NIKHEF
NATIONAAL INSTITUUT VOOR KERNFYSICA EN HOGE-ENERGIEFYSICA

DAMTP/93-36
NIKHEF-H/93-18

Brownian motion and the heat equation on superspace and anyspace

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Abstract

We use random walks to study diffusion on anyspace. Anyspace is characterized by co-ordinate ξ with $\xi^N = 0$ and statistics $\xi\xi' = e^{\frac{2\pi i}{N}}\xi'\xi$ between independent copies. Anyonic integration and anyonic Dirac δ-functions are introduced, and reduce to familiar results for supersymmetry when $N = 2$. These ingredients are then used to formulate and solve the resulting anyonic diffusion equation.

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1. In this note we use algebraic methods to formulate a general theory of random walks on anyspace. This is then used to define Brownian motion and diffusion on anyspace as the continuous-time limit of a suitable random walk. Anyspace itself is for us the algebra (in the one-dimensional case) generated by one co-ordinate function ξ and relations

$$\xi^N = 0. \tag{1}$$

This is an obvious generalization of Grassmann variable. Moreover, when one wants to add two independent anyonic variables we use the rule

$$\xi\xi' = q\,\xi'\xi, \qquad q = e^{2\pi i/N} \tag{2}$$

which ensures that $(\xi' + \xi)^N = 0$ also. Thus these anyonic variables are subject to a braid statistics of fractional charge and hence are of the general type encountered in some aspects of anyonic physics. This point of view and the algebraic structure here has been introduced in [1], to which this paper is a direct sequel.

In [1] we developed the above *anyonic line algebra* mathematically as a *braided Hopf algebra* [2]. These braided Hopf algebras or braided groups are a generalization of quantum groups and supergroups to the case where the tensor products of two copies of the algebra enjoy braid statistics. For a usual group or quantum group one uses the usual notion of tensor product in which the two independent copies are considered to mutually commute. For a super-group or super-quantum group there is a ±1 factor corresponding to Bose/Fermi statistics, while for anyonic groups or quantum groups there is the factor (2). Mathematically, the addition law of the anyonic line B is given by a coproduct $B \to B\underline{\otimes}B$ where $B\underline{\otimes}B$ is the braided tensor product algebra

$$(a \otimes b)(c \otimes d) = q^{|b||c|}(ac \otimes bd) \tag{3}$$

on homogeneous elements of degree $|\xi^n| = n$. A more compact notation is to label the elements of the first copy (say) of B with a prime. So $B\underline{\otimes}B$ is generated by $\xi' \equiv \xi \otimes 1, \xi \equiv 1 \otimes \xi$ with the relations $\xi^N = 0, \xi'^N = 0$ for each copy of B and the cross relations (2). For the anyonic line we take the linear coproduct [1]

$$\underline{\Delta}\xi = \xi \otimes 1 + 1 \otimes \xi, \quad \text{i.e.,} \quad \xi \mapsto \xi' + \xi \tag{4}$$

extended to products of as an algebra homomorphism $B \to B\underline{\otimes}B$. An introduction to the general theory of braided Hopf algebras is in [3].

One application of this anyonic addition or anyonic coproduct (4) is the notion of anyonic differentiation as infinitesimal addition [4] [5]. For f a function of ξ it is defined as the part in the expansion of $f(\xi' + \xi)$ which is linear in ξ', just as in the ordinary definition of differentiation. In the anyonic case however, the ordering matters and we keep ξ' to the left using the relations (2). As a result, one obtains a q-derivative rather than an ordinary one,

$$\partial_\xi f(\xi) = \text{coeff of } \xi' \text{ in } f(\xi' + \xi) = \frac{f(\xi) - f(q\xi)}{(1-q)\xi}. \tag{5}$$

This anyonic differentiation can also be characterised by the anyonic Leibniz rule

$$\partial_\xi(fg) = (\partial_\xi f)g + (L_q f)\partial_\xi g, \qquad L_q f(\xi) = f(q\xi). \tag{6}$$

When $N = 2$ we recover the usual super differentiation

$$\partial_\xi 1 = 0, \quad \partial_\xi \xi = 1$$

and its usual super-Leibniz rule. The q-derivative is well-known in the theory of quantum deformations but in the present approach [4] [5] it is derived from something deeper, namely the addition law (4) on anyspace.

In this paper we shall extend this systematic approach further. Firstly, in Section 2 we derive the corresponding left-invariant integration. The left-invariant integral on a braided-Hopf algebra is uniquely determined up to scale. What we find recovers Berezin integration for $N = 2$. We also introduce in the process the anyonic or q-Dirac delta function. In Section 3 we use algebraic techniques to do random walks on anyspace. The use of usual Hopf algebras and also super-ones to do random walks is standard since [6] and was recently used in [7]. We apply the same ideas now to the anyonic line. Finally we are ready in Section 4 to take the continuous time limit of the natural Brownian motion. This defines the anyonic heat equation or q-diffusion equation and also provides a method to solve it, which we do.

We note that some elements of our results are similar to formulae arising in attempts at anyonic path integrals [8] and anyonic quantum mechanics. However, our systematic development of Brownian motion is different and moreover relies heavily on the notion and properties of anyonic quantum groups from [1]. Further applications of our machinery to develop a systematic theory of classical and quantum mechanics on anyspace will be explored elsewhere.

2. On any quantum group or braided quantum group $(B,\underline{\Delta},\underline{\epsilon})$ one can look for a linear functional $\int : B \to \mathbb{C}$ called the left-invariant integral, and characterised by the property

$$(\mathrm{id}\otimes\int)\circ\underline{\Delta}=1\int. \tag{7}$$

This equation corresponds to translation-invariance because $\underline{\Delta}$ corresponds to the group law used for the translation. The same equation is familiar for Hopf algebras or quantum groups [9] and we use it now in the braided case. In the finite dimensional case it exists and is unique up to scale.

Some general theory of integrals on braided groups has been developed in [10] but we do not use it here. Instead we consider B the anyonic line generated by ξ as above and work with it directly. The coproduct is the one in (4) so our invariance requirement (7) reads now as

$$\int f(\xi'+\xi)=\int f(\xi) \tag{8}$$

in the compact notation explained above. The integral on the left is over the second (unprimed) copy of B in $B\underline{\otimes}B$.

We now compute the integral directly. To do this, we use the more concrete notation in which the tensor factors are written explicitly. We require

$$(\mathrm{id}\otimes\int)\underline{\Delta}\xi^n=(\mathrm{id}\otimes\int)(\xi\otimes 1+1\otimes\xi)^n=(\mathrm{id}\otimes\int)\sum_{r=0}^{n}\begin{bmatrix} n\\ r\end{bmatrix}_q \xi^r\otimes\xi^{n-r}$$

where the q-binomial coefficient is defined with q-integers $[r]_q=\frac{1-q^r}{1-q}$. This is needed because the two factors copies of B do not commute, $(1\otimes\xi)(\xi\otimes 1)=q(\xi\otimes\xi)=q(\xi\otimes 1)(1\otimes\xi)$, as explained in Section 1. Let $\int\xi^n=I_n$ say. Then we require

$$\sum_r \begin{bmatrix} n\\ r\end{bmatrix}_q \xi^r I_{n-r}=I_n.$$

Now consider $n=N-1$. Since q is a primitive N-th root of unity we know that $\begin{bmatrix} N-1\\ r\end{bmatrix}\neq 0$ for all r. But the elements $1,\xi,\cdots,\xi^{N-1}$ are linearly independent (since the only relation in the algebra of the anyonic line is $\xi^N = 0$). Hence we conclude that $I_{N-1-r} = 0$ for $r = 1, 2, \cdots, N-1$. We find therefore that our integral $\int$ is uniquely determined up to normalization as

$$\int \xi^n = 0,\ n = 0, 1, \cdots, N-2, \qquad \int \xi^{N-1} = 1. \tag{9}$$

For $N = 2$ we recover the usual Berezin integration, which we have therefore derived now using (braided) Hopf algebra methods.

We recall from Section 1 that the infinitesimal coproduct gives the notion of anyonic differentiation. Putting this into the general invariance property (7) implies at once that

$$\int \partial_\xi f = 0, \qquad \forall\ f(\xi). \tag{10}$$

This is because by definition $\partial_\xi = (\delta_1 \otimes \mathrm{id})\underline{\Delta}$ where $\delta_1(\xi^n) = \delta^n{}_1$ (the Kronecker delta-function). Then $\int(\delta_1 \otimes \mathrm{id})\underline{\Delta} = (\delta_1 \otimes \int)\underline{\Delta} = \delta_1(1) \int = 0$ by (7).

This property for the integral of a total derivative is essential in the applications to physics that we have in mind, and is also a characteristics property in the $N = 2$ case of the Berezin integral.

Once we have a good notion of integration, we can define the notion of a Dirac-delta function. This should be such that

$$\int \underline{\delta}_a f = f(a), \qquad \forall\ f(\xi) \tag{11}$$

where a can be a C-number or another anyonic variable that commutes with ξ so long as it does not involve ξ. To do this, consider arbitrary functions

$$\rho(\xi) = \sum_{i=0}^{N-1} \rho_i \xi^i, \quad f(\xi) = \sum_{i=0}^{N-1} f_i \xi^i \tag{12}$$

say. Then from (9) we have

$$\int \rho f = \sum_{i=0}^{N-1} \rho_{N-1-i} f_i. \tag{13}$$

Hence to fulfill (11) we need

$$\underline{\delta}_a(\xi) = \sum_{i=0}^{N-1} a^{N-1-i} \xi^i. \tag{14}$$

We call this the *anyonic delta function* at a and underline it to remind us that it is an anyonic function and not a usual delta function. For $N=2$ we have $\underline{\delta}_a = a + \xi$ which clearly gives the right answer against Berezin integration.

Finally, note that we have an obvious q-Taylors theorem

$$\int \underline{\delta}_a f = f(a) = \left(\sum_{r=0}^{N-1} \frac{a^r}{[r]_q!}\partial_\xi^r f\right)(0) = e_q^{a\partial_\xi} f|_{\xi=0} \tag{15}$$

where we use the usual q-exponential and note that $\partial_\xi^N = 0$.

3. We now use these mathematical tools to begin to do some physics. Namely, we study random walks on the anyonic line.

To do this we use the algebraic method explained in elementary terms for ordinary Hopf algebras in [7] but using appropriate modifications to describe the anyonic case. Our computation is in fact closely related to the 'unphysical' example in Section 2 of [7] and provides the correct physical interpretation of it, namely in an anyonic setting. One can also see [11] for a related theory of q-stochastic processes, though we shall not use any of that machinery here. The connection between the anyonic case and usual Hopf algebras is quite a precise one and explored in another context in [12].

Recall first that the notion of random walk is subordinate to a probability distribution ρ which governs the probability distribution of each step. If the particle begins at the origin then its probability distribution after n steps is given by the n-fold convolution of ρ.

In an algebraic setting we think of the distribution ρ as defining a state

$$\phi(f) = < f >_\phi = \int \rho f \tag{16}$$

which is the expectation value of the random variable f (any function of position) in the state ϕ corresponding to density ρ. We assume $\int \rho = 1$ or equivalently $\phi(1) = 1$. We do not worry here about positivity requirements or $*$-structure since these topics are not fully understood in an anyonic setting. Note that we can always recover ρ from ϕ as

$$\rho(\eta) = \phi(\underline{\delta}_\eta) \tag{17}$$

using the anyonic delta-function introduced above. Here η is another anyonic variable commuting with ξ. This works because $\int d\eta \rho(\eta) f(\eta) = \int d\eta \phi(\underline{\delta}_\eta(\xi)) f(\eta) = \phi(\int d\eta \underline{\delta}_\xi(\eta) f(\eta)) =$

$\phi(f)$ where we used that $\underline{\delta}$ in (14) is symmetric and where we explicitly wrote the measure $d\eta$ for clarity.

We then work with the state abstractly as a linear map from the algebra B of functions on the space to $\mathbb{C}$, i.e. we think of it as an element of B^*. The usual convolution of density functions in the classical case is then simply the product in B^* which in turn is given by the coproduct $\underline{\Delta}$ of B. In this algebraic language everything is defined and can be done without mentioning any actual points but working directly with B as generated by the co-ordinates ξ. This means that the algebraic approach works at once in the anyonic setting.

Let ϕ be such any normalised linear functional which we think of in this way as corresponding to a probability density, and consider the corresponding random walk in anyspace. The state ϕ^n after n steps is $(\phi\otimes\cdots\otimes\phi)\underline{\Delta}^{n-1}$ which we compute as follows. Firstly, it is convenient to set

$$\xi_i = 1\otimes\cdots\otimes\xi\otimes\cdots\otimes 1,\quad i=1,\cdots,n \tag{18}$$

with ξ in the i'th place and 1 elsewhere. These generate the braided tensor product algebra $B\underline{\otimes}B\underline{\otimes}\cdots\underline{\otimes}B$ as in (3) iterated, with cross relations

$$\xi_j\xi_i = q\xi_i\xi_j,\quad j>i. \tag{19}$$

This is the generalization of (2) to n *anyonically independent* random variables. It is the interpretation in the present setting of the anyonic or braided tensor product introduced in [1] [2].

Armed with this and the q-binomial theorem mentioned above (or indeed by iterating an expression for $\underline{\Delta}\xi^m$) we have at once

$$\underline{\Delta}^{n-1}\xi^m = (\xi_1+\cdots+\xi_n)^m = \sum_{i_1+\cdots+i_n=m}\frac{[m]_q!}{[i_1]_q!\cdots[i_n]_q!}\xi^{i_1}\otimes\cdots\otimes\xi^{i_n}. \tag{20}$$

Applying $\phi\otimes\cdots\otimes\phi$ to this gives

$$<\xi^m>_{\phi^n} = \sum_{i_1+\cdots+i_n=m}\frac{[m]_q!}{[i_1]_q!\cdots[i_n]_q!} <\xi^{i_1}>_\phi\cdots<\xi^{i_n}>_\phi \tag{21}$$

for the moments after n steps in terms of the moments after one step. For example,

$$<\xi>_{\phi^n} = n<\xi>_\phi,\quad <\xi^2>_{\phi^n} = n<\xi^2>_\phi + \binom{n}{2}(1+q)<\xi>_\phi^2.$$

If we define $\xi^{(n)} = n^{-1}\xi$ for the rescaled variable corresponding to ξ, then

$$< \xi^{(n)} >_{\phi^n} = < \xi >_\phi, \quad < \left(\xi^{(n)}\right)^2 >_{\phi^n} - \left(\frac{1+q}{2}\right) < \xi^{(n)} >^2_{\phi^n} = n^{-1}\left(< \xi^2 >_\phi - \left(\frac{1+q}{2}\right) < \xi >^2_\phi\right).$$

This means that the standard-deviation decreases with a factor $n^{-1/2}$ as usual but provided the variance is defined as shown with a $(1+q)/2$ factor. Note that for $N=2$ we cannot consider such questions since the second and higher moments are zero.

4. We are now ready to take the limit of an infinite number of steps of the above random walk and obtain in this way an anyonic diffusion equation governing anyonic Brownian motion. We use the algebraic approach of [7] but now in an anyonic context. Firstly, we take for our state

$$\phi(f) = pf(a) + (1-p)f(-a), \quad \text{i.e.,} \quad \rho = p\delta_a + (1-p)\delta_{-a} \tag{22}$$

corresponding to a probability p to step a and $1-p$ to step $-a$ in our anyspace. Iterating this and taking a limit exactly as in [7] we have

$$\phi^n = (\epsilon + 2a(p - \frac{1}{2})\partial_{\xi|_0} + \frac{a^2}{[2]_q}\partial^2_{\xi|_0} + \cdots)^n = (\epsilon + \frac{ct}{n}\partial_{\xi|_0} + \frac{t\alpha}{n}\partial^2_{\xi|_0} + \cdots)^n$$

where $2a(p-1/2) = ct/n$ and $a^2/[2]_q = t\alpha/n$. In the limit $n \to \infty, \delta, a \to 0$ with t, c, α fixed and $t = n\delta$ we obtain

$$\phi^\infty(f) = \left(e^{t\alpha\partial_\xi^2 + ct\partial_\xi} f\right)_{\xi=0}. \tag{23}$$

We make these computations in the algebra B^* with $\partial_\xi|_{\xi=0}$ regarded as a linear functional $B \to \mathbb{C}$. The counit ϵ is the identity functional in this algebra.

Finally, we write $\phi^\infty(f) = \int \rho^\infty f$ for some ρ^∞. To compute this it is enough to compute ϕ^∞ on an anyonic-delta function according to (17). This comes out easily as

$$\rho^\infty(t,\xi) = \sum_{i=0}^{N-1} \xi^{N-1-i} \sum_{j=0}^{j\le\frac{i}{2}} \frac{(ct)^{i-j}(\alpha/c)^j\,[i]_q!}{j!\,(i-2j)!} \tag{24}$$

which we call the *anyonic Gaussian* distribution. This is computed from

$$\begin{aligned} \phi^\infty(\xi^i) &= \sum_{r=0}^{N-1} \frac{t^r}{r!}(\alpha\partial_\xi^2 + c\partial_\xi)^r \xi^i{}_{|_0} \\ &= \sum_{r\ge\frac{1}{2}i}^{i} \frac{t^r}{(i-r)!\,(2r-i)!}\alpha^{i-r}c^{2r-i}\,[i]_q! \end{aligned}$$

since only the i'th power of ∂_ξ can contribute, and gives $[i]_q!$. Then $\rho^\infty(\eta) = \sum_i \eta^{N-1-i}\phi^\infty(\xi^i)$ from (17) gives the result (24) after a change of variables from r to a suitable variable j. Another form of the result is

$$\rho^\infty(t,\xi) = \sum_{r=0}^{N-1}(ct)^r \sum_{j=0}^{\min(r,N-1-r)} \frac{(\alpha/c)^j\,[j+r]_q!}{j!\,(r-j)!}\xi^{N-1-r-j} \tag{25}$$

obtained by changing the order of summation and a change of i to a suitable variable j. The zero-drift case is with $c=0$ and gives

$$\rho^\infty(t,\xi)_{|_{c=0}} = \sum_{r=0}^{r\leq\frac{N-1}{2}} \frac{(\alpha t)^r[2r]_q!}{r!}\xi^{N-1-2r}. \tag{26}$$

From (23) we can deduce that $\rho^\infty(t,\xi)$ is characterised as the solution of a differential equation. To do this we introduce the adjoint differential $\partial_\xi^\dagger$ defined by

$$\partial_\xi^\dagger = -\partial_\xi L_{q^{-1}} \tag{27}$$

where L is as in the anyonic Leibniz rule (6), which we write in the form

$$f\partial_\xi g = -\left(\partial_\xi L_{q^{-1}}f\right)g + \partial_\xi\left((L_{q^{-1}}f)g\right).$$

We use this and (10) to do integration by parts so that

$$\int f\partial_\xi g = \int(\partial_\xi^\dagger f)g \tag{28}$$

for any anyonic functions f,g. This is the abstract definition of $\partial_\xi^\dagger$ as adjoint to ∂_ξ. It is interesting to compare their explicit forms also on monomials. Noting that $q^N=1$ one finds $[m]_{q^{-1}} = -q[N-m]_q$ so that we have

$$\partial_\xi\xi^m = [m]_q\xi^{m-1}, \quad \partial_\xi^\dagger\xi^m = [N-m]_q\xi^{m-1} \tag{29}$$

which are indeed adjoint under (9). For the case $N=2$ we have $\partial_\xi = \partial_\xi^\dagger$.

We are now ready to derive the equation for ρ^∞ as

$$\begin{aligned}\int(\partial_t\rho^\infty)f \quad &= \left(e^{t\alpha\partial_\xi^2+ct\partial_\xi}(\alpha\partial_\xi^2+c\partial_\xi)f\right)(0)\\ &= \phi^\infty((\alpha\partial_\xi^2+c\partial_\xi)f)\\ &= \int\rho^\infty(\alpha\partial_\xi^2+c\partial_\xi)f\\ &= \int\left((\alpha\partial_\xi^{\dagger 2}+c\partial_\xi^\dagger)\rho^\infty\right)f\end{aligned}$$

using (28). This is for all $f(\xi)$ hence we conclude that ρ^∞ is characterised as the solution of the *anyonic diffusion equation*

$$\partial_t\rho = (\alpha\partial_\xi^{\dagger 2} + c\partial_\xi^\dagger)\rho; \qquad \rho(0,\xi) = \underline{\delta}_0 = \xi^{N-1}. \tag{30}$$

The initial conditions are the anyonic-delta function at the origin. Also, at all times t we have the normalization $\int \rho^\infty = 1$ as we must by our construction.

It is easy to pose this equation [7] but in our anyonic context we have given the solution for the relevant initial conditions, the anyonic delta-function (14). Given the solution (24)-(25) one can verify with a certain amount of work that it indeed obeys the anyonic diffusion equation as it must from our derivation. It is therefore some kind of anyonic-Gaussian corresponding conceptually to the heat kernel $(4\pi\alpha t)^{-\frac{1}{2}}e^{-\frac{(x-ct)^2}{4\alpha t}}$ in the usual or bosonic version of the above analysis. The anyonic version looks of course quite a bit different. Some small N cases are

$$\begin{aligned}
N=2: &\qquad \rho(t,\xi) = \xi + ct\\
N=3: &\qquad \rho(t,\xi) = \xi^2 + ct\xi + [2]_q(\frac{c^2t^2}{2} + \alpha t)\\
N=4: &\qquad \rho(t,\xi) = \xi^3 + ct\xi^2 + [2]_q(\frac{c^2t^2}{2} + \alpha t)\xi + [3]_q!(\frac{c^3t^3}{3!} + c\alpha t^2).
\end{aligned}$$

For $N = 2$ the heat equation is first order and the solution is easy to find. This is the case of a Grassmann variable. We see that this is generalised now to fractional or anyonic statistics in a natural way. It is determined by our choice of probability distribution (22) for the discrete walk and its scaling limit.

It is significant that we solve by these random walk methods an equation which equates a usual continuous time derivative to a q-deformed differential operator. This approach to the heat equation also suggests a version of anyonic quantum mechanics and anyonic path integrals. Some of the tools for this are developed above (such as integration and integration by parts) but the notions of anyonic $*$-structures and anyonic Hilbert spaces etc would also be needed and these remain for further work.

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