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LEVEL-RANK DUALITY OF WZW THEORIES AND ISOMORPHISMS OF $N = 2$ COSET MODELS

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Abstract. Mappings between certain infinite series of $N = 2$ superconformal coset models are constructed. They make use of level-rank dualities for B, C and D type WZW theories, which are described in some detail. The WZW level-rank dualities do not constitute isomorphisms of the theories; for example, for B and D type WZW theories, only simple current orbits rather than individual primary fields are mapped onto each other. Nevertheless they lead to level-rank dualities of $N = 2$ coset models that preserve the conformal field theory properties in such a manner that the coset models related by duality are expected to be in fact isomorphic as conformal field theories; in particular, the representation of the modular group on the characters and the ground states of the Ramond sector are shown to coincide. The construction also gives some further insight in the nature of the resolution of field identification fixed points of coset theories.

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1 Introduction and summary

Level-rank dualities relate objects that are present in two different structures that are connected to each other by exchanging the level (or possibly some simple function thereof) and the rank of an affine Lie algebra (or some closely related algebraic structure). They emerge in various areas of physics and mathematics: in WZW conformal field theories [1, 2, 3, 4, 5] and the theories obtained from them via the coset construction [6]; in three-dimensional Chern–Simons theories [4,7]; in the representation theory of quantum groups with deformation parameter a root of unity [8,9] and of Hecke algebras whose parameter is a root of unity [10]; and in the description of edge variables in fusion-RSOS models [11].

Usually, level-rank duality merely implies certain non-trivial relations among quantities of different theories, such as correlation functions or fusion rules of WZW models. In this paper, we describe several level-rank dualities which go much beyond such relations in that they provide an isomorphism between the respective theories. We show that there exist several such equivalences among infinite series of $N = 2$ superconformal coset theories. More specifically, we describe the identifications

$$\begin{aligned} &(B, 2n + 1, 2k + 1) \cong (B, 2k + 1, 2n + 1)\,, \\ &(B, 2n, 2k + 1) \cong (B, 2k + 1, 2n)_{|D}\,, \\ &(BB, n + 2, 1) \cong (CC, 2, 2n + 1)\,, \\ &(CC, n, k) \cong (CC, k + 1, n - 1)\,. \end{aligned} \tag{1.1}$$

Here the notations are taken from [12] and [13], compare also the tables 1 and 2 below. Let us note that isomorphisms between infinite series of coset conformal field theories have been observed previously. For instance, the $c < 1$ minimal conformal models can be described [14] as $\mathcal{C}[(A_1)_{m-2} \oplus (A_1)_1 \,/\, (A_1)_{m-1}]$, but also as $\mathcal{C}[(C_{m-1})_1 \,/\, (C_{m-2})_1 \oplus (C_1)_1]$; in this case the field contents is tightly constrained by the representation theory of the chiral algebra, so that it is relatively easy to construct an isomorphism as a mapping between primary fields. Our result (1.1) demonstrates for the first time the presence of such isomorphisms for $N = 2$ superconformal theories of arbitrarily high central charge.

The identifications (1.1) are constructed as one-to-one maps between the primary fields of the respective theories. Both at the level of the representation of the modular group, and hence for the fractional part of the conformal dimensions and for the fusion rules, and (by identifying Ramond ground states) at the level of the ring of chiral primary fields we verify that these maps possess the properties needed for an isomorphism of conformal field theories. Clearly one would like to extend the proof from the fusion rules to the full operator product algebra. Because of the technical difficulties arising in the conformal bootstrap (compare e.g. [15]), this would be a quite formidable task. However, it is reasonable to expect that any two $N = 2$ superconformal field theories that possess the same value of the conformal central charge, the same fusion rules, and the same conformal dimensions modulo integers are in fact isomorphic. [1] We are therefore convinced that the two coset theories in question furnish merely two different descriptions of one and the same conformal field theory. In this context note that in general the conformal dimensions of primary fields change with the 'moduli' of some class of conformal field theories. For compatibility with the fusion rules, the number of primary fields must then depend on the moduli as well (in fact, when deforming a rational conformal field theory by a massless modulus one generically obtains an irrational theory, compare the situation at $c = 1$). The arguments in favor of the interpretation of the relations (1.1) as isomorphisms seem to us already conclusive for any fixed choice of a pair of theories from the list (1.1); they become even more convincing when one realizes that our identifications always come in infinite series.

Similar remarks apply to the structure of the chiral ring. We can substantiate our expectation that there is not only a one-to-one map between the chiral primary fields of the theories, but that the sets of chiral primaries also possess isomorphic ring structures, by various arguments. First note that the identification of the sets of Ramond ground states of two $N = 2$ theories implies that they possess the same Poincaré polynomial. From the experience with coset constructions, the observation that there exist coset theories with

[1] In the non-supersymmetric case, examples are known [16] where conformal field theories for which these data coincide are nevertheless distinct theories. These theories have conformal central charge a multiple of 8 and contain only a single primary field.

coinciding Poincaré polynomials is not very spectacular. However, it has in fact been shown [17,13] that not only the ordinary Poincaré polynomials, but also the *extended* Poincaré polynomials (introduced in [17]) of the relevant theories appearing in (1.1) coincide; [2] note that the extended Poincaré polynomial describes explicitly part of the structure of the chiral ring, whereas the ordinary Poincaré polynomial essentially counts multiplicities. Second, the mapping between Ramond ground states, and thus also between chiral primary fields, leaves the superconformal charge q invariant. When proving this, it is important that (in contrast to the case of generic primary fields of a coset theory) for Ramond ground states not only can we easily compute the conformal weight exactly (and not just modulo integers), but also the superconformal charge q [13]. In addition, the ring product of the chiral ring is highly constrained by the fusion rules. Namely, since the ring product is defined as the operator product at coinciding points, the fusion rules (together with naturality [18]) determine which of the structure constants of the chiral ring are non-zero. Finally, the charge conjugation on the fusion ring is implemented by the fusion coefficients ${\mathcal N_{ij}}^0$, and thus our map respects charge conjugation, too. In particular, the charge conjugation behavior of the Ramond ground states is respected. Since conjugation on the chiral ring is induced by the ordinary charge conjugation on the Ramond ground states via spectral flow (which means that there is a highly non-trivial interplay between the chiral ring and the representation of the modular group on the characters), it follows that the map is compatible with the conjugation of the chiral ring.

As it turns out, the identifications (1.1) are also interesting in the context of the field identification problem that arises in coset conformal field theories. Namely, field identification fixed points are mapped on non-fixed points, so that the duality provides additional insight into the procedure of fixed point resolution. (The resolution procedure for field identification fixed points shows up in two different ways: for models of BB type, or of B type with rank and level odd, fixed points are mapped on longer orbits, while for

[2] Surprisingly, it seems that in fact for *all* $N=2$ coset theories for which the ordinary Poincaré polynomials are identical, the same holds for the extended Poincaré polynomials as well.

B type theories at odd level and even rank the resolution is accomplished by mapping on pairs of so-called spinor-conjugate orbits.)

The plan of our paper is as follows. The various level-rank dualities (1.1) of coset theories are consecutively dealt with in the sections 6 to 9 (the isomorphism statements are made in the equations (6.1), (7.1), (8.1), and (9.1), respectively). These sections make heavy use of underlying level-rank dualities for the WZW theories [19,4] the coset models are composed of. For the benefit of the reader we describe the relevant aspects of these dualities in some detail in the sections 3 to 5, in a formulation that is adapted to the needs in $N = 2$ theories, making in particular frequent use of modern simple current terminology. In addition, we present in section 2 a brief reminder about some results and formulæ from conformal field theory that are needed in the sequel.

To conclude this introduction to the subject, let us mention that level-rank dualities for $N = 2$ coset theories have first been conjectured, for hermitian symmetric cosets, in [20]; this conjecture just relied on the symmetry of the conformal central charges of the relevant coset theories. Calculations of the spectra of $N = 2$ coset theories were first performed in [12, 17] for hermitian symmetric cosets, and in [13] for non-hermitian symmetric cosets. The results of [17] provided some evidence that the dualities indeed exist; in particular, it was realized that for B type theories at odd level and even rank the D type modular invariant must be used rather than the diagonal one. In the present paper, we combine the level-rank dualities of WZW theories with the properties of simple current symmetries to construct a map between the primary fields of the $N = 2$ coset theories in question that makes the level-rank duality explicit and is expected to be an isomorphism of the two conformal field theories. It is worth to stress that the underlying level-rank dualities of WZW theories are definitely not isomorphisms of conformal field theories. In particular, these WZW dualities are typically *not* mappings between primary fields, but rather between simple current orbits of (part of) the primary fields. As we will see, this fits perfectly to the application to coset theories, because owing to the necessary field identifications the physical fields of a coset theory can be characterized in terms of combinations of simple current orbits only. In some cases this technical complication makes the formulation of the mapping somewhat awkward (and adds to the length of our paper), but, nonetheless, the mappings are based on simple current symmetries, and hence on natural objects of the underlying WZW theories. We shall show in the sequel that these mappings have the properties required for isomorphisms of conformal field theories.

In [20] it was conjectured that a relation between B type theories at even rank and even level should exist, too. In this case non-diagonal modular invariants must be chosen, but up to now it is not yet clear which of them could do the job. [3] Finally, based on a free field realization of the symmetry algebra, a level-rank duality for the A type hermitian symmetric cosets has been shown to be present at the level of symmetry algebras [20]. It would be interesting to explore these dualities by the techniques developed in the present paper.

2 Results from conformal field theory

2.1 Primary fields

The collection of fields of a two-dimensional conformal field theory carries the structure of a direct sum of irreducible highest weight modules $[\phi_i]$ of the symmetry algebra. The fields corresponding to the highest weights are the primary fields ϕ_i. Upon forming radially ordered products, the fields realize a closed associative operator product algebra. A large amount of information about the operator product algebra is contained in the fusion rules of primary fields, which can be written as formal products, $\phi_i \star \phi_j = \sum_k {\mathcal{N}_{ij}}^k \phi_k$; ${\mathcal{N}_{ij}}^k$ counts the number of times that $[\phi_k]$ appears in the operator product of ϕ_i and ϕ_j.

The characters $\chi_i(\tau) = \mathrm{tr}_{[\phi_i]} \exp(2\pi \mathrm{i} \tau (L_0 - c/24))$ associated to the modules $[\phi_i]$ span a unitary module of the group $\mathrm{SL}(2,\mathbf{Z})$, the twofold cover of the modular group $\mathrm{PSL}(2,\mathbf{Z})$. This group is generated freely by elements S and T, modulo the relations $S^2 = C = (ST)^3$, where $C^2 = 1\!\!1$; on the characters, these generators act as $S:\ \tau \mapsto -1/\tau$, $T:\ \tau \mapsto \tau + 1$. Thus

[3] Also, none of these $N = 2$ models is relevant to string compactification. For us this is another reason to refrain from investigating these dualities here.

in particular T is represented as a matrix whose entries are determined by the fractional part of the conformal dimensions Δ_i of the primary fields, $T_{ij} = \delta_{ij} \exp(2\pi\mathrm{i}(\Delta_i - c/24))$. Further, the fusion rules of the theory can be calculated from the matrix S via the Verlinde formula [21]. The largest eigenvalue of the matrix $\mathcal{N}_i$ with elements $(\mathcal{N}_i)_{jk} = \mathcal{N}_{ij}{}^k$, the *quantum dimension* of ϕ_i, is of paricular interest; it equals S_{i0}/S_{00}, where the index '0' refers to the identity field $\mathbf{1}$.

Among the primary fields of a conformal field theory there may be fields J with the property that $J\star J^+ = \mathbf{1}$. These fields are known as simple currents; the collection of simple currents of a conformal field theory forms an abelian group, with the product given by the fusion product and inversion given by conjugation. The fusion product of a simple current with an arbitrary primary field ϕ of the conformal field theory consists of only a single primary field, and correspondingly we will often use the notation $J\phi = J \star \phi$ for this field; ϕ and $J\phi$ are said to lie on the same *orbit* with respect to the simple current J. For simplicity we will also write multiple fusion products of simple currents as powers, $J^m = \underbrace{J \star J \star \ldots \star J}_{m \text{ factors}}$. Simple currents play an important role for the construction of non-diagonal modular invariants and (as we will discuss in subsection 2.3) for the description of the spectrum of coset conformal field theories [22]. In particular, the so-called D type modular invariants correspond, roughly speaking, to incorporating a simple current into the chiral algebra. The D type invariant relevant to us is induced by a simple current of order two (i.e., $J^2 = \mathbf{1}$) with integral conformal weight. Thus it is of the form $\sum (N_0/N_\phi)\,|\sum_{i=0}^{N_\phi - 1} \chi_{J^i\phi}|^2$, where N_ϕ denotes the length of the simple current orbit containing the primary field ϕ and $N_0 = 2$ the length of the orbit of the identity field $\mathbf{1}$, and were the first sum is restricted to orbits that have vanishing monodromy charge with respect to J.

In $N = 2$ superconformal field theories another interesting set of fields are the chiral primary fields. For unitary field theories (these are the only ones we are going to deal with) they can be characterized as those fields in the Neveu–Schwarz sector for which the relation $\Delta = q/2$ between the conformal dimension and the superconformal u_1-charge holds. Their operator product at coinciding points provides a ring structure different from the one defined by the fusion rules, the so-called chiral ring. The chiral ring plays a crucial role for many applications of $N = 2$ theories; for instance, it encodes interesting phenomenological information when one uses the theories as the inner sector of a superstring compactification, and also the relation to topological field theory is mainly through this ring. Via spectral flow [23] the chiral ring is in one-to-one correspondence to the set of ground states of the Ramond sector of the $N = 2$ theory, which in our context is easier to deal with.

2.2 WZW theories

A Wess–Zumino–Witten (WZW) theory is a conformal field theory whose chiral symmetry algebra is the semidirect sum of the Virasoro algebra with an untwisted affine Lie algebra, with the energy-momentum tensor being quadratic in the currents, i.e. in the generators of the affine algebra. Many quantities of interest of a WZW theory can be described entirely in terms of the horizontal subalgebra, i.e. the simple Lie algebra g that is generated by the zero mode currents, and of the level K which (for unitary theories) is a non-negative integer. For instance, the conformal central charge is $c(g, K) = K \dim(g)/(K + g^\vee)$, with $g^\vee$ the dual Coxeter number of g.

The primary fields of a left-right symmetric unitary WZW theory are in one-to-one correspondence with the integrable highest weights, i.e. with the dominant integral weights Λ of g that satisfy

$$(\Lambda, \theta) \leq K, \tag{2.2}$$

where θ is the highest root of g (normalized such that $(\theta, \theta) = 2$). The conformal dimension of a primary field with highest weight Λ is

$$\Delta_\Lambda \equiv \Delta_{(g)}(\Lambda) = \frac{(\Lambda, \Lambda + 2\rho)}{2(K + g^\vee)}, \tag{2.3}$$

where ρ is the Weyl vector $\rho = \sum_{i=1}^n \Lambda_{(i)}$, with $\Lambda_{(i)}$ the fundamental weights of g.

The situation is particularly simple for $g = D_d$ at level one. Then there are four primary fields corresponding to the singlet (0), vector (v), spinor (s), and conjugate spinor (c) representation of D_d, or, in other words, to the conjugacy classes of the D_d weight lattice; their conformal dimension is

$$\Delta = \begin{cases} 0 & \text{for} \quad 0\,, \\ 1/2 & \text{for} \quad \mathrm{v}\,, \\ d/8 & \text{for} \quad \mathrm{s,\ c}\,. \end{cases} \tag{2.4}$$

The modular matrix S of D_d reads

$$S((D_d)_1) = \tfrac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & \mathrm{i}^{-d} & -\mathrm{i}^{-d} \\ 1 & -1 & -\mathrm{i}^{-d} & \mathrm{i}^{-d} \end{pmatrix}. \tag{2.5}$$

Another important class of conformal field theories are those describing a single free boson compactified on a circle of rational radius squared, to which for simplicity we will refer as a WZW theory with horizontal subalgebra u_1. The primary fields ϕ_Q of these theories are labelled by u_1-charges $Q \in \{0, 1, , ... , \mathcal{N}-1\}$, where the number $\mathcal{N}$ of primaries is related to the radius of the circle. (The values of the integer $\mathcal{N}$ that appear in the $N = 2$ coset theories of our interest will be given in table 1.) The conformal dimension of a u_1-primary of charge Q is $Q^2/2\mathcal{N}$. The S-matrix elements of a u_1WZW theory are

$$S_{PQ} = \frac{1}{\sqrt{\mathcal{N}}} \exp(-2\pi\mathrm{i}\, PQ/\mathcal{N}). \tag{2.6}$$

To fix the notation, let us also list the simple currents of the WZW theories of our interest. For B and C type theories, there is a single simple current besides the identity primary field; this current will be denoted by J (the corresponding highest weight is $K\Lambda_{(1)}$ for B_r, and $K\Lambda_{(r)}$ for C_r theories). For D_r type theories, there are three non-trivial simple currents, corresponding to the highest weights $K\Lambda_{(1)}$, $K\Lambda_{(r)}$, and $K\Lambda_{(r-1)}$; they are denoted by J_v, J_s, and J_c, as their fusion rules are isomorphic to the multiplication of the vector (v), spinor (s), and conjugate spinor (c) conjugacy classes. Finally, for u_1 WZW theories, the fusion rules read $\phi_P \star \phi_Q = \phi_{P+Q \bmod \mathcal{N}}$, and hence any primary field is a simple current.

2.3 Coset theories

The idea of the coset construction [14] of conformal field theories is to associate to any pair g, h of reductive Lie algebras for which h is a subalgebra of g, a conformal field theory called the coset theory

$$\mathcal{C}[g/h]_K\,. \tag{2.7}$$

By definition [14], the Virasoro generators of the coset theory are obtained by subtracting the Virasoro generators of the WZW theory based on h from the ones of the WZW theory based on g. If g is simple, then the level K_i of any simple summand h_i of h is related to the level K of g by $K_i = I_i K$, where I_i is the Dynkin index of the embedding $h \hookrightarrow g$.

In order to check whether this definition of the coset Virasoro algebra leads to a well-defined conformal field theory, one also has to specify the spectrum of primary fields of the theory. As it turns out, to obtain these primary fields of the coset theory is a somewhat delicate issue. To some extent, the field contents can be read off the so-called branching functions b^Λ_λ, which are the coefficient functions in the decomposition

$$\mathcal{X}_\Lambda(\tau) = \sum_\lambda b^\Lambda_\lambda(\tau)\,\chi_\lambda(\tau) \tag{2.8}$$

of the characters $\mathcal{X}_\Lambda$ of g with respect to the characters χ_λ of h. (Here Λ and λ stand for integrable highest weights of g and h, respectively, if g and h are simple, and similarly in the general case.) The behavior of the branching functions under modular transformations suggests that the coset theory associated to the embedding $h \hookrightarrow g$ might be essentially something like $g \oplus h^*$, where the notation '$*$' indicates that the complex conjugates of the modular transformation matrices of the WZW theory based on h should be used. Note that if S and T generate a representation of the modular group, the same is true for S^* and T^*. If there exists a conformal field theory whose characters transform according to this complex conjugate representation, it is called the *complement* of the h theory [24].

However, closer inspection shows that the coset theory is in fact rather different from $g \oplus h^*$. Namely, in a coset theory the requirement that the characters span a unitary module of $\mathrm{SL}(2,\mathbf{Z})$ forces us to associate physical fields

Table 1: Some $N = 2$ superconformal coset theories and their Virasoro charges

name	$\mathcal{C}[g_K \oplus (D_d)_1 / \bigoplus_i (h_i)_{K_i} \oplus (\mathrm{u}_1)_{\mathcal{N}}]$	c
$(B, 2n+1, K)$	$\mathcal{C}[(B_{n+1})_K \oplus (D_{2n+1})_1 \,/\, (B_n)_{K+2} \oplus (u_1)_{4(K+2n+1)}]$	$\dfrac{3K(2n+1)}{K+2n+1}$
$(B, 2n, K)$	$\mathcal{C}[(D_{n+1})_K \oplus (D_{2n})_1 \,/\, (D_n)_{K+2} \oplus (u_1)_{4(K+2n)}]$	$\dfrac{6Kn}{K+2n}$
$(BB, 3, K)$	$\mathcal{C}[(B_3)_K \oplus (D_7)_1 \,/\, (A_1)_{2K+8} \oplus (A_1)_{K+3} \oplus (u_1)_{2(K+5)}]$	$21 - \dfrac{96}{K+5}$
$(BB, n, K)\,,\ n>3$	$\mathcal{C}[(B_n)_K \oplus (D_{4n-5})_1 \,/$ $(B_{n-2})_{K+4} \oplus (A_1)_{K+2n-3} \oplus (\mathrm{u}_1)_{2(K+2n-1)}]$	$12n - 15 - \dfrac{24(n-1)^2}{K+2n-1}$
(CC, n, K)	$\mathcal{C}[(C_n)_K \oplus (D_{2n-1})_1 \,/\, (C_{n-1})_{K+1} \oplus (u_1)_{2(K+n+1)}]$	$6n - 3 - \dfrac{6n^2}{K+n+1}$

not with individual branching functions, but rather with certain equivalence classes of them. This is commonly referred to as 'field identification' [25,22]. As already mentioned, the field identification can be understood in terms of simple current symmetries. Namely [22], technically it is convenient to implement the identification procedure by means of the action of appropriate simple currents, known as identification currents. These are specific tensor products, to be denoted as $(J_{(g)} \,/\, J_{(h)})$, of the simple currents of the WZW theories that underly the coset theory.

As long as all orbits with respect to the identification currents have equal size, the orbits are precisely the independent physical fields we are after.The situation is more involved if the orbits have different numbers of representatives. These numbers are divisors of the length N_0 of the orbit of the identity field. Orbits with less than N_0 representatives are referred to as 'fixed points' of the identification currents. Among the theories of our interest, only the cosets of CC type and $(B, 2n, 2k+1)$ do not possess any fixed points. If fixed points are present, one has to complement the previous prescription for

Table 2: Identification currents for $N = 2$ coset theories

name	N_0	Independent identification currents
$(B, 2n+1, 2k+1)$	4	$\begin{cases} J_{(1)} := (J, 0 \,/\, J, 0) \\ J_{(2)} := (J, J_{\rm v} \,/\, 0, \pm 4(k+n+1)) \end{cases}$
$(B, 2n+1, 2k)_{\vert D}$	8	$\begin{cases} J_{(1)} := (J, 0 \,/\, J, 0) \\ J_{(2)} := (J, J_{\rm v} \,/\, 0, \pm 2(2k+2n+1)) \\ J_{(3)} := (J, 0 \,/\, 0, 0)) \end{cases}$
$(B, 2n, K)$	8	$\begin{cases} J_{(1)} := (J_{\rm v}, 0 \,/\, J_{\rm v}, 0) \\ J_{(2)} := (J_{\rm s}, (J_{\rm v})^n \,/\, J_{\rm s}, (K+2n)) \end{cases}$
(BB, n, K)	4	$\begin{cases} J_{(1)} := (J, 0 \,/\, J, 0, 0) \\ J_{(2)} := (J, 0 \,/\, 0, J, \pm(K+2n-1)) \end{cases}$
(CC, n, K)	2	$J_{(1)} := (J, (J_{\rm v})^n \,/\, J, \pm(K+n+1))$

finding the physical fields by a so-called 'fixed point resolution.' Every fixed point of length N_f has to be resolved in N_0/N_f distinct physical fields.

As has been shown in [26], coset theories $\mathcal{C}[\tilde{g}/\tilde{h}]_K$ with

$$\tilde{g} = g \oplus D_d \qquad \tilde{h} = h \oplus {\rm u}_1 \tag{2.9}$$

with $2d = \dim g - \dim \tilde{h}$ and D_d at level one can possess $N = 2$ superconformal symmetry; further, all combinations of g and h for which this happens have been listed [26,27]. In table 1 we present those cases which are relevant for our present purposes.

For $N = 2$ coset theories, the fixed point resolution procedure has been worked out in [17] (for the so-called hermitian symmetric cosets) and in [13] (for non-hermitian symmetric $N = 2$ theories). The primary fields Φ of a $N = 2$ coset theory $\mathcal{C}[g \oplus D_d/h \oplus {\rm u}_1]_K$ may be labelled by the weights carried by the primaries of the WZW theories it is composed of, i.e.

$$\Phi \,\hat{=}\, (\Lambda, {\rm x} \,/\, \lambda, Q) \tag{2.10}$$

with Λ and λ integrable highest weights of the g and h algebras, x a conjugacy class of D_d, and $Q \in \{0, 1, , \dots, \mathcal{N}-1\}$ a u_1-charge. However, as a consequence of the necessary field identification, this labelling is not one-to-one. Rather, all combinations of labels that are connected via the action of the identification currents describe one and the same primary field; moreover, fixed points have to be resolved, which introduces an additional label i according to

$$\Phi_{\text{fix}} \mathrel{\hat{=}} (\Lambda, \mathrm{x} \,/\, \lambda, Q)_i \tag{2.11}$$

The identification currents (including the simple current that implements the D type modular invariant in the case of $(B, 2n+1, 2k)$) of the $N = 2$ theories of table 1 are displayed in table 2. [4]

The conformal dimension of the field Φ is modulo integers

$$\Delta(\Phi) = \Delta_{(g)}(\Lambda) + \Delta_{(D_d)}(\mathrm{x}) - \Delta_{(h)}(\lambda) - \Delta_{(\mathrm{u}_1)}(Q), \tag{2.12}$$

where $\Delta_{(g)}(\Lambda)$ and $\Delta_{(h)}(\lambda)$ are defined as in (2.3), $\Delta_{(D_d)}(\mathrm{x})$ is given in (2.4), and $\Delta_{(1)}(Q) = Q^2/2\mathcal{N}$. The superconformal charge q is modulo 2 given by

$$q(\Phi) = \sum_\alpha \mathrm{x}^\alpha - \frac{\xi Q}{K + g^\vee} \,. \tag{2.13}$$

Here x^α are the components of x in the orthonormal basis of the D_d weight space, and ξ is a rational number that can be calculated [13] with the help of the relation between the normalization of the u_1-charge and the length of $\rho_g - \rho_h$. For the theories of our interest, one has $\xi = n$ for $(B, 2n, K)$ and (CC, n, K), $\xi = n + \frac{1}{2}$ for $(B, 2n+1, K)$, and $\xi = 2(n-1)$ for (BB, n, K).

For the S-matrix elements in a coset theory with fixed points one makes the ansatz [17]

$$\tilde{S}_{e_i f_j} = \frac{N_e N_f}{N_0} S_{ef} + \Gamma^{ef}_{ij} \,, \tag{2.14}$$

[4] Non-trivial simple currents of WZW theories are denoted by the symbols J, J_{v} etc. introduced above.
We also note that in [17] a different value for $\mathcal{N}$ is used for $(B, 2n, K)$, namely $\mathcal{N} = 16(K + 2n)$. This is compensated by adding, as is done in [17], a further identification current which is trivial in all parts except the u_1 part, namely $(1, 1 \,/\, 1, \pm 8(K + 2n))$.

where $i = 1, 2, \ldots, N_0/N_e$ and $j = 1, 2, \ldots, N_0/N_f$ label the fields into which the naive fields ϕ_e and ϕ_f are to be resolved if they are fixed points. Modular invariance implies the sum rule

$$\sum_i \Gamma^{ef}_{ij} = 0 = \sum_j \Gamma^{ef}_{ij} \tag{2.15}$$

for the S-matrix elements of the fields ϕ_{f_i}. Note that if either ϕ_e or ϕ_f is not a fixed point, the sum rule tells us that Γ vanishes. So Γ is non-zero only for pairs of resolved fixed points, in which case it has also to be symmetric under simultaneous exchange of (e, i) and (f, j), since the total S-matrix $\tilde{S}$ must have this property.

To find a solution for Γ for $N = 2$ coset theories, we assume that with respect to the individual entries of the multi-index $(f, i) = (\Lambda, \mathrm{x} \,/\, \lambda, Q)_i$, Γ factorizes as

$$\Gamma^{\Lambda,\mathrm{x},\lambda,Q;\Lambda',\mathrm{x}',\lambda',Q'}_{i;j} = \Gamma^{\Lambda\,\Lambda'}_{(g)} \Gamma^{\mathrm{x}\,\mathrm{x}'}_{(D_d)} \Gamma^{\lambda\,\lambda'}_{(h)} \Gamma^{Q\,Q'}_{(\mathrm{u}_1)} P_{ij}\,, \tag{2.16}$$

where

$$P_{ij} = \delta_{ij} - \frac{N_f}{N_0} \tag{2.17}$$

if $N_e = N_f$. (For brevity we will focus here on fixed points with respect to identification currents of prime order, in which case $N_e = N_f = 1$ for all fixed points. If the order is not prime, an iterative procedure [24] must be applied to resolve fixed points with $N_f > 1$. Also note that for $N_0 = 2$ and $N_f = 1$ the factorization of P in (2.16) is a necessary consequence of the sum rule (2.15).) In [24] a general prescription to find the matrix factors $\Gamma_{(\cdot)}$ for WZW theories was given. These matrices, as well as the character modifications to be described below, can be viewed as the S-matrix and characters of some conformal field theory, which is called the 'fixed point theory.' In most cases the fixed point theory turns out to be another WZW theory. As we shall see below, this prescription to deal with fixed points is consistent with our level-rank duality in coset theories. In fact, level-rank duality will even provide additional insight in the nature of the fixed point resolution.

Resolving a fixed point amounts to considering fields having different characters χ_{f_i}, i.e. the naive branching function χ_f of the 'unresolved fixed point' must be modified by adding an appropriate multiple of a character

$\breve{\chi}_f$ of the fixed point theory. Again, modular invariance implies a sum rule, namely

$$\sum_i \chi_{f_i} = \frac{N_0}{N_f}\,\chi_f\,. \tag{2.18}$$

For $N_0/N_f = 2$, we denote the two resolved fixed points by $f_\pm$. According to (2.18), the corresponding characters read

$$\chi_{f_\pm} = \chi_f \pm v\breve{\chi}_f\,, \tag{2.19}$$

where without loss of generality we can assume that $v > 0$. It is easy to see that these characters transform indeed according to the resolved S-matrix (2.14), with $P_{++} = P_{--} = \frac{1}{2}$ and $P_{+-} = P_{-+} = -\frac{1}{2}$.

In order to identify the chiral rings of $N = 2$ coset theories, we will look at the Ramond ground states. Any Ramond ground state has at least one representative

$$\Phi_R = (\Lambda, \mathrm{x}, \tilde\lambda) \tag{2.20}$$

for which Λ and $\tilde\lambda$ are related through a Weyl group element $w \in W(g)$ according to [23]

$$\tilde\lambda + \rho_h = w(\Lambda + \rho_g)\,. \tag{2.21}$$

Here $\tilde\lambda$ incorporates both the weight λ of the semi-simple part h of $\tilde h$ and the u_1-charge Q, and

$$\mathrm{x} = \begin{cases} \mathrm{s} & \text{for sign}\,(w) = 1\,, \\ \mathrm{c} & \text{for sign}\,(w) = -1\,. \end{cases} \tag{2.22}$$

Furthermore, the Weyl group element w has to be chosen in such a manner that λ is a highest weight of h (this fixes uniquely one representative of each element of the coset $W(g)/W(h)$).

The superconformal charge q (including the integer part) of a Ramond ground state is conveniently computed from the formula [13]

$$q(\Phi_R) = \frac{d}{2} - l(w) - \frac{\xi Q}{K + g^\vee} \tag{2.23}$$

that relates q to the u_1-charge Q and to the length $l(w)$ of the Weyl group element w that appears in (2.21) (ξ is the number introduced in (2.13)). The length $l(w)$ can be obtained [28] as the number of negative roots α of g for which $w(\alpha)$ is a positive root,

$$l(w) = |\{\, \alpha{<}0 \mid w(\alpha){>}0 \,\}| \ . \tag{2.24}$$

For an $N=2$ coset theory $\mathcal{C}[g\oplus D_d/h\oplus \mathrm{u}_1]_K$ without fixed points, the number of chiral primary fields is [23]

$$\mu = \frac{N(g)}{|Z(g)|}\,\frac{|W(g)|}{|W(h)|}\,, \tag{2.25}$$

where N is the number of primary fields of the WZW theory based on g at level K, and $Z(g)$ is the center of the universal covering group whose Lie algebra is g (which is isomorphic to the group of simple currents of the WZW theory). The factor $1/|Z(g)|$ takes care of the necessary field identifications among representatives of the form (2.20), (2.21). In contrast, if an $N=2$ coset theory has fixed points, the number of Ramond ground states is larger than (2.25). Namely, each primary field of g still gives rise to $|W(g)/|W(h)|$ representatives of chiral primaries, but in addition for fixed points it is still true that (after resolution of fixed points) every Ramond ground state has a representative whose g- and h-weights fulfill (2.21), and that every equivalence class containing one representative with $\tilde\lambda = w(\Lambda+\rho_g)-\rho_h$ yields precisely one Ramond ground state.

3 B type WZW theories at odd level

In this section we will describe a map τ between the WZW theories $(B_n)_{2k+1}$ and $(B_k)_{2n+1}$ that has simple behavior with respect to the modular matrices T (i.e., with respect to conformal dimensions modulo integers) and S. Thus the two theories that are connected by τ are related by exchanging twice the rank plus one (recall that $B_n\cong \mathrm{so}(2n+1)$) with the level of a B type affine Lie algebra; a relation of this type is called *level-rank duality.* As mentioned in the introduction, such dualities emerge in various different contexts; here we will concentrate on those aspects that are needed for the identifications of $N=2$ coset theories in sections 6 to 8 below. The level-rank duality in question was first realized in [4]; in the notation of [4], our map τ corresponds to the map 'tilde' for B weights that are tensors, and to the map 'hat' for spinor weights, respectively. [5] To be more precise, τ will be a one-to-one map between orbits with respect to the relevant simple currents J of the two theories. Thus, to start, we note that the number of primaries of the $(B_n)_{2k+1}$ WZW theory, i.e. the number of integrable representations of the affinization of B_n at level $2k+1$, is

$$N^B_{n,2k+1} = \sum_{l=0}^{k} \binom{2k-2l+3}{2} \binom{l+n-3}{l} = (4k+3n)(n+k-1)!/n!k!\,; \tag{3.1}$$

of these,

$$F^B_{n,2k+1} = \binom{n+k-1}{k} \tag{3.2}$$

are fixed points, so that the number of orbits is $2\binom{n+k}{k}$. This is invariant under $n \leftrightarrow k$, so that indeed a one-to-one map between the respective sets of orbits is conceivable.

For any integrable highest weight $\Lambda = \sum_{i=1}^{n} \Lambda^i \Lambda_{(i)}$ of $(B_n)_{2k+1}$, denote by

$$c_\Lambda = \Lambda^n \bmod 2 \tag{3.3}$$

the conjugacy class of Λ. For brevity, we will often refer to Λ as a 'tensor' and as 'spinor' weight if $c_\Lambda = 0$ and $c_\Lambda = 1$, respectively. Consider now the components of Λ in the orthonormal basis of the weight space; they read

$$\ell_i(\Lambda) = \sum_{j=i}^{n-1} \Lambda^j + \tfrac{1}{2}\,\Lambda^n. \tag{3.4}$$

Adding to these numbers the components of the Weyl vector as well as a term $\frac{1}{2}\,(1-c_\Lambda)$ such as to make the result integer-valued, one defines

$$\tilde{\ell}_i(\Lambda) := \ell_i(\Lambda+\rho) + \tfrac{1}{2}\,(1-c_\Lambda) = \sum_{j=i}^{n-1} \Lambda^j + n + 1 - i + \tfrac{1}{2}\,(\Lambda^n - c_\Lambda). \tag{3.5}$$

[5] We are grateful to the authors of [4] for extensive explanation of their notation.

Under the action of the simple current J that carries the highest weight $(2k+1)\Lambda_{(1)}$, the numbers ℓ_i, $i=2,3,...,n$, are invariant, while ℓ_1 gets replaced by $2k+1-\ell_1$. As a consequence, we may characterize any orbit of J by a set of n positive integers $\tilde\ell_i$, $i=1,2,...,n$ subject to $\tilde\ell_i>\tilde\ell_j$ for $i<j$ as well as $\tilde\ell_1\le k+n$, or in other words, by a subset M_Λ of cardinality $|M_\Lambda|=n$ of the set

$$M:=\{1,2,\ldots,k+n\}\,. \tag{3.6}$$

Each such subset describes precisely one tensor and one spinor orbit (in particular, there are as many spinor orbits as tensor orbits if the level of B_n is odd), and conversely, any integrable highest weight of $(B_n)_{2k+1}$ corresponds to precisely one of these subsets.

We are now in a position to present the map τ. First consider spinor weights Λ of $(B_n)_{2k+1}$. Given the associated subset $M_\Lambda\subset M$, define the complementary set

$$\{\tilde\ell_i^{(\tau)}\}\equiv M_{\tau(\Lambda)}:=M\setminus M_\Lambda, \tag{3.7}$$

where the numbers $\tilde\ell_i^{(\tau)}$ are to be ordered according to $\tilde\ell_i^{(\tau)}>\tilde\ell_j^{(\tau)}$ for $i<j$. Since this subset of M again satisfies $\tilde\ell_1^{(\tau)}\le k+n$, and is of cardinality k, it describes precisely one orbit $\{\tau(\Lambda),J\star\tau(\Lambda)\}$ of integrable highest spinor weights of $(B_k)_{2n+1}$. Also note that M_Λ describes a spinor fixed point iff $k+n\in M_\Lambda$ (in contrast, there do not exist tensor fixed points at odd level); thus spinor fixed points are mapped to spinor orbits of size two, and vice versa.

Let us now check how the modular matrix T transforms under the map τ. By combining the formulæ (2.3) and (3.7), and inserting the strange formula for the length of the Weyl vectors, one finds

$$\begin{aligned}\Delta_\Lambda+\Delta_{\tau(\Lambda)} &= [\textstyle\sum_{j\in M_\Lambda} j^2-(\rho,\rho)+\sum_{j\in M_{\tau(\Lambda)}} j^2-(\rho^{(\tau)},\rho^{(\tau)})]/[4(k+n)]\\ &= [\textstyle\sum_{j=1}^{k+n} j^2-\frac{1}{12}\,(4n^3-n+4k^3-k)]/[4(k+n)]\\ &= \tfrac18\,(k+n+2kn+\tfrac12)\,,\end{aligned} \tag{3.8}$$

where ρ and $\rho^{(\tau)}$ denote the Weyl vectors of B_n and B_k respectively. (Recall that we choose the representatives Λ and $\Lambda^{(\tau)}$ such that $\tilde\ell_1\le k+n$ and ${\tilde\ell_1}^{(\tau)}\le k+n$; as the conformal dimensions of the elements of a spinor orbit differ by an integer, this means that for the other member of a length-two orbit, the formula holds true modulo $\mathbf{Z}$).

For tensors we will have to consider a definition of τ that is different from that for spinors [4]. Namely, while again the complement of M_Λ in M plays a role, we now define $M_{\tau(\Lambda)}$ by

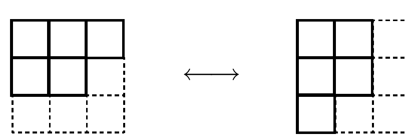
$$\{\tilde{\ell}_i^{(\tau)}\} \equiv M_{\tau(\Lambda)} := \{k+n+1-l \mid l \in M \setminus M_\Lambda\}. \tag{3.9}$$

By definition, this maps tensor orbits to tensor orbits, and again the image covers all such orbits of $(B_k)_{2n+1}$ precisely once. For the sum of conformal dimensions we now obtain

$$\begin{aligned} \Delta_\Lambda + \Delta_{\tau(\Lambda)} \;&=\; [\textstyle\sum_{j\in M_\Lambda}(j-\frac12)^2 - (\rho,\rho) + \sum_{j\in M\setminus M_\Lambda}(k+n+\frac12-j)^2 \\ &\qquad\qquad\qquad - (\rho^{(\tau)},\rho^{(\tau)})]\,/[4(k+n)] \\ &=\; \tfrac14\,k(k+2n+1) - \tfrac12 \textstyle\sum_{j\in M\setminus M_\Lambda} j\,, \end{aligned} \tag{3.10}$$

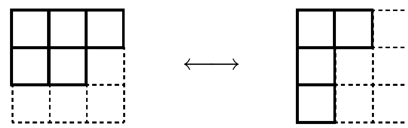
which is a half integer. (Again this result is true for Λ such that $\tilde{\ell}_1 \le k+n$, and analogously for $\tau(\Lambda)$; the conformal dimensions of the elements of a tensor orbit differ by $\frac12$ plus an integer, so that for the other members of the orbits, the formula still holds modulo $\mathbf{Z}/2$).

One can visualize the map τ in terms of Young tableaux $Y(\Lambda)$, defined as having $\ell_i(\Lambda) - \frac12 c_\Lambda$ boxes in the ith row. The prescription (3.7) corresponds to forming the complement with respect to the rectangular Young tableau $Y(k\Lambda_{(n)})$, followed by reflection at an axis perpendicular to the main diagonal. Similarly, the map (3.9) corresponds just to reflection at the main diagonal. For example, consider the following mapping between tensor orbits of the (self-dual) $(B_3)_7$ WZW theory (for better readability, we display, with dotted lines, also the missing boxes that are needed to extend a tableau $Y(\Lambda)$ to $Y(k\Lambda_{(n)})$):



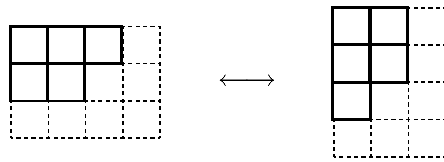
(3.11)

According to the previous prescriptions, the corresponding orbits are $\{(1,2,0),(2,2,0)\}$ for the left hand side, and $\{(0,1,2),(3,1,2)\}$ for the right hand side (here we write the weights in the basis of fundamental highest weights), and indeed these orbits are mapped onto each other by (3.9). Considering, instead, the left hand side as a Young tableau for a spinor orbit, namely for the fixed point $(1,2,1)$, it gets mapped via (3.7) to the spinor orbit $\{(1,0,3),(3,0,3)\}$, i.e.



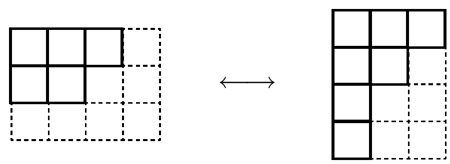
(3.12)

As further examples, consider the mappings



(3.13)

and



(3.14)

between orbits of $(B_3)_9$ (left) and $(B_4)_7$ (right). The first of these corresponds to the tensor orbits $\{(1,2,0),(4,2,0)\} \leftrightarrow \{(0,1,1,0),(3,1,1,0)\}$, and the second to the spinor orbits $\{(1,2,1),(3,2,1)\} \leftrightarrow \{(1,1,0,3)\}$.

Above, we have already obtained all information that we need about the modular matrix T. Next we want to determine the behavior of the S-matrix under the map τ. We first recall that the Weyl group W of B_n acts in the orthonormal basis by all possible permutations and sign changes of the components. This implies that

$$\sum_{w\in W} \operatorname{sign}(w)\, \exp\big[\frac{\pi\mathrm{i}}{k+n}\,(w(\Lambda+\rho),\Lambda'+\rho)\big] = (2\mathrm{i})^n \operatorname{det}_{i,j}\mathcal{M}_{ij}(\Lambda,\Lambda'), \tag{3.15}$$

where

$$\mathcal{M}_{ij}(\Lambda,\Lambda') := \sin\big[\frac{\pi\,\ell_i(\Lambda+\rho)\,\ell_j(\Lambda'+\rho)}{k+n}\big]\,. \tag{3.16}$$

Inserting this identity into the Kac–Peterson formula [29,30] for the S-matrix, one arrives at

$$S_{\Lambda,\Lambda'} = (-1)^{n(n-1)/2} 2^{n/2-1} (k+n)^{-n/2} \operatorname{det}_{i,j}\mathcal{M}_{ij}(\Lambda,\Lambda'). \tag{3.17}$$

Now of course this result for the S-matrix refers to particular highest weights Λ and Λ'. However, what we really would like to compare are not the S-matrix elements for individual weights, but S-matrix elements for orbits with respect to simple currents. Now within an orbit, the sign of S depends on the choice of the representative (except if only tensor weights are involved). Thus if we want to interpret (3.17) as an equation for orbits, we have to keep in mind that when evaluating the equation we have to employ specific representatives (namely, those with the smaller value of ℓ_1). For the application to coset theories it will be crucial that the sign in (3.17) is correlated with the alternative whether the relation (3.10) between conformal weights holds exactly or only modulo $\frac{1}{2}\mathbf{Z}$.

An analogous computation as for (3.17) yields

$$S_{\tau(\Lambda),\tau(\Lambda')} = (-1)^{k(k-1)/2} 2^{k/2-1} (k+n)^{-k/2} \det_{i,j} \tilde{\mathcal{M}}_{ij}(\Lambda,\Lambda') \tag{3.18}$$

with

$$\tilde{\mathcal{M}}_{ij}(\Lambda,\Lambda') := \sin\Big[\frac{\pi\, \ell_i^{(\tau)}(\Lambda^{(\tau)}+\rho^{(\tau)})\, \ell_j^{(\tau)}(\Lambda'^{(\tau)}+\rho^{(\tau)})}{k+n}\Big]\,. \tag{3.19}$$

To relate the numbers (3.17) and (3.18), we first note that $\mathcal{M}_{ij}(\Lambda,\Lambda')$ can be viewed as a $n\times n$ sub-matrix of the $(k+n)\times(k+n)$ matrix

$$A_{ij} := \begin{cases} A^{(\rm tt)}_{ij} := \sin[(\pi\,(i-\frac12)(j-\frac12)/(k+n)] & \text{for} \quad c_\Lambda = c_{\Lambda'} = 0\,, \\ A^{(\rm ts)}_{ij} := \sin[(\pi\,(i-\frac12)j)/(k+n)] & \text{for} \quad c_\Lambda = 0,\ c_{\Lambda'} = 1\,, \\ A^{(\rm st)}_{ij} := \sin[(\pi\,i(j-\frac12)/(k+n)] & \text{for} \quad c_\Lambda = 1,\ c_{\Lambda'} = 0\,, \\ A^{(\rm ss)}_{ij} := \sin[(\pi\,ij)/(k+n)] & \text{for} \quad c_\Lambda = c_{\Lambda'} = 1\,, \end{cases} \tag{3.20}$$

$i,j = 1,2,\ldots,k+n$. Similarly, $\tilde{\mathcal{M}}_{ij}(\Lambda,\Lambda')$ is a $k\times k$ sub-matrix of

$$\tilde A_{ij} := \begin{cases} \sin[(\pi\,(k+n+\frac12-i)(k+n+\frac12-j))/(k+n)] \\ \qquad\qquad = (-1)^{i+j+k+n+1}\,A^{(\rm tt)}_{ij} & \text{for} \quad c_\Lambda = c_{\Lambda'} = 0\,, \\ \sin[(\pi\,(k+n+\frac12-i)j)/(k+n)] = (-1)^{j+1}\,A^{(\rm ts)}_{ij} & \text{for} \quad c_\Lambda = 0,\ c_{\Lambda'} = 1\,, \\ \sin[(\pi\,i(k+n+\frac12-j))/(k+n)] = (-1)^{i+1}\,A^{(\rm st)}_{ij} & \text{for} \quad c_\Lambda = 1,\ c_{\Lambda'} = 0\,, \\ A^{(\rm ss)}_{ij} & \text{for} \quad c_\Lambda = c_{\Lambda'} = 1\,. \end{cases} \tag{3.21}$$

More precisely, the two submatrices are such that together they cover each value of i and j precisely once. As a consequence, one can use (a simple case of) the so-called Jacobi-theorem [31, 4] to relate $S_{\Lambda,\Lambda'}$ to $S_{\tau(\Lambda),\tau(\Lambda')}$. The theorem states that for any invertible matrix A whose rows and columns are labelled by a set H, one has for $I,\ J\subset H$ with $I\cup J=H$, $I\cap J=\emptyset$, that

$$\det\,[(A^{-1})^t]_{IJ} = (-1)^{\Sigma_I+\Sigma_J}(\det A)^{-1}\,(\det A)_{\overline{IJ}} \tag{3.22}$$

with $\overline{I} = H\setminus I$, $\overline{J} = H\setminus J$, and

$$\Sigma_I = \sum_{j\in I} j, \quad \Sigma_J = \sum_{j\in J} j. \tag{3.23}$$

Writing $S_{\Lambda,\Lambda'} = \alpha\det A_{IJ},\ S_{\tau(\Lambda),\tau(\Lambda')} = \beta\det A_{\overline{IJ}}$ and $\det\,[(A^{-1})^t]_{IJ} = \delta\det A_{IJ}$, application of this theorem yields

$$S_{\Lambda,\Lambda'} = (-1)^{\Sigma_I+\Sigma_J}\alpha\,(\beta\gamma\delta)^{-1}S_{\tau(\Lambda),\tau(\Lambda')} \tag{3.24}$$

with $I = M_\Lambda$, $J = M_{\Lambda'}$, and A as defined in (3.20). (Actually, the definition of δ implies the assumption that $\det A_{IJ}\neq 0$ for all choices of I and J. This turns out to be true for all cases we are interested in. Moreover, in some cases in fact δ does not depend on the choice of I and J at all.)

An explicit expression for the number α can be read off (3.17), while when determining the parameters β, γ, δ, one has to distinguish between tensors and spinors. If both Λ and Λ' are tensors, then by straightforward calculation one finds

$$\begin{aligned} &\beta = (-1)^{k(k-1)/2}(-1)^{k(k+n+1)+\Sigma_{\overline{I}}+\Sigma_{\overline{J}}}\, 2^{k/2-1}(k+n)^{-k/2}, \\ &\gamma = (-1)^{(k+n)(k+n-1)/2}((k+n)/2)^{(k+n)/2}, \qquad \delta = (2/(k+n))^n \,. \end{aligned} \tag{3.25}$$

When inserted into (3.24), this yields, upon use of the identity $\Sigma_{\overline{I}} + \Sigma_I = \sum_{j=1}^{k+n} j = (k+n)(k+n+1)/2$ [4],

$$S_{\Lambda,\Lambda'} = S_{\tau(\Lambda),\tau(\Lambda')}. \tag{3.26}$$

Note that this implies that τ connects tensor orbits with identical quantum dimension. (Since simple currents have quantum dimension 1 and quantum dimensions behave multiplicatively under the fusion product, the quantum dimension is constant on simple current orbits.)

If Λ is a tensor and Λ' a spinor, one obtains [6]

$$\begin{aligned} &\beta = (-1)^{k+\Sigma_{\overline{I}}} 2^{k/2-1}(k+n)^{-k/2}, \\ &\gamma = (-1)^{(k+n)(k+n-1)/2}\, 2^{(1-k-n)/2}(k+n)^{(k+n)/2}, \qquad \delta = 2^{n-f(\Lambda')}\,(k+n)^{-n}\,, \end{aligned} \tag{3.27}$$

where

$$f(\Lambda') := \begin{cases} 1 & \text{for} \quad \Lambda' \text{ a fixed point,} \\ 0 & \text{for} \quad \Lambda' \text{ an orbit of length two.} \end{cases} \tag{3.28}$$

Thus in this case [4]

$$S_{\Lambda,\Lambda'} = (-1)^{\Sigma_I + n(n+1)/2}\, 2^{f(\Lambda')-1/2}\, S_{\tau(\Lambda),\tau(\Lambda')}. \tag{3.29}$$

---

[6] Notice that if Λ is a tensor, then the order of the rows of $\tilde{A}_{ij}$ is actually to be read backwards such as to satisfy the requirement that the numbers obey $\tilde{\ell}_i^{(\tau)} > \tilde{\ell}_j^{(\tau)}$ for $i < j$; this contributes a factor $(-1)^{k(k-1)/2}$ to β. If Λ' is a tensor, the same factor arises from an analogous re-ordering of columns. In particular, for both Λ and Λ' tensors, these factors cancel out.

(Again, the sign depends on the choice of representative of the tensor orbit. It is as given in (3.29) if the representative with smaller value of ℓ_1 and $\tilde{\ell}_1$ is taken.) In particular, for spinors the quantum dimensions of the orbits of Λ and $\tau(\Lambda)$ differ by a factor $\sqrt{2}$ for orbits of length 2, and by a factor $1/\sqrt{2}$ for fixed points.

Analogously, for Λ a spinor and Λ' a tensor, one obtains (3.29) with Σ_J replaced by Σ_I and $f(\Lambda')$ replaced by $f(\Lambda)$. Finally, if both Λ and Λ' are spinors, we again have to distinguish between several cases. Observing that Λ is a fixed point iff $k+n \in M_\Lambda$, and that $A^{(\mathrm{ss})}_{j,k+n} = A^{(\mathrm{ss})}_{k+n,j} = \sin(\pi j) = 0$, we conclude that

$$S_{\Lambda,\Lambda'} = S_{\tau(\Lambda),\tau(\Lambda')} = 0 \tag{3.30}$$

if Λ is a fixed point and Λ' belongs to a length-two spinor orbit, or vice versa. In contrast, if both Λ and Λ' are fixed points, $S_{\Lambda,\Lambda'}$ vanishes but $S_{\tau(\Lambda),\tau(\Lambda')}$ does not, and the other way round for both Λ and Λ' belonging to length-two spinor orbits.

4 B type theories at even level versus D type at odd level

In this section we present a map τ relating $(B_k)_{2n}$ and $(D_n)_{2k+1}$ that behaves similarly as the one described in the previous section. However, for $(B_k)_{2n}$ we now have to restrict ourselves to tensor weights; for these, we define ℓ_i and $\tilde{\ell}_i$ as in (3.4) and (3.5). In contrast to odd level, now the map is no longer one-to-one on the simple current orbits. Rather, some of the orbits of $(B_k)_{2n}$ (namely, those tensors which are fixed points; in contrast to odd level, fixed points now must be tensors) correspond to two distinct orbits of $(D_n)_{2k+1}$.

For $(D_n)_{2k+1}$, the components of a weight Λ in the orthonormal basis are

$$\begin{aligned} &\ell_i(\Lambda) = \textstyle\sum_{j=i}^{n-2} \Lambda^j + \frac{1}{2}\,(\Lambda^{n-1} + \Lambda^n) \quad \text{for } i = 1,2,\dots,n-2\,, \\ &\ell_{n-1} = \tfrac{1}{2}\,(\Lambda^{n-1} + \Lambda^n)\,, \qquad \ell_n = \tfrac{1}{2}\,(-\Lambda^{n-1} + \Lambda^n)\,. \end{aligned} \tag{4.1}$$

At odd level, all orbits (with respect to the full set of simple currents, which is generated by J_s for odd n, and by J_s and J_v for even n) consist of four fields. Each such orbit of integrable highest weights contains precisely one representative that satisfies $\Lambda^0 \geq \Lambda^1$ and $\Lambda^{n-1} - \Lambda^n \in 2\mathbf{Z}$, implying that $\ell_i(\Lambda) \in \mathbf{Z}$ and $k > \ell_1 \geq \ell_2 \geq \ldots \geq \ell_n$. From now on we restrict our attention to this particular representative. Thus the numbers

$$\tilde{\ell}_i(\Lambda) := \ell_i(\Lambda + \rho) = \ell_i(\Lambda) + n - i \tag{4.2}$$

satisfy $0 < \tilde{\ell}_i(\Lambda) < k + n - 1$ for $i = 1, 2, ..., n - 1$, and $|\tilde{\ell}_n| < k$. As it turns out, a special role is played by those orbits for which $\ell_n = 0$; we will refer to such orbits as *spinor-symmetric*. Analogously, orbits that are transformed into each other upon changing the sign of ℓ_n ($\neq 0$) are called 'spinor-conjugate' to each other.

We define now a map τ between the orbits of $(D_n)_{2k+1}$ and the tensor orbits of $(B_k)_{2n}$ as follows. To an orbit of $(D_n)_{2k+1}$ with representative Λ we associate the subset M_Λ of $M = \{1, 2, \ldots, k + n\}$ by

$$M_\Lambda := \{\, |\tilde{\ell}_i(\Lambda)| + 1 \mid i = 1, 2, ..., n\} \,. \tag{4.3}$$

Then the (tensor) weight $\tau(\Lambda)$ of $(B_k)_{2n}$ is defined by the requirement that the set $M_{\tau(\Lambda)}$ (with the connection between Λ and M_Λ for $(B_k)_{2n}$ defined in the same way as for $(B_k)_{2n+1}$ in section 3) is given by

$$\{\tilde{\ell}_i^{(\tau)}\} \equiv M_{\tau(\Lambda)} := \{k + n + 1 - l \mid l \in M \setminus M_\Lambda\} \,. \tag{4.4}$$

Note that we have chosen our conventions for $(D_n)_{2k+1}$ (in particular the constant term '+1' in (4.3)) in such a manner that the prescription (4.4) is formally the same as (3.9) in section 3. Furthermore, $\tau(\Lambda)$ is a fixed point iff $k + n \in M_{\tau(\Lambda)}$, i.e. iff $1 \notin M_\Lambda$, i.e. iff Λ is not spinor-symmetric. Note also that this map is *not* one-to-one on the orbits. Rather, non-spinor-symmetric $(D_n)_{2k+1}$weights which transform into each other upon interchanging ℓ^{n-1} and ℓ^n get mapped on the same weight of $(B_k)_{2n}$. (As we will see later on, this is precisely the behavior we need in coset theories in order to implement the fixed point resolution.)

We now consider the behavior of the modular matrices T and S under the map τ. For the sum of conformal dimensions one finds

$$\begin{aligned}\Delta_\Lambda + \Delta_{\tau(\Lambda)} &= [\textstyle\sum_{j\in M_\Lambda} j^2 - (\rho,\rho) + \sum_{j\in M_{\tau(\Lambda)}} j^2 - (\rho^{(\tau)},\rho^{(\tau)})] \,/ [4(k+n-\frac12)] \\ &= \tfrac14\, k(k+2n+1) - \tfrac12 \textstyle\sum_{j\in M\setminus M_\Lambda} j\,,\end{aligned} \tag{4.5}$$

which is always a half integer. The Weyl group of $(D_n)_{2k+1}$ corresponds in the orthonormal basis to permutations and to even numbers of sign changes of the components, so that the Kac–Peterson formula for the S-matrix leads to

$$S_{\Lambda,\Lambda'} = (-1)^{n(n-1)/2} 2^{n/2-2} (k+n-\tfrac12)^{-n/2} \, [\mathrm{det}_{i,j}\mathcal{M}^+_{ij}(\Lambda,\Lambda') + \mathrm{i}^n \, \mathrm{det}_{i,j}\mathcal{M}^-_{ij}(\Lambda,\Lambda')]\,, \tag{4.6}$$

where

$$\begin{aligned}\mathcal{M}^+_{ij}(\Lambda,\Lambda') &:= \cos\,[\frac{2\pi\,\ell_i(\Lambda+\rho)\,\ell_j(\Lambda'+\rho)}{2k+2n-1}], \\ \mathcal{M}^-_{ij}(\Lambda,\Lambda') &:= \sin\,[\frac{2\pi\,\ell_i(\Lambda+\rho)\,\ell_j(\Lambda'+\rho)}{2k+2n-1}]\,.\end{aligned} \tag{4.7}$$

Note that $\mathcal{M}^-_{ij}(\Lambda,\Lambda') = 0$ whenever Λ or Λ' are spinor-symmetric. For later convenience we denote by $S^{(+)}_{\Lambda,\Lambda'}$ the numbers obtained from (4.6) when replacing $\mathcal{M}^-_{ij}(\Lambda,\Lambda')$ by zero, i.e.

$$S^{(+)}_{\Lambda,\Lambda'} = (-1)^{n(n-1)/2} 2^{n/2-2} (k+n-\tfrac12)^{-n/2} \, \mathrm{det}_{i\in M_\Lambda, j\in M_{\Lambda'}} \cos\,[\frac{\pi\,(i-1)(j-1)}{k+n-\frac12}]\,. \tag{4.8}$$

The S-matrix of $(B_k)_{2n}$ can be calculated analogously as described in the previous section for $(B_k)_{2n+1}$. The result is

$$\begin{aligned}S_{\tau(\Lambda),\tau(\Lambda')} &= (-1)^{k(k-1)/2} 2^{k/2-1} (k+n-\tfrac12)^{-k/2} \\ &\quad \cdot \mathrm{det}_{i\in M\setminus M_\Lambda, j\in M\setminus M_{\Lambda'}} \sin\,[\frac{\pi\,(k+n+\frac12-i)(k+n+\frac12-j)}{k+n-\frac12}]\,.\end{aligned} \tag{4.9}$$

Combining (4.8) with (4.9), we can use the Jacobi-theorem together with the identity $\sin[\pi\,(k{+}n{+}\frac12{-}i)(k{+}n{+}\frac12{-}j)/(k{+}n{-}\frac12)] = (-1)^{i+j+k+n+1}\cos[\pi\,(i{-}1)(j-1)/(k+n-\frac12)]$ to obtain again a relation like (3.24), namely

$$S^{(+)}_{\Lambda,\Lambda'} = (-1)^{\Sigma_I+\Sigma_J} \alpha\,(\beta\gamma\delta)^{-1} S_{\tau(\Lambda),\tau(\Lambda')}. \tag{4.10}$$

The parameters are this time calculated as [4]

$$\begin{aligned}
\alpha &= (-1)^{k(k-1)/2}(-1)^{k(n+k+1)+\Sigma_I+\Sigma_J}2^{k/2-1}(k+n-\tfrac{1}{2})^{-k/2},\\
\beta &= (-1)^{n(n-1)/2}2^{n/2-2}(k+n-\tfrac{1}{2})^{-n/2},\\
\gamma &= 2(-1)^{(k+n)(k+n-1)/2}((k+n-\tfrac{1}{2})/2)^{(k+n)/2},\\
\delta &= 2^{-\mathrm{s}(\Lambda)-\mathrm{s}(\Lambda')}\,(2/(k+n-\tfrac{1}{2}))^{k}\,,
\end{aligned} \tag{4.11}$$

where

$$\mathrm{s}(\Lambda) := \begin{cases} 0 & \text{if } \Lambda \text{ is spinor-symmetric,} \\ 1 & \text{else.} \end{cases} \tag{4.12}$$

This leads to

$$S^{(+)}_{\Lambda,\Lambda'} = 2^{\mathrm{s}(\Lambda)+\mathrm{s}(\Lambda')} S_{\tau(\Lambda),\tau(\Lambda')}. \tag{4.13}$$

(When interpreting this equation as a relation between simple current orbits, one must take the specific representative of the orbit of the D type WZW theory described above. Otherwise (4.13) gets modified by a phase. However, as only tensors of the B type WZW theory are involved, the phase does not depend on the representative of the orbits of the B theory.) Recalling that $\mathcal{M}^{-}_{ij}(\Lambda,\Lambda')=0$, i.e. $S_{\Lambda,\Lambda'} = S^{(+)}_{\Lambda,\Lambda'}$, if Λ or Λ' are spinor-symmetric, this means in more detail that

$$S_{\tau(\Lambda),\tau(\Lambda')} = \begin{cases} S_{\Lambda,\Lambda'} & \text{for } \Lambda \text{ and } \Lambda' \text{ spinor-symmetric,} \\ 2\,S_{\Lambda,\Lambda'} & \text{for } \Lambda \text{ spinor-symmetric, } \Lambda' \text{ non-spinor-symmetric,} \\ & \qquad\qquad\qquad\qquad\qquad \text{or vice versa,} \\ 4\,S^{(+)}_{\Lambda,\Lambda'} & \text{for } \Lambda \text{ and } \Lambda' \text{ non-spinor-symmetric.} \end{cases} \tag{4.14}$$

5 C type WZW theories

When considering C type WZW theories, we are in a more convenient position than previously. Namely, one can construct a map τ between individual fields, and not just between simple current orbits. In the notation of [4], our map τ is the composition of the maps 'ρ' of section 2 and 'tilde' of section 1 of [4].

We consider again the components of Λ in an orthogonal basis of the weight space. However, for convenience we multiply the components of the ortho*normal* basis by a factor $\sqrt{2}$, because we then have to deal with integral coefficients only. The components of a weight Λ in this non-normalized basis read $\ell_i(\Lambda) = \sum_{j=i}^{n} \Lambda^j$. Again we add to these numbers the components of the Weyl vector, i.e. define

$$\tilde{\ell}_i(\Lambda) := \ell_i(\Lambda+\rho) = \sum_{j=i}^{n} \Lambda^j + n + 1 - i. \tag{5.15}$$

This time the integrability condition (2.2) implies, for $(C_n)_k$, that

$$k+n \geq \ell_1 > \ldots > \ell_i > \ell_{i+1} > \ldots > \ell_n \geq 1\,. \tag{5.16}$$

Thus we can describe every weight Λ uniquely by a set of n positive integers $\tilde{\ell}_i$, $i=1,2,...,n$, subject to $\tilde{\ell}_i > \tilde{\ell}_j$ for $i<j$ as well as $\tilde{\ell}_1 \leq k+n$, that is, by a subset M_Λ of cardinality n of the set $M=\{1,2,\ldots,k+n\}$. Given such a subset M_Λ, we define $\tau(\Lambda)$ through the complementary set $\{\tilde{\ell}_i^{(\tau)}\} \equiv M_{\tau(\Lambda)} := M \setminus M_\Lambda$, where again the numbers $\tilde{\ell}_i^{(\tau)}$ are to be ordered according to $\tilde{\ell}_i^{(\tau)} > \tilde{\ell}_j^{(\tau)}$ for $i<j$. Since this subset of M again satisfies $\tilde{\ell}_1^{(\tau)} \leq k+n$, and is of cardinality k, it describes precisely one integrable highest weight $\tau(\Lambda)$ of $(C_k)_n$. (In terms of Young tableaux, the map corresponds to forming the complement with respect to the rectangular Young tableau $Y(k\Lambda_{(n)})$, followed by reflection at an axis perpendicular to the main diagonal.)

As in the previous sections, it is straightforward to calculate the quantity $\Delta_\Lambda + \Delta_{\tau(\Lambda)}$. Taking care of the extra factor $\frac{1}{2}$ in the scalar product that is caused by our normalization of the ℓ_i, one obtains

$$\begin{aligned} \Delta_\Lambda + \Delta_{\tau(\Lambda)} &= [\tfrac{1}{2}\textstyle\sum_{j\in M_\Lambda} j^2 - (\rho,\rho) + \tfrac{1}{2}\sum_{j\in M_{\tau(\Lambda)}} j^2 - (\rho^{(\tau)},\rho^{(\tau)})]\,/[2(k+n+1)] \\ &= [\tfrac{1}{2}\textstyle\sum_{j=1}^{k+n} j^2 - \tfrac{1}{12}\,(2n^3+3n^2+n+2k^3+3k^2+k)]\,/[2(k+n+1)] \\ &= \tfrac{1}{4}\,kn\,, \end{aligned} \tag{5.17}$$

where ρ and $\rho^{(\tau)}$ denote the Weyl vectors of C_n and C_k, respectively.

Proceeding to the modular matrix S, we note that the Weyl group W of C_n acts in the orthogonal basis by permutations and arbitrary sign changes, implying that

$$\sum_{w\in W} \mathrm{sign}\,(w)\, \exp\big[\frac{\pi \mathrm{i}}{k+n}\,(w(\Lambda+\rho),\Lambda'+\rho)\big] = (2\mathrm{i})^n\, \mathrm{det}_{i,j}\mathcal{M}_{ij}(\Lambda,\Lambda') \tag{5.18}$$

with

$$\mathcal{M}_{ij}(\Lambda,\Lambda') := \sin\big[\frac{\pi\, \ell_i(\Lambda+\rho)\, \ell_j(\Lambda'+\rho)}{k+n+1}\big]\,. \tag{5.19}$$

Thus the Kac–Peterson formula for the S-matrix yields

$$S_{\Lambda,\Lambda'} = (-1)^{n(n-1)/2} 2^{n/2} (k+n+1)^{-n/2}\, \mathrm{det}_{i,j}\mathcal{M}_{ij}(\Lambda,\Lambda'), \tag{5.20}$$

and similarly,

$$S_{\tau(\Lambda),\tau(\Lambda')} = (-1)^{k(k-1)/2} 2^{k/2} (k+n+1)^{-k/2}\, \mathrm{det}_{i,j}\tilde{\mathcal{M}}_{ij}(\Lambda,\Lambda'). \tag{5.21}$$

Now $\mathcal{M}_{ij}(\Lambda,\Lambda')$ can be viewed as a $n\times n$ sub-matrix, and $\tilde{\mathcal{M}}_{ij}(\Lambda,\Lambda')$ as a $k\times k$ submatrix, of the $(k+n)\times(k+n)$ matrix $A_{ij} := \sin[(\pi\, ij/(k+n+1)]$, $i,j \in \{1,2...,k+n\}$, such that the two submatrices together cover each value of i and j precisely once. As a consequence, the Jacobi-theorem is again applicable, leading to the relation (3.24) between $S_{\Lambda,\Lambda'}$ and $S_{\tau(\Lambda),\tau(\Lambda')}$. The numbers α, β, γ, δ in that relation are this time found to be

$$\begin{aligned} &\alpha = (-1)^{n(n-1)/2}\, 2^{n/2} (k+n+1)^{-n/2}, \qquad \beta = (-1)^{k(k-1)/2} 2^{k/2} (k+n+1)^{-k/2}, \\ &\gamma = (-1)^{(k+n)(k+n-1)/2} ((k+n+1)/2)^{(k+n)/2}, \qquad \delta = (2/(k+n+1))^n\,. \end{aligned} \tag{5.22}$$

When inserting this into (3.24), we make use of the identities $\Sigma_I = n(n+1)/2 + r(\Lambda)$ and $\Sigma_{\overline{I}} = k(k+1)/2 + r(\Lambda')$, where

$$r(\Lambda) := \sum_{i=1}^{n} \ell_i(\Lambda)\,, \tag{5.23}$$

which is modulo 2 the conjugacy class of the C_n-weight Λ (also, r equals the number of boxes in the Young tableau $Y(\Lambda)$ that is associated to Λ). One then obtains

$$S_{\Lambda,\Lambda'} = (-1)^{r(\Lambda)+r(\Lambda')+kn}\, S_{\tau(\Lambda),\tau(\Lambda')}. \tag{5.24}$$

6 $N=2$ coset models of type B; odd values of rank and level

6.1 The map $\mathcal{T}$

We are now going to describe a one-to-one map $\mathcal{T}$ between the primary fields of the $N=2$ superconformal coset theories $(B, 2n+1, 2k+1)$ and $(B, 2k+1, 2n+1)$. We will show that this map leaves the modular matrices S and T invariant, and, moreover, provides a one-to-one map between chiral primary fields. Correspondingly we consider the two coset theories as isomorphic conformal field theories and write

$$(B, 2n+1, 2k+1) \stackrel{\mathcal{T}}{\cong} (B, 2k+1, 2n+1)\,. \tag{6.1}$$

This is in contrast to the level-rank duality of the underlying WZW theories which is far from providing an isomorphism of conformal field theories.

To start, let us mention two simple necessary requirements for such an identification to exist. First, from table 1 we read off that the Virasoro central charge of $(B, 2n+1, 2k+1)$ is $c_{2n+1,2k+1} = \frac{3(2k+1)(2n+1)}{2(k+n+1)}$, which is invariant under exchanging n and k. It was precisely this observation [20] that led to the idea of level-rank duality of these theories. Second, we see that the two theories possess the same number of (Virasoro and u_1) primary fields. Namely, for the coset theory $(B, 2n+1, 2k+1)$ the number of primaries can be expressed as

$$\begin{aligned}\nu^{BB}_{2n+1,2k+1} = N^{D}_{2n+1,1}\, N^{1}_{8(k+n+1)} \big(\tfrac{1}{16}\, [N^{B}_{n+1,2k+1}\, N^{B}_{n,2k+3} - F^{B}_{n+1,2k+1}\, F^{B}_{n,2k+3}] \\ + 2 \cdot \tfrac{1}{4}\, F^{B}_{n+1,2k+1}\, F^{B}_{n,2k+3} \big)\end{aligned} \tag{6.2}$$

in terms of the numbers $N^{B}_{m,K}$ of primary fields and $F^{B}_{m,K}$ of fixed points of the B type WZW theories. Here the first two factors come from D_{2n+1} at level one and from the u_1 theory, respectively. The numbers in the bracket refer to the theories B_{n+1} at level $2k+1$ and B_n at level $2k+3$; the term in square brackets corresponds to the orbits of length four, with the factor $\frac{1}{16}$ taking care of the selection rule and the identification of order four (one quarter of the possible combinations of quantum numbers of the individual theories gets projected out, and each identification orbit has four members), and the second term corresponds to the fixed points, the factor of 2 being due to the resolution procedure (for the fixed points, the factor of $\frac{1}{16}$ gets replaced by $\frac{1}{4}$ because none of the fixed points is projected out by the selection rule encoded in $J_{(1)}$). Inserting $N^D_{d,1}=4$ and $N^1_{\mathcal{N}}=\mathcal{N}$ as well as the formulæ (3.1) and (3.2) for $N^B_{m,K}$ and $F^B_{m,K}$, (6.2) becomes

$$\nu^{BB}_{2n+1,2k+1} = 2\,\big(4n+4k+3-\frac{2kn}{k+n+1}\big) \left(\begin{array}{c} k+n+1 \\ k \end{array}\right) \left(\begin{array}{c} k+n+1 \\ n \end{array}\right) . \tag{6.3}$$

Obviously, for $(B,2k+1,2n+1)$ one obtains the same number of primaries.

After these preliminaries, we now present the map $\mathcal{T}$ alluded to above. Suppose we are given a specific representative $(\Lambda,\mathrm{x}\,/\,\lambda,Q)$ of a field Φ as described in (2.10); then we map the simple current orbits of Λ and λ on their images under the map τ that was introduced in section 3. Thus

$$\mathcal{T}(\Phi) \hat{=} (\tau(\lambda),\mathrm{x}_{\mathcal{T}}\,/\,\tau(\Lambda),Q_{\mathcal{T}})\,, \tag{6.4}$$

with τ as defined in (3.7) and (3.9), and with $\mathrm{x}_{\mathcal{T}}$ and $Q_{\mathcal{T}}$ to be specified below. Now the objects on the right hand side of (6.4) are just representatives of primary fields, and not yet the primary fields themselves. In particular, the quantities $\mathrm{x}_{\mathcal{T}}$ and $Q_{\mathcal{T}}$ are to be considered as orbits, and only after fixing representatives of the orbits of $\tau(\lambda)$ and $\tau(\Lambda)$, they are fixed as well so that $\mathrm{x}_{\mathcal{T}}$ becomes an element of $\{0,\mathrm{v},\mathrm{s},\mathrm{c}\}$ and $Q_{\mathcal{T}}$ an integer between 0 and $\mathcal{N}$. To describe the physical fields, we have to implement the identification currents. According to table 2, in the present case there are two independent identification currents $J_{(1)}$ and $J_{(2)}$. As $J_{(1)}=(J,1\,/\,J,0)$ acts trivially on the D_d and u_1 parts, it is convenient to first restrict the attention to $J_{(1)}$-orbits, and implement $J_{(2)}$ later on. Provided that no fixed points are present, [7] for fixed choice of $\mathrm{x}_{\mathcal{T}}$ and $Q_{\mathcal{T}}$ we have to deal with a total of four representatives of two $J_{(1)}$-orbits.

Now observe that, owing to the selection rule implemented by $J_{(1)}$, the conjugacy classes of Λ and λ coincide, so that we only need to consider

[7] Note that in order to have a fixed point of the coset theory, we must have a fixed point in all WZW theories that make up the coset.

combinations of tensors with tensors, or of spinors with spinors. In the case of tensors of both $(B_{n+1})_{2k+1}$ and $(B_n)_{2k+3}$, fixed points do not occur. Further, modulo $\mathbf{Z}$, the conformal dimensions of the two $J_{(1)}$-orbits differ by $\frac{1}{2}$, precisely as the conformal dimensions of the corresponding fields of $(B, 2n+1, 2k+1)$. To start with the definition of $\mathcal{T}$, we now simply choose the $J_{(1)}$-orbit that has conformal weight equal to $\Delta_\Lambda - \Delta_\lambda$ modulo $\mathbf{Z}$. Due to the identification current $J_{(2)}$, this choice actually does not constitute any loss of generality (but it simplifies some formulæ further on). Namely, each of the $J_{(1)}$-orbits $\mathcal{O}$ lies on the same orbit with respect to $J_{(2)}$ as another $J_{(1)}$-orbit whose values of $\mathrm{x}_{\mathcal{T}}$ and $Q_{\mathcal{T}}$ differ from those of $\mathcal{O}$ in such a manner that the values of $\Delta_\Lambda - \Delta_\lambda$ differ by $\frac{1}{2}$ mod $\mathbf{Z}$.

For spinors, both $J_{(1)}$-orbits in question have identical conformal weight. The freedom to choose one of the orbits turns out to be closely connected with the issue of fixed point resolution. Namely, the property of τ to map WZW fixed points on WZW-orbits of length two and vice versa, translates into the following property of $\mathcal{T}$: any 'unresolved fixed point' is mapped on two distinct fields, and vice versa, such that the non-fixed points of one theory precisely describe the resolved fixed points of the other theory. In case that just one of the orbits in either the 'numerator' or the 'denominator' of the coset theory is a fixed point, we have exactly the reversed situation in the dual theory.

Having fixed the B parts of the theory, we extend the definition of $\mathcal{T}$ to the u_1 and D_d parts by the following definitions: the D_d part remains unchanged, i.e. $\mathrm{x}_{\mathcal{T}} = \mathrm{x}$, while the u_1-charge is transformed according to

$$Q_{\mathcal{T}} = -Q + \begin{cases} QL & \text{for} \quad c_\Lambda = 0 \quad \text{and} \quad \mathrm{x} \in \{0, \mathrm{v}\}\,, \\ Q(2n+1)L & \text{for} \quad c_\Lambda = 0 \quad \text{and} \quad \mathrm{x} \in \{\mathrm{s}, \mathrm{c}\}\,, \\ Q(2k+1)L & \text{for} \quad c_\Lambda = 1 \quad \text{and} \quad \mathrm{x} \in \{0, \mathrm{v}\}\,, \\ (2n-2k-Q)L & \text{for} \quad c_\Lambda = 1 \quad \text{and} \quad \mathrm{x} \in \{\mathrm{s}, \mathrm{c}\}\,. \end{cases} \tag{6.5}$$

Here, for convenience, we use the abbreviation

$$L = 2(k+n+1)\,, \tag{6.6}$$

and all u_1-charges are understood modulo $\mathcal{N} = 4L$. (Thus L is one quarter of the u_1-charge of the primary field that extends the chiral algebra of the u_1 theory, and hence the appearance of this number in (6.5) is quite as natural.)

The definition of $\mathcal{T}$ is not yet complete, of course, as we still have to make precise its meaning when acting on, or mapping to, resolved fixed points. Nevertheless already at this stage we can verify that $\mathcal{T}$ as defined above satisfies the following properties:

1. The result is independent of the particular choice of the representative of the original field Φ. [8]

2. The conformal weights Δ of fields related by $\mathcal{T}$ are equal modulo $\mathbf{Z}$, which implies that the modular T-matrices of the two theories coincide. This is in fact already the maximal information about conformal dimensions that we could hope to prove in the general case, because for primary fields of a coset theory (other than Ramond ground states of an $N = 2$ theory) it is very hard to compute the integer part of the conformal weight.

3. The superconformal u_1-charges coincide modulo 2 (again, except for Ramond ground states it is hard to show that the charges coincide exactly).

Actually, the two last-mentioned properties (together with a prescribed choice of the orbits of $\tau(\Lambda)$ and $\tau(\lambda)$, such as the one discussed above) already specify uniquely $\mathrm{x}_{\mathcal{T}}$ and $Q_{\mathcal{T}}$ for fields that are not fixed points. Thus our choice $\mathrm{x}_{\mathcal{T}} = \mathrm{x}$ and $Q_{\mathcal{T}}$ as in (6.5) is the only possibility that allows for $\mathcal{T}$ to possess the required properties.

4. The elements of the modular S-matrices corresponding to non-fixed points coincide after properly taking into account the field identification. As we will show in the next subsection, the same is true for fixed points; it follows that both theories possess the same fusion rules, and, together with the first observation, that their characters realize isomorphic representations of $\mathrm{SL}(2,\mathbf{Z})$. If the B weights of one field are tensors and those of the other field are spinors, (2.14) implies that the corresponding S-matrix element of the full theory is simply the product of the respective WZW S-matrix elements if the spinors are fixed points, and twice this product if the spinors are not fixed points.

[8] Also, applying the analogous prescription $\mathcal{T}_{\mathcal{T}}$ to the transformed field $\mathcal{T}(\Phi)$ brings us back to the field Φ of the original theory, thus justifying the name *du*ality.

For the dual theory, the corresponding factor of two is provided by our map $\mathcal{T}$ through the factor $\sqrt{2}$ that appears (both for the 'numerator' and the 'denominator' of the coset theory) in the transformation (3.29) of S-matrix elements of the B type WZW theories under τ.

5. $\mathcal{T}$ maps the unique Ramond ground state $\Phi_R^{\rm max}$ with highest superconformal charge $q=\frac{c}{6}$ of one theory to the corresponding Ramond ground state of the dual theory. (This check is particularly important, as this field is the simple current that generates spectral flow.) Namely, for this field there is a standard representative [13] $\Phi_R^{\rm max} \hat{=} (0, {\rm s} \,/\, \rho_g - \rho_h)$, and $\mathcal{T}$ maps this particular representative to the analogous representative of the highest Ramond ground state of the dual theory.

6.2 Fixed points

In order to prove that these statements pertain to the full coset theories, [9] we now come to the more detailed description of the action of $\mathcal{T}$ on fixed points, as promised. (The fixed point resolution will be interesting also from a different point of view, see the remarks after (6.13) below.) As it turns out, this is a somewhat subtle issue. We will first deal with the case when an unresolved fixed point is mapped on a pair of non-fixed points. In fact, we have so far only specified on what pair of fields a fixed point gets mapped, and noticed that the number of the fields is the right one. But each unresolved fixed point gives rise to two distinct physical fields, and so we have to describe which of the resolved fixed points is mapped to which field. To settle this question, it is not sufficient to look at the fractional part of the conformal dimensions Δ and superconfomal charges q, because for the two resolved fixed points the conformal dimensions and superconformal charges must coincide modulo $\mathbf{Z}$ and $2\mathbf{Z}$, respectively. Thus we have to resort to the modular matrix S.

In order to simplify notation, we first look at those parts of the theory

[9] Recall that only after fixed point resolution, we are allowed to interpret the object $\mathcal{C}[\tilde{g}/\tilde{h}]_K$ as a genuine conformal field theory.

which behave non-trivially under the identification current that has fixed points, which is $J_{(1)} = (J, 1\,/\,J, 0)$. In other words, we restrict our attention to the theory $(B_{n+1})_{2k+1}[(B_n)_{2k+3}]^*$, where we use the symbol '*' to indicate that the complex conjugates of the modular S- and T-matrices are to be considered (compare the remarks after (2.8)). As has been shown in [24], the matrices $\Gamma_{(\cdot)}$ appearing in (2.14) and in the factorization (2.16) are given, up to certain phases, by the S-matrices of the WZW theories $(C_n)_k$ and $(C_{n-1})_{k+1}$. We denote these phases, to be determined below, by ω_n and ω_{n-1}, respectively.

In terms of the components $\tilde{\ell}_i$, the relation between fixed points and the corresponding fields of the fixed point theory is given by

$$\tilde{\ell}_i^{(C)} = \tilde{\ell}_{i+1}^{(B)} \tag{6.7}$$

for $i = 1, 2, ..., n$. In other words, for the S-matrices the resolution of fixed points amounts to simply deleting the row and the column with $i = k+n+1$ of the matrix A as defined in (3.20). But it was precisely this row that made the S-matrix elements vanish if fixed points were involved. Now once more we can use the Jacobi-theorem for the $(k+n) \times (k+n)$ matrix $M_{ij} = \sin[(\pi ij)/(k+n+1)]$ to relate the S-matrix of the fixed point resolution to the S-matrix of the images of the fixed points. We find that

$$\tilde{S}_{\Lambda\Lambda'}\tilde{S}_{\lambda\lambda'} = \varepsilon\omega_{n-1}\omega_n\,(-1)^{\Sigma_{\Lambda\Lambda'}+\Sigma_{\lambda\lambda'}+1}S_{\tau(\Lambda)\tau(\Lambda')}S_{\tau(\lambda)\tau(\lambda')}\,. \tag{6.8}$$

Here $\tilde{S}$ denotes the S-matrix of the fixed point resolution, while

$$\Sigma_{\Lambda\Lambda'} = \sum_{i\in M_\Lambda} i + \sum_{i\in M_{\Lambda'}} i\,, \tag{6.9}$$

and $\Sigma_{\lambda\lambda'}$ is the sum of the analogous numbers for the theory in the 'denominator.' Further, $\varepsilon \equiv \varepsilon_{\Lambda\lambda\Lambda'\lambda'} \in \{1, -1\}$ depends on the particular action of $\mathcal{T}$ on resolved fixed points. Namely, the left hand side of (6.8) is to be multiplied with the matrix $P = \frac{1}{2}\left(\begin{smallmatrix} 1 & -1 \\ -1 & 1 \end{smallmatrix}\right)$. On the right hand side of (6.8) this is reflected by the fact that the subscripts actually do not refer to an orbit, but to a specific representative; the sign of the right hand side changes when one changes from one representative to the other representative of the orbit.

The two representatives, which will be denoted by $\tau(\Lambda,\lambda)_>$ and $\tau(\Lambda,\lambda)_<$, can be described as follows. For any orbit $\{\phi_\Lambda, J\phi_\Lambda\}$ of a B type WZW theory denote by $\Lambda_<$ the representative with smaller values of $\tilde{\ell}_1$, and by $\Lambda_>$ the other one; then $\tau(\Lambda,\lambda)_> := (\tau(\Lambda)_<, \tau(\lambda)_<) \cong (\tau(\Lambda)_>, \tau(\lambda)_>)$, while $\tau(\Lambda,\lambda)_< := (\tau(\Lambda)_<, \tau(\lambda)_>) \cong (\tau(\Lambda)_>, \tau(\lambda)_<)$, with the two equivalent states mapped onto one another by the action of the identification current $J_{(1)}$. Now the value of ε depends on whether the first of the resolved fixed points is mapped to $\tau(\Lambda,\lambda)_>$ and the second to $\tau(\Lambda,\lambda)_<$, or the other way round. As we will see, a consistent prescription for this choice can be given for which ε precisely cancels the further possible signs in (6.8).

To compute the phases ω_n and ω_{n-1}, we first note that, given a representation $(ST)^3 = S^2$, $S^4 = \mathbb{1}$ of $\mathrm{SL}(2,\mathbf{Z})$, the only rescalings of S and T which again lead to a representation of $\mathrm{SL}(2,\mathbf{Z})$ are

$$T \mapsto \mathrm{e}^{\pi\mathrm{i}m/6}\, T\,, \qquad\qquad S \mapsto \mathrm{e}^{-\pi\mathrm{i}m/2}\, S. \tag{6.10}$$

We can determine the integer m in the first of these rescalings from the global shift in the conformal dimensions that is present in the fixed point theories as compared to the C type WZW theories. In the case of our interest we have for $(B_{n+1})_{2k+1}$ the shift $\Delta_{(B)} - \Delta_{(C)} = (6k+2n+3)/24$, and analogously for $(B_n)_{2k+3}$. Subtracting the two shifts, one finds $m = -2$. With (6.10), this implies that for the resolution one should take minus the product of the S-matrices of the C type theories rather than simply their product. In other words, $\omega_{n-1}\omega_n = -1$, and hence (6.8) reduces to

$$\tilde{S}_{\Lambda\Lambda'} \tilde{S}_{\lambda\lambda'} = \varepsilon(-1)^{\Sigma_{\Lambda\Lambda'}+\Sigma_{\lambda\lambda'}} S_{\tau(\Lambda)\tau(\Lambda')} S_{\tau(\lambda)\tau(\lambda')}\,. \tag{6.11}$$

To complete the construction of $\mathcal{T}$, we first investigate the restrictions that are obtained from requiring that the S-matrix is left invariant. Let us choose an arbitrary fixed point $\Phi_f \mathrel{\hat=} (\Lambda,\lambda)$ to start with, and denote the resolved fixed points by $\Phi_{f_\pm}$, as in (2.19). We can now map Φ_{f_+} either to $\tau(\Lambda,\lambda)_>$ or to $\tau(\Lambda,\lambda)_<$ (and, correspondingly, Φ_{f_-} to $\tau(\Lambda,\lambda)_<$ and to $\tau(\Lambda,\lambda)_>$, respectively). After fixing this choice, the requirement that the S-matrix should be invariant already fixes $\mathcal{T}(\Phi_{f'})$ for any fixed point $\Phi_{f'}$ uniquely. Namely, assume that the first possibility, $\Phi_{f_+} \mapsto \tau(\Lambda,\lambda)_>$, is chosen; then we have to map $\Phi_{f'_+} \mapsto \tau(\Lambda',\lambda')_>$, $\Phi_{f'_-} \mapsto \tau(\Lambda',\lambda')_<$ if the number

$\Sigma_{ff'} \equiv \Sigma_{\Lambda\Lambda'} + \Sigma_{\lambda\lambda'}$ computed according to (6.9) is even, while if $\Sigma_{ff'}$ is odd, the map must be $\Phi_{f'_+} \mapsto \tau(\Lambda',\lambda')_<$, $\Phi_{f'_-} \mapsto \tau(\Lambda',\lambda')_>$. With this prescription, one obtains $\varepsilon_{ff'} = (-1)^{\Sigma_{ff'}}$, and hence (6.11) reduces to the desired equality

$$\tilde{S}_{\Lambda\Lambda'}\tilde{S}_{\lambda\lambda'}\,P_{ij} = \Big(S_{\tau(\Lambda)\tau(\Lambda')}S_{\tau(\lambda)\tau(\lambda')}\Big)_{\mathcal{T}(i)\,\mathcal{T}(j)}\,, \tag{6.12}$$

where on the left hand side $i,j \in \{+,-\}$, while on the right hand side $\mathcal{T}(i),\mathcal{T}(j) \in \{<,>\}$. This not only works for any fixed choice of f', but also for all S-matrix elements $S_{f'f''}$, because $\Sigma_{f'f''} = \Sigma_{ff'} + \Sigma_{ff''}$. The latter identity also implies that the choice of reference fixed point Φ_f is immaterial.

As long as we only take care of the S-matrix, the alternative to choose $\Phi_{f_+} \mapsto \tau(\Lambda,\lambda)_>$ or $\Phi_{f_+} \mapsto \tau(\Lambda,\lambda)_<$ means that there are two different allowed mappings on the fixed points. But according to (2.19) the characters of Φ_{f_+} and Φ_{f_-} are different; Φ_{f_+} has more states with minimal conformal weight. Therefore by looking at the characters one can remove the ambiguity in the definition of $\mathcal{T}$. However, since this reasoning can be applied to any fixed point, it has also to be checked whether the constraints obtained from different fixed points are compatible. In practice, this is quite difficult to check, as it requires a detailed analysis of the characters. But there is a rather general argument that the consistency conditions coming from the characters are compatible with those originating from the S-matrix. Namely, defining for any fixed point f the function

$$\mathcal{X}_{\tau(f)} := \chi_{\tau(f)_>} - \chi_{\tau(f)_<}\,, \tag{6.13}$$

it is easy to verify that the functions $\mathcal{X}$ transform under the modular group exactly like the character modifications $\breve{\chi}_f \equiv (\chi_{f_+} - \chi_{f_-})/2v$. In itself, this does not yet imply that $\mathcal{X}_{\tau(f)}$ and $\breve{\chi}_f$ are necessarily equal, but the fact that the result holds for an infinite series is a rather strong hint that they indeed coincide. (Note that it directly follows from (2.19) that only $\mathcal{X}$ as defined in (6.13), and not $-\mathcal{X}$ can be a sensible character; thus there is in particular no sign ambiguity in defining $\mathcal{X}$.)

In principle, we should perform the same kind of reasoning as above also for resolved fixed points that occur as the images of non-fixed points. However, due to the duality property of the map $\mathcal{T}$ the arguments needed for this analysis closely parallel the arguments given above, so that we refrain from repeating them here.

At this point it is worth to recall that there does not exist a general proof that the fixed points of a coset theory can be resolved in a unique way [24]. In the present case, the manner in which the resolution procedure described in [17] fits to the duality map $\mathcal{T}$ is however so non-trivial, that it is hard to imagine that there could exist another prescription for the resolution that would be compatible with duality as well. Note that the extended Poincaré polynomials of the theories considered here should obey level-rank duality for *any* possible resolution, because according to quite general arguments [17,13] the extended Poincaré polynomial of an $N=2$ coset theory does not depend on the details of the resolution procedure.

6.3 Ramond ground states

Finally we turn our attention to the chiral ring of the theories. According to the formula (2.25), the number of representatives of Ramond ground states with a fixed $(B_{n+1})_{2k+1}$ weight is given by the relative size

$$\frac{|W(g)|}{|W(h)|} = \frac{2^{n+1}\,(n+1)!}{2^n\,n!} = 2\,(n+1) \tag{6.14}$$

of the Weyl groups. After implementing the resolution of fixed points, one finds that the dimension of the ring is indeed invariant under the exchange of n and k; this is a direct consequence of the much stronger result [17] that the (ordinary, and also even the extended) Poincaré polynomials of the theories coincide.

Our goal is now to show that the map $\mathcal{T}$ defined above maps every Ramond ground state to a Ramond ground state of the dual theory with identical superconformal charge. To do so, we first note that the relation (2.21) between Λ and λ can be reformulated in terms of the sets M_Λ and M_λ, and of the charge Q, as follows. Take a highest g-weight Λ described by the set M_Λ, and consider it as ordered with respect to the magnitude of the elements. The action of any Weyl group element w is then to permute the elements of M_Λ and to multiply them with a sign: the $2(n+1)$ special elements of the classes of $W(g)/W(h)$ that appear in (2.21) are characterized by the property that they choose among the $n+1$ elements of M_Λ a particular element $\tilde{\ell}_i$ which gets placed before all the other elements, and change its sign or not, leaving all other signs unchanged. We will denote such a Weyl group element that maps the ith basis vector e_i of the orthonormal basis on $\pm e_1$ and respects the ordering of all other basis vectors by $w_i^{(\pm)}$. By inserting the explicit form of the roots α in the orthonormal basis into (2.24), it is straightforward to calculate the length of the elements $w_i^{(\pm)}$. We find

$$l(w_i^{(+)}) = i-1 \qquad \text{and} \qquad l(w_i^{(-)}) = 2(n+1)-i\,, \tag{6.15}$$

where $n+1$ is the rank of the algebra. This result reflects the linear structure of the associated Hasse diagram of the embedding $B_n \hookrightarrow B_{n+1}$ [32].

For the Ramond ground state defined by acting with $w_i^{(\pm)}$ on Λ, the u_1-charge Q is given by $\pm 2\tilde{\ell}_i$ for spinors and $\pm(2\tilde{\ell}_i - 1)$ for tensors. Opposite sign choices correspond to choosing charge-conjugate Ramond ground states. As a consequence, the map $\mathcal{T}$ automatically respects the charge conjugation properties of the Ramond ground states, and hence is compatible with the conjugation isomorphisms of the chiral rings of the theories. As mentioned in the introduction, this compatibility must in fact hold on rather general grounds.

Next we remark that not all representatives of a Ramond ground state are of the form (2.21) (recall that (2.21) is a formula for representatives, and not for physical fields). To be able to employ the relation (2.21), we therefore pick a specific representative of any combination of simple current orbits of weights Λ and λ that describes a Ramond ground state. After applying the map $\mathcal{T}$ in the form (6.4), (6.5) to this specific representative of a Ramond ground state Φ_R, we obtain a specific representative of the primary field $\mathcal{T}(\Phi_R)$ of the dual theory. What we have to show is that $\mathcal{T}(\Phi_R)$ is again a Ramond ground state, and we will do this by employing the formula (2.21). Of course, generically the particular representative of $\mathcal{T}(\Phi_R)$ with which we are dealing in the first place cannot be expected to be of the form (2.21). As we will see, it is indeed sometimes not of this form, but as was shown in [23] there is always at least one representative of the Ramond ground state fulfilling (2.21).

Suppose, to start with, that Λ and λ are both spinor weights, and that the Ramond ground state is given by the Weyl group element $w_i^{(+)}$ acting on Λ. Recalling that the index i of $w_i^{(+)}$ refers to the fact that $M_\Lambda \setminus M_\lambda = \{\tilde\ell_i\}$, and observing that via the map τ on the WZW theories, i.e. upon forming the complement relative to $\{1,2,\ldots,k+n+1\}$, this is transformed to the relation $M_{\tau(\lambda)} \setminus M_{\tau(\Lambda)} = \{\tilde\ell_i\}$, we learn that there exists a Weyl group element $w_{\mathcal T}$ of the dual theory that relates $\tau(\lambda)$ and $\tau(\Lambda)$ in the correct manner and is given by one of the two elements $w_{i_{\mathcal T}}^{(\pm)}$, with $i_{\mathcal T}$ determined by the requirement $\tilde\ell_{i_{\mathcal T}}^{(\tau)} = \tilde\ell_i$. To decide which of these two elements is the correct one, we observe that owing to the latter relation $Q_{\mathcal T}$ must be equal either to Q or to $-Q$; from (6.5) (together with the explicit form of the identifiation currents) it follows that in fact $Q_{\mathcal T} = -Q$. In summary, using the sets $M_{\tau(\lambda)}$ and $M_{\tau(\Lambda)}$, and the sign of $Q_{\mathcal T}$ relative to the sign of Q, we fix a unique Weyl group element $w_{\mathcal T}$ of $W(B_{k+1})$; in fact, a more detailed analysis shows that $i_\tau = k+n-Q/2-i+3$, i.e. $w_{\mathcal T} = w_{k+n-Q/2-i+3}^{(-)}$. To verify that this Weyl group element indeed provides us with a Ramond ground state, the only thing that we still have to do is to check that it yields the proper D_d part. [10] While in the foregoing discussion we fixed the representative with respect to $J_{(2)}$ by $\mathrm{x}_{\mathcal T} = \mathrm{x}$, the present choice of representative for the charge $Q_{\mathcal T}$ implies that $\mathrm{x}_{\mathcal T}$ must be given by

$$\mathrm{x}_{\mathcal T} = (J_\mathrm{v})^{n-k-Q/2}\mathrm{x}\,. \tag{6.16}$$

Now the formulæ (6.15) for the length of Weyl group elements tell us that

$$l(w) - l(w_{\mathcal T}) = n-k-Q/2\,, \tag{6.17}$$

and hence, recalling that the sign of w is equal to $(-1)^{l(w)}$,

$$\mathrm{sign}\,(w)\,\mathrm{sign}\,(w_{\mathcal T}) = (-1)^{l(w)+l(w_{\mathcal T})} = (-1)^{k+n+Q/2}. \tag{6.18}$$

In view of (2.22), this shows that (6.16) is indeed fulfilled. Furthermore, plugging (6.17) into the formula (2.23) for the superconformal charge of Ramond

[10] In some cases we also must show that the correct $J_{(1)}$-orbit out of two possibilities is chosen. This happens when an 'unresolved fixed point' gets resolved into two fields whose conformal weights differ by an integer. The discussion of fixed points in the previous subsection shows that indeed the right orbit is chosen.

ground states, it follows that Φ_R and $\mathcal{T}(\Phi_R)$ have the same superconformal charge (exactly, and not just modulo 2).

The reasoning above applies also to the case $w = w_i^{(-)}$, as the two cases are clearly dual to each other. If both Λ and λ are tensor weights, the situation is slightly more complicated. This is because $\tilde{\ell}_i$ gets mapped under τ to $\tilde{\ell}^{(\tau)}_{i_\mathcal{T}} = \frac{L}{2}+1-\tilde{\ell}_i$. If the Ramond ground state is defined by $w = w_i^{(+)}$, this shows that $Q = 2\tilde{\ell}_i - 1$ should be mapped on $Q_\mathcal{T} = L - Q$, implying that $w_\mathcal{T}$ involves no minus sign. While in the foregoing discussion we always chose the representative of the field by requiring that $\Delta_\Lambda - \Delta_\lambda$ should be an integer, we now have to fix the representative by requiring that $Q_\mathcal{T} = -Q + L$, which, owing to the second identification current $J_{(2)}$, is always possible. This choice of representative leads to

$$\mathrm{x}_\mathcal{T} = (J_\mathrm{v})^{n-(Q-1)/2}\mathrm{x}\,. \tag{6.19}$$

Again, a Weyl group element $w_\mathcal{T}$ for the dual theory is completely fixed, and can be shown to be given by $w_\mathcal{T} = w^{(+)}_{(Q+1)/2-n+i-1}$. It follows that $l(w) - l(w_\mathcal{T}) = n - \frac{Q+1}{2} + 1$, so that

$$\mathrm{sign}\,(w)\,\mathrm{sign}\,(w_\mathcal{T}) = (-1)^{n+(1-Q)/2}\,, \tag{6.20}$$

implying that the correct mapping (6.19) of the D_d-weights is reproduced, and also that the superconformal charge is left invariant. It is also clear that we have chosen the right $J_{(1)}$-orbit, because Δ is conserved modulo $\mathbf{Z}$ under $\mathcal{T}$ and because the relevant different $J_{(1)}$-orbits differ in their conformal weight by $\frac{1}{2}$ modulo $\mathbf{Z}$.

For $w = w_i^{(-)}$, the discussion must be slightly changed. This time $Q = -(2\tilde{\ell}_i - 1)$ is mapped on $Q_\mathcal{T} = -L - Q$, i.e. we have to choose a different representative, leading to $\mathrm{x}' = (J_\mathrm{v})^{n+2k+(Q+1)/2}\,\mathrm{x}$. Explicit calculation shows that $w_\mathcal{T} = w^{(-)}_{i-n-(Q+1)/2}$, leading to $l(w) - l(w_\mathcal{T}) = n - 2k - \frac{Q+1}{2}$, which gives the right transformation of the D_d part and implies identity of superconformal charges.

Thus we have proven that $\mathcal{T}$ always maps Ramond ground states to Ramond ground states with identical superconformal charge.

7 Type B coset models with level and rank not congruent modulo 2

In the same spirit as before, we can deal with the other level-rank dualities mentioned in the introduction. As the discussion often closely parallels the one of the previous section, we will usually be rather brief and shall only mention some new features. In the present section we use the map τ for B type algebras at even level to relate the coset theory $(B, 2k+1, 2n)$ with the D type modular invariant to $(B, 2n, 2k+1)$ with the diagonal modular invariant, i.e.

$$(B, 2k+1, 2n)_{|D} \stackrel{\mathcal{T}}{\cong} (B, 2n, 2k+1)\,. \tag{7.1}$$

According to subsection 2.2, taking the D-invariant amounts to incorporating the integer spin simple current $J_{(3)} := (J, 1\,/\,1, 0)$ into the chiral algebra. This introduces further fixed points which can have order 2 or 4 and which have to be resolved, but it also has the crucial advantage that it leaves us with tensors of the B algebras only, so that the map τ constructed in section 4 is applicable.

The choice of the $J_{(1)}$-orbits is now immaterial. This is because the presence of $J_{(3)}$ implies that $\tau(\Lambda,\lambda)_< \cong \tau(\Lambda,\lambda)_>$, so that any pair of tensor orbits of the B type WZW theories, combined with a D_d-weight and a u_1-charge, corresponds to a single physical field. However, we still have to take into account the additional identification current $J_{(2)} \star J_{(3)} = (1, J_{\mathrm{v}}\,/\,1, \pm 2L)$, where $L := 2k+2n+1$.

Again the general form of the map $\mathcal{T}$ is given by (6.4) (recall that on the right hand side of (6.4) only a representative of $\mathcal{T}(\Phi)$ is given). Starting from a fixed representative of a field or a fixed point Φ of the B type coset theory at even level and odd rank, we obtain all representatives of $\mathcal{T}(\Phi)$ by using the map τ and the identification currents of the coset theory at even rank and odd level. Moreover, with the help of the identification currents we can also fix uniquely a representative of $\mathcal{T}(\Phi)$ for which $\tau(\Lambda)$ and $\tau(\lambda)$ are tensors and which has the same conformal weight as the chosen representative of Φ. Note that fixed points are mapped on a spinor-conjugate pair of orbits, which reflects the resolution of fixed points. In particular fixed points of order two and of order four are mapped on two and four fields, respectively.

One can now show again that there is a unique mapping $\mathcal{T}$ that preserves both the superconformal charge q modulo 2 and the conformal dimension Δ modulo integers; it is given by

$$Q_{\mathcal{T}} = \begin{cases} -Q + QL & \text{for} \quad \mathrm{x} \in \{0, \mathrm{v}\}\,, \\ -Q + (2k+1)QL & \text{for} \quad \mathrm{x} \in \{\mathrm{s}, \mathrm{c}\}\,, \end{cases} \tag{7.2}$$

and

$$\mathrm{x}_{\mathcal{T}} = \begin{cases} (J_{\mathrm{v}})^{Q/2}\mathrm{x} & \text{for} \quad \mathrm{x} \in \{0, \mathrm{v}\}\,, \\ (J_{\mathrm{v}})^{-k+(Q-1)/2}\mathrm{x} & \text{for} \quad \mathrm{x} \in \{\mathrm{s}, \mathrm{c}\}\,. \end{cases} \tag{7.3}$$

To check this, one has to make use of the fact that the representatives of the orbits of the D type WZW theories that were chosen above always have vanishing monodromy charge relative to $(J_{\mathrm{s}}, 1 \,/\, J_{\mathrm{s}}, 0)$.

Of course, again $\mathcal{T}$ must be complemented by a prescription on the fixed points. This time the fixed point theory is not a WZW theory; rather, it is closely related to certain conformal field theories, denoted by the symbol $\mathcal{B}$, that were described in [24]. In fact, the existence of the map $\mathcal{T}$ suggests that the S-matrix and characters of the $\mathcal{B}$ theories are related to a D type WZW theory, and it should be interesting to explore the level-rank duality further to gain deeper insight in the structure of these peculiar conformal field theories. Finally, it is again possible to prove that the modular S-matrices are identical and that Ramond ground states are mapped on Ramond ground states with equal superconformal charge.

8 BB versus CC theories

In this section we present the isomorphism

$$(BB, n+2, 1) \overset{\mathcal{T}}{\cong} (CC, 2, 2n+1)\,. \tag{8.1}$$

To relate the non-hermitian symmetric cosets $(BB, n+2, 1)$ and $(CC, 2, 2n+1)$ we first notice the isomorphism $C_2 \cong B_2$ of simple Lie algebras. This allows to make use once again of the map τ of section 3 to relate the $(B_n)_5$ theory appearing in $(BB, n+2, 1)$ with the $(B_2)_{2n+1} \cong (C_2)_{2n+1}$ part of

$(CC, 2, 2n+1)$. The $(B_{n+2})_1$ part, on the other hand, is comparatively easy to deal with, because it has only three integrable highest weights, and because the identification current $J_{(1)}$ strongly restricts their combination with weights of the other parts. Namely, $(B_n)_5$-weights that are tensors must be combined with either the tensor weight $\Lambda = 0$ or the tensor weight $\Lambda = \Lambda_{(1)}$ of $(B_{n+2})_1$, while spinors are to be combined with the spinor weight $\Lambda_{(n+2)}$ of $(B_{n+2})_1$; furthermore, $J_{(1)}$ introduces an additional identification, implying that in the case of tensors we can characterize the B part completely by a $(B_n)_5$-weight and by the difference $\Delta_\Lambda - \Delta_\lambda$ of the conformal dimensions. Also, by using the identification current $J_{(1)}$ of the CC models, we can choose without loss of generality for a fixed representative of Φ the representative of the C_2-orbit in such a way that it has conformal dimension $\Delta_\Lambda - \Delta_\lambda$ modulo integers. For spinor fixed points we have again an ambiguity which is connected to the issue of fixed point resolution.

This time, the mapping τ has to be complemented not only by a mapping on the D_d and u_1 parts, but also on the $(A_1)_{2n+2}$ part of the theory. Thus

$$\begin{aligned} &\Phi \,\hat{=}\, (\Lambda, \mathrm{x}\,/\,\lambda, \mu, Q)\,, \\ &\mathcal{T}(\Phi) \,\hat{=}\, (\tau(\lambda), \mathrm{x}_\mathcal{T}\,/\,\mu_\mathcal{T}, Q_\mathcal{T})\,, \end{aligned} \tag{8.2}$$

where μ and $\mu_\mathcal{T}$ are A_1-weights (recall that $C_1 \cong A_1$). It is easy to see that equality of the superconformal charges modulo 2 is equivalent to the relation $\mathrm{x}_\mathcal{T} = (J_\mathrm{v})^Q \mathrm{x}$. In fact one can show again that there is a unique mapping that preserves the fractional part of Δ, as well as q modulo 2. Namely, choosing the weights of the B parts in the manner decribed above, for tensors in the B parts one needs

$$Q_\mathcal{T} = \left\{ \begin{array}{lll} -Q + QL & \text{for} & \mathrm{x} \in \{0, \mathrm{v}\}\,, \\ -Q + (Q+1)L & \text{for} & \mathrm{x} \in \{\mathrm{s}, \mathrm{c}\} \end{array} \right. \tag{8.3}$$

with $L = 2n + 4$, while for spinor weights in the B parts we must set

$$Q_\mathcal{T} = \left\{ \begin{array}{lll} -Q + L & \text{for} & \mathrm{x} \in \{0, \mathrm{v}\}\,, \\ -Q & \text{for} & \mathrm{x} \in \{\mathrm{s}, \mathrm{c}\}\,. \end{array} \right. \tag{8.4}$$

The corresponding prescription for the weight μ of $(A_1)_{2m+2}$ is, independent of the value of x,

$$\mu_{\mathcal{T}} = \begin{cases} J^\mu \mu & \text{for } c_\Lambda = c_\lambda = 0\,, \\ \mu & \text{for } c_\Lambda = c_\lambda = 1\,. \end{cases} \tag{8.5}$$

Fixed points have to be dealt with more carefully again. Using general simple current arguments, it is easy to see that the S-matrix element between a fixed point and *any* other spinor has to vanish. At first sight, this might seem inconsistent, because the S-matrix element between two non-fixed point spinors of $(B_n)_5$ does not vanish in general, whereas both are mapped on fixed points with respect to $J_{(1)}$ of the C_2-theory, and the S-matrix of the image vanishes. However, spinors of $(B_n)_5$ are always combined with the spinor weight $\Lambda_{(n+2)}$ of $(B_{m+2})_1$; now $S_{\Lambda_{(n+2)}\Lambda_{(n+2)}}$ vanishes, and hence the same is true for the corresponding S-matrix element of the coset theory.

We can use the Jacobi-theorem to relate the S-matrix arising in the resolution of the fixed points to the S-matrix of the CC theory. The resolution is this time accomplished by mapping the fixed point on an orbit of length two. Calculation shows that the product of the S-matrix elements of A_1, D_d, and u_1 differs from the corresponding S-matrix-element of the CC coset theory by a factor of $\varepsilon(-1)^{P+Q}$, where P and Q are the u_1-charges of the BB theory, and where the sign ε depends on the specific action of $\mathcal{T}$ on fixed points analogously as discussed after (6.9). In a similiar manner as we dealt with the factor $(-1)^\Sigma$ in section 6, it can be shown that the action of $\mathcal{T}$ can be chosen in such a way that $\varepsilon(-1)^{P+Q}$ is the correct sign for obtaining equality of the full S-matrices. A parallel argument also shows that this definition of $\mathcal{T}$ reproduces the correct identification between the characters of the resolved fixed points and those of the corresponding fields of the CC theory. Let us also mention that the factors stemming from the S-matrix of $(B_{m+2})_1$ precisely compensate the different size of the identification group in the case of non-fixed points; for fixed points they assure, together with the factors of $\sqrt{2}$ appearing in (3.29), the equality of the S-matrices.

It is by now not too difficult to verify that the mapping $\mathcal{T}$ fulfills the same properties as in the cases treated in the previous sections. Besides preserving q and Δ as well as the modular S-matrix, we see that $\mathcal{T}$ maps again the Ramond ground states with highest superconformal charge onto each other, proving again the isomorphisms of the spectral flows. It is also possible to check that Ramond ground states are mapped on Ramond ground states with the same superconformal charge quite in the same way we did before. Owing to the presence of the additional A_1-subalgebra, the arguments are, however, slightly more complicated, and we refrain from presenting the technical details here.

9 Duality in the CC series

Here we construct a map $\mathcal{T}$ between the $N=2$ superconformal coset models (CC,n,k) and $(CC,k+1,n-1)$, which as in the previously discussed cases leaves S and T invariant and identifies the rings of chiral primary fields,

$$(CC,n,k) \stackrel{\mathcal{T}}{\cong} (CC,k+1,n-1)\,. \tag{9.1}$$

The definition of the map $\mathcal{T}$ will be such that

$$\mathcal{T}((\Lambda,\mathrm{x}\,/\,\lambda,Q)) = (\tau(\lambda),\mathrm{x}_{\mathcal{T}}\,/\,\tau(\Lambda),Q_{\mathcal{T}})\,. \tag{9.2}$$

This is formally very similar to the analogous definition (6.4) in section 6, but its contents is quite different. Namely, this time the underlying map τ of the C type WZW theories was defined on representatives of simple current orbits rather than on the orbits themselves (see section 5). Correspondingly, (9.2) is a map between representatives as well, and hence we will have to check that the relevant quantities of the coset theories do not depend on the choices of representatives. Therefore we will be a bit more explicit than in the two previous sections.

We begin again by checking the conformal central charge and the number of primaries. According to table 1, the Virasoro charge of (CC,n,k) is equal to $-3+6n(k+1)/(k+n+1)$, and hence is invariant under exchanging $n\leftrightarrow k+1$. The number of primaries of the $(C_n)_k$ WZW theory is $N^C_{n,k}=\binom{n+k}{k}$. Furthermore, the coset theory does not have any fixed points, and hence the number of primary fields of (CC,n,k) is

$$\nu^{CC}_{n,k} = \frac{1}{4}\, N^C_{n,k}\, N^D_{2n-1,1}\, N^C_{n-1,k+1}\, N^1_{2(k+n+1)} = 2(k+n+1) \begin{pmatrix} k+n \\ n \end{pmatrix} \begin{pmatrix} k+n \\ n-1 \end{pmatrix}, \tag{9.3}$$

where the first factor $\frac{1}{4}$ takes care of the selection rule and the identification of order two. Obviously, the number of primaries of the $(CC,k+1,n-1)$ theory is given by (9.3), too.

Next we present the map $\mathcal{T}$. In (9.2), τ is to be taken as the map defined after (5.16), and $\mathrm{x}_\mathcal{T}$ and $Q_\mathcal{T}$ are defined by

$$\mathrm{x}_\mathcal{T} = \begin{cases} (J_\mathrm{v})^{k+n+Q+1}\mathrm{x} & \text{for} \quad \mathrm{x} \in \{\mathrm{s},\mathrm{c}\}\,, \\ (J_\mathrm{v})^{k+1-Q}\mathrm{x} & \text{for} \quad \mathrm{x} \in \{\mathrm{v},0\}\,, \end{cases} \tag{9.4}$$

and

$$Q_\mathcal{T} = \begin{cases} -Q & \text{for} \quad \mathrm{x} \in \{\mathrm{s},\mathrm{c}\}\,, \\ -Q+k+n+1 & \text{for} \quad \mathrm{x} \in \{\mathrm{v},0\}\,. \end{cases} \tag{9.5}$$

Note that already in terms of representatives, the map $\mathcal{T}$ squares to the identity, $\mathcal{T}_\mathcal{T} \circ \mathcal{T} = id$. Also, combining the expression (2.12) for the conformal dimension of Φ with the result (5.17) for the conformal dimensions of the C type WZW theories, one can again show that $\mathcal{T}$ is the only map that preserves q modulo 2 and the fractional part of the conformal weight Δ, as well as the S-matrix. To check the last-mentioned property, it is important make use of the selection rules encoded in the identification current $J_{(1)}$.

As already emphasized, the map $\mathcal{T}$ must provide a mapping between fields rather than only a mapping between formal combinations of weights of the underlying Lie algebras. The following remarks show that the mapping is indeed well defined on physical fields.

1. The map (9.2) is consistent with the selection rules, i.e. it maps allowed fields to allowed fields. Note that the dependence of $Q_\mathcal{T}$ on x is necessary to fulfill the selection rule encoded in $J_{(1)}$, (explicitly, the selection rule reads $r(\Lambda)+r(\lambda)+n\sigma+Q \equiv 0 \bmod 2$, where $r(\Lambda)$ is the number defined in (5.23), which modulo 2 is equal to the conjugacy class of Λ, and where σ is 0 in the Neveu–Schwarz sector and 1 in the Ramond sector).

2. Identification currents are mapped onto identification currents: [11]

$$
\begin{aligned}
&(0,0\,/\,0,0) \stackrel{\mathcal{T}}{\longleftrightarrow} ((n-1)\Lambda_{(k+1)},(J_{\rm v})^{k+1}\,/\,n\Lambda_{(k)},\pm(k+n+1))\,,\\
&(k\Lambda_{(n)},(J_{\rm v})^{n}\,/\,(k+1)\Lambda_{(n-1)},\pm(k+n+1)) \stackrel{\mathcal{T}}{\longleftrightarrow} (0,0\,/\,0,0)\,.
\end{aligned}
\tag{9.6}
$$

Computation shows that the products of S-matrices of the respective WZW theories coincide (one has to make use once again of the selection rules, which imply cancellation of the factors $(-1)^{r(\Lambda)}$ that are present in equation (5.24)). This implies that in fact the two representatives of one physical field Φ are mapped on the representatives of the corresponding physical field $\mathcal{T}(\Phi)$ of the dual theory, or, in other words, that we can interpret $\mathcal{T}$ also as a mapping of physical fields.

3. The two representatives of the Ramond ground state with highest u_1-charge get exchanged:

$$
\begin{aligned}
&(0,{\rm s}\,/\,0,n) \stackrel{\mathcal{T}}{\longleftrightarrow} ((n-1)\Lambda_{(k+1)},(J_{\rm v})^{k+1}{\rm s}\,/\,n\Lambda_{(k)},-n)\,,\\
&(k\Lambda_{(n)},(J_{\rm v})^{n}{\rm s}\,/\,(k+1)\Lambda_{(n-1)},-k-1) \stackrel{\mathcal{T}}{\longleftrightarrow} (0,{\rm s}\,/\,0,k+1)\,.
\end{aligned}
\tag{9.7}
$$

In other words, in terms of fields we have proven compatibility of the map $\mathcal{T}$ with spectral flow.

To show that $\mathcal{T}$ maps Ramond ground states on Ramond ground states, again we first check the dimension of the chiral ring. We have to use the formula (2.25) with $N = N^C_{n,k}$, $|Z| = |Z(C_n)| = 2$, and

$$
\frac{|W(g)|}{|W(h)|} = \frac{2^n\,n!}{2^{n-1}(n-1)!} = 2n\,. \tag{9.8}
$$

Thus $\mu^{CC}_{n,k} = n\,N^C_{n,k} = (n+k)!/((n-1)!\,k!)$, which is invariant under $n \leftrightarrow k+1$. Of course, this also follows from the fact, observed in [13], that the (ordinary and extended) Poincaré polynomials of the theories (CC,n,k) and $(CC,k+1,n-1)$ are identical.

[11] This does not furnish a group isomorphism between the groups that describe the fusion rules of the identification currents. Since these groups are isomorphic to $\mathbf{Z}_2$, such an isomorphism would necessarily be trivial.

To analyse the Ramond ground states in more detail, first recall that in the orthogonal basis the action of the Weyl group is given by permuting the components and multiplying them with a sign, and has thus the same structure as in the case of B type Lie algebras. This allows us to use the same notation for Weyl group elements as in section 6. Furthermore, the roots of B type and C type algebras differ only by normalization factors, and these are irrelevant for the determination of the length of Weyl group elements. As a consequence, the formulæ (6.15) are valid for C type Lie algebras, too (and the Hasse diagram of the embedding $C_{n-1} \hookrightarrow C_n$ is again linear [13]). Correspondingly, the reasoning below will be very similar to the one of section 6. The relation (2.21) between the weights Λ and $\tilde{\lambda}$ implies that in terms of the numbers $\tilde{\ell}_i$ introduced in (5.15), the C_{n-1}-weight λ of a Ramond ground state Φ_R is related to the C_n-weight Λ by

$$\tilde{\ell}_i(\lambda) = \tilde{\ell}_{i+1}(w(\Lambda))\,, \tag{9.9}$$

and also

$$|Q| = \tilde{\ell}_1(w(\Lambda)) \tag{9.10}$$

for some Weyl group element w. When we characterize Λ and λ by the sets M_Λ and M_λ, this translates into

$$M_\lambda = M_\Lambda \setminus \{\tilde{\ell}_\circ\}, \tag{9.11}$$

where $\tilde{\ell}_\circ = \pm Q$ is an arbitrary element of M_Λ (recall that $\tilde{\ell}_\circ > 0$). Again the freedom in the choice of the sign of Q reflects the invariance of the set of Ramond ground states under charge conjugation. An analogous description applies to the image $\mathcal{T}(\Phi_R)$ of the Ramond ground state. Now $\mathcal{T}$ fixes uniquely the transformation of all weights, and

$$M_{\tau(\lambda)} = M \setminus M_\lambda = (M \setminus M_\Lambda) \cup \{\tilde{\ell}_\circ\} = M_{\tau(\Lambda)} \cup \{\tilde{\ell}_\circ\}, \tag{9.12}$$

so that $\tau(\Lambda)$ and $Q_{\mathcal{T}}$ are related to $\tau(\lambda)$ by the formula (2.21) with a suitably chosen Weyl group element $w_{\mathcal{T}}$.

To verify that $\mathcal{T}(\Phi_R)$ is again a Ramond ground state, it is now sufficient to check that $\mathcal{T}$ gives the correct weight in the D_d part of the theory. The

Weyl group elements w and $w_{\mathcal{T}}$ are uniquely fixed by the weights Λ and $\tilde{\lambda}$, respectively by their images under τ; for $w = w_i^{(+)}$, the Weyl group element $w_{\mathcal{T}}$ is given by $w^{(-)}_{k+n-Q-i+2}$, which implies that $l(w) - l(w_{\mathcal{T}}) = n - k - 1 - Q$. From this equation we can derive not only the equality of superconformal charges, but also the behavior on the D_d part; we have

$$\mathrm{sign}\,(w)\,\mathrm{sign}\,(w_{\mathcal{T}}) = (-1)^{l(w)+l(w_{\mathcal{T}})} = (-1)^{k+n+1+Q}, \tag{9.13}$$

which reproduces the prescription given in (9.4). This shows that $\mathcal{T}$ maps Ramond ground states on Ramond ground states, as claimed, and thus completes our arguments that the map $\mathcal{T}$ fulfills the requirements for the isomorphism (9.1) of conformal field theories, analogously as for the other isomorphisms of (1.1).

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