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Bäcklund Transformations and Zero-Curvature Representations of Systems of Partial Differential Equations

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I Introduction

During the past three decades so-called integrable systems of nonlinear partial differential equations (PDEs) have attracted much interest both in physics and in mathematics. This interest is owing to the numerous applications which such equations have in many different branches of physics and to the rich mathematical structures which showed up behind them. Such structures are the existence of Lax pairs, Miura maps, Bäcklund transformations, infinitely many local conservation laws, (bi-)Hamiltonian structures and the applicability of inverse scattering methods.

Celebrated examples of integrable nonlinear PDEs which have some or all of these remarkable properties are the Korteweg-de Vries (KdV) equation, the modified Korteweg-de Vries (mKdV) equation, the Sine-Gordon equation, the Liouville equation and the nonlinear Schrödinger equation. Meanwhile one knows infinitely many systems of nonlinear PDEs with these properties. A famous example of an infinite set of nonlinear integrable PDEs is given by the KdV hierarchy which in fact itself is just one member of an infinite set of related hierarchies [1].

However our knowledge about structures related with integrability as those mentioned above is still incomplete and in many respects gives the impression of an accumulation of examples and methods whose deeper origin, connection or range of applicability are not completely understood yet. In particular methods are lacking which allow to test a given system of PDEs for integrability and to find the related mathematical structures systematically.

A key for progress in this field may be provided by an improved understanding of a particular property which many, if not all known integrable systems of PDEs have. This property is the existence of a zero-curvature representation (ZCR). A

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ZCR of a system of PDEs for functions u_a consists of a set of Lie algebra valued 'gauge fields' $\mathcal{A}_\mu(x, u, \partial u, \partial^2 u, \ldots)$ constructed of the u_a, their partial derivatives and the coordinates x^μ of the underlying manifold such that the vanishing of the field strengths $\partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu - [\mathcal{A}_\mu, \mathcal{A}_\nu]$ encodes the system of PDEs for the u_a (this will be made more precise later).

A ZCR may be regarded as a master property of an integrable system of PDEs since often other important properties of the system and methods for its investigation can be derived from it. For instance the inverse scattering method [2] for solving special systems of nonlinear PDEs is based on ZCRs of these PDEs [3], and the Lax representation [4] of a system of nonlinear PDEs essentially is nothing but a ZCR [1] (in fact a ZCR of a system of PDEs may be regarded as a generalized Lax representation). In some cases a ZCR itself may have interesting physical interpretation, see e.g. [5].

Another remarkable property of many integrable systems of PDEs is the existence of Bäcklund transformations (BTs). The latter have proved to be particularly useful for the construction of solutions of systems of nonlinear PDEs. For instance a BT may relate a system of PDEs to a simpler one whose solutions can be used to construct solutions of the more complicated system by means of the BT. Or a BT may relate different solutions of the same system. Then it is called an auto-BT (or self-BT) for this system and may be used to construct complicated solutions of the system from simpler solutions by an algebraic method based on the so-called permutability of auto-BTs [6, 7, 8]. Other useful applications may arise if a BT contains a Miura map whose exponent has been found in [9] and has been used to prove the existence of an infinite set of local conservation laws for the KdV equation and to construct these conservation laws explicitly [10]. Miura maps have interesting applications especially for Hamiltonian systems of evolution equations if they relate different Hamiltonian structures [11]. Such 'Hamiltonian Miura maps' may be useful even for the quantization of conformal field theories [12]. This application originates in the connection of conformal field theories with (bi-)Hamiltonian systems of PDEs via their (second) Hamiltonian Poisson bracket structure [13] which provides a realization of so-called classical $\mathcal{W}$–algebras whose investigation was initiated by [14].

Interrelations between BTs and ZCRs have been noticed by several authors. For instance a connection of the pioneering work [15] with zero-curvature conditions has been pointed out already in [16]. The present paper works out close relationships of BTs and ZCRs. Namely it turns out that a BT of a certain (rather general) type for a given system of PDEs induces a corresponding ZCR of this system and, conversely, that a ZCR of a system can be used to construct BTs for it.

The connection between ZCRs and BTs is established by means of gauge transformations which relate different BTs and represent the gauge group underlying the ZCR in general nonlinearly on an infinite dimensional function space. In particular this allows to define gauge equivalence of BTs. A second, more technical ingredient used to relate BTs with ZCRs is an approach which can be formalized using the jet bundle theory. The latter provides a suitable mathematical framework for an investigation of algebraic aspects of PDEs in general, see e.g. [17]. We shall only need some very elementary ideas underlying the jet-bundle theory. A systematic and more formal application of this theory to BTs can be found e.g. in [18].

The paper is organized as follows. In section II the type of BTs is defined which are considered in this paper and the basic notation is introduced. Furthermore some celebrated examples of BTs are recalled which are used later for exemplifications. In section III the connection between BTs and ZCRs is worked out. Section IV introduces the above-mentioned gauge transformations. In section V nonlinear representations of Lie groups are discussed which are supposed to be particular interesting in this context since they forge links to inverse scattering techniques. The general procedure for the construction of BTs from given ZCRs is exemplified in section VI for the generalized KdV hierarchies. Finally in section VII a method is outlined and exemplified which allows a systematic search for a ZCR of a given system of PDEs.

II Bäcklund transformations

Let me first introduce some notation. The BTs considered later will generally relate two sets of functions $\{u_a\}$, $\{v_i\}$

$$u_a = u_a(x), \quad a = 1, \ldots, N_u, \quad v_i = v_i(x), \quad i = 1, \ldots, N_v \tag{2.1}$$

where N_u and N_v are not necessarily equal. The argument x of these functions refers to some coordinate system

$$x^\mu, \quad \mu = 1, \ldots, D \tag{2.2}$$

of the underlying basis manifold whose dimension D will not be fixed in the general case. However since all examples will refer to the case $D = 2$ we shall also use the customary notation

$$D = 2: \quad t = x^1, \quad x = x^2 \tag{2.3}$$

hoping this will not cause confusion with the collective notation x for arguments of functions in the general (D-dimensional) case as in (2.1).

$[u]$ denotes collectively the functions u_a and their partial derivatives, i.e. the whole set of variables

$$\left\{ \frac{\partial^n u_a}{\partial x^{\mu_1} \ldots \partial x^{\mu_n}} : \quad a = 1, \ldots, N_u, \quad n = 0, 1, 2, \ldots \right\}. \tag{2.4}$$

In fact a suitable subset of (2.4) will mostly be regarded as a set of independent variables instead of regarding its elements as functions of the coordinates x^μ. This approach is formalized in the above-mentioned jet bundle theory.

According to this remark it should be clear that it will be important what are the relevant variables in a present context. This will be indicated mostly by the arguments of a function which usually are written omitting any indices. Thus $f(v, x, [u])$ will in the general case denote a function of the variables x^μ, v_i and the elements of (2.4).

Let us now define the BTs which we shall deal with. A system of D-dimensional partial differential equations of the form

$$\frac{\partial v_i}{\partial x^\mu} = A_{\mu i}(v, x, [u]) \tag{2.5}$$

will be called a BT if its integrability conditions hold owing to a system of PDEs satisfied by the u_a which we denote by

$$P_A(x, [u]) = 0, \quad A = 1, \ldots, N_P. \tag{2.6}$$

We do not insist on $N_P = N_u$ which is of course the most important case (however also 'overdetermined' systems of PDEs with $N_P > N_u$ may be interesting, see e.g. [19, 20]). Regarding $A_{\mu i}(v, x, [u])$ as functions $B_{\mu i}(x)$ of the coordinates x^μ, the integrability conditions for (2.5) read of course

$$\frac{\partial B_{\mu i}(x)}{\partial x^\nu} - \frac{\partial B_{\nu i}(x)}{\partial x^\mu} = 0, \quad B_{\mu i}(x) = A_{\mu i}(v, x(x), [u(x)]) \tag{2.7}$$

but this is not a useful point of view in the present context since it does not make contact with (2.6). Therefore we shall introduce more useful versions of the integrability conditions for (2.5) in section III. Notice that by assumption $A_{\mu i}(v, x, [u])$ does not depend on derivatives of v, i.e. we consider only systems (2.5) of first order in the derivatives of v. However since higher order systems can be rewritten in first order form, (2.5) is more general than it may appear at first sight.

Furthermore it is stressed that generally we do not require that (2.5) implies a system of PDEs

$$Q_B(x, [v]) = 0, \quad B = 1, \ldots, N_Q \tag{2.8}$$

for the v_i as well. This should be kept in mind since the definition of BTs of the form (2.5) often is restricted to the cases where (2.5) and its integrability conditions imply both (2.6) and (2.8). In particular (2.5) is called an auto-BT if these systems are equal ($P = Q$).

Examples:

Let us conclude this section with some celebrated examples of BTs for the simplest case $N_u = N_v = 1$, $D = 2$. We shall use the notation (2.3) and denote differentiations with respect to x or t by subscripts ($v_x = \partial v/\partial x$ etc.). Furthermore we use $v = v_1$ and $u = u_1$ resp. $w = u_1$, and in all examples λ denotes an arbitrary constant (spectral parameter) and (2.5) and its integrability conditions lead to decoupled PDEs for u (resp. w) and v denoted by $P([u]) = 0$ (resp. $P([w]) = 0$) and $Q([v]) = 0$.

a) The classical example for a BT is the auto-BT for the Sine-Gordon equation which in light-cone coordinates reads

$$v_x = -u_x + \lambda \sin\frac{u-v}{2}, \quad v_t = u_t - \frac{4}{\lambda}\sin\frac{u+v}{2}, \tag{2.9}$$

$$P([u]) = Q([u]) = u_{xt} - \sin u = 0. \tag{2.10}$$

b) A BT which relates the Liouville equation and the two-dimensional d'Alembert equation is in light-cone coordinates given by

$$v_x = u_x + \lambda \exp\frac{u+v}{2}, \quad v_t = -u_t - \frac{2}{\lambda}\exp\frac{u-v}{2}, \tag{2.11}$$

$$P([u]) = u_{xt} - \exp u = 0, \quad Q([v]) = v_{xt} = 0. \tag{2.12}$$

c) A BT which relates the KdV equation and the (λ-dependent) mKdV equation is given by [21]

$$\begin{aligned} v_x &= \lambda + u + v^2, \\ v_t &= -u_{xx} - 2u^2 + 2\lambda u + 4\lambda^2 - 2u_x v + v^2(4\lambda - 2u), \end{aligned} \tag{2.13}$$

$$P([u]) = u_t + u_{xxx} + 6uu_x = 0, \tag{2.14}$$

$$Q([v]) = v_t + v_{xxx} - 6v^2 v_x - 6\lambda v_x = 0. \tag{2.15}$$

The 'space part' of the BT (2.13) is the famous Miura map [9].

d) An auto-BT for the 'potential KdV' (pKdV) equation is given by [22]

$$\begin{aligned} v_x &= -w_x - 2\lambda - \tfrac{1}{2}(w-v)^2, \\ v_t &= w_{xxx} + (w_x)^2 - 4\lambda w_x - 8\lambda^2 + 2w_{xx}(w-v) + (w_x - 2\lambda)(w-v)^2, \end{aligned} \tag{2.16}$$

$$P([w]) = Q([w]) = w_t + w_{xxx} + 3(w_x)^2 = 0. \tag{2.17}$$

(2.17) is called the pKdV equation since its solutions serve as potentials for solutions of the KdV equation ($u = w_x$ solves (2.14) if w solves (2.17)).

e) An auto-BT for the 'potential mKdV' (pmKdV) equation reads [21]

$$\begin{aligned} v_x &= w_x + \lambda \sinh(v+w), \\ v_t &= -w_{xxx} + 2(w_x)^3 - 2\lambda^2 w_x - 2\lambda w_{xx}\cosh(v+w) \\ &\quad +\lambda\left(2(w_x)^2 - \lambda^2\right)\sinh(v+w), \end{aligned} \tag{2.18}$$

$$P([w]) = Q([w]) = w_t + w_{xxx} - 2(w_x)^3 = 0. \tag{2.19}$$

Remarks:

(i) The BTs (2.13) and (2.16) can both be obtained from a 'mother-BT' which we shall construct in section VII (see eq. (7.32)) and are gauge equivalent, see section IV.

(ii) The auto-BTs (2.16) and (2.18) are usually written in forms which have a symmetry under exchange of v and w rather than in the form (2.5). For instance using (2.19) and the 'space part' of (2.18) one can write its 'time part' in the form

$$v_t - w_t = \lambda\left((v_x)^2 + (w_x)^2\right)\sinh(v+w) - \lambda(v_{xx} + w_{xx})\cosh(v+w)$$

which, as the 'space part', is invariant under $v \leftrightarrow w$, $\lambda \to -\lambda$ (the same symmetry occurs in (2.9)).

III The connection between Bäcklund transformations and zero-curvature representations

In order to establish the connection between BTs of the form (2.5) and a ZCR of a system (2.6) we first write the integrability condition (2.7) in a more useful form by regarding $A_{\mu i}$ not as a function of x^μ as in (2.7) but as a function of the variables x^μ, v_i and (2.4) as in (2.5). Taking advantage of (2.5) we represent the partial derivatives on these variables by the operators

$$D_\mu = \partial_\mu + A_{\mu i}(v,x,[u])\,\frac{\partial}{\partial v_i} \tag{3.1}$$

where Einstein's summation convention is used (summation over i) and the piece ∂_μ acts nontrivially only on the variables (2.4) and on the x^μ:

$$\partial_\mu\,\frac{\partial^n u_a}{\partial x^{\mu_1}\ldots\partial x^{\mu_n}} = \frac{\partial^{n+1} u_a}{\partial x^{\mu}\partial x^{\mu_1}\ldots\partial x^{\mu_n}},\quad \partial_\mu x^\nu = \delta_\mu^\nu,\quad \partial_\mu v_i = 0. \tag{3.2}$$

On functions $f(v,x,[u])$ we of course define D_μ as first order differential operator satisfying the product rule

$$D_\mu(XY) = (D_\mu X)Y + X(D_\mu Y). \tag{3.3}$$

We can now easily calculate the commutator of two 'partial derivatives':

$$[D_\mu, D_\nu] = F_{\mu\nu i}(v,x,[u])\frac{\partial}{\partial v_i}\,, \tag{3.4}$$

$$F_{\mu\nu i}(v,x,[u]) = D_\mu A_{\nu i}(v,x,[u]) - D_\nu A_{\mu i}(v,x,[u]). \tag{3.5}$$

Requiring that the commutator (3.4) vanishes expresses the integrability condition for (2.5) since they read in terms of the variables $x, v, [u]$:

$$0 = [D_\mu, D_\nu]\,v_i = F_{\mu\nu i}(v,x,[u]). \tag{3.6}$$

(3.6) has already the form of a zero-curvature condition imposed on the 'field strengths' $F_{\mu\nu i}$ of the 'gauge fields' $A_{\mu i}(v,x,[u])$. Notice however that (3.6) involves the v_i. According to the previous section we assume that (3.6) holds by virtue of a system of PDEs (2.6) for the u_a which does *not* involve the v_i. This will be used now to extract from (2.5) a ZCR of (2.6). To this end we decompose the right-hand sides of (2.5) according to

$$A_{\mu i}(v,x,[u]) = A_\mu{}^I(x,[u])\,R_{Ii}(v) \tag{3.7}$$

where again summation over I is understood and $\{R_{Ii}\}$ denotes a set of linearly independent functions. The differential operators

$$\delta_I = -R_{Ii}(v)\,\frac{\partial}{\partial v_i} \tag{3.8}$$

span a Lie algebra $\mathcal{G}$ whose structure constants are denoted by $f_{IJ}{}^K$:

$$[\delta_I, \delta_J] = f_{IJ}{}^K \, \delta_K. \tag{3.9}$$

If possible one of course chooses the R_{Ii} such that $\mathcal{G}$ is finite. The operators D_μ now take the familiar form of covariant derivatives in Yang-Mills theories

$$D_\mu = \partial_\mu - A_\mu{}^I(x,[u])\, \delta_I \tag{3.10}$$

which suggests to call the $A_\mu{}^I(x,[u])$ gauge fields. This terminology will be justified in the next section. (3.4) now takes the form

$$[D_\mu, D_\nu] = -F_{\mu\nu}{}^I(x,[u])\, \delta_I \tag{3.11}$$

where the $F_{\mu\nu}{}^I(x,[u])$ are the field strengths constructed of the gauge fields $A_\mu{}^I(x,[u])$ according to

$$F_{\mu\nu}{}^I(x,[u]) = \partial_\mu A_\nu{}^I - \partial_\nu A_\mu{}^I - f_{JK}{}^I A_\mu{}^J A_\nu{}^K, \quad A_\mu{}^I = A_\mu{}^I(x,[u]). \tag{3.12}$$

The $F_{\mu\nu}{}^I$ are just the coefficients occurring in the decomposition of the $F_{\mu\nu i}$ analogously to (3.7):

$$F_{\mu\nu i}(v,x,[u]) = F_{\mu\nu}{}^I(x,[u])\, R_{Ii}(v). \tag{3.13}$$

Since by assumption the R_{Ii} are linearly independent and (3.6) holds by virtue of (2.6) the latter implies

$$F_{\mu\nu}{}^I(x,[u]) = 0. \tag{3.14}$$

More precisely this requires

$$F_{\mu\nu}{}^I = \sum_n r_{\mu\nu}{}^{IA\rho_1\ldots\rho_n}(x,[u])\, \partial_{\rho_1}\ldots\partial_{\rho_n} P_A(x,[u]) \tag{3.15}$$

with generally nontrivial functions $r_{\mu\nu}{}^{IA\rho_1\ldots\rho_n}(x,[u])$. Thus the $A_\mu{}^I(x,[u])$ indeed provide a ZCR of the system (2.6) in the sense of the definition outlined in the introduction.

Examples:
The BTs a)–e) listed in the previous section provide ZCRs of the respective PDEs for $\mathcal{G} = sl(2)$. The $sl(2)$-generators are chosen such that algebra reads

$$[\delta_I, \delta_J] = (I-J)\, \delta_{I+J}, \quad I, J = -1, 0, 1. \tag{3.16}$$

A choice of the generators $\delta_I = -R_I(v)\partial/\partial v$ which satisfies (3.16) is respectively given by

$$\begin{array}{rlll} a): & R_{-1} = 2 + 2\cos\frac{v}{2}, & R_0 = 2\sin\frac{v}{2}, & R_1 = 2 - 2\cos\frac{v}{2}, \\ b): & R_{-1} = 2\exp(-\frac{v}{2}), & R_0 = 2, & R_1 = 2\exp\frac{v}{2}, \\ c), d): & R_{-1} = 1, & R_0 = v, & R_1 = v^2, \\ e): & R_{-1} = 1 + \cosh v, & R_0 = \sinh v, & R_1 = -1 + \cosh v. \end{array} \tag{3.17}$$

One easily reads off the ZCR of the respective PDE for the choice (3.17) and may use it to check (3.15). For instance in the case d) one obtains

$$\begin{aligned}
&{A_2}^{-1} = -w_x - 2\lambda - \tfrac{1}{2}w^2, \quad {A_2}^0 = w, \qquad\qquad {A_2}^1 = -\tfrac{1}{2}, \\
&{A_1}^{-1} = w_{xxx} + (w_x)^2 - 4\lambda w_x - 8\lambda^2 + 2ww_{xx} + w^2(w_x - 2\lambda), \\
&\qquad\qquad {A_1}^0 = -2w_{xx} - 2w(w_x - 2\lambda), \quad {A_1}^1 = w_x - 2\lambda, \\
&{F_{12}}^{-1} = -(w + \partial_x)(w_t + w_{xxx} + 3(w_x)^2), \\
&\qquad\qquad {F_{12}}^0 = w_t + w_{xxx} + 3(w_x)^2, \qquad {F_{12}}^1 = 0.
\end{aligned} \tag{3.18}$$

Remark:

It is no surprise that $sl(2)$ occurs in all examples a)–e) since it is the unique finite Lie algebra which can occur in the case $N_v = 1$. This is easily seen as follows: Assume that $\delta_I = -R_I(v)\partial/\partial v$ and $\delta_J = -R_J(v)\partial/\partial v$ are two commuting elements of $\mathcal{G}$. Due to $N_v = 1$ this implies

$$R_I(v)\frac{\partial R_J(v)}{\partial v} - R_J(v)\frac{\partial R_I(v)}{\partial v} = 0 \;\Rightarrow\; R_I(v) = c\, R_J(v)$$

where c is an arbitrary constant. Thus δ_I and δ_J are linearly dependent, i.e. the Cartan subalgebra of $\mathcal{G}$ is 1-dimensional and thus $\mathcal{G}$ indeed is unique.

IV Gauge transformations

The formulae of the previous section suggest the presence of gauge transformations associated with the gauge fields $A_{\mu i}(v, x, [u])$ resp. ${A_\mu}^I(x, [u])$. Indeed they can be found. Namely consider transformations of the form

$$\tilde{v}_i = \tilde{v}_i(v, x, [u]). \tag{4.1}$$

We call (4.1) a gauge transformation if the functions $\tilde{v}_i(v, x, [u])$ are local in a suitable sense[1] and if (4.1) is invertible in a sufficiently large neighborhood of the point $(v_1, \ldots, v_{N_v})$ with an inverse

$$v_i = v_i(\tilde{v}, x, [u]) \tag{4.2}$$

which is local too. The corresponding transformation of the 'gauge fields' $A_{\mu i}$ and their 'field strengths' $F_{\mu\nu i}$ are given by

$$\tilde{A}_{\mu i}(\tilde{v}, x, [u]) = \partial_\mu \tilde{v}_i(v, x, [u]) + A_{\mu j}(v, x, [u])\,\frac{\partial \tilde{v}_i(v, x, [u])}{\partial v_j}, \tag{4.3}$$

$$\tilde{F}_{\mu\nu i}(\tilde{v}, x, [u]) = F_{\mu\nu j}(v, x, [u])\,\frac{\partial \tilde{v}_i(v, x, [u])}{\partial v_j}\;. \tag{4.4}$$

[1]Usually we have in mind here a definition of locality which requires polynomial dependence on the derivatives of the u_a. However the definition of locality generally should be adapted to the particular problem (2.6).

(4.3) follows from evaluating the left-hand side of

$$\frac{\partial \tilde{v}_i}{\partial x^\mu} = \tilde{A}_{\mu i}(\tilde{v}, x, [u]) \tag{4.5}$$

using (4.1) and (2.5) (and the chain rule). (4.5) itself is nothing but (2.5) for the transformed quantities. (4.4) follows from (4.3) since the $\tilde{F}_{\mu\nu i}$ are defined in terms of the $\tilde{A}_{\mu i}$ analogously to (3.4):

$$[\tilde{D}_\mu, \tilde{D}_\nu] = \tilde{F}_{\mu\nu i}(\tilde{v}, x, [u]) \frac{\partial}{\partial \tilde{v}_i}, \quad \tilde{D}_\mu = \partial_\mu + \tilde{A}_{\mu i}(\tilde{v}, x, [u]) \frac{\partial}{\partial \tilde{v}_i}. \tag{4.6}$$

The presence of the inhomogeneous term $\partial_\mu \tilde{v}_i$ in the transformation law (4.3) justifies to call the $A_{\mu i}$ gauge fields. Notice that the $F_{\mu\nu i}$ indeed transform covariantly and are therefore rightly called field strengths. In particular $\tilde{F}_{\mu\nu i}$ vanishes if $F_{\mu\nu i}$ vanishes. This expresses of course just the fact that the integrability conditions for (4.5) follow from those for (2.5). BTs which are related by gauge transformations (4.1) are called gauge equivalent. Notice however that a system of PDEs (2.8) for the v_i which might be implied by (2.5) of course is not gauge invariant, i.e. the system for the $\tilde{v}_i$ generally differs from the system for the v_i. In particular it is not guaranteed that (4.5) implies decoupled systems of PDEs for $\{u_a\}$ and $\{\tilde{v}_i\}$ if (2.5) implies decoupled systems of PDEs for $\{u_a\}$ and $\{v_i\}$ and vice versa.

Let us now inspect how gauge transformations (4.1) affect the ZCR ${A_\mu}^I(x, [u])$. The decomposition of the transformed gauge fields (4.3) and the corresponding generators are denoted by

$$\tilde{A}_{\mu i}(\tilde{v}, x, [u]) = \tilde{A}_\mu{}^I(x, [u])\, \tilde{R}_{Ii}(\tilde{v}), \tag{4.7}$$

$$\tilde{\delta}_I = -\tilde{R}_{Ii}(\tilde{v}) \frac{\partial}{\partial \tilde{v}_i}. \tag{4.8}$$

For an arbitrary gauge transformation (4.1) it of course does not make much sense to define individual transformations of the R_{Ii} and ${A_\mu}^I$ since in general it is not possible to find a decomposition (4.7) such that $\tilde{R}_{Ii}$ and $\tilde{A}_\mu{}^I$ are related to their counterparts R_{Ii} and ${A_\mu}^I$ in a simple way. However there are two subgroups of gauge transformations (4.1) which allow such decompositions and deserve special attention.

One of these subgroups consists of those gauge transformations which are generated by the operators δ_I themselves:

$$\tilde{v}_i(v, x, [u]) = \exp\left(g^I(x, [u])\delta_I\right)\, v_i\ . \tag{4.9}$$

Notice that the 'parameters' g^I generally depend not only on the x^μ but on the u_a and their derivatives as well. They are therefore not completely arbitrary but must be chosen such that (4.9) satisfies the above-mentioned requirements imposed by locality. The fact that the transformations (4.9) are generated by the δ_I implies that these gauge transformations allow a decomposition (4.7) given by

$$\tilde{A}_{\mu i}(\tilde{v}, x, [u]) = \tilde{A}_\mu{}^I(x, [u])\, R_{Ii}(\tilde{v}) \tag{4.10}$$

with the same functions R_{Ii} as in (3.7):

$$\tilde{R}_{Ii}(\tilde{v}) = R_{Ii}(\tilde{v}) \quad \forall\, I, i. \tag{4.11}$$

In order to verify that (4.11) indeed is compatible with (4.9) one may calculate the Lie derivative of $A_{\mu i}$ using (4.10). To this end one considers 'infinitesimal' gauge transformations

$$g^I(x,[u]) = \varepsilon\, G^I(x,[u]), \quad \varepsilon \ll 1.$$

Taking advantage of

$$R_{Ij}\frac{\partial R_{Ji}}{\partial v_j} - R_{Jj}\frac{\partial R_{Ii}}{\partial v_j} = -f_{IJ}{}^K\, R_{Ki}$$

which holds by assumption according to (3.9) one easily checks that (3.7), (4.9) and (4.10) imply

$$\tilde{A}_\mu{}^I(x,[u]) = A_\mu{}^I(x,[u]) + \partial_\mu g^I(x,[u]) + A_\mu{}^J(x,[u])g^K(x,[u])f_{JK}{}^I + O(\varepsilon^2). \tag{4.12}$$

This proves on the one hand the compatibility of (4.11) and (4.9) and shows on the other hand that the $A_\mu{}^I$ indeed have the standard transformation of Yang-Mills fields under gauge transformations (4.9) if the decomposition (4.7) is chosen according to (4.11).

The second above-mentioned subgroup of gauge transformations (4.1) consists of those transformations which do not depend on the u's or the coordinates:

$$\tilde{v}_i = \tilde{v}_i(v). \tag{4.13}$$

These 'rigid' gauge transformations allow the decomposition

$$\tilde{A}_{\mu i}(\tilde{v},x,[u]) = A_\mu{}^I(x,[u])\, \tilde{R}_{Ii}(\tilde{v}) \tag{4.14}$$

with the same 'components' $A_\mu{}^I$ as in (3.7) and functions $\tilde{R}_{Ii}$ which are related to the R_{Ii} according to

$$\tilde{R}_{Ii}(\tilde{v}) = R_{Ij}(v)\,\frac{\partial \tilde{v}_i}{\partial v_j}. \tag{4.15}$$

Notice that (4.15) is a 'contravariant' transformation compared to the 'covariant' transformation of the derivatives $\partial/\partial v_i$ under a gauge transformation (4.13). Therefore the generators (4.8) span the same Lie algebra $\tilde{\mathcal{G}} = \mathcal{G}$ with the same structure constants as the δ_I for the choice (4.15):

$$[\delta_I, \delta_J] = f_{IJ}{}^K\delta_K \quad \Rightarrow \quad [\tilde{\delta}_I, \tilde{\delta}_J] = f_{IJ}{}^K\tilde{\delta}_K. \tag{4.16}$$

Remark:

It is essential to realize that the gauge fields $A_\mu{}^I(x,[u])$ are not pure gauges in the space of local functions since their field strengths vanish only modulo (2.6). Of course, if the $u_a(x)$ solve (2.6), then the gauge field matrices

$$\mathcal{B}_\mu(x) = A_\mu{}^I(x,[u(x)])\, T_I$$

are, at least in a neighborhood of x, pure gauges of the form

$$\mathcal{B}_\mu(x) = -G(x)\frac{\partial G^{-1}(x)}{\partial x^\mu}, \quad G(x) = \exp(\varepsilon^I(x)T_I) \tag{4.17}$$

where $\{T_I\}$ denotes a suitable matrix representation of $\mathcal{G}$ (T_I has constant entries). But generally the entries of $G(x)$ and $G^{-1}(x)$ are not local functions of the form $f(x,[u])$ since (4.17) holds only for solutions of (2.6) but not for arbitrary functions u_a.

Examples:
We consider again the simplest case, namely $N_v = 1$. In this case a 'standard form' of the generators of $\mathcal{G} = sl(2)$ is given by

$$\delta_{-1} = -\frac{\partial}{\partial v}, \quad \delta_0 = -v\,\frac{\partial}{\partial v}, \quad \delta_1 = -v^2\,\frac{\partial}{\partial v} \tag{4.18}$$

which satisfy (3.16). The corresponding form of the gauge fields A_μ is

$$A_\mu = A_\mu{}^{-1}(x,[u]) + A_\mu{}^0(x,[u])\,v + A_\mu{}^1(x,[u])\,v^2 \tag{4.19}$$

and the gauge transformations (4.9) are Möbius transformations given by

$$\tilde{v} = \frac{\alpha(x,[u])\,v + \beta(x,[u])}{\gamma(x,[u])\,v + \delta(x,[u])}, \quad \alpha\delta - \beta\gamma \in \{1,-1\} \tag{4.20}$$

where $\alpha\delta - \beta\gamma \in \{1,-1\}$ can be replaced by $\alpha\delta - \beta\gamma = 1$ without loss of generality if $\alpha,\ldots,\delta$ are complex[2]. The corresponding finite transformations of the $A_\mu{}^I(x,[u])$ under the gauge transformations (4.20) are conveniently written in matrix form. To this end we use the following matrix representation of (3.16)

$$\sigma_{-1} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \sigma_0 = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}, \quad \sigma_1 = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \tag{4.21}$$

and define the matrices

$$\mathcal{A}_\mu = \sum_{I=-1}^{1} A_\mu{}^I \sigma_I = \begin{pmatrix} \frac{1}{2}A_\mu{}^0 & A_\mu{}^{-1} \\ -A_\mu{}^1 & -\frac{1}{2}A_\mu{}^0 \end{pmatrix}, \quad \mathcal{M} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}. \tag{4.22}$$

The transformed components $\tilde{A}_\mu{}^I$ are related to the original components via

$$\tilde{\mathcal{A}}_\mu = \sum_{I=-1}^{1} \tilde{A}_\mu{}^I \sigma_I = \mathcal{M}\,(\mathcal{A}_\mu - \partial_\mu)\,\mathcal{M}^{-1}. \tag{4.23}$$

Examples of BTs which are of the form (4.19) are given by (2.13) and (2.16). They also provide an example of two BTs which are related by a gauge transformation

[2] In the case of real $\alpha,\ldots,\delta$ a restriction to $\alpha\delta - \beta\gamma = 1$ would exclude for instance the gauge transformation $\tilde{v} = -v$.

(4.20). Namely denoting the function v which occurs in (2.13) by $\tilde{v}$ and identifying $u = w_x$, the gauge transformation which relates (2.13) and (2.16) is given by

$$\tilde{v} = \tfrac{1}{2}\,(w - v). \tag{4.24}$$

Examples for rigid gauge transformations (4.13) which cast the BTs given in examples a), b) and e) of section II in the form (4.19) are respectively given by

$$a) \quad z = \tan\frac{v}{4}, \quad b) \quad z = \exp\frac{v}{2}, \quad e) \quad z = \tanh\frac{v}{2}. \tag{4.25}$$

(4.25) can also be used to construct closed forms of the gauge transformations (4.9) for the $sl(2)$-representations arising from (3.17) in the cases a), b) and e). They are simply given by

$$\tilde{v}(v, x, [u]) = \frac{\alpha\, z(v) + \beta}{\gamma\, z(v) + \delta}$$

with $\alpha, \ldots, \delta$ as in (4.20) and $z(v)$ as in (4.25).

V Construction of Bäcklund transformations from zero-curvature representations

The results of the previous sections show that a ZCR $A_\mu{}^I(x, [u])$ of a system of PDEs can be used to construct BTs for this system according to

$$\frac{\partial v_i}{\partial x^\mu} = A_\mu{}^I(x, [u])\, R_{Ii}(v) \tag{5.1}$$

where the R_{Ii} are obtained from a representation (3.8) of $\mathcal{G}$. The question arises which representations are suited for the construction of BTs. In this section we discuss nonlinear representations which are linked with linear representations of $\mathcal{G}$ and therefore can relate BTs with inverse scattering methods. This makes them particularly interesting in the present context.

These representations are constructed by means of $n \times n$-matrix representations $\{T_I\}$ of $\mathcal{G}$ which satisfy[3]

$$[T_I, T_J] = f_{IJ}{}^K T_K. \tag{5.2}$$

A set of differential operators of the form (3.8) which represent $\mathcal{G}$ linearly is obviously given by

$$l_I = -r_{I\alpha}(\varphi)\,\frac{\partial}{\partial \varphi_\alpha}, \quad r_{I\alpha}(\varphi) = T_{I\alpha}{}^\beta \varphi_\beta, \quad \alpha, \beta = 1, \ldots, n \tag{5.3}$$

where $T_{I\alpha}{}^\beta$ are the entries of T_I (α labels the rows, β the columns). BTs of the form (2.5) are then obtained from a given ZCR by

$$\frac{\partial \varphi_\alpha}{\partial x^\mu} = A_\mu{}^I(x, [u])\, T_{I\alpha}{}^\beta \varphi_\beta. \tag{5.4}$$

[3]For the applications we have in mind one has to choose the representation $\{T_I\}$ such that the structure constants $f_{IJ}{}^K$ in (5.2) agree with those which occur in (3.12).

In a more common terminology (5.4) would not be called a 'BT' but rather a 'scattering problem' since it has the form of the linear problems which are used in the inverse scattering theory [2]. Nonlinear representations of $\mathcal{G}$ can be obtained from the linear representations (5.3) by means of a set of functions

$$v_i = v_i(\varphi), \quad i = 1, \ldots, N_v \tag{5.5}$$

whose l_I-variations can be written completely in terms of the v_i again:

$$l_I\, v_i(\varphi) = -T_{I\alpha}{}^{\beta}\varphi_\beta \frac{\partial v_i}{\partial \varphi_\alpha} = -R_{Ii}(v). \tag{5.6}$$

(5.6) represents a nontrivial requirement since (5.5) is not assumed to be invertible. In particular in general one has $N_v \neq n$. (5.6) implies immediately that the operators

$$\delta_I = -R_{Ii}(v)\, \frac{\partial}{\partial v_i} \tag{5.7}$$

represent $\mathcal{G}$ with the same structure constants which occur in (5.2):

$$[T_I, T_J] = f_{IJ}{}^K T_K \;\Rightarrow\; [l_I, l_J] = f_{IJ}{}^K l_K \;\Rightarrow\; [\delta_I, \delta_J] = f_{IJ}{}^K \delta_K. \tag{5.8}$$

Thus each choice (5.5) which satisfies (5.6) provides representations (3.8) of $\mathcal{G}$ and can be used to construct a BT (2.5) from a given ZCR according to (5.1). The φ_α then may be regarded only as 'auxiliary variables' introduced to construct representations (3.8) of $\mathcal{G}$. However it is useful and quite instructive to assume

$$v_i = v_i(\varphi(x)) \tag{5.9}$$

and impose (5.4) on the φ_α. Namely then (5.1) follows from (5.4) since (5.9), (5.4) and (5.6) imply

$$\frac{\partial v_i}{\partial x^\mu} = \frac{\partial v_i}{\partial \varphi_\alpha}\frac{\partial \varphi_\alpha}{\partial x^\mu} = \frac{\partial v_i}{\partial \varphi_\alpha} A_\mu{}^I\, T_{I\alpha}{}^\beta\, \varphi_\beta = A_\mu{}^I(x,[u]) R_{Ii}(v).$$

Furthermore (5.6) implies that the gauge transformations (4.9) of the v_i generated by the δ_I are induced by the transformations of the φ_α which transform according to a linear representation of the group:

$$\tilde v_i \;=\; \exp\left(g^I(x,[u])\,\delta_I\right)\, v_i = \exp\left(g^I(x,[u])\, l_I\right)\, v_i(\varphi) = v_i(\tilde\varphi), \tag{5.10}$$

$$\tilde\varphi_\alpha \;=\; g_\alpha{}^\beta(x,[u])\,\varphi_\beta, \quad g(x,[u]) = \exp\left(-g^I(x,[u])\,T_I\right) \tag{5.11}$$

where the $g_\alpha{}^\beta$ denote the entries of the matrix g. (5.4), (5.9) and (5.10) imply that the finite gauge transformations of the gauge fields $A_\mu{}^I$ read in matrix forms

$$\tilde{\mathcal{A}}_\mu(x,[u]) = g(x,[u])\,(\mathcal{A}_\mu(x,[u]) - \partial_\mu)\, g^{-1}(x,[u]) \tag{5.12}$$

where $\mathcal{A}_\mu(x,[u])$ and $\tilde{\mathcal{A}}_\mu(x,[u])$ are defined according to

$$\mathcal{A}_\mu(x,[u]) = A_\mu{}^I(x,[u])\,T_I, \quad \tilde{\mathcal{A}}_\mu(x,[u]) = \tilde{A}_\mu{}^I(x,[u])\,T_I. \tag{5.13}$$

Using these matrices (3.14) takes the form

$$\partial_\mu \mathcal{A}_\nu(x,[u]) - \partial_\nu \mathcal{A}_\mu(x,[u]) - [\mathcal{A}_\mu(x,[u]), \mathcal{A}_\nu(x,[u])] = 0. \tag{5.14}$$

Remark:
As mentioned already in the previous section the gauge field matrix $\mathcal{A}_\mu(x,[u(x)])$ is (at least in some neighborhood of x) a pure gauge of the form (4.17), provided $\{u_a(x)\}$ solves (2.6). Then (5.4) implies

$$\frac{\partial}{\partial x^\mu}\Big((G^{-1})_\alpha{}^\beta(x)\,\varphi_\beta(x)\Big) = 0 \quad \Rightarrow \quad \varphi_\alpha(x) = G_\alpha{}^\beta(x)\,\Lambda_\beta \tag{5.15}$$

where $\{\Lambda_\alpha\}$ is a set of constants and $G(x)$ is the representation matrix occurring in (4.17). Thus the functions $v_i(x)$ can be constructed by means of the matrices $G(x)$ according to (5.9). This uncovers further connections between BTs, ZCRs and inverse scattering techniques. Notice however that the connection between $G(x)$ and the corresponding solution of (2.6) is rather involved since the entries of $G(x)$ are not local functions $f(x,[u])$ as has been pointed out in the previous section.

Examples:
Nontrivial examples of representations (3.8) constructed by means of the above-described procedure are obtained for arbitrary $\mathcal{G}$ by means of 'projective coordinates in φ-space'

$$v_i = \frac{\varphi_i}{\varphi_n}, \quad i = 1,\ldots,n-1. \tag{5.16}$$

Notice that $N_v = n-1$ in this case. One easily checks that (5.6) holds and reads:

$$R_{Ii}(v) = T_{Ii}{}^n - T_{In}{}^n v_i + \sum_{j=1}^{n-1}(T_{Ii}{}^j v_j - T_{In}{}^j v_j v_i), \quad i = 1,\ldots,n-1 \tag{5.17}$$

where $T_{In}{}^n$ denotes the particular entry of T_I (not its trace). The generators δ_I constructed by means of (5.17) generalize the $sl(2)$ representation (4.18) since (5.17) is a polynomial of degree 2 in the v_i—in fact (5.17) reproduces (4.18) in the case $\mathcal{G} = sl(2)$ for the choice $T_I = \sigma_I$ with σ_I as in (4.21). The gauge transformations (4.9) arising from (5.17) generalize the Möbius transformations (4.20) since (5.10) in this case gives

$$\tilde{v}_i = \frac{\sum_{j=1}^{n-1} g_i{}^j v_j + g_i{}^n}{\sum_{k=1}^{n-1} g_n{}^k v_k + g_n{}^n} \tag{5.18}$$

where $g_i{}^j$ are the entries of the matrix $g(x,[u])$ occurring in (5.11).

For a given ZCR ${A_\mu}^I(x,[u])$ of a system of PDEs for functions u_a one now can use the functions (5.17) to construct a BT of the form (2.5). The result can be written in the form

$$\frac{\partial v_i}{\partial x^\mu} = {A_\mu}^I(x,[u])\, R_{Ii}(v) = (0,\ldots,0,1,0,\ldots,0,-v_i)\, \mathcal{A}_\mu(x,[u]) \begin{pmatrix} v_1 \\ \vdots \\ v_{n-1} \\ 1 \end{pmatrix} \tag{5.19}$$

where the 1 in the vector $(0,\ldots,0,1,0,\ldots,-v_i)$ occurs at the ith position.

VI Construction of Bäcklund transformations for generalized KdV systems

This section exemplifies the construction of BTs from ZCRs for the generalized KdV systems. The latter are defined by their Lax representation [4]

$$\partial_t L^{(n)} = [B^{(n,k)}, L^{(n)}], \quad n = 2,3,\ldots, \quad k = 1,2,\ldots \tag{6.1}$$

where $L^{(n)}$ denotes the nth order Lax operator

$$L^{(n)} = \partial^n + \sum_{i=0}^{n-2} u_{i+1}\partial^i, \quad \partial = \frac{\partial}{\partial x} \tag{6.2}$$

which depends on $n-1$ functions $u_i(x,t)$. $B^{(n,k)}$ is an operator of the form

$$B^{(n,k)} = \partial^k + \sum_{i=0}^{k-2} a_i(\{u\})\partial^i \tag{6.3}$$

where $\{u\}$ denotes collectively the u_i and their x-derivatives. $B^{(n,k)}$ can be constructed for instance by means of pseudo-differential operators [1]. (6.1) yields the members of the nth generalized KdV hierarchy in the form

$$P_i^{(n,k)}([u]) = \partial_t u_i + p_i^{(n,k)}(\{u\}) = 0, \quad i = 1,\ldots,n-1. \tag{6.4}$$

(6.1) is the integrability condition for the generalized Schrödinger problem

$$L^{(n)}\psi = \lambda\psi, \quad \psi_t = B^{(n,k)}\psi. \tag{6.5}$$

A ZCR of (6.4) can be obtained by writing (6.5) in matrix form

$$\partial_x \Psi = \mathcal{L}^{(n)}\Psi, \quad \partial_t \Psi = \mathcal{B}^{(n,k)}\Psi, \quad \Psi = (\partial^{n-1}\psi, \partial^{n-2}\psi, \ldots, \partial\psi, \psi)^\top \tag{6.6}$$

where $\mathcal{L}^{(n)}$ denotes the $n \times n$-matrix

$$\mathcal{L}^{(n)} = \begin{pmatrix} 0 & -u_{n-1} & -u_{n-2} & \dots & -u_2 & -u_1 + \lambda \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix} \quad (6.7)$$

$\mathcal{B}^{(n,k)}$ is the $n \times n$-matrix which represents ∂_t on Ψ according to $\psi_t = B^{(n,k)}\psi$. Its construction is also straightforward but somewhat involved since one has to use the first equation (6.5) in order to eliminate partial derivatives $\partial^m \psi$ of order $m \geq n$. The Lax representation (6.1) now indeed takes the form of a zero-curvature-condition (5.14):

$$\partial_t \mathcal{L}^{(n)} - \partial_x \mathcal{B}^{(n,k)} - [\mathcal{B}^{(n,k)}, \mathcal{L}^{(n)}] = 0 \quad (6.8)$$

i.e. the matrices $\mathcal{L}^{(n)}$ and $\mathcal{B}^{(n,k)}$ are a ZCR of (6.4) for $\mathcal{G} = sl(n)$ (both $\mathcal{L}^{(n)}$ and $\mathcal{B}^{(n,k)}$ are traceless). Using the defining representation $\{T_I\}$ of $sl(n)$ (traceless $n \times n$-matrices) (5.17) thus allows to construct BTs with $N_v = N_u = n - 1$. They are obtained from (5.19) with

$$\mathcal{A}_1 = \mathcal{B}^{(n,k)}, \quad \mathcal{A}_2 = \mathcal{L}^{(n)}. \quad (6.9)$$

The 'space part' of these BTs reads explicitly

$$\frac{\partial v_1}{\partial x} = \lambda - u_1 - \sum_{i=2}^{n-1} u_{n+1-i} v_i - v_1 v_{n-1}, \quad 1 < i < n: \quad \frac{\partial v_i}{\partial x} = v_{i-1} - v_i v_{n-1}. \quad (6.10)$$

One can use these BTs for instance to construct BTs whose space part can be written as a generalized Miura map of the form

$$u_i = u_i(\{\tilde{v}\}), \quad i = 1, \dots, n-1. \quad (6.11)$$

Namely simple examples of gauge transformations which lead to such generalized Miura maps are given by

$$\tilde{v}_i = v_i + \sum_{j=0}^{n-2-i} \alpha_{i,j} \partial^j u_{i+j+1}, \quad \alpha_{i,0} \neq 0 \quad (6.12)$$

where $\alpha_{i,j}$ are constant coefficients. Namely (6.12) and (6.10) imply

$$\frac{\partial \tilde{v}_i}{\partial x} = -\alpha_{i-1,0}\, u_i + Y_i(\tilde{v}, \{u_{i+1}\}, \dots, \{u_{n-1}\}), \quad \alpha_{0,0} := 1 \quad (6.13)$$

and one easily makes sure that this implies (6.11). The BTs obtained in this way relate the generalized KdV system (6.4) to a similar system of evolution equations for the $\tilde{v}_i$ which has the form

$$\tilde{Q}_i([\tilde{v}]) = \partial_t \tilde{v}_i + q_i(\{\tilde{v}\}) = 0, \quad i = 1, \dots, n-1. \quad (6.14)$$

This is easily seen combining (5.19), (6.12) and (6.11).

One may also look for gauge transformations which allow to construct auto-BTs for the generalized KdV systems. To this end it seems reasonable to introduce a potential w_i for each u_i such that

$$u_i = \partial w_i \tag{6.15}$$

and look for auto-BTs for the systems of PDEs which arise from (6.4) for the w_i and are therefore called generalized pKdV systems. The introduction of the potentials w_i is suggested by the fact that w_i and v_i have the same dimension $n-i$ (this follows if one assigns dimension 1 to ∂ and requires that the operators and equations given above have definite dimension). Therefore one may hope to find auto-BTs for the generalized pKdV systems by means of appropriate gauge transformations $\tilde{v}_i = \tilde{v}_i(v,[w])$ chosen such that $dim(\tilde{v}_i) = dim(v_i)$. Simple examples of such gauge transformations are given by

$$\tilde{v}_i = \gamma_i v_i + \sum_{j=0}^{n-1-i} \beta_{i,j} \partial^j w_{i+j} \tag{6.16}$$

where γ_i and $\beta_{i,j}$ are dimensionless constants (in particular they do not depend on λ). Of course (6.16) may be generalized by allowing for nonlinear contributions with dimension $n-i$ on the right-hand sides.

Examples:

(i) The simplest nontrivial KdV system arises for $(n,k) = (2,3)$ and is given by the KdV equation itself since in this case (6.4) reads (for $u = u_1$)

$$u_t - \tfrac{1}{4} u_{xxx} - \tfrac{3}{2} u u_x = 0 \tag{6.17}$$

which takes the form (2.14) after the rescaling $t \to -t/4$. One may check that (5.19) then yields precisely the BT (2.13) after the replacements $v \to -v$, $\lambda \to -\lambda$. We know already that the auto-BT (2.16) for the pKdV equation (2.17) is related to (2.13) via the gauge transformation (4.24) which indeed is of the form (6.16).

(ii) The simplest nontrivial generalized KdV system for $n = 3$ (Boussinesq hierarchy) is given by $(n,k) = (3,2)$. The Lax pair, the system of PDEs and its ZCR in matrix form read

$$L^{(3)} = \partial^3 + u_2 \partial + u_1, \quad B^{(3,2)} = \partial^2 + \tfrac{2}{3} u_2, \tag{6.18}$$

$$\partial_t u_2 = -\partial^2 u_2 + 2\partial u_1, \quad \partial_t u_1 = \partial^2 u_1 - \tfrac{2}{3}\partial^3 u_2 - \tfrac{2}{3} u_2 \partial u_2, \tag{6.19}$$

$$\mathcal{L}^{(3)} = \begin{pmatrix} 0 & -u_2 & -u_1 + \lambda \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \tag{6.20}$$

$$\mathcal{B}^{(3,2)} = \begin{pmatrix} -\frac{1}{3}u_2 & \frac{1}{3}\partial u_2 - u_1 + \lambda & \frac{2}{3}\partial^2 u_2 - \partial u_1 \\ 0 & -\frac{1}{3}u_2 & \frac{2}{3}\partial u_2 - u_1 + \lambda \\ 1 & 0 & \frac{2}{3}u_2 \end{pmatrix} \tag{6.21}$$

and one now constructs easily a BT by means of (5.19). It reads

$$
\begin{aligned}
\partial v_1 &= \lambda - u_1 - u_2 v_2 - v_1 v_2, \quad \partial v_2 = v_1 - (v_2)^2, \\
\partial_t v_1 &= \tfrac{2}{3}\partial^2 u_2 - \partial u_1 - u_2 v_1 + v_2(\lambda - u_1 + \tfrac{1}{3}\partial u_2) - (v_1)^2, \\
\partial_t v_2 &= \lambda - u_1 + \tfrac{2}{3}\partial u_2 - u_2 v_2 - v_1 v_2.
\end{aligned} \tag{6.22}
$$

The pKdV system which corresponds to (6.19) can be written in the form

$$
w_t = \bar{w}_x, \quad \bar{w}_t = -\tfrac{1}{3} w_{xxx} - \tfrac{2}{3}(w_x)^2 \tag{6.23}
$$

where w and $\bar{w}$ are appropriately defined in terms of the potentials w_i of the u_i:

$$
w = w_2, \quad \bar{w} = 2w_1 - \partial w_2 \quad \Leftrightarrow \quad u_1 = \tfrac{1}{2}(\bar{w}_x + w_{xx}), \quad u_2 = w_x. \tag{6.24}
$$

Using the notations $v := \tilde{v}_2$, $\bar{v} := \tilde{v}_1$ a gauge transformation (6.16) which yields an auto-BT for (6.23) is given by

$$
v = 3v_2 + w, \quad \bar{v} = 3v_1 + 2w_x + \bar{w}. \tag{6.25}
$$

In order to check this one may insert (6.25) and its derivatives into

$$
v_t = \bar{v}_x, \quad \bar{v}_t = -\tfrac{1}{3} v_{xxx} - \tfrac{2}{3}(v_x)^2
$$

and verify that this yields an identity using (6.22) and (6.23). The auto-BT itself is easily obtained from (6.22) and (6.25). I remark that its space part can be written in a form which is invariant under $v \leftrightarrow w$, $\bar{v} \leftrightarrow -\bar{w}$, $\lambda \to -\lambda$ (such discrete symmetries are typical properties of auto-BTs written in an appropriate form, see remark at the end of section II):

$$
\begin{aligned}
(v+w)_x &= (\bar{v} - \bar{w}) - \tfrac{1}{3}(v-w)^2, \\
(\bar{v} + \bar{w})_x &= \tfrac{1}{3}(w-v)_{xx} - \tfrac{2}{3}(v-w)(\bar{v}-\bar{w}) + \tfrac{2}{27}(v-w)^3 + 4\lambda.
\end{aligned} \tag{6.26}
$$

VII Construction of zero-curvature representations

It has been shown how one can construct BTs for systems of PDEs which have a ZCR. The question arises how to find a ZCR for a given system. In special cases the ZCRs may be obtained from the definition of the respective system of PDEs as in the case of the generalized KdV systems discussed in the previous section whose ZCRs can be constructed from their Lax representation. In fact a ZCR of a given system of PDEs can be viewed as a generalized Lax representation and one may hope to find new interesting systems of PDEs by imposing zero-curvature conditions on appropriately chosen 'gauge fields'. This procedure has been applied recently by various authors, see e.g. [23, 24, 19].

In general however the construction of a ZCR for a given system represents a very nontrivial problem and of course it may turn out to be impossible. In the following a method is outlined which allows a systematic search for a ZCR of a given system after the Lie algebra $\mathcal{G}$ has been fixed. The choice of $\mathcal{G}$ is left as an open problem in the general case. The outlined method may be regarded as a systematization and generalization of procedures used for instance in [25].

Let me first describe the procedure in general and then exemplify it by applying it to the pKdV equation (2.17). When looking for a method to determine ZCRs one is faced with the problem that the zero-curvature conditions (3.15) contain arbitrary functions $r_{\mu\nu}{}^{IA\rho_1\ldots\rho_n}(x,[u])$. This arbitrariness reflects of course the fact that the zero-curvature conditions do not have to hold identically in the variables (2.4) but only modulo (2.6). The idea is now to choose a suitable subset $\{w_k\}$ of (2.4) and a corresponding representation of the operators ∂_μ on the w_k such that the $F_{\mu\nu}{}^I$ must vanish *identically* in these variables. Both the choice of the w_k and the representation of the ∂_μ are obtained from the system (2.6) itself and encode it. The outlined method may be formalized using the jet-bundle formalism. However I found it more instructive to explain it by exemplifying it for a simple example.

Let me add some remarks before dealing with this example. The choice of the $\{w_k\}$ eliminates infinitely many variables (2.4) but the number of remaining variables w_k is still infinite. However the requirement that the gauge fields $A_\mu{}^I$ are local functions means that they actually depend only on a finite subset of $\{w_k\}$ which is denoted by $\{W_k,\ k=1,\ldots,N_W\}$. Generally there is a minimal choice of such a finite subset which can lead to a nontrivial ZCR. This minimal choice depends on the order of the system (2.6) resp. on the induced representation of the ∂_μ (see example below). Together with a choice of $\mathcal{G}$ this converts (3.15) into a well-defined problem in finitely many variables W_k. However we know in advance that this problem does not have a unique solution. Namely still one has a 'gauge freedom' corresponding to those transformations (4.1) which leave invariant the space of gauge fields $A_\mu{}^I(x,W_1,\ldots,W_{N_W})$. Thus one has to choose a gauge at some stage of the investigation. However I stress that the gauge cannot be chosen independently of the particular system (2.6). For instance one cannot impose some standard gauge fixing condition like the Lorentz gauge $\partial^\mu A_\mu{}^I=0$ from the beginning since generally it is not possible to perform a *local* gauge transformation (4.1) such that a given ZCR takes a form satisfying such a standard condition, i.e. it depends decisively on the system (2.6) which gauge fixing conditions are compatible with locality.

I remark that the method itself characterizes the systems of PDEs to which it is applicable. Namely such systems must allow the choice of a subset $\{w_k\}$ and a corresponding representation of the ∂_μ with the above-mentioned properties. Examples for systems of PDEs to which the method is applicable are Cauchy-Kowalewski systems (see e.g. [26]).

Example:
The outlined method will now be explained and exemplified by applying it to the pKdV equation in the form (2.17):

$$w_t + w_{xxx} + 3(w_x)^2 = 0. \tag{7.1}$$

First we use (7.1) to choose a subset $\{w_k\}$ and establish the corresponding representation of the $\partial_\mu \in \{\partial_t, \partial_x\}$: By means of (7.1) we eliminate all derivatives of w which contain a derivative with respect to t, i.e. the remaining variables in which (3.14) has to hold identically are given by

$$w_k := (\partial_x)^k w, \quad k = 0, 1, 2, \dots. \tag{7.2}$$

The representation of ∂_t and ∂_x on these variables which is induced by (7.1) reads

$$\partial_x w_k = w_{k+1}, \quad \partial_t w_k = -w_{k+3} - 3 \sum_{m=0}^{k} \binom{k}{m} w_{m+1} w_{k+1-m}. \tag{7.3}$$

Locality of the ${A_\mu}^I$ requires that they do not depend on derivatives of w of higher order than some maximal value N, i.e.:

$$\frac{\partial {A_\mu}^I}{\partial w_k} = 0 \quad \forall\, k > N. \tag{7.4}$$

One easily verifies that (7.3) and (7.4) imply

$$\begin{aligned} \partial_t {A_2}^I - \partial_x {A_1}^I - {f_{JK}}^I {A_1}^J {A_2}^K &= -\left(w_{N+3} + O(N+1)\right) \frac{\partial {A_2}^I}{\partial w_N} \\ -w_{N+2} \frac{\partial {A_2}^I}{\partial w_{N-1}} - w_{N+1} \frac{\partial {A_2}^I}{\partial w_{N-2}} &- w_{N+1} \frac{\partial {A_1}^I}{\partial w_N} + O(N) \end{aligned} \tag{7.5}$$

where $O(N)$ collects terms which do not depend on the w_k, $k > N$. Since (7.5) has to hold identically in the variables w_k the coefficients of w_{N+3}, w_{N+2} and w_{N+1} have to vanish separately which gives

$${A_2}^I = f^I(w_0, \dots, w_{N-2}), \quad {A_1}^I = -w_N \frac{\partial f^I(w_0, \dots, w_{N-2})}{\partial w_{N-2}} + g^I(w_0, \dots, w_{N-1}). \tag{7.6}$$

The smallest value of N which can lead to a nontrivial ZCR obviously is

$$N = 2 \tag{7.7}$$

which reflects the fact that (7.1) is of order 3 in the partial derivatives. The corresponding minimal set $\{W_k\}$ is given by

$$N = 2: \quad \{W_k : \ k = 1, 2, 3\} = \{w, w_x, w_{xx}\} \tag{7.8}$$

where we returned to the more familiar notation which denotes x-derivatives by subscripts. We introduce the notation $\vec{X}$ for a vector with components X^I and $\vec{X} \times \vec{Y}$ for the vector whose components are given by ${f_{JK}}^I X^J Y^K$. For the case

$$\mathcal{G} = sl(2) \tag{7.9}$$

and a basis satisfying (3.16) we obtain

$$\vec{X} = (X^{-1}, X^0, X^1),\ \vec{Y} = (Y^{-1}, Y^0, Y^1)\ \Rightarrow$$
$$\vec{X}\times\vec{Y} = (X^0Y^{-1} - X^{-1}Y^0, 2X^1Y^{-1} - 2X^{-1}Y^1, X^1Y^0 - X^0Y^1). \quad (7.10)$$

Using this notation we have to determine $\vec{A}_\mu(w, w_x, w_{xx})$ such that

$$0 = \partial_t\vec{A}_2 - \partial_x\vec{A}_1 - \vec{A}_1\times\vec{A}_2 \quad (7.11)$$

holds identically in the variables w_k with ∂_x and ∂_t given in (7.3). In the case $N = 2$ (7.6) gives

$$\vec{A}_2 = \vec{f}(w), \quad \vec{A}_1 = -w_{xx}\vec{f}\,'(w) + \vec{g}(w, w_x) \quad (7.12)$$

where the prime denotes differentiation with respect to w:

$$X' := \frac{\partial X}{\partial w}.$$

Inserting (7.12) into (7.11) and omitting the arguments of the functions one obtains

$$0 = -3(w_x)^2\vec{f}\,' + w_{xx}w_x\vec{f}\,'' - w_{xx}\frac{\partial\vec{g}}{\partial w_x} - w_x\vec{g}\,' + w_{xx}\vec{f}\,'\times\vec{f} - \vec{g}\times\vec{f}. \quad (7.13)$$

Vanishing of the terms containing w_{xx} requires

$$\frac{\partial\vec{g}}{\partial w_x} = w_x\vec{f}\,'' + \vec{f}\,'\times\vec{f} \quad \Rightarrow \quad \vec{g} = \tfrac{1}{2}(w_x)^2\vec{f}\,'' + w_x\vec{f}\,'\times\vec{f} + \vec{k}(w) \quad (7.14)$$

where the function $\vec{k}(w)$ is not determined so far. If we now insert (7.14) into (7.13) all terms containing w_{xx} cancel and we obtain the following equations by requiring the coefficients of $(w_x)^n$, $n = 3, 2, 1, 0$ to vanish:

$$0 = \vec{f}\,''', \quad (7.15)$$
$$0 = \vec{f}\,' + \tfrac{1}{2}\vec{f}\,''\times\vec{f}, \quad (7.16)$$
$$\vec{k}\,' = -(\vec{f}\,'\times\vec{f})\times\vec{f}, \quad (7.17)$$
$$0 = \vec{k}\times\vec{f}. \quad (7.18)$$

(7.15) immediately implies

$$\vec{f} = \vec{a} + \vec{b}\,w + \vec{c}\,w^2 \quad (7.19)$$

where $\vec{a}, \vec{b}, \vec{c}$ are constant vectors which must be determined. (7.16) then requires

$$\vec{b} = \vec{a}\times\vec{c}, \quad 2\vec{c} = \vec{b}\times\vec{c}. \quad (7.20)$$

Using (7.20) one may verify that

$$\vec{f}\,'\times\vec{f} = -2\vec{f} + \vec{d}, \quad \vec{d} = 2\vec{a} + \vec{b}\times\vec{a}. \quad (7.21)$$

By means of (7.21) one easily makes sure that (7.17) gives

$$\vec{k} = \vec{l} + \vec{a}\times\vec{d}\,w + \tfrac{1}{2}\,\vec{b}\times\vec{d}\,w^2 + \tfrac{1}{3}\,\vec{c}\times\vec{d}\,w^3 \tag{7.22}$$

where $\vec{l}$ is a constant vector. $\vec{a}$, $\vec{b}$, $\vec{c}$ and $\vec{l}$ now have to be determined from (7.18) and (7.20). An obvious solution of the second equation (7.20) is given by $\vec{c} = 0$. However one easily verifies that

$$\vec{c} = 0 \;\Rightarrow\; \vec{b} = 0 \;\Rightarrow\; \vec{A}_2 = \vec{a} \;\Rightarrow\; \ldots \;\Rightarrow\; \vec{A}_1 = \rho\vec{a} \tag{7.23}$$

with an arbitrary constant ρ. Thus the case $\vec{c} = 0$ leads to gauge fields ${A_\mu}^I$ which do not depend on w and its derivatives and thus to an uninteresting result. We now consider the case $\vec{c} \neq 0$. As mentioned above there cannot be a unique nontrivial solution since we have the freedom of gauge transformations. Therefore we now choose a gauge. Notice that the gauge field matrix $\mathcal{A}_2$ occurring in (4.22) is in our case of the simple form

$$\mathcal{A}_2 = A + B\,w + C\,w^2, \quad A = \begin{pmatrix} \frac{1}{2}a^0 & a^{-1} \\ -a^1 & -\frac{1}{2}a^0 \end{pmatrix},$$
$$B = \begin{pmatrix} \frac{1}{2}b^0 & b^{-1} \\ -b^1 & -\frac{1}{2}b^0 \end{pmatrix}, \quad C = \begin{pmatrix} \frac{1}{2}c^0 & c^{-1} \\ -c^1 & -\frac{1}{2}c^0 \end{pmatrix} \tag{7.24}$$

where the entries of A, B, C are the components of $\vec{a}$, $\vec{b}$, $\vec{c}$. Since C does not vanish due to $\vec{c} \neq 0$ one can always find a constant matrix $\mathcal{M}$ with determinant ± 1 such that

$$\mathcal{M}\,C\,\mathcal{M}^{-1} = \begin{pmatrix} 0 & 1 \\ -\eta & 0 \end{pmatrix} \tag{7.25}$$

(since η is the determinant of C it cannot be fixed by such a transformation). Therefore we can assume without loss of generality that $\vec{c}$ is of the form

$$\vec{c} = (1, 0, \eta). \tag{7.26}$$

Notice that the constancy of $\mathcal{M}$ guarantees that our original requirement (7.7) still holds. We now evaluate (7.20) explicitly for $\vec{c}$ given by (7.26). The second equation (7.20) requires

$$(b^0, 2b^1 - 2b^{-1}\eta, -b^0\eta) = 2(1, 0, \eta) \quad\Leftrightarrow\quad b^0 = 2, \quad b^1 = \eta = 0. \tag{7.27}$$

Notice that this means

$$B = \begin{pmatrix} 1 & b^{-1} \\ 0 & -1 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}. \tag{7.28}$$

This result can be simplified by a gauge transformation which does not change the form of C but simplifies B:

$$\mathcal{M} = \begin{pmatrix} 1 & \frac{1}{2}b^{-1} \\ 0 & 1 \end{pmatrix} \quad\Rightarrow\quad \mathcal{M}\,B\,\mathcal{M}^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Thus we can assume without loss of generality that

$$\vec{c} = (1,0,0), \quad \vec{b} = (0,2,0). \tag{7.29}$$

One now straightforwardly solves (7.18) and the first equation (7.20) with the result

$$\vec{a} = (\lambda,0,1), \quad \vec{l} = 4\lambda\,\vec{a} \tag{7.30}$$

where λ remains undetermined and cannot be fixed by a gauge transformation. The resulting ZCR reads in vector notation

$$\begin{aligned} \vec{A}_2 &= (\lambda + w^2, 2w, 1), \\ \vec{A}_1 &= -w_{xx}(2w,2,0) + (4\lambda w_x + (w_x)^2, 0, 0) + (4\lambda - 2w_x)(\lambda + w^2, 2w, 1). \end{aligned} \tag{7.31}$$

Let us finally use (7.31) to construct a BT of the form (2.5) with $N_v = 1$ using the representation (4.18) of $sl(2)$. The resulting BT reads

$$v_x = \lambda + (v+w)^2, \quad v_t = (4\lambda - 2w_x)(v+w)^2 - 2w_{xx}(v+w) + (w_x)^2 + 2\lambda\, w_x + 4\lambda^2. \tag{7.32}$$

Notice that this BT is not among those given in the section II. Namely both (2.13) and (2.16) contain derivatives of w up to order 3, i.e. they are ZCRs with $N = 3$ for the pKdV equation (recall that in (2.13) one has to identify $u = w_x$). Solving the 'space part' of (7.32) for w and inserting the result into the 'time part' one may check that the BT (7.32) relates the pKdV equation to the following evolution equation for v:

$$v_t = \frac{3\,(v_{xx})^2}{4\,(v_x - \lambda)} + 3(v_x)^2 - v_{xxx}. \tag{7.33}$$

(7.32) may be viewed as the mother of the BTs (2.13) and (2.16) since both can be obtained from it by a gauge transformation (4.1). Namely one can easily check that the BT (2.13) which relates the KdV to the MKdV equation arises from (7.32) through the gauge transformation

$$\tilde{v} = v + w \tag{7.34}$$

and the identification $u = w_x$. The auto-BT (2.16) for the pKdV equation arises from (7.32) through the gauge transformation

$$\tilde{v} = -2v - w \tag{7.35}$$

where in (7.34) and (7.35) v denotes the function which occurs in (7.32). These gauge transformations depend on w and change the value of N from 2 to 3.

Notice that for $N = 2$ and $\mathcal{G} = sl(2)$ we have obtained a ZCR of the pKdV equation which is unique up to gauge transformations which do not change this value of N. This uniqueness gets lost for higher values of N since for each odd value of N (and each choice of $\mathcal{G}$) there is among others a nontrivial solution of (7.11) which can be written in the form

$$N = 2m+1: \quad \vec{A}_1 = (T(w_k), 0, 0, \ldots) \quad \vec{A}_2 = (Q(w_k), 0, 0, \ldots) \tag{7.36}$$

and corresponds to a local conservation law of the KdV equation

$$\partial_t Q(w_k) = \partial_x T(w_k). \tag{7.37}$$

This suggests a close relationship of BTs and local conservation laws in the case $D = 2$ and shows that in two dimensions local conservation laws can also be determined by the method outlined in this section. Since in (7.36) only the first component of ${A_\mu}^I$ is non-zero the Lie algebra of course in this case is actually abelian, i.e. from this point of view local conservation laws (7.37) are abelian ZCRs of a system of PDEs.

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