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Charge sum rules in N=2 theories

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ABSTRACT

We derive sum rules involving moments of the $U(1)$ charge in the Ramond sector of $N = 2$ super-conformal field theories. These charge sum rules are obtained by analyzing the modular properties of the elliptic genus. The lowest order sum rule, $\langle Q^2 \rangle = \frac{\hat{c}}{12}$, pertains to the average of the charge squared over the Ramond ground ring. The higher sum rules contain information on the null state structure of the underlying chiral algebra.

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1. Introduction

Superconformal field theories (SCFTs) with $N = 2$ have been the subject of a large number of studies in the last few years, mainly due to their role in the construction of space–time supersymmetric vacua in string theory. Much is known about the structure of such theories, in particular for unitary models, to which we will restrict ourselves here. Most of the relevant information resides in an algebraic structure known as the chiral ring, introduced in [1], to which paper we refer for details and definitions. Although the algebraic structure of $N = 2$ SCFTs and in particular of their chiral rings is tightly constrained, a complete classification of all such SCFTs is not yet available.

One of the important quantities which characterize an $N = 2$ SCFT is a torus partition function defined as [2]

$$Z(q, z, \bar{z}) = \text{Tr}[(-1)^F q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{c}{24}} e^{2\pi i(zJ_0 + \bar{z}\bar{J}_0)}] , \quad (1.1)$$

where L_0 and J_0 are the Virasoro and $U(1)$ zero mode generators and bars refer to rightmovers. The trace is over all states in the Ramond sector. The computation of the partition function for $N = 2$ SCFTs is generally very difficult, and possible only for simple cases like the $N = 2$ minimal models.

The situation is better for the elliptic genus [3,4,2], which is the partition function taken at $\bar{z} = 0$,

$$Z(z|\tau) = \text{Tr}[(-1)^F q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{c}{24}} e^{2\pi izJ_0}] . \quad (1.2)$$

In this case, a simple supersymmetry argument shows that all states of non–zero $\bar{L}_0 - \frac{c}{24}$ cancel in pairs, so that the elliptic genus is a holomorphic function of $q = e^{2\pi i\tau}$. The elliptic genus is invariant under smooth deformations of the theory which preserve the right–moving supersymmetry, and in particular it is a topological invariant of the target space for supersymmetric sigma models. At $z = 0$ the elliptic

genus equals the Witten index $\text{Tr}[(-1)^F]$, which gives the Euler characteristic in the case of sigma models. The elliptic genus can quite easily be computed for Landau-Ginzburg models [2,5] and for their orbifolds [6,7], and has also been calculated for some sigma models [7] and for the supersymmetric $SU(2)/U(1)$ coset model [8].

Another essential property of the elliptic genus is its behavior under modular transformations. This behavior can be derived using the same argument that was used in [4] for the anomaly generating function. First one observes that $Z(0, \tau)$ is obtained from a path integral on the torus with periodic boundary conditions along both cycles, often referred to as the ‘‘PP-sector’’ (note that the operator $(-1)^F$ acts on left- as well as on right-movers). Under $\tau \rightarrow -\frac{1}{\tau}$ the characters in this sector transform among themselves, and we consider a combination of these characters so that $Z(0|\tau)$ is modular invariant, for example the diagonal invariant. Since $Z(0|\tau)$ is a constant this may seem trivial, but modular invariance is in general a non-trivial property when we write Z in terms of $N = 2$ characters, and becomes even more interesting when we include the z -dependence. To do so, note that including a z -dependence for the $U(1)$ characters affects their transformation under $\tau \rightarrow -\frac{1}{\tau}$ in only two ways: a rescaling of the argument z by $\frac{1}{\tau}$, and a factor $e^{\pi i N z^2 / \tau}$, where N is the normalization of the $U(1)$ current, $J(z)J(w) = \frac{N}{(z-w)^2}$ (the standard normalization in $N = 2$ models is $N = \hat{c} \equiv \frac{c}{3}$). Both effects are the same for all characters, and hence they only affect the partition function in a global way. Hence we obtain (see also [7]):

$$\begin{aligned} Z(z|\tau + 1) &= Z(z|\tau) \\ Z\left(\frac{z}{\tau} \middle| -\frac{1}{\tau}\right) &= e^{\pi i \hat{c} \frac{z^2}{\tau}} Z(z|\tau) . \end{aligned} \tag{1.3}$$

Holomorphic modular functions with similar transformations have appeared at least twice before in the literature. The first was the anomaly generating function of [4], whose transformation rule differs from (1.3) by a space-time dimension dependent weight factor, and the second was the ‘‘character valued’’ partition of the meromorphic conformal field theories at $c = 24$ [9] whose partition function is the

absolute modular invariant $J(\tau)$. In both cases the combination of holomorphicity and modular invariance allowed the derivation of general rules governing the content of the theory. These rules are respectively the Green-Schwarz factorization of space-time chiral anomalies, and a relation between the levels and dual Coxeter numbers of the Kac-Moody algebras forming a meromorphic $c = 24$ modular invariant.

The main purpose of this paper is to investigate whether any such general rules can be derived in the present case as well. We will indeed find a universal quadratic sum rule for the $U(1)$ charges of any $N = 2$ theory. In addition we find many other sum rules, but they always involve states of higher excitation levels, whose multiplicities depend on the chiral algebra. Only for minimal models, whose chiral algebra is nothing but the $N = 2$ algebra, can these sum rules be checked directly, but of course nothing new can be learned there. In all other cases the quadratic sum rule imposes a non-trivial constraint on the chiral ring, and all other sum rules impose rather complicated constraints on the action of the chiral algebra on the Ramond ground states.

Another motivation for this work is a recent speculation that the elliptic genus might be determined completely by the Poincaré polynomial. Under an assumption regarding the analytic behavior of the zeroes of the elliptic genus at $\tau = 0$ this was shown in [5]. However, from the present analysis it follows that uniqueness cannot be derived from modular invariance alone, indicating that the assumption made in [5] is not in general satisfied. Nevertheless, we will find that deviations from the conjecture must be such that they only affect sums of the twelfth power (or higher) of the charges, and that all lower order sum rules are indeed determined by the Poincaré polynomial.

In the next section we present the general parametrization of the $N = 2$ elliptic genus as dictated by modular invariance. Then, in section 3, we derive the sum rules. In section 4 a different analysis of the elliptic genus is presented that focuses on its zeroes, and from which the quadratic sum rule is obtained in a different

way. In at least one case this method seems to be more powerful than the one given in section 2. In section 5 we discuss the application of the sum rule to the special cases of Calabi-Yau, Landau-Ginzburg and coset models. In the latter case we investigate the effects of field identification fixed points. Finally we discuss the restrictions imposed by the sum rule on the null-vector structure of super W-algebras.

2. Eisenstein expansion of the elliptic genus

To remove the exponential factor in the transformation rule (1.3) we consider the following modification

$$\tilde{Z}(z|\tau) = e^{\frac{1}{2}\hat{c}z^2G_2(\tau)} Z(z|\tau) \quad (2.1)$$

where $G_2(\tau)$ is the Eisenstein function defined by

$$G_2(\tau) = 2\zeta(2) + 2(2\pi i)^2 \sum_{n=1}^{\infty} \sigma(n)q^n \quad (2.2)$$

and $\sigma(n) = \sum_{d|n} d$. Then, since $G_2(\tau+1) = G_2(\tau)$ and $G_2(-\frac{1}{\tau}) = \tau^2G_2(\tau) - 2\pi i\tau$, we find that $\tilde{Z}(z|\tau) = \tilde{Z}(z|\tau+1) = \tilde{Z}(\frac{z}{\tau} | -\frac{1}{\tau})$. Expanding \tilde{Z} now in a power series in z ,

$$\tilde{Z}(z|\tau) = \sum_{k=0}^{\infty} z^{2k} D_{2k}(\tau), \quad (2.3)$$

and inserting the transformations of \tilde{Z} , we find that $D_{2k}(\tau)$ must be a modular form of weight $2k$, meaning that $D_{2k}(\tau+1) = D_{2k}(\tau)$ and $D_{2k}(-\frac{1}{\tau}) = \tau^{2k} D_{2k}(\tau)$. These transformation properties, in addition to the knowledge that these functions do not have poles at $q = 0$, allow us to determine them up to a few unknown parameters. They can be expressed as polynomials in the Eisenstein functions G_4 and G_6 , for example

$$\begin{aligned}
D_0(q) &= c_0 & D_8(q) &= c_8(G_4(q))^2 \\
D_2(q) &= 0 & D_{10}(q) &= c_{10}G_4(q)G_6(q) \\
D_4(q) &= c_4G_4(q) & D_{12}(q) &= c_{12}^1(G_4(q))^3 + c_{12}^2(G_6(q))^2 \\
D_6(q) &= c_6G_6(q) & D_{14}(q) &= c_{14}(G_4(q))^2G_6(q) ,
\end{aligned}$$

etc. Functions with higher weights all have at least two parameters, and the number of parameters continues to increase in an obvious way.

3. Sum rules

The foregoing results enable us to express $Z(z|\tau)$ in terms of a set of real parameters c_{2k} . By expanding Z in z and/or q , and considering linear combinations from which these parameters cancel, we obtain sum rules for certain powers of the charges. Obviously $c_0 = \text{Tr}[(-1)^F]$, the Witten index. From the vanishing of D_2 we get

$$\text{Tr}[(-1)^F q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{c}{24}} J_0^2] = \frac{\hat{c}E_2(q)}{12} \text{Tr}[(-1)^F] , \quad (3.1)$$

where for convenience we define normalized Eisenstein functions E_{2k} whose constant term is equal to 1. Then $G_2 = \frac{\pi^2}{3}E_2$, and $E_2(q) = 1 - 24q + \dots$. To leading order in q we find thus the following sum rule for the charges of the Ramond ground states :

$$\text{Tr}[(-1)^F J_0^2] = \frac{\hat{c}}{12} \text{Tr}[(-1)^F] . \quad (3.2)$$

This is the main result of this paper. If we apply it to the diagonal invariant of an $N = 2$ conformal field theory whose Ramond ground states have multiplicity one[★]

★ This means each Ramond module has at most one state with $h = \frac{c}{24}$. This is true for example in coset theories as long as one does not extend their chiral algebra by, for example, simple currents. If there are more Ramond ground states per module, their relative sign of $(-1)^F$ is relevant, and then the more general formula (3.2) should be used.

the operator $(-1)^F$ acts in the same way on left- and right-movers, and we get the interesting relation

$$\langle J_0^2 \rangle = \frac{\hat{c}}{12} ,$$

where the average is over the chiral ring. This simplified form of the sum rule is valid, for example, for any coset model, and in particular for the minimal models where it can be verified easily. This sum rule is quite surprising since in unitary $N = 2$ theories the lowest and highest charges are plus or minus $\frac{\hat{c}}{2}$ so that one would perhaps expect $\langle J_0^2 \rangle$ to be quadratic in \hat{c} , but it turns out to be linear.

Note that (3.1) upon expansion in q yields a quadratic sum rule for every excitation level. It is perhaps instructive to compare this result with the analogous situation for chiral anomalies. In that case the argument is very similar, but first of all the presence of an extra weight factor in the modular transformations shifts the order of J_0 upward, so that one gets relations among higher order traces. Secondly, the elliptic genus of a superstring can have a pole at $q = 0$. In that case the analog of D_2 does not vanish, but it is still true that its constant term vanishes.[†] This observation was sufficient to prove Green-Schwarz factorization for the chiral anomalies in string theory [4].

Returning now to $N = 2$ models, let us consider the expansion of (3.1) to higher orders in q . For the first excited level one obviously gets

$$\text{Tr}[(-1)^F J_0^2]_{\text{level 1}} = -2\hat{c}\mathcal{N} ,$$

where $\mathcal{N} = \text{Tr}[(-1)^F]$ denotes the total number of Ramond ground states. In this case the $(-1)^F$ factor clearly *is* important, since some of the states at the first level are created by the $N = 2$ supercurrent, and hence contribute with the opposite sign. It is interesting to inspect this sum rule for the special case of the minimal $N = 2$ models. Minimality implies that excitations can only be produced by acting

[†] Furthermore the full q -dependence can be determined from the residue of the pole.

with generators of the $N = 2$ algebra itself. Acting on a generic Ramond ground state with charge Q and $(-1)^F = 1$, the supercurrents produce two states with $(-1)^F = -1$ and charges $Q \pm 1$, except for the maximal and minimal charge states (those with $Q = \frac{\hat{c}}{2}$). For the latter two states one can show, using the $N = 2$ algebra, that the states with charge $\pm|\frac{\hat{c}}{2} + 1|$ are in fact null states. Because of supersymmetry these $(-1)^F = -1$ states are matched by an equal number of states with $(-1)^F = 1$, generated from the ground states by L_{-1} and J_{-1} (modulo null states). These states have thus the same charge as the ground state, Q . Hence the sum rule reads

$$\sum_i [2Q_i^2 - (Q_i - 1)^2 - (Q_i + 1)^2] - 2[(\frac{\hat{c}}{2})^2 - (\frac{\hat{c}}{2} + 1)^2] = -2\hat{c}\mathcal{N} .$$

Here the sum is over the entire chiral ring, and the last term removes the null states. This sum rule leads to the following relation between \hat{c} and \mathcal{N} :

$$\hat{c} = \frac{\mathcal{N} - 1}{\mathcal{N} + 1} .$$

This relation is indeed satisfied for minimal models (note that $\mathcal{N} = k + 1$). Conversely, this is a very simple and direct way of obtaining the set of allowed central charges for minimal unitary $N = 2$ models. When we apply the sum rule to non-minimal models it yields a constraint on the squares of the charges of those excited states that are created by the operators that extend the chiral algebra.

One can also consider higher powers of J_0 . After some straightforward algebra, we obtain the following sum rule for J_0^4 :

$$\text{Tr}[(-1)^F q^{L_0 - \frac{\hat{c}}{24}} \bar{q}^{\bar{L}_0 - \frac{\hat{c}}{24}} J_0^4] = c_4 E_4(q) + \frac{1}{48} \hat{c}^2 (E_2(q))^2 \text{Tr}[(-1)^F] \quad (3.3)$$

(here c_4 has been redefined by an irrelevant factor). The unknown constant c_4 can be determined at level 0, and once it is known we can predict the quartic traces at all the higher levels. In a similar way slightly more complicated sum rules for traces of order 6, 8 and 10 can be derived, as well as for a particular linear combination of traces of order 14 and 12.

4. Analysis of the zeroes of the elliptic genus

In this section we present a different way of analyzing the elliptic genus, by parametrizing the zeroes in the first argument, z . In the theories we are interested in, the $U(1)$ charges of all states are (in the Neveu-Schwarz sector) integer multiples of $\frac{1}{h}$ for some integer h , and we will find it convenient to change variables to $u = z/h$, and to define $t = e^{2\pi i u}$. Obviously, $\hat{c}h$ is then an integer, since the highest charge state has charge \hat{c} in this sector.

The modular transformations of the elliptic genus (which we will now consider as a function of u and τ) are as follows. The transformations in the τ variable follow directly from (1.3):

$$\begin{aligned} Z(u|\tau + 1) &= Z(u|\tau) \\ Z\left(\frac{u}{\tau} \middle| -\frac{1}{\tau}\right) &= e^{\pi i \hat{c} h^2 \frac{u^2}{\tau}} Z(u|\tau). \end{aligned} \tag{4.1}$$

Using the last equality twice we can easily get the charge conjugation symmetry

$$Z(-u|\tau) = Z(u|\tau). \tag{4.2}$$

The u transformations of the elliptic genus are [7]:

$$\begin{aligned} Z(u + 1|\tau) &= (-1)^{\hat{c}h} Z(u|\tau) \\ Z(u + \tau|\tau) &= (-1)^{\hat{c}h} e^{-\pi i \hat{c} h^2 (\tau + 2u)} Z(u|\tau). \end{aligned} \tag{4.3}$$

The Poincaré polynomial of the theory is defined as $P(t) = \lim_{q \rightarrow 0} Z(u|\tau)$ and receives contributions only from the ground ring of the $N = 2$ theory.

To analyze the general form of the elliptic genus, we will use the Jacobi theta functions, defined as in [10]. Our first claim is that the elliptic genus of an $N = 2$

theory may always be written as

$$Z(u|\tau) = A(\tau) \left(\Theta_1\left(u - \frac{1}{2}|\tau\right) \Theta_1\left(u - \frac{\tau}{2}|\tau\right) \Theta_1\left(u + \frac{\tau+1}{2}|\tau\right) \right)^p \times \prod_{i=1}^n \Theta_1(u - a_i(\tau)|\tau) \Theta_1(u + a_i(\tau)|\tau) \quad (4.4)$$

where $p = 1$ if $\hat{c}h^2$ is an odd integer and $p = 0$ otherwise, $n = \frac{1}{2}(\hat{c}h^2 - 3p)$ and $A(\tau)$ and $a_i(\tau)$ are some functions (obviously the $a_i(\tau)$ are zeroes of the elliptic genus). We will assume that the zeroes are well-behaved functions of τ (except perhaps in singular limits such as $\tau \rightarrow 0$), a reasonable assumption since the function $Z(u|\tau)$ is well-behaved. Because of equation (4.2), if a is a zero of Z for some τ then so is $-a$ (defined modulo 1 and τ). Therefore, all zeroes of the elliptic genus come in pairs except perhaps for zeroes at the half-integer lattice points $0, \frac{1}{2}, \frac{\tau}{2}$ and $\frac{\tau+1}{2}$. Double zeroes at these points can also be added to the a_i 's (a zero at $u = 0$ must be of even order because of the charge conjugation symmetry). However, if there is a single zero at one of these points for some τ , there must be a single zero there in a neighborhood of this τ since it cannot move and still obey the $u \rightarrow -u$ symmetry. Therefore, assuming the zeroes are well behaved we get that there must be such a zero for all τ . We know that for all τ , $Z(0|\tau) = \text{Tr}[(-1)^F]$ is the Witten index. A zero at one of the other half-integer lattice points for all τ necessarily gives a zero at all three points, since $Z(\frac{\tau}{2}|\tau) = e^{-\pi i \hat{c} h^2 \frac{\tau}{4}} Z(\frac{1}{2} | -\frac{1}{\tau})$ and $Z(\frac{\tau+1}{2}|\tau) = Z(\frac{\tau+1}{2}|\tau+1)$. Therefore, either for all τ there is a single zero (after removing double zeroes) at all three half-integer lattice points, or there is no single zero at any of them for all τ .

Let us now show that the total number of zeroes is always equal to $\hat{c}h^2$. For every τ , $Z(u|\tau)$ is an analytic function of u , so that by Cauchy's theorem the number of zeroes of Z inside the parallelogram connecting the points $(0, 1, \tau+1, \tau)$ is $\frac{1}{2\pi i}$ times the du integral of $\frac{Z'(u|\tau)}{Z(u|\tau)}$ along the sides of the parallelogram (Z' denotes the derivative of Z with respect to u). However, from equation (4.3) it is easy to see that $\frac{Z'(u+1|\tau)}{Z(u+1|\tau)} = \frac{Z'(u|\tau)}{Z(u|\tau)}$ and $\frac{Z'(u+\tau|\tau)}{Z(u+\tau|\tau)} = \frac{Z'(u|\tau)}{Z(u|\tau)} - 2\pi i \hat{c} h^2$ and

therefore the integral can easily be computed (since the parallel sides cancel up to a constant) and equals $2\pi i \hat{c} h^2$, and therefore $Z(u|\tau)$ indeed has $\hat{c} h^2$ zeroes inside the parallelogram for every τ . The claim (4.4) is now obvious if we define the $a_i(\tau)$ to be the zeroes which are not at half-integer lattice points (choosing one from each pair of such zeroes). Since both sides of equation (4.4) (taking the right side without $A(\tau)$) have the same zeroes and no poles and furthermore the same $u \rightarrow u + 1$ and $u \rightarrow u + \tau$ transformations, their ratio is (for every τ) an elliptic function with no zeroes or poles and therefore a constant, which we denote by $A(\tau)$. Actually, in proving that the u transformations of both sides are the same we need to use the equality $(-1)^{\hat{c}h} = (-1)^{\hat{c}h^2}$, meaning that if $\hat{c}h$ is odd then h must be odd as well. This is correct, since if $\hat{c}h$ is odd we get from (4.2) and (4.3) that $Z(\frac{1}{2}|\tau) = 0$. Therefore Z has a zero at the half-integer lattice points, $p = 1$ and $\hat{c}h^2$ which is the total number of zeroes must be odd as well (the zero at $u = \frac{1}{2}$ must be of odd order since $Z(\frac{1}{2} + u|\tau) = Z(-\frac{1}{2} - u|\tau) = -Z(\frac{1}{2} - u|\tau)$). It is not clear how this constraint may be obtained directly from the $N = 2$ algebra.

The next step of our proof uses well-known identities of theta functions which state that

$$\Theta_1(u - a_i(\tau))\Theta_1(u + a_i(\tau)) = (\Theta_4)^{-2}[\Theta_1(u)^2\Theta_4(a_i(\tau))^2 - \Theta_4(u)^2\Theta_1(a_i(\tau))^2] \quad (4.5)$$

(where we define $\Theta_i(u) = \Theta_i(u|\tau)$ and $\Theta_i = \Theta_i(0|\tau)$) to transform (4.4) into

$$Z(u|\tau) = (\Theta_2(u)\Theta_3(u)\Theta_4(u))^p \sum_{k=0}^n A_k(\tau)(\Theta_1(u))^{2k}(\Theta_4(u))^{2(n-k)} \quad (4.6)$$

where all functions depending only on τ and not on u were swallowed into the arbitrary functions $A_k(\tau)$.

Next we shall analyze the $\tau \rightarrow \tau + 1$ and $\tau \rightarrow -\frac{1}{\tau}$ transformations of (4.6), and get the transformations of the $A_k(\tau)$ functions. Let us first look at the $\tau \rightarrow \tau + 1$ transformation, under which $Z(u|\tau) = Z(u|\tau + 1)$. Inserting the τ transformations

of the theta functions we get

$$\begin{aligned}
Z(u|\tau) &= Z(u|\tau + 1) = \\
&= (\Theta_2(u)\Theta_3(u)\Theta_4(u)e^{\frac{1}{4}i\pi})^p \sum_{k=0}^n A_k(\tau + 1) (\Theta_1(u))^{2k} e^{\frac{1}{2}i\pi k} (\Theta_3(u))^{2(n-k)}.
\end{aligned} \tag{4.7}$$

Now, by using the identity

$$(\Theta_3(u))^2 = (\Theta_4)^{-2} [(\Theta_4(u))^2 (\Theta_3)^2 - (\Theta_1(u))^2 (\Theta_2)^2] \tag{4.8}$$

and equating the coefficients on both sides, we find that the transformation of A_k is given by

$$A_k(\tau) = \sum_{j=0}^k A_j(\tau + 1) e^{\frac{1}{4}i\pi(p-2j)} \binom{n-j}{n-k} (-1)^k (\Theta_4)^{2(j-n)} (\Theta_2)^{2(k-j)} (\Theta_3)^{2(n-k)}. \tag{4.9}$$

The computation of the $\tau \rightarrow -\frac{1}{\tau}$ transformation is similar. After some algebra, using the identity

$$(\Theta_2(u))^2 = (\Theta_4)^{-2} [(\Theta_4(u))^2 (\Theta_2)^2 - (\Theta_1(u))^2 (\Theta_3)^2], \tag{4.10}$$

we get the transformation equation

$$A_k(\tau) = \sum_{j=0}^k A_j\left(-\frac{1}{\tau}\right) (-i\tau)^{\frac{1}{2}\hat{c}h^2} \binom{n-j}{n-k} (-1)^k (\Theta_4)^{2(j-n)} (\Theta_3)^{2(k-j)} (\Theta_2)^{2(n-k)}. \tag{4.11}$$

Note that in both transformations of A_k only A_j 's with j less than or equal to k participate.

Let us now start by looking at $A_0(\tau)$: this is easily determined by taking

$u = 0$, and we find that

$$Z(0|\tau) = (\Theta_2\Theta_3\Theta_4)^p A_0(\tau)(\Theta_4)^{2n} . \quad (4.12)$$

Now, using the fact that $Z(0|\tau) = \text{Tr}[(-1)^F]$ for all τ , we find that

$$A_0(\tau) = \text{Tr}[(-1)^F](\Theta_4)^{-2n}(\Theta_2\Theta_3\Theta_4)^{-p}. \quad (4.13)$$

Hence A_0 is the same up to a multiplicative constant in all theories with the same \hat{c} , h , and is obviously determined by the $q \rightarrow 0$ limit of the elliptic genus (the Poincaré polynomial) alone. It can be checked that the expression (4.13) obeys the equations (4.9) and (4.11) for the case of $k = 0$. If $\text{Tr}[(-1)^F] = 0$ we find that $A_0(\tau) = 0$.

In general the $q \rightarrow 0$ limit of equation (4.6) can be computed to be

$$\lim_{q \rightarrow 0} Z(u|\tau) = (2 \cos(\pi u))^p \sum_{k=0}^n 2^{2k} (\sin(\pi u))^{2k} \lim_{q \rightarrow 0} (A_k(\tau) q^{\frac{k}{4} + \frac{p}{8}}) \quad (4.14)$$

and therefore, since we are dealing with theories with finite Poincaré polynomials, the low q behavior of the functions A_k is of the form

$$A_k(\tau) = C_k q^{-\frac{p}{8} - \frac{k}{4}} (1 + O(q)) . \quad (4.15)$$

The Poincaré polynomial is then

$$P(t) = (2 \cos(\pi u))^p \sum_{k=0}^n C_k 2^{2k} (\sin(\pi u))^{2k}. \quad (4.16)$$

Note that there are $\frac{1}{2}\hat{c}h^2$ terms in this expansion, but in unitary theories there are only $\frac{1}{2}\hat{c}h$ non-zero terms in the Poincaré polynomial, so that for h not equal to one the higher coefficients C_k all vanish.

4.1. CONSTRAINTS ON THE CHIRAL RING FROM THE ELLIPTIC GENUS

Let us now try to solve equations (4.9) and (4.11) for the A_k 's. The solution we found above for A_0 is of course unique, both because of the Witten index argument and also directly since the ratio B of any other solution divided by (4.13) would have to satisfy $B(\tau) = B(\tau + 1) = B(-\frac{1}{\tau})$ and have no poles, and therefore must be a constant. The equations for the other A_k 's are linear, so that their general solution is the sum of a special solution of the equations, plus a general solution to the homogeneous equations obtained by setting all A_j 's with $j < k$ to zero in equation (4.11). Looking at these homogeneous equations

$$\begin{aligned} A_k(\tau) &= e^{\frac{1}{4}i\pi(p+2k)}(\Theta_4)^{2(k-n)}(\Theta_3)^{2(n-k)}A_k(\tau+1) \\ A_k(\tau) &= (-i\tau)^{\frac{1}{2}ch^2}(-1)^k(\Theta_4)^{2(k-n)}(\Theta_2)^{2(n-k)}A_k(-\frac{1}{\tau}) \end{aligned} \quad (4.17)$$

we see that the equation for A_k is very similar to the equation for A_0 , taken with $n - k$ instead of n . So, let us denote the unique solution of that equation by

$$F_n(\tau) = (\Theta_4)^{-2n}(\Theta_2\Theta_3\Theta_4)^{-p} \quad (4.18)$$

and look at the ratio $B_k(\tau) = A_k(\tau)/F_{n-k}(\tau)$. The equations it satisfies are

$$\begin{aligned} B_k(\tau) &= B_k(\tau+1)e^{\frac{1}{2}i\pi k} \\ B_k(\tau) &= B_k(-\frac{1}{\tau})(i\tau)^k \end{aligned} \quad (4.19)$$

and the solution should behave as $q^{-\frac{k}{4}}$ as $q \rightarrow 0$ for consistency. Let us now define $H_k(\tau) = (\eta(\tau))^{6k}B_k(\tau)$. The transformations of H_k are now easily found to be

$$\begin{aligned} H_k(\tau+1) &= H_k(\tau) \\ H_k(-\frac{1}{\tau}) &= \tau^{2k}H_k(\tau) \end{aligned} \quad (4.20)$$

so that H_k is a modular function of weight $2k$, and since it has no poles at finite τ it must be a sum of Eisenstein functions as in chapter 2. The H_k functions then

also have the correct $q \rightarrow 0$ behavior because of the $\eta(\tau)^{6k}$ factor. From the fact that there exist no non-zero modular forms of weight 2 we find that H_1 must be zero, so that there is no non-trivial solution to the homogeneous equation for A_1 , and A_1 is completely determined by A_0 (their ratio is a constant function).

Let us examine the consequences of this fact - it implies a linear restriction on the Poincaré polynomial of such theories, since the ratio $\frac{C_1}{C_0}$ (with the coefficients C_i as defined above appearing in the expression for the Poincaré polynomial) must also be a constant for every \hat{c} and h . In fact it is simpler to expand the Poincaré polynomial in a power series in u , $P(t) = c_0 + c_1(\pi u)^2 + c_2(\pi u)^4 + \dots$, and then we found that c_1/c_0 must be the same for all theories with the same transformations. We can relate this ratio to the average of J_0^2 in the chiral ring. We defined

$$\text{Tr}[(-1)^F e^{2\pi i u h J_0}] = c_0 + c_1(\pi u)^2 + \dots \quad (4.21)$$

and then by taking $u = 0$ we immediately find $\text{Tr}[(-1)^F] = c_0$, while

$$\text{Tr}[(-1)^F J_0^2] = -\frac{1}{4h^2} \frac{\partial^2}{\partial(\pi u)^2} \text{Tr}[(-1)^F e^{2\pi i u h J_0}] = -\frac{c_1}{2h^2} \quad (4.22)$$

(where the derivative is taken at $u = 0$). Now it is clear that the resulting relation must be precisely (3.2). This allows us to determine the ratio of c_1 and c_0 : $\frac{c_1}{c_0} = -\frac{\hat{c}h^2}{6}$.

The analysis in this section has thus provided us with a second method for deriving the quadratic sum rule (and in fact all other sum rules as well). It is clear that the two methods are closely related, since they both give expansions in terms of Eisenstein functions. Indeed, by expanding (4.6) in u one can directly relate the functions A_k to the functions D_{2k} introduced in section 2. The foregoing discussion shows that the number of free parameters is identical for each order $k \leq n$. Nevertheless there is an important difference between the two approaches: (4.6) expresses the elliptic genus in terms of a finite number of functions A_k , whereas the sum in (2.3) is infinite. This is a consequence of the fact that in the present

section we put in some extra information, namely the exact quantization of the $U(1)$ charge in units of $\frac{1}{h}$. In the approach of section 2 the same information would eventually also emerge, because for a given charge quantization there is only a finite number of independent traces. Hence for a sufficiently large power k , $\text{Tr}(-1)^F J_0^k$ is fully determined by the lower order traces, and hence there are no more free parameters. However, it seems nearly impossible to carry this out in practice and demonstrate explicitly – without using (4.6) – that the number of free parameters in (2.3) is actually finite. The number of parameters in the second expansion is exactly the number of linearly independent modular forms of degrees less than or equal to $2n = \hat{c}h^2 - 3p$. For each degree one parameter may be determined from the Poincaré polynomial.

5. Applications

5.1. CALABI-YAU COMPACTIFICATIONS

Looking at specific theories, the fact that $\langle J_0^2 \rangle = \frac{\hat{c}}{12}$ has various implications. We first consider Calabi-Yau models, which have $\hat{c} = d$ (the complex dimension) and charges quantized according to $h = 1$ (h is defined in chapter 4). The total contribution to $\text{Tr}[(-1)^F]$ from states of charge $p - \frac{d}{2}$ is given by $N_p = \sum_{q=0}^d (-1)^{p+q} h_{p,q}$ where $d = \hat{c}$ and $h_{p,q}$ are the Hodge numbers of the surface. The sum rule yields the strange relationship

$$\sum_{p,q=0}^d h_{p,q} (-1)^{p+q} (p - \frac{d}{2})^2 = \frac{d}{12} \sum_{p,q=0}^d h_{p,q} (-1)^{p+q} = \frac{d}{12} \chi(M) \quad (5.1)$$

where χ is the Euler characteristic of the surface. This can be used, with the help of Poincaré duality, *i.e.* $N_p = N_{d-p}$, to determine the Poincaré polynomials for $d = 1, 2$ and 3 almost completely.

For $\hat{c} = 1$ the sum rule reads $\frac{1}{2}N_0 = \frac{1}{6}N_0$, so that $N_0 = 0$. This reflects the well-known fact that the only 1-dimensional “Calabi-Yau” manifold is a torus.

For $\hat{c} = 2$ we find $2N_0 = \frac{1}{6}(2N_0 + N_1)$, which immediately results in all Poincaré polynomials being proportional to the one of the $K3$ surface, $t^{-1}(2+20t+2t^2)$. The overall factor is of course never determined by considerations of modular invariance. It is interesting to note that the same result can be obtained from the requirement of Green-Schwarz factorization of the chiral gauge and gravitational anomalies in six-dimensional strings. This factorization was derived in [4] from a different, but closely related elliptic genus, the character valued chiral partition function. This function contains at least the same information as the elliptic genus considered in the present paper, since the $U(1)$ charge of the $N = 2$ algebra is absorbed in the E_7 gauge group, and contributes to the chiral anomaly. The computations yielding the spectrum are a bit more cumbersome, since they involve quartic traces rather than quadratic ones, but on the other hand one can get a little more information from the gravitational anomalies, which determine the number of gauge singlets in the string spectrum. In this approach, the overall normalization of the Poincaré polynomial is fixed by space-time supersymmetry considerations, *i.e.* the number of gravitinos in the spectrum.

For $\hat{c} = 3$ we get the relation $\frac{9}{2}N_0 + \frac{1}{2}N_1 = \frac{1}{2}N_0 + \frac{1}{2}N_1$. This implies that $N_0 = 0$, so that all Poincaré polynomials must be proportional to $t^{-\frac{1}{2}}(1+t)$. This is again not an unknown result, but the derivation is quite interesting.

In all these cases (or generally whenever $\hat{c}h^2$ equals 1,[★] 2, 3 or 5) n in equation (4.6) is less than 2, so that there is no freedom in determining the elliptic genus. For example, in the $\hat{c} = 3$ Calabi-Yau case we find that the elliptic genus is given by $Z(z|\tau) = \frac{1}{2} \text{Tr}[(-1)^F] \frac{\Theta_1(2z|\tau)}{\Theta_1(z|\tau)}$ (this follows from the same argument as given in [5]; for a different approach see [14]). For larger \hat{c} , the elliptic genus is not determined just from the modular transformations. Note that for $\hat{c} = 5$ the quadratic sum rule itself is not sufficient. Furthermore for $\hat{c} = 2, 3$ and 5 it is not at all manifest that

★ For $\hat{c}h^2 = 1$ the the analysis of chapter 4 shows that $p = 1$ and hence $n = -1$, which does not seem to make sense. However in this case the elliptic genus vanishes identically, and therefore the analysis of section 4 does not apply. The Eisenstein expansion and hence the sum rules remain valid, however.

the higher order sum rules determine the elliptic genus. In this case the analysis of section 4, that uses charge integrality in the Neveu-Schwarz sector, appears to be more powerful than the sumrules alone. Nevertheless careful analysis of the higher sum rules should lead to the same conclusion.

5.2. LANDAU-GINZBURG THEORIES

The elliptic genus of Landau-Ginzburg theories, is known [2,5] to be of the form $\prod_{i=1}^l \frac{\Theta_1((d-d_i)u|\tau)}{\Theta_1(d_i u|\tau)}$, where $\hat{c}h = \sum_{i=1}^l (d - 2d_i)$, and, as before, $u = z/h$ (in this case $h = d$). The $q \rightarrow 0$ limit is $\prod_{i=1}^l \frac{\sin((d-d_i)\pi u)}{\sin(d_i \pi u)}$, and expanding in a power series in πu we find that it behaves as

$$\left(\prod_{i=1}^l \frac{d-d_i}{d_i}\right) \left(1 - \frac{1}{6}(\pi u)^2 \sum_{i=1}^l ((d-d_i)^2 - d_i^2) + \dots\right) \quad (5.2)$$

which equals

$$\left(\prod_{i=1}^l \frac{d-d_i}{d_i}\right) \left(1 - \frac{1}{6}(\pi u)^2 \hat{c}h^2 + \dots\right) . \quad (5.3)$$

This trivially satisfies the quadratic sum rule. This is not surprising since the expression for the elliptic genus is manifestly modular invariant. Hence the sum rules do not yield any useful information about such models.

5.3. COSET MODELS

In coset models the sum rule gives a non-trivial equality between expressions involving the weights of the groups involved. In such models the equation (3.2) may also be used to constrain the possible modular invariants of a group G - the diagonal elements appearing in such a modular invariant must satisfy $\langle J_0^2 \rangle = \frac{\hat{c}}{12}$ for every coset G/H (where J_0 is the $U(1)$ charge of that coset for the relevant representations and \hat{c} is its central charge). For G/H coset models it is natural to

take $h = k + g$ where g is the second Casimir invariant of G , and then the central charge is given by [11]

$$\hat{c}h = h \frac{d_G - d_H}{2} - 4\rho_G \cdot (\rho_G - \rho_H). \quad (5.4)$$

The primary chiral states of these models are in a one to one correspondence with pairs (Λ, ω) , where Λ is a weight of the group G at level k and $\omega \in W(G/H)$ (meaning that ω is a Weyl transformation of G that preserves the positivity of the positive roots of H), identified by the action of the outer automorphism σ of the algebra G [1,11]. When this automorphism has no fixed points, all of its orbits have a length $|Z(G)|$ (where $Z(G)$ is the center of the group G), so that ignoring the identification would just cause us to count each state $|Z(G)|$ times and would not change $\langle J_0^2 \rangle$. The charge of the state corresponding to (Λ, ω) in the Neveu-Schwarz sector is given by [11]

$$hQ_\omega^\Lambda = hl(\omega) + 2(\omega^{-1}(\Lambda + \rho_G) - \rho_G) \cdot (\rho_G - \rho_H) \quad (5.5)$$

where $l(\omega)$ is the length of the Weyl transformation ω . The charges in the Neveu-Schwarz sector are shifted by $\hat{c}/2$ relative to those in the Ramond sector, leading to $\langle Q^2 \rangle = \frac{\hat{c}}{12} + \frac{\hat{c}^2}{4}$ in this sector. Inserting the formulas above, and denoting by N_k^G the number of weights of G_k , we find for the case in which σ has no fixed points the formula

$$\begin{aligned} \sum_{\Lambda \in G_k} \sum_{\omega \in W(G/H)} (hl(\omega) + 2(\omega^{-1}(\Lambda + \rho_G) - \rho_G) \cdot (\rho_G - \rho_H))^2 = \\ = N_k^G \frac{|W(G)|}{|W(H)|} \left[\frac{h}{12} \left(h \frac{d_G - d_H}{2} - 4\rho_G \cdot (\rho_G - \rho_H) \right) + \right. \\ \left. \frac{1}{4} \left(h \frac{d_G - d_H}{2} - 4\rho_G \cdot (\rho_G - \rho_H) \right)^2 \right]. \end{aligned} \quad (5.6)$$

This formula may easily be checked to be correct in the $k = 1$ case, where the

Poincaré polynomial is given by [1,11]

$$t^{\frac{\hat{c}}{2}} P_{G/H}(t) = \prod_{\alpha \in \Delta_{G/H}} \frac{1 - t^{k+g-\alpha \cdot \rho_G}}{1 - t^{\alpha \cdot \rho_G}}. \quad (5.7)$$

It is also possible to prove (5.6) directly for the $A(n, m, k)$ case, in which $G/H = SU(n+m)_k / (SU(n) \times SU(m) \times U(1))$. The formula turns out to be correct for all n, m, k , even though there are fixed points when n, m, k have a common factor.

5.4. FIXED POINTS

When the field identification has fixed points, the correct rule for Ramond ground states is to count one such state for each identification orbit [12]. In the analysis of the previous subsection we have thus overcounted the non-fixed points by a factor p , the order of the identification, and we still satisfied the sum rule. This can be explained as follows. Let us assume for the moment that p is prime, so that there are only simple fixed points. The resolution of fixed points requires a matrix \hat{S} , defined only on the fixed points, which together with the matrix T of the fixed points defines a representation of the modular group [13]. In the case of $A(n, m, k)$ this representation turns out to be precisely equal to the one of another such theory, namely $A(n/p, m/p, k/p)$ [12]. The charges of the "fixed point conformal field theory" turn out to be $\frac{1}{p}$ times the charges of the corresponding fixed points of $A(n, m, k)$, and its central charge is \hat{c}/p^2 . Since the sum rule is satisfied for the fixed point CFT, it follows that the fixed points of $A(n, m, k)$ have an average charge-squared of precisely $\hat{c}/12$, and therefore counting them with any multiplicity, even the wrong one, will not change the result. If the fixed point CFT itself has field identification fixed points one can just iterate this argument, so that we may conclude that the average of the squares of the charges should be equal to $\frac{\hat{c}}{12}$ for all sets of fixed points of the same order.

This "fixed point independence" is empirically also true for KS-models of type B , C and D , for which the fixed point resolution matrix \hat{S} cannot be identified with

a known $N = 2$ theory. This can be understood in the following way. Before one resolves the fixed point and normalizes the partition function, one has a perfectly well-defined elliptic genus for the coset branching functions. Since the partition function contains the identity more than once the partition function and the elliptic genus do not correspond to a sensible $N = 2$ theory, but all the conditions for the sum rules (essentially holomorphicity and modular invariance) are satisfied. This explains why even without dealing correctly with fixed points one gets the correct average of Q^2 . Furthermore the sum rule is of course also satisfied if one deals correctly with the fixed points. Hence the average of the squares of the fixed point charges must separately be equal to $\frac{\hat{c}}{12}$. Unlike the previous argument this one does not immediately generalize to multiple fixed points, but in models of type B , C and D these do not occur anyway.

5.5. LEVEL-1 NULL VECTORS

As a final application of the sum rules we show how they can be used to get information about the null-vector structure of $N = 2$ W -algebras, information that would be rather difficult to get directly from the algebra due to its non-linearity. The $N = 2$ structure implies that extra generators must come in quartets consisting of two bosonic generators of charge zero and two fermionic ones of charge ± 1 . Hence a Ramond ground state with charge q and $(-1)^F = 1$ can have, in addition to its $N = 2$ excitations, four additional excited states generated by the -1 modes of each such quartet. Supersymmetry requires these states to come in pairs with opposite sign of $(-1)^F$, and charges $(q, q + 1)$ or $(q, q - 1)$. These states appear unless they are null states of the extended algebra. The presence of these states is controlled by the first level sum rules, as explained in section 3. If $|q| < \frac{1}{2}$ (which is always true if $\hat{c} < 1$) each pair contributes with a negative sign, so that no cancellations are possible. In this case the sum rules are clearly more powerful than for $\hat{c} > 1$, which is unfortunate since this is precisely where extensions of the algebra become important. Although the sum rules impose non-trivial restrictions also for $\hat{c} > 1$, we will restrict ourselves here to the simpler case $\hat{c} \leq 1$.

Consider first $A(2, 2, 1)$. This theory has $\hat{c} = \frac{4}{5}$ and hence belongs to the minimal series. It is in fact a D-type invariant of $A(1, 1, 8)$, whose chiral algebra is extended by the simple current of $SU(2)_8$ and its $N = 2$ partners. Hence the $A(2, 2, 1)$ charges are a subset of those of $A(1, 1, 8)$, and are $\pm\frac{2}{5}, \pm\frac{1}{5}$ each occurring once and 0 occurring twice. At the first excited level the following $N = 2$ doublets appear:

$$a \times \left(-\frac{2}{5}, -\frac{7}{5}\right) \cup (1+b) \times \left(-\frac{2}{5}, \frac{3}{5}\right) \cup (1+c) \times \left(-\frac{1}{5}, \frac{4}{5}\right) \\ \cup (1+d) \times \left(-\frac{1}{5}, -\frac{6}{5}\right) \cup (1+e) \times (0, 1) \cup (1+f) \times (0, -1) + \text{c.c.}$$

The first member of each pair contributes with $(-1)^F = 1$ and the second with the opposite sign. The parameters $a \dots f$ indicate the unknown excitations due to the extra current quartet. Since there is only one such quartet, these coefficients can only be 0 or 1, but one does not actually need this limitation to solve the equations. The excitations of the $N = 2$ algebra itself are known and are as indicated.

The quadratic sum rule at the first level is $\text{Tr}[(-1)^F J_0^2] = -2\hat{c} \text{Tr}[(-1)^F] = -\frac{48}{5}$. This yields the following equation for the unknown coefficients

$$-42 - 2(9a - b - 3c - 7d - 5e - 5f) = -48 .$$

The only solution is $c = 1$, and all other parameters equal to zero. The quartic sum rule may be used as a check. If one does not restrict the range of the parameters to 0 and 1, the quartic sum rule fixes any remaining ambiguity. It is interesting that most of the expected excitations are null states.

A second example we have studied is $A(3, 2, 1)$, which has $\hat{c} = 1$ and 10 chiral primary fields with charges $\pm\frac{1}{2}, \pm\frac{1}{3}, 2 \times (\pm\frac{1}{6})$ and 2×0 . The excited state can be described by 8 integer parameters. In this case the quadratic equation has many degeneracies, but the quartic one determines all parameters except one that cannot be determined by any sum rule, namely the multiplicity of a charge $(-\frac{1}{2}, \frac{1}{2})$ pair. Apart from the known $N = 2$ excitations, we find the following pairs at the

first level: $(-\frac{1}{3}, \frac{2}{3}), (-\frac{1}{6}, \frac{5}{6}), (0, 1)$ plus an unknown number of $(-\frac{1}{2}, \frac{1}{2})$ pairs, plus complex conjugates.

Although this information about first level excitations may not be extremely important in itself, it does give interesting information about the null state structure of $N = 2$ W-algebras, which apparently is rather subtle. If on the other hand one could understand the null state structure directly from the algebra, one could use the sum rules as a classification tool for the chiral rings that may belong to such an algebra. In that case one would start with a set of chiral primaries whose charges are the free parameters, and use the sum rules to determine the central charge and the allowed charges. For minimal models this would certainly give us the first members of the series, if we had no other way of getting them. Unfortunately our present knowledge about W-algebras is too limited to contemplate such a classification programme seriously.

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