trivial topological properties of the orbit space \mathcal{M} make it difficult to describe \mathcal{M} directly. Accordingly for various purposes it is necessary to resort to the covering space \mathcal{A} of \mathcal{M} and identify in \mathcal{A} an appropriate region that upon dividing out \mathcal{G} projects bijectively on some open subset of \mathcal{M} . As we will see below, one can find subsets of \mathcal{A} which are isomorphic to \mathcal{M} modulo certain boundary identifications; any such subset of \mathcal{A} will be called a *fundamental modular domain* for \mathcal{M} .

A necessary requirement for identifying a fundamental region is to ‘fix a gauge’ in \mathcal{A} , i.e. to choose, in a continuous manner, representatives out of the gauge orbits \mathcal{O} . The gauge fixing we are going to use is implemented by means of a background gauge, which is the most natural

gauge condition from a geometric point of view. That is, we choose some arbitrary, but fixed, connection $\bar{A} \in \mathcal{A}$ as the background, and keep only those connections that belong to the affine subspace

$$\Gamma \equiv \Gamma_{\bar{A}} := \{A \in \mathcal{A} \mid \nabla_{\bar{A}}^*(A - \bar{A}) = 0\}. \quad (2.1)$$

Here ∇_A denotes the covariant derivative with respect to A , which acts on one-forms B as $\nabla_A(B) = dB + [A, B]$.

The restriction to Γ certainly reduces the ‘number’ of degrees of freedom of the system (which remains infinite, of course). However, as it turns out, the gauge fixing is not complete, i.e. generically more than one element of a gauge orbit \mathcal{O} satisfies the gauge condition (this is not an artefact specific to background gauges, but in fact happens for any continuous gauge fixing procedure [2, 10]). Thus the subspace Γ of \mathcal{A} together with the natural projection to orbits fails to provide a coordinate system of \mathcal{M} , i.e. Γ cannot yet serve as a fundamental modular domain. This can be made more precise; namely, gauge copies appear at least outside a certain convex subset $\Omega \equiv \Omega_{\bar{A}}$ of Γ that can be characterized [11] as the set of all minima of the functionals

$$\Phi_A \equiv \Phi_{\bar{A}; A} : \mathcal{G} \rightarrow \mathbf{R}_{\geq 0}, \quad \Phi_A[g] := |A^g - \bar{A}|^2 \quad (2.2)$$

for $A \in \mathcal{A}$. (The minima of Φ_A carry information about the topology of the gauge orbit \mathcal{O}_A through A . Accordingly, the functional Φ_A is called a Morse function [12].) Ω is known as the *Gribov* [13] *region*, and its boundary $\partial\Omega$ as the *Gribov horizon*. One can show that any gauge orbit \mathcal{O} intersects Ω at least once [11, 14], and that Ω is convex and bounded [11].

To establish some equations that characterize the set Ω more concretely, it is convenient to look [6] at a local one parameter family $g_t = e^{tw}$ of elements of the gauge group \mathcal{G} (thus $w = w(x)$ takes values in the Lie algebra \mathfrak{g} of G). For the first variation one finds

$$\frac{d}{dt} \Phi_A[g_t]_{t=0} = 2(A - \bar{A}, \nabla_A w) = -2(\nabla_{\bar{A}}^*(A - \bar{A}), w), \quad (2.3)$$

(the last expression is, in a first step, to be understood in the sense of weak derivatives). Thus the first variation of Φ_A is

$$\frac{\delta \Phi_A}{\delta g} = -2 \nabla_{\bar{A}}^*(A - \bar{A}) = -2 \nabla_{\bar{A}}^*(A - \bar{A}). \quad (2.4)$$

Thus in particular the definition (2.1) of Γ guarantees that the points $A \in \Gamma$ are characterized by the property that $g_t|_{t=0} \equiv \mathbf{1} \in \mathcal{G}$ is a stationary point of Φ_A .

Similarly we obtain for the second variation

$$\frac{d^2}{dt^2} \Phi[g_t]_{t=0} = 2(\nabla_A w, \nabla_A w) + 2(A - \bar{A}, \nabla_A w) = 2(\nabla_{\bar{A}} w, \nabla_A w) = -2(w, \nabla_{\bar{A}}^* \nabla_A w), \quad (2.5)$$

where again the last equality is to be read in the weak sense. The Hessian of the variation is thus given by the Faddeev–Popov operator

$$\Delta_{\text{FP}} \equiv \Delta_{\text{FP}}^{(\bar{A}, A)} := -\nabla_{\bar{A}}^* \nabla_A. \quad (2.6)$$

Note that

$$\nabla_A^* \nabla_{\bar{A}} = \nabla_{\bar{A}}^* \nabla_A \quad (2.7)$$

for all $A \in \Gamma$, and hence Δ_{FP} is symmetric. The determinant of Δ_{FP} is the usual Faddeev–Popov gauge fixing determinant, which is closely related [15] to a natural Riemannian metric on \mathcal{M} . According to (2.5) the positivity of the Faddeev–Popov operator Δ_{FP} ensures that the connection A possesses the property that $\mathbf{1} \in \mathcal{G}$ is a minimum of Φ_A .

3 Reducible connections

While the previous statements are valid for any choice of the background \bar{A} , some of the properties of Γ and Ω do depend on this choice. Namely, we have to distinguish between irreducible and reducible backgrounds. Here by an *irreducible* connection A we mean ¹ a connection for which the stabilizer (or isotropy subgroup)

$$\mathcal{S}_A = \{g \in \mathcal{G} \mid A^g = A\} \quad (3.1)$$

of the action of the gauge group is trivial, i.e. equal to the center $Z(G)$ of the structure group G (clearly, the constant gauge transformations corresponding to the elements of Z are in the stabilizer of any connection). It is a standard result in the theory of principal bundles that the stabilizer group \mathcal{S}_A of any connection A is isomorphic to the centralizer of the holonomy group $H \equiv H_A$ of A relative to the finite-dimensional structure group G [16, Lemma 4.2.8]. (The centralizer of a subgroup \tilde{G} of G consists of all elements of G that commute with all elements in \tilde{G} . It is again a subgroup of G and contains $Z(G)$ as a subgroup.) For any $A \in \mathcal{A}$, the holonomy group H is a Lie subgroup of the finite-dimensional structure group G [16, p. 132], and hence any stabilizer \mathcal{S}_A is isomorphic to a closed subgroup of G [7].

It is known that the set \mathcal{M} of orbits of all connections forms a connected, separable and metrizable (and hence in particular Hausdorff) topological space, and that this space has the structure of a stratified variety, i.e. as a set is the disjoint sum of certain strata which are smooth manifolds; the set of orbits of reducible connections is a closed subset of this variety which is nowhere dense [7, 17, 18, 19]. It has also been shown that each stratum carries the structure of a Hilbert manifold, i.e. an infinite-dimensional C^∞ manifold modelled on a Hilbert space, and its pre-image in \mathcal{A} is a smooth G -invariant submanifold of \mathcal{A} [4, 5, 7, 18].

¹ In the literature the term ‘irreducible’ is sometimes (see e.g. [10]) only used for connections with maximal holonomy; such connections have in particular a trivial stabilizer.

Each stratum of the variety \mathcal{M} consists of all orbits of connections that are of the same ‘symmetry type’ in the sense [18] that their stabilizers are conjugate subgroups of \mathcal{G} . (In particular, the stabilizers of all connections belonging to a fixed stratum are all isomorphic when considered as abstract Lie groups. For the so-called main stratum which consists of the orbits of irreducible connections, the stabilizer is just the set of constant gauge transformations with values in $Z(G)$.) Thus the strata of \mathcal{M} can be labelled by conjugacy classes of closed subgroups of the gauge group \mathcal{G} that are isomorphic to Lie subgroups of the finite-dimensional structure group G . The set of such classes, and hence the set of strata of \mathcal{M} , is countable [7,18]. (Depending on the topological properties of the space-time M and of G , the number of strata may be finite; this will be the situation in the case of $M = S^4$ and $G = \text{SU}(2)$ that will be treated in section 7.) The inclusion relation among conjugacy classes of stabilizers in \mathcal{G} induces a partial ordering of the stabilizers. As the latter are used to label the strata, this carries over to a partial ordering of strata. One can show [7] that the stratum with stabilizers conjugated to some given stabilizer \mathcal{S} is dense in the union of all strata that have stabilizers containing \mathcal{S} . In particular the main stratum, which has trivial stabilizer, is dense in the configuration space \mathcal{M} , so that the singular strata can be approximated arbitrarily well by irreducible connections.

For practical purposes one is often interested in *reducible* (i.e., non-irreducible) background connections. (In section 7 below we will describe in some detail the reducible connections in the case where the structure group is $G = \text{SU}(2)$.) In particular, the background $\bar{A} = \overset{\circ}{A}$, where $\overset{\circ}{A}$ stands for the ‘vacuum’, i.e. the configuration

$$\overset{\circ}{A}_\mu(x) \equiv 0, \quad (3.2)$$

is reducible, and of course many calculations are simplified by this choice of background. Now when fixing the gauge around a reducible background, various subtleties have to be taken into account.

First of all, only if the background connection \bar{A} is irreducible, then [20] any two connections A, B ($A \neq B$) in Γ that are sufficiently close to \bar{A} in the topology induced by the scalar product (1.2) belong to distinct gauge orbits $\mathcal{O}_A \neq \mathcal{O}_B$. In contrast, for reducible background, Γ does not have this property. Further, for irreducible background \bar{A} , the Gribov region Ω can be defined as the subspace of Γ on which the Faddeev–Popov operator Δ_{FP} is positive,

$$\Omega \equiv \Omega_{\bar{A}} = \{A \in \Gamma_{\bar{A}} \mid (A, \Delta_{\text{FP}}A) \geq 0\}; \quad (3.3)$$

in particular, the Gribov horizon $\partial\Omega$ can be characterized by the vanishing of the Faddeev–Popov determinant $\det(\Delta_{\text{FP}})$. On the contrary, for reducible connections there always exists [16, p. 132] a covariantly constant global section σ of the vector bundle $P \times_G \mathfrak{g}$ (the ‘adjoint bundle’) over M , where G acts on $\mathfrak{g} = \text{Lie}(G)$ in the adjoint representation, i.e. a global section satisfying $\nabla_{\bar{A}}\sigma = 0$. To derive the existence of such a section, we have to keep in mind that g is a section of the bundle $P \times_G G$, and to note that the gauge transformation (1.5) can also be written in

the form

$$A^g = A + g^{-1} \nabla_A g. \quad (3.4)$$

When considered for the background \bar{A} , this equation shows that we can describe the elements of the stabilizer $\mathcal{S}_{\bar{A}}$ by covariantly constant sections in $P \times_G G$. Now any stabilizer is a finite-dimensional Lie subgroup of \mathcal{G} ; in particular, if the stabilizer is non-trivial (so that the dimension of this Lie subgroup is not zero), we can consider a smooth family $\{g_t(x) \mid t \in (-1, 1)\}$ of such sections that are connected to the unit element $\mathbf{1}$ of \mathcal{G} , i.e. $g_{t=0}(x) = \mathbf{1}$ for all $x \in M$. By differentiating this with respect to t , we see that $\sigma := \frac{d}{dt} g_t(x)|_{t=0}$ is a covariantly constant section of $P \times_G \mathfrak{g}$.

Because of the symmetry (2.7) of the Faddeev–Popov operator, such a section σ obeys

$$\nabla_{\bar{A}}^* \nabla_A \sigma = \nabla_A^* \nabla_{\bar{A}} \sigma = 0, \quad (3.5)$$

i.e. σ is in the kernel of Δ_{FP} , from which it follows that $\det(\Delta_{\text{FP}}) = 0$. Therefore in the case of reducible backgrounds the Gribov horizon $\partial\Omega$ cannot be characterized as the set of all points of Γ where the Faddeev–Popov determinant vanishes. However, it is still possible to describe Ω as the set of minima of the functionals (2.2).

Finally we note that if the background is reducible, then the Morse functionals Φ_A possess a systematic degeneracy. Namely, if h is an element of the stabilizer $\mathcal{S}_{\bar{A}}$ of the background \bar{A} , then we have

$$\Phi_A[gh] = |A^{gh} - \bar{A}|^2 = |A^{g^h} - \bar{A}^h|^2 = |(A^g - \bar{A})^h|^2 = |(A^g - \bar{A})|^2 = \Phi_A[g]. \quad (3.6)$$

Here one uses the fact that the difference of two connections transforms homogeneously, and that the norm $|\cdot|$ is invariant under the action of the gauge group. Conversely, if for a given $h \in \mathcal{G}$ (3.6) holds for all $A \in \mathcal{A}$ and all $g \in \mathcal{G}$, then h is in fact an element of the stabilizer of \bar{A} .

4 Fundamental modular domains

In general the subset $\Omega = \Omega_{\bar{A}}$ of \mathcal{A} contains absolute minima as well as relative minima of the functionals Φ_A . The subset

$$\Lambda \equiv \Lambda_{\bar{A}} := \{A \in \mathcal{A} \mid \Phi_A(g) \geq \Phi_A(\mathbf{1}) \text{ for all } g \in \mathcal{G}\} \subseteq \Omega_{\bar{A}} \quad (4.1)$$

of *absolute* minima of Φ_A contains at least one representative of any gauge orbit \mathcal{O} [6, 14] and for topologically simple space-times is properly contained in Ω [21].

Using again repeatedly the fact that the difference of two connections transforms homogeneously, and that the inner product (1.2) is invariant under gauge transformations, we can

deduce that

$$\begin{aligned}
& |(\xi B + (1 - \xi)C)^g - \bar{A}|^2 - |(\xi B + (1 - \xi)C) - \bar{A}|^2 \\
&= |\xi(B - C)^g + (C^g - \bar{A})|^2 - |\xi(B - C) + (C - \bar{A})|^2 \\
&= 2\xi((B - C)^g, C^g - \bar{A}) - 2\xi(B - C, C - \bar{A}) + |C^g - \bar{A}|^2 - |C - \bar{A}|^2 \quad (4.2) \\
&= 2\xi(B - C - B^g + C^g, \bar{A}) + |C^g - \bar{A}|^2 - |C - \bar{A}|^2 \\
&= \xi(|B^g - \bar{A}|^2 - |B - \bar{A}|^2) + (1 - \xi)(|C^g - \bar{A}|^2 - |C - \bar{A}|^2)
\end{aligned}$$

for any $\bar{A}, B, C \in \mathcal{A}$ and for any $\xi \in \mathbf{R}$. As a consequence, if both B and C belong to $\Lambda_{\bar{A}}$ so that both $|B^g - \bar{A}|^2 - |B - \bar{A}|^2$ and $|C^g - \bar{A}|^2 - |C - \bar{A}|^2$ are positive, then for all $\xi \in [0, 1]$ the left hand side of (4.2) is positive as well, and hence the connection $\xi B + (1 - \xi)C$ belongs to $\Lambda_{\bar{A}}$, too. This shows that, just as $\Omega_{\bar{A}}$, the subset $\Lambda_{\bar{A}}$ of \mathcal{A} is convex. Furthermore, as $\Lambda_{\bar{A}}$ is contained in the bounded set $\Omega_{\bar{A}}$, it is bounded, too. (However, just as e.g. the infinite-dimensional unit sphere, $\Lambda_{\bar{A}}$ is not compact.)

Now as Λ is a convex subset of an affine space and hence topologically trivial, the topological non-triviality of \mathcal{M} must stem from the fact that upon projection onto \mathcal{M} some points of Λ must be identified in a non-trivial manner. A priori such identifications may take place for boundary points of Λ as well as in the interior of Λ . We will now show that the identification in the interior precisely amounts to dividing out the stabilizer $\mathcal{S}_{\bar{A}}$.

For reducible backgrounds the systematic degeneracy (3.6) of the functionals Φ_A implies that in particular their absolute minima are degenerate. Thus in order to obtain a fundamental modular domain one must divide out at least the action of the stabilizer of the background from Λ . Actually, the latter procedure is already sufficient to obtain a modular domain. To see this, define $\tilde{\Lambda}$ as the subset of Λ that consists of all connections in Λ for which the only gauge copies contained in Λ are precisely those related by elements of the stabilizer of the background. Owing to (3.4) one has the identity

$$\Phi_{\bar{A}; \bar{A}}[g] = |\bar{A}^g - \bar{A}|^2 = |g^{-1} \nabla_{\bar{A}} g|^2, \quad (4.3)$$

which implies that on the orbit of \bar{A} the minimum of the Morse functional is zero. Any gauge transformation g for which $\Phi_{\bar{A}; \bar{A}}[g]$ has this minimal value is covariantly constant with respect to \bar{A} and thus in the stabilizer $\mathcal{S}_{\bar{A}}$ of the background. This shows that $\tilde{\Lambda}$ contains at least the background \bar{A} and is thus not empty. In addition, the difference $\Lambda \setminus \tilde{\Lambda}$ is contained in the boundary of Λ ,

$$\Lambda \setminus \tilde{\Lambda} \subseteq \partial\Lambda. \quad (4.4)$$

Namely (compare [12]), let C be a connection in $\Lambda \setminus \tilde{\Lambda}$. When taking $B = \bar{A}$ in (4.2), the first term on the right hand side is equal to $\xi |\bar{A}^g - \bar{A}|^2$ and is thus strictly positive for any $\xi > 0$ and all $g \in \mathcal{G} \setminus \mathcal{S}_{\bar{A}}$, while the term proportional to $1 - \xi$ is non-negative for all $\xi \in [0, 1]$ since C

corresponds to a minimum of $\Phi_{\bar{A},C}$. As a consequence, any point on the straight line between \bar{A} and C is an absolute minimum and is in fact contained in $\tilde{\Lambda}$. Only for $\xi = 0$, i.e. at the connection C itself, the minimum has further degeneracies. We conclude that $\Lambda \setminus \tilde{\Lambda}$ cannot have inner points, since otherwise the straight line between C and \bar{A} had to contain minima with larger degeneracy for some $\xi \in (0, 1)$. This proves (4.4).

Furthermore, the stabilizer of any connection in the interior of $\Lambda_{\bar{A}}$ has to be contained, as a subgroup of \mathcal{G} , in the stabilizer of the background. This holds because the interior of Λ is contained in $\tilde{\Lambda}$, and because by definition in $\tilde{\Lambda}$ the degeneracy of the Morse functional is trivial, namely equal to the stabilizer of the background \bar{A} . Now if g is in the stabilizer of A , then

$$\Phi_A[gh] = |A^{gh} - \bar{A}|^2 = |A^h - \bar{A}|^2 = \Phi_A[h] \quad (4.5)$$

for all $h \in \mathcal{G}$. Thus if the stabilizer of A is not contained in the stabilizer of \bar{A} , then there is an additional degeneracy, and hence $A \in \Lambda \setminus \tilde{\Lambda} \subseteq \partial\Lambda$. In particular, if $A \in \Lambda$ and \bar{A} belong to the same stratum, then their stabilizers are in fact identical rather than only conjugated subgroups of \mathcal{G} . Given the partial ordering of strata induced by the inclusions of conjugacy classes of the stabilizers, our result implies that the interior of Λ contains only elements of strata with equal or ‘less’ symmetry as the background. Specializing to the case of irreducible background, it follows that reducible connections are necessarily on the boundary of Λ .

With the above result it is easy to give locally a more precise description of the pre-images of the strata of \mathcal{M} in Λ . Suppose we are dealing with a connection \bar{A} as the background which has non-trivial stabilizer $\mathcal{S}_{\bar{A}}$. Let U be a neighbourhood of \bar{A} that is contained in the interior of Λ , and \tilde{U} the intersection of the stratum of \bar{A} with U . Since all elements A in \tilde{U} have identical stabilizer, any connection

$$\xi\bar{A} + (1 - \xi)A, \quad \xi \in [0, 1], \quad (4.6)$$

has the same stabilizer as well, and is hence contained in \tilde{U} . This shows that \tilde{U} is the intersection of a linear subspace of Γ with U . (This description applies only locally, and in general the same stratum may also have additional points, with stabilizer conjugate but not identical to $\mathcal{S}_{\bar{A}}$, on the boundary $\partial\Lambda$. We will see in section 7 that this is in fact the case for the U(1) stratum of a SU(2) gauge theory over S^4 .)

We are now in a position to specify the true fundamental modular domains: for any background \bar{A} , they are given by $\Lambda_{\bar{A}}/\mathcal{S}_{\bar{A}}$. For the objects that are obtained from these fundamental modular domains by boundary identification, such that they are isomorphic to the configuration space \mathcal{M} , we will use the notation $\check{\Lambda}$:

$$\check{\Lambda} \equiv \check{\Lambda}_{\bar{A}} = \Lambda_{\bar{A}}|_{\text{bound.id.}}/\mathcal{S}_{\bar{A}}, \quad (4.7)$$

which reduces to

$$\check{\Lambda} = \Lambda|_{\text{bound.id.}}/Z(G) = \Lambda|_{\text{bound.id.}} \quad (4.8)$$

in the case of an irreducible background \bar{A} . After taking into account the boundary identifications in this manner, $\check{\Lambda}$ can be shown to be paracompact [22]. As $Z(G)$ leaves all connections fixed, in the sequel we will no longer mention the presence of $Z(G)$ explicitly. Note that as soon as the stabilizer of the background is non-trivial, the action we have to divide out is non-trivial as well, and in fact it possesses at least one fixed point, namely the background itself. When fixing the gauge around a reducible connection \bar{B} (e.g. $\bar{B} = \check{A}$), this property ensures that \bar{B} which is a smooth inner point of $\Lambda_{\bar{B}}$ becomes a singular point of $\Lambda_{\bar{B}}/\mathcal{S}_{\bar{B}}$, and in fact a singular point of $\check{\Lambda}_{\bar{B}}$.

Let us for the moment assume now that the background \bar{A} is irreducible. Then in the transition to $\check{\Lambda}$ in fact any irreducible connection on the boundary $\partial\Lambda_{\bar{A}}$ of $\Lambda_{\bar{A}}$ has to be identified with other irreducible connections that lie on $\partial\Lambda_{\bar{A}}$. This can be seen as follows. Assume that $B \in \partial\Lambda_{\bar{A}}$ is irreducible. Then instead of $\Lambda_{\bar{A}}$ we can alternatively consider the corresponding subset $\Lambda_B \subset \mathcal{A}$ that is obtained by taking B as a background. Now the same argument that was used in the proof of (4.4) shows that B is a smooth inner point of Λ_B . In addition, as just mentioned, upon projection to \mathcal{M} inner points do not get identified and hence are projected to smooth inner points of \mathcal{M} . But the only way in which the connection B , now considered again as a point of $\partial\Lambda_{\bar{A}} \subset \Lambda_{\bar{A}}$, can project to a smooth inner point of \mathcal{M} is that a neighbourhood of B on the boundary (which can be taken to consist only of irreducible connections because the reducible connections are nowhere dense) has to be identified with a different neighbourhood on the boundary of $\Lambda_{\bar{A}}$. (The latter neighbourhood has to consist of boundary points, because inner points do not have non-trivial copies in $\Lambda_{\bar{A}}$.) In contrast, if B is reducible, then the identifications of boundary points have to take place in such a manner that the point remains singular. Roughly speaking, there are ‘less’ identifications for reducible than for irreducible connections.

The previous results imply in particular that $\check{\Lambda}$ does not have any boundary points except for reducible connections. The latter are, strictly speaking, no boundary points either, because by the usual definition a boundary of a manifold is another manifold of codimension one that is patched to the manifold in a specific manner; in the case of singular connections the codimension is in general infinite, and the patching is much more complicated.

It is worth emphasizing that even if the subset of reducible connections is ‘small’, in the context of quantum field theory its presence cannot simply be ignored. (For instance, Green functions are distributions, and hence one has to analyze carefully whether a set of measure zero can be disregarded.) Unfortunately, only little is known about possible effects on quantum physics that are due to the presence of singularities in the configuration space. Indeed we expect that the reducible connections have to be treated with still more care. For instance, it might be necessary to resolve the singularities, in a consistent manner analogous to the blowing up of orbifold singularities of finite-dimensional complex [23] or symplectic [24, 25] manifolds. This would induce an additional source of non-triviality for the topology of \mathcal{M} , and thus also for the anomaly structure if the theory is coupled to chiral fermions. Now in a Hamiltonian

formulation it can be shown [26] that classical trajectories are always contained in one fixed stratum of the configuration space, so that when dealing with the classical field theory we can restrict ourselves to a fixed stratum, which is in itself a smooth infinite-dimensional manifold. Therefore we expect any influence on the main stratum arising from the singularities of \mathcal{M} to be a genuine *quantum* effect.

Finally we recall from the introduction that by the choice of a specific (isomorphism class of) principal bundle P we have fixed the instanton number. Thus when allowing for arbitrary instanton number k , i.e. considering the Yang–Mills theory with arbitrary asymptotic conditions at infinity, the fundamental modular domain is in fact the disjoint sum over $k \in \mathbf{Z}$ of the modular domains for each value of k . (The collection of the orbit spaces for all $k \in \mathbf{Z}$ is sometimes referred to as the extended orbit space [22].)

5 The pointed gauge group

In this section we investigate the relation between the region $\hat{\Lambda}_{\mathring{A}} := \Lambda_{\mathring{A}}|_{\text{bound.id.}}$ of absolute minima of the Morse functionals $\Phi_{\mathring{A};\mathcal{A}}$ for the background \mathring{A} , and another object, namely the orbit space $\mathcal{A}/\mathcal{G}_0$ with respect to the so-called *pointed gauge group* \mathcal{G}_0 . The latter group plays an important rôle for detailed investigations of the Riemannian geometry of the configuration space [1, 4, 5, 7, 17]; it is defined as

$$\mathcal{G}_0 := \{g \in \mathcal{G} \mid g(x_0) = \mathbb{1}\}, \quad (5.1)$$

with $x_0 \in M$ an arbitrary, but fixed, space-time point. Suppose we have performed all boundary identifications in $\Lambda_{\mathring{A}}$, thereby obtaining an infinite-dimensional variety $\hat{\Lambda}_{\mathring{A}}$, which in contrast to $\Lambda_{\mathring{A}}$ is not embedded in an affine space. We claim that

$$\hat{\Lambda}_{\mathring{A}} \cong \mathcal{A}/\mathcal{G}_0, \quad (5.2)$$

i.e. that $\hat{\Lambda}_{\mathring{A}}$ is isomorphic, as a manifold endowed with a G -action, to the space \mathcal{A} of all connections divided by the pointed gauge group. The fact that both objects are in particular diffeomorphic shows that $\hat{\Lambda}_{\mathring{A}}$ is not only a variety, but even a smooth infinite-dimensional manifold.

To prove (5.2), we first note that a set of representatives for

$$\mathcal{G}/\mathcal{G}_0 \cong G \quad (5.3)$$

is given by the constant gauge transformations, i.e. by the stabilizer of the background \mathring{A} . This is simply because any gauge transformation $g \in \mathcal{G}$ can be written as

$$g = g_c (g_c^{-1} g), \quad (5.4)$$

where g_c denotes the constant gauge transformation with value $g_c(x) = g(x_0) \in \text{SU}(2)$ independent of x . Also, the only constant gauge transformation that is an element of \mathcal{G}_0 is the unit element $\mathbf{1}$. We now map any element of $\hat{\Lambda}_{\hat{\lambda}}$ (or more precisely, its pre-image in $\Lambda_{\hat{\lambda}}$) on its equivalence class modulo \mathcal{G}_0 . We have to prove that this map is injective and surjective. To show that the map is injective, suppose that $A, B \in \hat{\Lambda}_{\hat{\lambda}}$ are mapped on the same element of $\mathcal{A}/\mathcal{G}_0$. Then A and B are in the same equivalence class with respect to \mathcal{G}_0 , i.e. $A = B^g$ with g an element of \mathcal{G}_0 . On the other hand, the only gauge copies left in $\hat{\Lambda}_{\hat{\lambda}}$ are those related by constant gauge transformations; therefore g must be the identity, and hence $A = B$, which proves injectivity. To show that the map is surjective we start with an arbitrary connection $A \in \mathcal{A}$. Since the orbit of any connection has a representative in $\Lambda_{\hat{\lambda}}$ (and hence in $\hat{\Lambda}_{\hat{\lambda}}$) we can find an element $B \in \hat{\Lambda}_{\hat{\lambda}}$ and an element $g \in \mathcal{G}$ such that $B^g = A$. Now decompose g as $g = g_c g_0$ with $g_0 \in \mathcal{G}_0$ and g_c a constant gauge transformation. Then

$$B^{g_c} = A^{g_0^{-1}}. \quad (5.5)$$

Now due to $g_c \in \mathcal{S}_{\hat{\lambda}}$ the left hand side of (5.5) is also in $\hat{\Lambda}_{\hat{\lambda}}$, while the right hand side is in the same equivalence class modulo \mathcal{G}_0 as A . Thus for each element of $\mathcal{A}/\mathcal{G}_0$ there exists an element of $\hat{\Lambda}_{\hat{\lambda}}$ that gets mapped to it, which proves surjectivity.

Furthermore, the spaces $\hat{\Lambda}_{\hat{\lambda}}$ and $\mathcal{A}/\mathcal{G}_0$ both carry a group action of the finite-dimensional structure group G . In the case of $\hat{\Lambda}_{\hat{\lambda}}$ this action is defined by applying a constant gauge transformation g_c on $A \in \hat{\Lambda}_{\hat{\lambda}}$; this is well-defined because A^{g_c} is an element of $\hat{\Lambda}_{\hat{\lambda}}$, too. For $\mathcal{A}/\mathcal{G}_0$ the G -action is again defined by use of constant gauge transformations: the orbit \mathcal{Q}_A containing A is mapped to the orbit $\mathcal{Q}_{A^{g_c}}$ of A^{g_c} , in other words $(\mathcal{Q}_A)^{g_c} := \mathcal{Q}_{A^{g_c}}$. This is well-defined because \mathcal{G}_0 is a normal subgroup: choosing a different representative $A^g \in \mathcal{Q}_A$ with $g \in \mathcal{G}_0$, the fact that \mathcal{G}_0 is normal in \mathcal{G} means that there is a $g' \in \mathcal{G}_0$ such that $gg_c = g_c g'$, and hence $(A^g)^{g_c} = (A^{g_c})^{g'} \in \mathcal{Q}_{A^{g_c}}$. The fact that for any $A \in \hat{\Lambda}_{\hat{\lambda}}$, A^{g_c} is mapped under the isomorphism on $\mathcal{Q}_{A^{g_c}} = (\mathcal{Q}_A)^{g_c}$ shows that both group actions coincide, or, more precisely, that the isomorphism intertwines the group actions.

Finally, let us note that analogous arguments as given above for the background \mathring{A} show that a similar relation as (5.2) holds whenever the background is a pure gauge. Namely, if

$$\bar{A} = \mathring{A}^{\bar{g}} = \bar{g}^{-1} d\bar{g}, \quad (5.6)$$

then

$$\Lambda_{\bar{A}}|_{\text{bound.id.}} \cong \mathcal{A}/\mathcal{G}_0^{(\bar{g})}, \quad (5.7)$$

with

$$\mathcal{G}_0^{(\bar{g})} := \{g \in \mathcal{G} \mid (\bar{g}g\bar{g}^{-1})(x_0) = \mathbf{1}\} = \bar{g}^{-1}\mathcal{G}_0\bar{g}, \quad (5.8)$$

and the action of the structure group is via the stabilizer $\mathcal{S}_{\bar{A}} = \bar{g}G\bar{g}^{-1}$ rather than via the constant gauge transformations.

6 Geodesic convexity

We are now going to discuss the consequences of the convexity of $\Lambda_{\bar{A}}$ for the configuration space. First note that, in contrast to $\Lambda_{\bar{A}}$, due to the division by $\mathcal{S}_{\bar{A}}$, the fundamental modular domain $\Lambda_{\bar{A}}/\mathcal{S}_{\bar{A}}$ is in general not a subset of an affine space, and hence we do not have the notion of convexity any more. However, we can show that the main stratum in $\Lambda_{\bar{A}}/\mathcal{S}_{\bar{A}}$ is still *geodesically convex*, i.e. any two non-singular points in $\Lambda_{\bar{A}}/\mathcal{S}_{\bar{A}}$ can be joined by a geodesic which only contains non-singular points of $\Lambda_{\bar{A}}/\mathcal{S}_{\bar{A}}$. (Of course, the fact that a space is geodesically convex does not mean at all that it is topologically trivial [1].) This can be seen as follows.

Suppose that $\mathcal{P}_B, \mathcal{P}_C$ are two non-singular points in $\Lambda_{\bar{A}}/\mathcal{S}_{\bar{A}}$, and let B be a representative of \mathcal{P}_B in $\Lambda_{\bar{A}}$. Then consider B instead of \bar{A} as the background connection, and let C be a representative of \mathcal{P}_C in Λ_B . It follows [10] that any geodesic through \mathcal{P}_B in the orbit space, and hence in particular any geodesic joining \mathcal{P}_B with \mathcal{P}_C , is given by the projection of a straight line through B in \mathcal{A} . Now since Λ_B is convex, the straight line connecting B and C is contained in Λ_B , and it projects down to a geodesic in $\Lambda_{\bar{A}}/\mathcal{S}_{\bar{A}}$. Furthermore, no reducible connection is contained in the straight line that connects B with C so that the corresponding geodesic does not hit a singularity. This is because, as seen above, for irreducible background B any reducible connection in Λ_B lies on the boundary $\partial\Lambda_B$. Note that this is still true if C is a point on the boundary of Λ_B (the straight line from B to C then meets $\partial\Lambda_B$, but only in the single point C which by assumption is irreducible). This implies that the result still holds after boundary identifications are taken into account, i.e. for $\Lambda_{\bar{A}}/\mathcal{S}_{\bar{A}}$.

It is natural to ask whether strata other than the main stratum are geodesically convex as well, and one may try to investigate this problem along similar lines as above. Inspection shows, however, that for non-main strata the situation is generically much more complicated than above. At a technical level, the main obstacle is that one cannot easily verify whether to any two points in the stratum one can find representatives in a fundamental modular domain for which the stabilizers are identical rather than only conjugate subgroups of \mathcal{G} .

To conclude, let us comment on the physical meaning of geodesic convexity. When interpreting our result in a Hamiltonian picture, it provides information about the obstructions that the geometry of the configuration space imposes on the classical motion. Notice that in pure Yang–Mills theory classical trajectories can be characterized by their instanton number which is a topological quantity, so that requiring the instanton number to have a fixed value amounts to a *kinematical* restriction to the motion. There is also a *dynamical* obstruction stemming from the fact that classical trajectories are contained in a stratum of fixed stabilizer type. Our result shows that within the main stratum there is no further obstruction of the latter type. The only possible further restrictions on the motion must then stem from the fact that the geodesics are *not* the classical trajectories, because in the Hamiltonian approach also a potential must be taken into account [10]. In the case of the Yang–Mills action (1.1), the relevant potential is $V = \frac{1}{4g^2} \int_{\mathcal{C}} d^3x \operatorname{tr} F_{ij} F^{ij}$, where integration is over some time slice \mathcal{C} .

7 Singularity structure of the SU(2) configuration space

7.1 Reducible connections

As an application, we now consider in detail the case $G = \text{SU}(2)$. As already mentioned in section 3, the holonomy group H is a Lie subgroup of the structure group G . For $G = \text{SU}(2)$, this requirement leaves the following possibilities. First, the holonomy group H can be $\text{SU}(2)$ or $\text{SO}(3)$. Then the centralizer of H is just the center $\mathbf{Z}_2 = \{\pm 1\}$ of $\text{SU}(2)$. Thus the stabilizer is trivial and hence the connection is irreducible. Secondly, the holonomy group may be $H = \text{U}(1)$. Then the stabilizer is also isomorphic to $\text{U}(1)$ as an abstract Lie group, and the analysis is less trivial than in the previous case. Finally, the holonomy group may be trivial so that the stabilizer is isomorphic to $\text{SU}(2)$; as we will see, this situation is best described as a special case of the connections with stabilizer isomorphic to $\text{U}(1)$.

To enter the discussion of connections with trivial or $\text{U}(1)$ holonomy, we recall that we assume the space-time manifold M to be S^4 , so that in particular its second cohomology group vanishes, $H^2(M, \mathbf{Z}) = 0$. It can be shown [2] that for $G = \text{SU}(2)$ the vanishing of $H^2(M, \mathbf{Z})$ implies that the instanton number of any reducible connection vanishes. Namely, let $V \cong \mathbf{C}^2$ be a two-dimensional complex vector space carrying the defining representation of $G = \text{SU}(2)$. If the connection has only $\text{U}(1)$ holonomy, the associated vector bundle $E := P \times_{\text{SU}(2)} V$ splits into the direct sum of two complex line bundles over M ,

$$E = P \times_{\text{SU}(2)} V_1 \oplus P \times_{\text{SU}(2)} V_2. \quad (7.1)$$

As in our case the second real cohomology of the space-time M is trivial, both line bundles and hence also E and P have to be trivial, i.e. $P = M \times G$. In particular, gauge transformations $g \in \mathcal{G}$ can be considered as functions from M to the structure group G .

It must also be noted that the space of maps from S^4 to $\text{SU}(2)$ is topologically non-trivial, $\pi_4(\text{SU}(2)) = \mathbf{Z}_2$. As a consequence, for structure group $\text{SU}(2)$ the gauge group \mathcal{G} is *not* connected. However, there are no fixed points with respect to the non-identity component of \mathcal{G} . This holds because any stabilizer is isomorphic to a Lie subgroup of $\text{SU}(2)$, and all these subgroups are connected.

7.2 The U(1) stratum

Assume now that $A \in \mathcal{A}$ is a connection with nontrivial stabilizer, i.e. that the stabilizer \mathcal{S}_A contains at least a $\text{U}(1)$ subgroup of \mathcal{G} . This $\text{U}(1)$ subgroup of \mathcal{G} in general does not consist of constant gauge transformations. However, we are only interested in gauge orbits, and the stabilizers of different points on an orbit are related via conjugation by gauge group elements. We will now show that we can always find a representative of the orbit for which the stabilizer $\text{U}(1)$ does consist of constant gauge transformations. We first notice that because of

the direct product structure $P = M \times G$, we can write down all quantities of interest in terms of functions over space-time. In particular, we can write a covariantly constant section σ in the adjoint bundle $P \times_{\text{SU}(2)} \text{su}(2) \cong M \times \text{su}(2) = S^4 \times \text{su}(2)$ as $\sigma(x) = \sum_{a=1}^3 \sigma_a(x) \tau^a$, where $\{\tau^a \mid a = 1, 2, 3\}$ is a basis of the Lie algebra $\text{su}(2)$. The fact that σ is covariantly constant implies that the ‘length squared’ $\text{tr}(\sigma(x)^\dagger \sigma(x))$ of $\sigma(x)$ is in fact independent of $x \in S^4$. Further, G acts transitively on elements τ of the same length; this suggests to relate σ to some fixed arbitrary element τ of $\text{su}(2)$ that has the same length as σ . Thus we set

$$\sigma(x) = g(x) \tau g^{-1}(x). \quad (7.2)$$

Now G does not act freely on elements of equal length; rather, the stabilizer of τ is the subgroup H that is spanned by all elements of the form $e^{i\tau\phi(x)}$ for some function ϕ . As a consequence, for any $x \in M$ the equation (7.2) does not determine uniquely an element of G , but rather specifies one coset of H in G . This coset is isomorphic to the circle S^1 , and hence we are dealing with a bundle over S^4 with fiber S^1 . Now any 1-sphere bundle over S^4 is a Cartesian product $S^4 \times S^1$ [27, section 26.5]. This implies that we can find a global section $g(x)$ in this bundle, which can be interpreted as an element of \mathcal{G} . Thus (7.2) provides us with a well-defined gauge transformation $g \in \mathcal{G}$. (Note that this is in contrast to the situation in two dimensions [19], where there is a different bundle corresponding to each integer [27, section 26.2].)

Applying the gauge transformation $g(x)$ that is determined in this manner by the section σ , the fact that σ is covariantly constant, $\nabla_A \sigma = 0$, yields

$$0 = \nabla_{A^{g^{-1}}} \sigma^{g^{-1}} = \nabla_{A^{g^{-1}}} \tau = [A^{g^{-1}}, \tau]. \quad (7.3)$$

This implies that we can always find one gauge equivalent representative for a reducible connection that is of the form

$$A_\mu(x) = A_\mu^{(\tau)}(x) := i\tau a_\mu(x). \quad (7.4)$$

The field strength corresponding to this connection is $F_{\mu\nu}(x) = i\tau f_{\mu\nu}(x)$, where $f_{\mu\nu} := \partial_\mu a_\nu - \partial_\nu a_\mu$. In (7.4), the quantities a_μ are arbitrary (sufficiently smooth) real-valued functions $a_\mu(x)$ on S^4 ; in the sequel we assume that a fixed choice for these functions has been made. It is easy to verify that the gauge configuration (7.4) is invariant at least under the $U(1)$ -subgroup of constant gauge transformations $\{e^{i\theta\tau} \mid \theta \in [0, 2\pi)\} \subset \text{SU}(2)$ that is generated by τ . As promised, the stabilizer $U(1)$ of the representative (7.4) of the gauge orbit consists of constant gauge transformations. The fact that we can always find such a representative means that the orbits of connections with stabilizer isomorphic to $U(1)$ form one single stratum of the configuration space \mathcal{M} .

Let us consider the configurations (7.4) more carefully and investigate at which point(s) the background gauge condition $\nabla_A^*(A^g - \bar{A}) = 0$ is satisfied. We have $A_\mu^g = g^{-1} A_\mu g + g^{-1} \partial_\mu g =$

$ig^{-1}\tau g a_\mu + g^{-1}\partial_\mu g$, and hence

$$\begin{aligned}\partial^\mu A_\mu^g &= i\partial^\mu(g^{-1}\tau g) a_\mu + ig^{-1}\tau g \partial^\mu a_\mu + \partial^\mu(g^{-1}\partial_\mu g) \\ &= ig^{-1}\tau g \partial^\mu a_\mu + i(-g^{-1}\partial_\mu g g^{-1}\tau g + g^{-1}\tau\partial_\mu g) a_\mu - g^{-1}\partial^\mu g g^{-1}\partial_\mu g + g^{-1}\partial_\mu\partial^\mu g.\end{aligned}\quad (7.5)$$

In the following we take the background to be the reducible connection $\bar{A} = \mathring{A}$. (The results will be independent of the choice of background, but explicit calculations are much more involved for any other background.) Thus the background gauge requirement is $\partial^\mu A_\mu^g = 0$; with (7.5), this reads $i\tau \partial^\mu a_\mu = i(\partial_\mu g g^{-1}\tau - \tau\partial_\mu g g^{-1}) a_\mu + \partial^\mu g g^{-1}\partial_\mu g g^{-1} - \partial_\mu\partial^\mu g g^{-1}$, i.e.

$$i\tau \partial^\mu a_\mu = i[\partial_\mu g g^{-1}, \tau] a_\mu - \partial^\mu(\partial_\mu g g^{-1}). \quad (7.6)$$

To determine the general solution to this second order differential equation for g would be a difficult task. However, we can find a particular solution by simply recalling the analogous problem in electrodynamics; thus we make the ansatz that g is of the special form

$$g(x) = \exp(i\tau\gamma(x)) \quad (7.7)$$

(thus in particular g is connected to the identity). Then $\partial_\mu g g^{-1} = i\tau \partial_\mu \gamma$, so that

$$A_\mu^g = i\tau (a_\mu + \partial_\mu \gamma). \quad (7.8)$$

Also, (7.6) becomes $\tau \partial^\mu a_\mu = -\tau \partial^\mu \partial_\mu \gamma$; we thus end up with the differential equation

$$\partial^2 \gamma = -\partial^\mu a_\mu \quad (7.9)$$

for the function γ . The general solution of (7.9) is $\gamma(x) = \int_M d^4y H(x, y) \partial^\mu a_\mu(y) + c$, where H denotes the Green function of the Laplacian ∂^2 on M , i.e. $\partial_x^2 H(x, y) = -\delta(x - y)$ with δ the delta function on M ; for S^4 , it reads $H(x, y) = \frac{1}{2} |x - y|^{-2} / \text{Vol}(S^3)$. The constant contribution c to $\gamma(x)$ is the general solution of the homogeneous equation $\partial^2 \gamma = 0$, as follows from the fact that our space-time manifold is compact and without boundary.

For completeness we note that in fact g as defined by (7.7) and (7.9) is an element of the relevant Sobolev gauge group \mathcal{G}^k with norm $|g|_k = \sum_{l=0}^k \int_M d^4x \text{tr}((D^l g)^*(D^l g))$. Here D^l is a shorthand for some multiple derivative of order l (also, we assume that $k > \dim(M)/2$ so that we can use the Sobolev inequality, compare [5]). Standard regularity results for the Poisson equation (7.9) immediately give estimates on a suitable Sobolev norm of γ . This can in turn be used to get similar estimates for the norm of g . (For instance, for $l = 0$ we have to integrate $\text{tr}(g^*g) = \text{tr} \mathbb{1} = 2$ over the compact space-time M , which gives a finite result. Furthermore, since any derivative of g can be written as a linear combination of unitary matrices with coefficients being polynomials in the derivatives of γ , we can control the norms of the derivatives of g , too.)

Thus we have shown that on any gauge orbit of reducible connections we can identify a connection which both lies in the gauge slice Γ and is of the specific form (7.4). Of course, not all of these connections will also belong to the fundamental modular domain $\Lambda_{\bar{A}}/SU(2)$ which is only a subset of the solutions to the gauge condition. To decide which of the above solutions belongs in fact to the modular domain, we would have to look for the absolute minima of the functionals (2.2) for $\bar{A} = \bar{A}$, i.e. of $\Phi_A[g] = |A^g|^2$. For $A = A^{(\tau)}$ (7.4) and $g = e^{i\tau\gamma}$ (7.7), one has

$$\Phi_A[g_{\min}] = \text{tr}(\tau^2) \int_M d^4x \{(a_\mu + \partial_\mu\gamma)(a^\mu + \partial^\mu\gamma)\}. \quad (7.10)$$

Upon partial integration and use of the differential equation (7.9) we can rewrite (7.10) as $\Phi_A[g_{\min}] = \Phi_A[\mathbf{1}] - \text{tr}(\tau^2) \int_M d^4x \partial_\mu\gamma \partial^\mu\gamma$, or equivalently as

$$\Phi_A[g_{\min}] = \Phi_A[\mathbf{1}] + \text{tr}(\tau^2) \iint_M d^4x d^4y \partial^\mu a_\mu(x) H(x, y) \partial^\nu a_\nu(y), \quad (7.11)$$

i.e. in the form of a self-energy. Obviously, for generic functions a_μ there is no simple way to tell whether this value is the absolute minimum of Φ_A or not. Hence in spite of the fact that we can identify on each singular gauge orbit a connection that belongs to the gauge slice, it would be very difficult to determine the U(1) stratum of the configuration space via the analysis presented above. Fortunately we can bypass this problem by employing the relation with electrodynamics in a different manner; this will be described in subsection 7.4. However, before coming to that, in the next subsection we will deal with a particularly simple situation where the above analysis can indeed be applied.

7.3 The stratum with SU(2) stabilizer

Namely, let us consider the case where the holonomy group is $H = \{\mathbb{1}\}$; then the stabilizer is isomorphic to SU(2). Thus in particular the stabilizer contains a U(1) subgroup, and therefore we can use the result of the previous subsection to conclude that the gauge slice contains representatives of the form (7.4). In addition, however, the fact that the holonomy vanishes implies that the field strength $i\tau f_{\mu\nu}$ for such connections vanishes, i.e. that the connection is a pure gauge. In this case we may integrate (7.9) to

$$a_\mu + \partial_\mu\gamma = c_\mu = \text{const}; \quad (7.12)$$

conversely, this solution exists only if $f_{\mu\nu}(x) \equiv 0$. Inserting the result (7.12) into the formula for the gauge transform of A yields

$$A_\mu^g = \tau (a_\mu + \partial_\mu\gamma) = \tau c_\mu. \quad (7.13)$$

This means that, while any constant gauge transformation out of the subgroup corresponding to τ leaves A invariant, there is an analogous non-constant gauge transformation such that A^g is

a *constant* multiple of τ . Further, the derivation of the result does not determine the constants c_μ ; thus *any* choice of constant $A_\mu = c_\mu \tau$ satisfies the gauge condition (7.5). In particular, $c_\mu = 0$ yields a solution, i.e. $A_\mu(x) \equiv 0$ belongs to the orbit of $A_\mu^{(\tau)}$. Thus any configuration of the type (7.4) with vanishing field strength is in fact gauge equivalent to the configuration \mathring{A} . We also note that any value of the constants c_μ leads to a connection on the orbit of \mathring{A} that lies in the region Γ . In contrast, there is only one point on this orbit which is also contained in Λ , namely just the connection \mathring{A} itself. Namely, combining (7.12) and (7.10), we see that we have to set all constants c_μ equal to zero, i.e. $(A^{(\tau)})^g = \mathring{A}$, in order for $|(A^{(\tau)})^g|^2$ to be an absolute minimum. (Clearly, this absolute minimum is at zero, which is just a special case of the analogous statement for the functional (4.3).) It is also clear from the general arguments of section 4 that the gauge transformations which preserve the absolute minimum are precisely the constant ones.)

In short, we have shown that the only connections with maximal stabilizer in a SU(2) Yang–Mills theory on the space-time S^4 are the pure gauges, and that after fixing the gauge around the configuration \mathring{A} , the only such connection contained in $\mathring{\Lambda}_{\mathring{A}}$ is \mathring{A} itself.

We can also give explicitly the specific non-constant gauge transformations that combine with the U(1) to form a SU(2) which leaves the original connection invariant. Namely, the stabilizer is simply the conjugate of the stabilizer of \mathring{A} by $e^{\gamma(x)\tau} \in \mathcal{G}$, with $\gamma(x)$ a solution of (7.12) with $c_\mu = 0$; its elements are of the form $e^{\gamma(x)\tau} g e^{-\gamma(x)\tau}$, where g is an arbitrary constant gauge transformation.

7.4 The reducible strata as a \mathbf{Z}_2 -orbifold

We now describe the set of all orbits with non-trivial stabilizer from another point of view. Our starting observation is that the equations (7.4) and (7.8) suggest that this set is related to the configuration space of a U(1) gauge theory, i.e. of electrodynamics. Indeed, we can construct a map from the orbits of vector potentials a_μ of a U(1) gauge theory on the orbits of reducible connections of the SU(2) theory by mapping the orbit containing a_μ on the orbit containing $i a_\mu \tau$. This is well-defined because (7.8) ensures that if a_μ and b_μ are on the same orbit, i.e. $b_\mu = a_\mu + \partial_\mu \gamma$, then also the images of a_μ and b_μ are related by a gauge transformation, namely by (7.7).

In a second step we analyze which orbits of the U(1) theory are mapped on the same orbit of the SU(2) theory. Thus assume that $A_\mu = i\tau a_\mu$ is related to $B_\mu = i\tau b_\mu$ by some element g in the gauge group of the SU(2) theory (that is not necessarily of the form (7.7)). Then in particular the associated field strength tensors are related by

$$F_{(A)}^{\mu\nu}(x) = i\tau f_{(a)}^{\mu\nu}(x) = g^{-1}(x) F_{(B)}^{\mu\nu}(x) g(x) = i g^{-1}(x) \tau g(x) f_{(b)}^{\mu\nu}(x). \quad (7.14)$$

Without loss of generality let us choose $\tau \in \mathfrak{su}(2)$ to be the Pauli matrix τ^3 ; then for any fixed $x \in M$ we have to look for all elements g of SU(2) for which $g^{-1}\tau^3 g$ is proportional to

τ^3 . For this it is necessary that either g is an element of the $U(1)$ group generated by τ^3 , i.e. $g = \exp(i\theta\tau^3)$, or that g is of the form

$$g = \begin{pmatrix} 0 & e^{i\theta} \\ -e^{-i\theta} & 0 \end{pmatrix} = i\tau^2 e^{i\theta\tau^3}. \quad (7.15)$$

Thus g must be an element of the following subgroup H of $SU(2)$: topologically, H is the disjoint union of two circles and hence not connected. Algebraically, H contains a normal $U(1)$ subgroup spanned by all elements of the form $\exp(i\theta\tau^3)$; the two cosets relative to that subgroup have representatives $\mathbb{1}$ and $i\tau^2$ and form thus a \mathbf{Z}_2 subgroup; hence H is a semi-direct product

$$H = \mathbf{Z}_2 \ltimes U(1). \quad (7.16)$$

If a gauge transformation $g \in \mathcal{G}$ connects gauge potentials of the form (7.4), then for any $x \in M$ it must lie in H , i.e. $g(x)$ is of the form described above, with the real number θ replaced by some real function $\gamma(x)$. Now our space-time $M = S^4$ is connected, and the elements of the gauge \mathcal{G} correspond to *continuous* mappings from M to $SU(2)$. Therefore the value of $g(x)$ lies in one and the same component of H for all space-time points. If it is in the component connected to the identity, then the gauge transformed connection is as already obtained in equation (7.8). In the other case, i.e. for g of the form (7.15) with θ replaced by $\gamma(x)$, we get instead

$$A_\mu^g = -i\tau(a_\mu + \partial_\mu\gamma). \quad (7.17)$$

This implies that the orbits of the $U(1)$ gauge theory that correspond to the functions a_μ and $-a_\mu$ are mapped on one and the same orbit of the $SU(2)$ theory. We conclude that the set of orbits of trivial or $U(1)$ holonomy is a \mathbf{Z}_2 -orbifold of the configuration space of electrodynamics. The only singular point of this orbifold is given by the single orbit with trivial holonomy which corresponds to $a_\mu \equiv 0$, i.e. by the vacuum \mathring{A} . This illustrates nicely that the vacuum – due to its enlarged $SU(2)$ symmetry, represented by the constant gauge transformations – is ‘more singular’ than ordinary connections with $U(1)$ holonomy.

From the results derived above we learn that $\Lambda_{\mathring{A}}/SU(2)$ (and the fundamental modular domain $\Lambda_{\mathring{A}}/\mathcal{S}_{\mathring{A}}$ in general) is not a very convenient tool for the global description of reducible connections. Namely, we can show that it is not possible to find for *all* reducible connections a representative that is both of the form (7.4) and contained in $\Lambda_{\mathring{A}}$. This is essentially a consequence of the fact that there is no Gribov effect in abelian gauge theories. Let A be of the form (7.4), i.e. $A_\mu = i\tau a_\mu$. Then the only gauge copies of A of the form (7.4) are $A_\mu^g = \pm i\tau(a_\mu + \partial_\mu\gamma)$. Now if A is contained in $\Lambda_{\mathring{A}}$, we have $\partial_\mu a^\mu = 0$; if A^g is in $\Lambda_{\mathring{A}}$ as well, it follows that γ has to be harmonic, $\partial^2\gamma = 0$. As all harmonic functions on S^4 are constant, we see that the only elements on the orbit of A that are contained in $\Lambda_{\mathring{A}}$ and are of the form (7.4) are $\pm A$. Therefore the fact that $\Lambda_{\mathring{A}}$ is bounded shows that, for any choice of the functions a_μ ,

the connection κA with κ large enough is not contained in $\Lambda_{\tilde{A}}$ and that the gauge orbit through κA does not contain a representative of the form (7.4) in $\Lambda_{\tilde{A}}$.

From our results it also follows that the singularities of the configuration space are conical. More precisely, \mathcal{M} has a ‘cone over cones’ structure. (A cone over a base space \mathcal{B} is by definition a space which is diffeomorphic to the direct product $[0, 1] \times \mathcal{B}$, where for $t = 0$ all points of \mathcal{B} are identified. If the base \mathcal{B} is endowed with a metric $d\Omega$, then the cone has a metric of the form $ds^2 = dt^2 + t^2 d\Omega$. Namely, the set of reducible connections forms the first cone; its tip is the ‘vacuum’ \mathring{A} and its base space \mathcal{B}_0 can be described as the real projective space $\mathcal{B}_0 = \mathbf{RP}^\infty$ that is obtained by the \mathbf{Z}_2 identification of antipodal points on the infinite-dimensional unit sphere

$$S := \{a_\mu \mid \int d^4x a_\mu a^\mu = 1\}. \quad (7.18)$$

To describe the geometric situation around a connection \bar{A} with $U(1)$ stabilizer we use \bar{A} as the background and introduce a neighbourhood U as we have done in section 4 for general G . The intersection \tilde{U} of U with some linear subspace of Γ contains locally all other connections with $U(1)$ stabilizer. Without loss of generality we can assume that U is a direct sum $U = \tilde{U} \oplus U^\perp$, where U^\perp is contained in the orthogonal complement of \tilde{U} . To get the true configuration space we now have to divide out the residual part of the gauge group, i.e. $U(1) \cong \mathcal{S}_{\bar{A}} \subset \mathcal{G}$. This group acts on U in the following way: it leaves \tilde{U} pointwise fixed and freely maps U^\perp on U^\perp . To make this more explicit, decompose any $A \in U$ as $A = \tilde{A} + A^\perp$ where $\tilde{A} \in \tilde{U}$ transforms as a connection and $A^\perp \in U^\perp$ transforms homogeneously under \mathcal{G} . Then a gauge transformation $g \in \mathcal{G}$ maps A to $A^g = \tilde{A} + g^{-1}A^\perp g$, where on the last part the action of \mathcal{G} is homogeneous and free. This shows that around every point of \tilde{U} again the structure of \mathcal{M} is that of a cone whose tip now lies in \tilde{U} . The base space of this cone is

$$\mathcal{B} = [S_r^\infty \cap U^\perp] / \mathcal{S}_A, \quad (7.19)$$

where S_r^∞ is a sphere of some radius r in the infinite-dimensional space Γ . \mathcal{B} is a smooth manifold since the action of \mathcal{S}_A on U^\perp is free. Since any point of \tilde{U} is itself part of the cone whose tip is \mathring{A} , we have indeed identified a ‘cone over cones’ structure of \mathcal{M} .²

7.5 Reducible connections on the lattice

Analogous considerations as for the configuration space of continuum gauge theories apply when the theory is considered on an arbitrary lattice. In particular, reducible connections also arise in lattice gauge theories.³ As it turns out, most of the results of the continuum theory can be

² In a Hamiltonian formulation, for any Yang–Mills theory a similar ‘cones over cones’ structure is present [26,28] in the subset of the phase space that consists of solutions to the constraint equations. It will be interesting to explore the relation between this cone structure and the singularity structure of \mathcal{M} .

³ Various other aspects of the configuration space of lattice theories have been treated in the literature. See, for instance, [29, 12, 30, 31].

related rather directly to the lattice. For example, the analogue of the configuration (7.4) on an arbitrary lattice reads

$$U_{x,y} = \exp(i\tau f_{x,y}), \quad (7.20)$$

where $U_{x,y}$ denotes the element of $SU(2)$ that is attached to the link joining the vertices x and y , τ is an arbitrary element of the Lie algebra $\mathfrak{su}(2)$, and where f is an arbitrary real-valued function on the set of links of the lattice. In the lattice formulation an element of the gauge group can be described by a map from the vertices x into the structure group G . The group variables attached to the links then transform according to

$$U_{x,y} \mapsto g_x^{-1} U_{x,y} g_y. \quad (7.21)$$

One can easily check that any configuration of the form (7.20) is invariant under the $U(1)$ generated by the constant gauge transformations $g_x = e^{i\theta\tau} = \text{const.}$ But we can also show the converse: given any configuration $U_{x,y}$ with non-trivial stabilizer of an $SU(2)$ gauge theory on the lattice, it is gauge equivalent to some configuration of the form (7.20). Namely, first, the fact that g is in the stabilizer means that

$$g_x^{-1} U_{x,y} g_y = U_{x,y}. \quad (7.22)$$

We can extend this formula by iterating it in such a manner that it applies to any two vertices x and y , with $U_{x,y} \equiv U_{x,y}^{(P_{xy})}$ the transporter for an arbitrary but fixed path P_{xy} joining the vertices. We then fix some reference point x_0 , which, due to the fact that g is non-trivial, we can choose such that $g_{x_0} \neq \mathbb{1}$. Then

$$g_x = U_{x,x_0} g_{x_0} U_{x,x_0}^{-1} \quad (7.23)$$

for any lattice point x and any path joining x with x_0 ; in particular, while U_{x,x_0} depends on the path chosen, the combination on the right hand side does not. As g describes a gauge transformation and hence transforms in the adjoint representation, (7.23) is just the discrete version of the statement that g is covariantly constant. Define now the map σ on the set of vertices by $\sigma_x := U_{x,x_0}$, and denote by \tilde{U} the configuration

$$\tilde{U}_{x,y} := \sigma_x^{-1} U_{x,y} \sigma_y. \quad (7.24)$$

By definition, \tilde{U} is gauge equivalent to U , and $g_x = \sigma_x g_{x_0} \sigma_x^{-1}$. In addition, for any x and y we have

$$\tilde{U}_{x,y} g_{x_0} \tilde{U}_{x,y}^{-1} = \sigma_x^{-1} U_{x,y} \sigma_y g_{x_0} \sigma_y^{-1} U_{x,y}^{-1} \sigma_x = \sigma_x^{-1} U_{x,y} g_y U_{x,y}^{-1} \sigma_x = \sigma_x^{-1} g_x \sigma_x = g_{x_0}. \quad (7.25)$$

Thus *all* link variables $\tilde{U}_{x,y}$ commute with the non-trivial element g_{x_0} of $SU(2)$. This shows that they have to be contained in the $U(1)$ subgroup generated by g_{x_0} , and hence for any

configuration with non-trivial stabilizer there is a gauge equivalent representative of the form (7.20).

In contrast to the situation in the continuum theory, in a lattice gauge theory reducible connections do not seem to cause any problems as long as one works at fixed finite lattice spacing a ; whether a link variable is part of a reducible (as in the above example) or an irreducible configuration seems to be irrelevant. A crucial difference to the continuum theory is that in the lattice theory one is not forced to fix a gauge. The partition function is just the sum over all configurations of link variables. More precisely, the integration measure is a product of Haar measures of the finite-dimensional structure group G at each lattice link. With respect to this measure, reducible connections are a set of measure zero. However, the interest in the lattice theory comes mostly from the desire to consider it as a regularization of the continuum theory, and for making contact to observational data one has ultimately to perform the continuum limit $a \rightarrow 0$. This limit is far from being trivial, and hence it is not at all clear what the influence of reducible connections might possibly be. The lattice approach suggests that the measure is concentrated at the reducible connections (which, nonetheless, are of measure zero) in the sense that any reducible connection is included in the partition function with a multiplicity corresponding to its stabilizer. In our opinion this is an additional hint that these connections should be ‘blown up’ in the true configuration space.

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