

**Simple currents versus orbifolds with
discrete torsion – a complete classification**

M. Kreuzer

CERN, 1211 Geneva 23, Switzerland

and

A. N. Schellekens

NIKHEF-H, P.O. Box 41882, 1009 DB Amsterdam, The Netherlands

Abstract

We give a complete classification of all simple current modular invariants, extending previous results for $(\mathbf{Z}_p)^k$ to arbitrary centers. We obtain a simple explicit formula for the most general case. Using orbifold techniques to this end, we find a one-to-one correspondence between simple current invariants and subgroups of the center with discrete torsions. As a by-product, we prove the conjectured monodromy independence of the total number of such invariants. The orbifold approach works in a straightforward way for symmetries of odd order, but some modifications are required to deal with symmetries of even order. With these modifications the orbifold construction with discrete torsion is complete within the class of simple current invariants. Surprisingly, there are cases where discrete torsion is a necessity rather than a possibility.

NIKHEF-H/93-13
June 1993

1. Introduction

The problem of classifying and enumerating all modular invariant partition functions of a given conformal field theory has been studied intensively during the last five years, but is still far from solved. However, there is one subclass of invariants that is almost under control, namely the simple current invariants. Simple currents [1] correspond to primary fields that upon fusion with any other field yield just one field. It is easy to see that the presence of simple currents implies that the conformal field theory has an abelian discrete symmetry called the center. A modular invariant partition function is called a simple current invariant if all fields that are paired non-diagonally are related by simple currents.

Although not all modular invariants are of this type, experience suggests that exceptions are rare. Hence by enumerating all of them one has probably listed most of the possible invariants of a given conformal field theory. The total number of such invariants grows very rapidly with the number of simple abelian factors in the center, a situation that is typical for tensor products of basic building blocks. Investigations of large classes of modular invariants that can be obtained with simple currents have been presented for example in [2-4].

The systematic study of simple current invariants was only partly complete up to now. In [5] all the pure automorphisms have been classified for any center. In [6] all invariants have been classified for centers of the form $(\mathbf{Z}_p)^k$ (and products thereof), where p is a prime. What is still missing is a complete classification for centers containing factors \mathbf{Z}_{p^n} .

Completing this classification is one of the goals of this paper. Another goal is to find an explanation for a phenomenon that was hard to understand from the point of view of [6]. In that paper the modular invariants were constructed by (a) classifying all possible extensions of the chiral algebra, (b) determining the allowed 'heterotic' combinations of different algebras and (c) superimposing all allowed automorphisms determined in [5]. Then the total number of invariants was calculated by adding up all these different kinds of solutions, a rather laborious computation. It turned out that the total number of invariants is only a function of the group structure of the center, and does not depend on the spins and relative monodromies of the currents.

To formulate the latter statement more precisely, consider a center generated by currents $J_i, i = 1, \dots, k$. The monodromies of these currents define a symmetric matrix

$$\tilde{R}_{ij} = Q_i(J_j) = Q_j(J_i) ,$$

where $Q_i(a)$ is the monodromy phase of the current J_i with respect to a field labelled a , which we will call the charge of a (for more details we refer to [1] or the review [7]). The matrix elements of \tilde{R} are quantized as

$$\tilde{R}_{ij} = \frac{\tilde{r}_{ij}}{N_i}, \quad \tilde{r}_{ij} \in \mathbf{Z} .$$

Note that \tilde{r}_{ij} is defined modulo N_i . Furthermore \tilde{r}_{ij} must be quantized in units of $N_i/\text{GCD}(N_i, N_j)$ because of the symmetry of \tilde{R} . This is equivalent to $\tilde{r}_{ij}N_j = 0 \pmod{N_i}$. The total number of invariants was found to be independent of \tilde{R} , provided one reduces the center to a subgroup, the ‘effective center’. This is obtained by removing all currents whose spin, multiplied by the order of the current, is not an integer. This eliminates for example the simple currents of $SU(2)$ at odd levels. In general it eliminates all currents that cannot preserve T -invariance, and hence cannot play a rôle in constructing modular invariants. The conformal weight of a current combination $J_1^{\alpha_1} \dots J_k^{\alpha_k}$ (henceforth denoted $[\vec{\alpha}]$) can be expressed in terms of a slight generalization of the matrix \tilde{R} :

$$h([\vec{\alpha}]) = \frac{1}{2} \sum_{ij} \alpha_i R_{ij} \alpha_j + \frac{1}{2} \sum_i r_{ii} \alpha_i , \pmod{1} \tag{1.1}$$

where $\tilde{R} = R \pmod{1}$. Note that the conformal weight changes by $\frac{1}{2}\alpha_i(\alpha_i + N_i)$ if we change R_{ii} by 1. If N_i is even this may be a change by a half-integer, which is not equivalent. Thus R contains more information than \tilde{R} , and its diagonal matrix elements are defined modulo 2, not 1. A necessary and sufficient condition for an effective center is that the diagonal elements r_{ii} are even (note that if N_i is odd all matrix elements r_{ij} can be chosen even since they are defined modulo N_i). Henceforth the word ‘center’ always means ‘effective center’.

For effective centers $(\mathbf{Z}_p)^k$, p prime, it was found that, even though the separate numbers of different kinds of invariants (see (a), (b) and (c) above) depend strongly on

R , the total number does not, and is given by the simple formula

$$N_{\text{tot}} = \prod_{i=0}^{k-1} (1 + p^i) .$$

Empirically, this phenomenon appears to hold also for more general centers (*i.e.* with \mathbf{Z}_{p^n} factors), but there are two difficulties with pursuing this further. The first is that although points (a) and (c) have already been solved in general, point (b) has not, *i.e.* no general rule is known for the allowed heterotic combinations of different algebras (a necessary condition is that they must have the same size). The second difficulty is that in the process of adding up all invariants for all possible monodromies more and more different cases have to be considered separately. Clearly a simpler approach is needed.

Such an approach is already available for \mathbf{Z}_N , for any N . In this case the classification of [6] still applies, since a subgroup is uniquely defined by its size, so that there cannot be any heterotic invariants. One finds that the number of invariants is in one-to-one correspondence with the subgroups of the center. Furthermore, a universal formula exists that gives all possible invariants [8]. To write down this formula one specifies a subgroup \mathcal{H} of the center, which is generated by a current J . Then the non-zero values of M_{ab} , the multiplicity of the module $|a\rangle \otimes |b\rangle$, are given by[★]

$$M_{a, J^n a} = \text{Mult}(a) \delta^1(Q(a) + \frac{1}{2}nQ(J)) . \tag{1.2}$$

Here a labels a primary field, $J^n a$ is the field a acted upon n times by the current J , and Q is the charge with respect to J . If J acts without fixed points on a $\text{Mult}(a) = 1$, and otherwise $\text{Mult}(a)$ is equal to the number of copies of a that one encounters on a standard-length orbit. This formula does depend on the monodromies via $Q(J)$, but the *number* of solutions clearly does not.

Formula (1.2) was originally obtained by applying orbifold twists to the discrete symmetries of the center, and hence it is natural to look in that direction for a more general formula. Orbifolds are not limited to \mathbf{Z}_N groups, and can in fact be written down for any subgroup of the center. This is still not enough because the number of

★ This formula is valid provided that r is even, as discussed above, and that Q is defined modulo 2 in terms of R .

invariants is in general larger than the number of subgroups. However, as Vafa [9] has shown, there are more general orbifolds one can write down because one can allow phases known as ‘discrete torsion’.[†]

It is instructive to count the number of orbifold invariants including discrete torsion. Suppose $\mathcal{N}_a(k)$ is the number of $(\mathbf{Z}_p)^a$ subgroups in $(\mathbf{Z}_p)^k$. This quantity satisfies the recursion relation $\mathcal{N}_a(k+1) = p^a \mathcal{N}_a(k) + \mathcal{N}_{a-1}(k)$. For each subgroup with a generators the number of discrete torsion coefficients according to [9] is equal to $p^{a(a+1)/2}$. Hence the total number of invariants is

$$\sum_a \mathcal{N}_a(k) p^{\frac{a(a+1)}{2}} = \prod_{i=0}^{k-1} (1 + p^i) .$$

This is precisely the result of [6]. Since the latter was shown to be complete, and since the orbifold invariants are all different, this establishes a one-to-one relation between the two approaches for centers $(\mathbf{Z}_p)^k$.

If the aforementioned monodromy-independence holds, we can already go much further by considering the case of trivial monodromy (all currents are mutually local and have integral spin). In that case all subgroups \mathcal{H} of the center \mathcal{C} can occur as extensions of the algebra. We denote the total number of subgroups isomorphic to \mathcal{H} in \mathcal{C} as $\mathcal{N}(\mathcal{H}, \mathcal{C})$. Once the algebra is extended, the quotient group \mathcal{C}/\mathcal{H} survives as the center of the new theory. The number of pure automorphism of a theory with trivial monodromy, center \mathcal{H} , and generated by currents of order N_i is given by [10]

$$A(\mathcal{H}) = \prod_{i < j} \text{GCD}(N_i, N_j) . \tag{1.3}$$

If all currents are mutually local, the left and right algebras must be the same, since there is no way to project anything out. The total number of invariants is thus equal to

$$\sum_{\mathcal{H}} \mathcal{N}(\mathcal{H}, \mathcal{C}) A(\mathcal{C}/\mathcal{H}) .$$

To get the number of orbifold invariants one also considers all possible subgroups, but then one counts the number of allowed torsions *within* each group. This number is again

[†] Historically the name refers to discrete values of a B_{ij} background field. Such an interpretation is not always available in a straightforward way in arbitrary conformal field theories, but it is natural to use the same name in general.

given by (1.3). Hence in this case we get a total which is equal to

$$\sum_{\mathcal{H}} \mathcal{N}(\mathcal{H}, \mathcal{C}) A(\mathcal{H}) .$$

Although this looks different, obviously $\mathcal{N}(\mathcal{H}, \mathcal{C}) = \mathcal{N}(\mathcal{C}/\mathcal{H}, \mathcal{C})$, so that the result is the same. Once again this establishes a one-to-one correspondence.

Although roughly correct, there are still some serious flaws in the foregoing arguments. First of all, it is not straightforward to define an orbifold (without torsion) for arbitrary subgroups of the center. Although the center defines a symmetry of the theory, not every symmetry defines an orbifold. Furthermore the center is *a priori* a symmetry of a chiral half of the theory. It is not completely trivial to find a related symmetry that acts on the complete theory, and satisfies level matching [9]. It turns out that currents of even order are especially hard to deal with. In some cases, an orbifold description in the sense of [9] does not seem to exist, although we can write down something similar. There are also cases where one cannot really define an orbifold without torsion, and where torsion is needed to write down any non-trivial invariant. Secondly, since the simple current classification for \mathbf{Z}_{p^n} is not complete for arbitrary monodromies, and the monodromy independence only conjectured, there remains a possibility that something is overlooked.

We will overcome the second point by proving directly that orbifolds with discrete torsion produce all possible simple current invariants. This requires some refinements of the arguments of [9], that were not intended to be a proof of completeness, but only of existence, and that in addition were presented for theories built out of free bosons or fermions, which is not the case for us. Of course the issue of completeness cannot be addressed in a practical manner if one studies orbifolds of tori, since one moves between different conformal field theories by the introduction of twist fields. In our case, however, we are dealing with a fixed set of primary fields from which all the invariants are constructed.

The final result is an extremely simple formula that generalizes (1.2), and yields all simple current invariants in all cases.

In the next section we will present the completeness proof using orbifold techniques. In section 3 we analyse the resulting invariants to find extensions of the algebra or pure automorphism, and we will make the relation with the results of [5] and [6] more precise.

2. Orbifolds

The main task in this section is to find a translation of [9] that generalizes it beyond the free boson/fermion theories for which it was written. To illustrate this we start with the simplest case, a theory with single orbit \mathbf{Z}_N , N odd but not necessarily prime. Suppose there exists a non-diagonal modular invariant. If it is a simple current invariant, any non-diagonal term must be of the form

$$\mathcal{X}_a \mathcal{X}_{La}^* ,$$

where L is some simple current. If N is odd, L can be written as a square of a simple current $K = L^{\frac{N+1}{2}}$. Furthermore the field a can be written as $K^c b$ for some definite b . Hence the off-diagonal term takes the form

$$\mathcal{X}_{K^c b} \mathcal{X}_{Kb}^* .$$

Inspired by this and by [9], we consider now the following set of "twisted" partition functions

$$P[0, n](\tau) = \sum_a \mathcal{X}_{(J^c)^{n_a}}(\tau) \mathcal{X}_{J^{n_a}}^*(\bar{\tau}) ,$$

where the sum is over all fields. These partition functions correspond to $(0, h)$ in the notation of [9]. To find the analog of the full (g, h) is now simply a matter of making modular transformations. For example

$$P[0, n]\left(-\frac{1}{\tau}\right) = \sum_{a,b,c} S_{(J^c)^{n_a}, b} S_{J^{n_a}, c}^* \mathcal{X}_b(\tau) \mathcal{X}_c^*(\bar{\tau}) ,$$

using the modular transformation properties of the characters. Now we use the formula for S from [8] and [11]:

$$S_{J^{n_a}, b} = e^{2\pi i n Q(b)} S_{ab} .$$

Furthermore we use $Q_c = -Q$ to get

$$P[0, n]\left(-\frac{1}{\tau}\right) = \sum_a e^{-2\pi i n (2Q(a))} \mathcal{X}_a(\tau) \mathcal{X}_a^*(\bar{\tau})$$

The result should be equal to $P[-n, 0](\tau)$. The same computation can be done for the transformation $\tau \rightarrow \tau + 1$. The results of combining these transformations can be

summarized by defining

$$P[m, n](\tau) = \sum_a e^{2\pi i(2mQ(a))} \mathcal{X}_{(J^c)^n a}(\tau) \mathcal{X}_{J^n a}^*(\bar{\tau}) ,$$

It is easy to verify that this transforms as

$$P[m, n]\left(\frac{a\tau + b}{c\tau + d}\right) = P[am + bn, cm + dn](\tau) .$$

Note that $2Q(a)$ is simply the charge of a field $\Phi_{(J^c)^n a}(z)\Phi_{J^n a}(z^*)$ with respect to the current JJ^c , and that this does not depend on n :

$$2Q(a) = Q((J^c)^n a) - Q_c(J^n a) .$$

This is a consequence of the fact that JJ^c is an integer spin current. This remark will become relevant in a moment. Obviously, by summing $P[m, n]$ over a modular orbit one gets a modular invariant partition function. This argument is exactly the one used when simple current invariants were originally constructed in [1], where the operator JJ^c was used as a twist operator. Here the formalism of [9] can be taken over literally.

Generalizing this to $\mathbf{Z}_{N_1} \times \dots \times \mathbf{Z}_{N_k}$ (still with all N_i odd) is essentially straightforward. We just replace m and n by vectors $\vec{\alpha}$ and $\vec{\beta}$. Then we define

$$P[\vec{\alpha}, \vec{\beta}] = \sum_a e^{4\pi i \vec{\alpha} \cdot \vec{Q}(a)} \mathcal{X}_{[-\vec{\beta}]a} \mathcal{X}_{[\vec{\beta}]a}^* ,$$

where $[\vec{\alpha}]a$ denotes the field obtained by acting with $J_1^{\alpha_1} \dots J_k^{\alpha_k}$ on the field a . These functions transform under modular transformations in the obvious way,

$$P[\vec{\alpha}, \vec{\beta}]\left(\frac{a\tau + b}{c\tau + d}\right) = P[a\vec{\alpha} + b\vec{\beta}, c\vec{\alpha} + d\vec{\beta}](\tau) .$$

The most general partition function that we can write down using these functions is

$$P_C = \sum_{\vec{\alpha}, \vec{\beta}} C(\vec{\alpha}, \vec{\beta}) P[\vec{\alpha}, \vec{\beta}] , \tag{2.1}$$

where the C 's are arbitrary complex numbers. Here the sum is over all values of $\vec{\alpha}$ and $\vec{\beta}$ covered by the modular group, modulo equivalences, *i.e.* precisely over all group elements of the center.

The crucial question is: can we write the most general simple current partition function in this way, or in other words, are the functions $P[\vec{\alpha}, \vec{\beta}]$ a complete basis for the space of simple current partition functions.* It is clear that they are not, because each function P contains a sum over all fields, and hence over all orbits. Therefore we cannot modify the behavior of individual orbits by changing the coefficients. However, it is not hard to see that we *can* get any simple current partition function that satisfies the additional requirement that the off-diagonal elements for each field a depend only on the charge of a . Fortunately it was shown in [5] and [6] that any simple current partition function that is modular invariant must have that property, provided the matrix S is reasonably well behaved (not too many unexpected zeroes and not too many fixed points fields in comparison to normal fields). This can be expressed in terms of a few regularity conditions, which in practice are usually satisfied. Since we have nothing new to say about this we will not dwell on this point, and refer the reader to [6] for a more detailed discussion and some examples of pathologies. Apart from such pathologies, we can show that by introducing the coefficients C we are able to get all possible simple current partition functions that have a chance of being modular invariant.

To see this explicitly, note that the sum over $\vec{\alpha}$ in (2.1) is simply a Fourier transformation of the set of coefficients. Indeed, consider the most general candidate invariant

$$\sum_a \sum_{\vec{\beta}} M(a, \vec{\beta}) \mathcal{X}_{[-\vec{\beta}]a} \mathcal{X}_{[\vec{\beta}]a}^* .$$

Now suppose that M depends on a only via the charge $\vec{Q}(a) \equiv \vec{q}$. This allows us to rewrite the previous expression as

$$\sum_{\vec{q}} \sum_{a, \vec{Q}(a)=\vec{q}} M(\vec{q}, \vec{\beta}) \mathcal{X}_{[-\vec{\beta}]a} \mathcal{X}_{[\vec{\beta}]a}^* .$$

Using a Fourier transform in the periodic variable \vec{q} we can write[†]

$$M(\vec{q}, \vec{\beta}) = \sum_{\vec{\alpha}} e^{4\pi i \vec{\alpha} \cdot \vec{q}} C(\vec{\alpha}, \vec{\beta}),$$

* By this we mean the set of functions $M_{ab} \mathcal{X}_a \mathcal{X}_b^*$, with M_{ab} a set of positive integers that vanish if a and b are not connected by simple currents. Note that we are not (yet) requiring modular invariance here.

† Note that although one would normally put 2π rather than 4π in the exponent, this makes no difference as long as the orders N_i are odd. It merely reorganizes the sum.

so that the partition takes the form

$$\sum_{\vec{q}} \sum_{a, \vec{Q}(a)=\vec{q}} \sum_{\vec{\alpha}} C(\vec{\alpha}, \vec{\beta}) e^{4\pi i \vec{\alpha} \cdot \vec{q}} \mathcal{X}_{[-\vec{\beta}]a} \mathcal{X}_{[\vec{\beta}]a}^* ,$$

which indeed is precisely (2.1).

To determine the complex numbers C we follow [9]. In this paper only phases were considered, but that makes little difference. To determine the coefficients higher loop modular invariance and factorization was used in [9]. On the other hand, the work in [5] and [6] uses one-loop modular invariance and positivity. Of course all these conditions are necessary, and even though positivity is not imposed as a condition, we will be able to verify it afterwards. We return to this point at the end of this section. The higher genus generalization of $C(\vec{\alpha}, \vec{\beta})$ is denoted $C(\vec{\alpha}_1, \vec{\beta}_1; \vec{\alpha}_2, \vec{\beta}_2; \dots)$, exactly as in [9] except that we use an additive notation.

One loop modular invariance implies

$$C(a\vec{\alpha} + b\vec{\beta}, c\vec{\alpha} + d\vec{\beta}) = C(\vec{\alpha}, \vec{\beta}) .$$

Factorization requires

$$C(\vec{\alpha}_1, \vec{\beta}_1; \vec{\alpha}_2, \vec{\beta}_2) = C(\vec{\alpha}_1, \vec{\beta}_1) C(\vec{\alpha}_2, \vec{\beta}_2) .$$

The Dehn twist around a curve connecting adjacent handles yields

$$C(\vec{\alpha}_1, \vec{\beta}_1; \vec{\alpha}_2, \vec{\beta}_2) = C(\vec{\alpha}_1 + \vec{\beta}_2 - \vec{\beta}_1, \vec{\beta}_1; \vec{\alpha}_2 + \vec{\beta}_1 - \vec{\beta}_2, \vec{\beta}_2) ,$$

and finally we can normalize $C(0, 0) = 1$. From these conditions one derives

$$C(\vec{\alpha}_1, \vec{\beta}_1) C(\vec{\alpha}_2, \vec{\beta}_2) = C(\vec{\alpha}_1 + \vec{\beta}_2, \vec{\beta}_1) C(\vec{\alpha}_2 + \vec{\beta}_1, \vec{\beta}_2) .$$

Setting $\vec{\alpha}_1 = \vec{\alpha}_2 = \vec{\beta}_2 = 0$ we get

$$C(0, \vec{\beta}_1) C(0, 0) = C(0, \vec{\beta}_1) C(\vec{\beta}_1, 0) .$$

We would like to conclude from this that $C(\vec{\beta}_1, 0) = C(0, 0) = 1$, but of course this is only true if $C(0, \vec{\beta}_1) \neq 0$. This possibility is rejected in [9] because all C 's are assumed

to be phases, but in fact there are perfectly valid solutions in which some coefficients vanish. It is not hard to see that these coefficients vanish in such a way that the non-vanishing ones span only a subgroup of the center. Of course this is exactly as it should be. If indeed by introducing C 's one can get all possible modular invariants, one must in particular be able to get all the invariants corresponding to subgroups, including the diagonal one. From now on we will assume that the C 's don't vanish.

The rest of the argument proceeds as in [9]. We find

$$C(\vec{\beta} + \vec{\alpha}, \vec{\gamma}) = C(\vec{\alpha}, \vec{\gamma})C(\vec{\beta}, \vec{\gamma}) ,$$

which implies

$$C(\vec{\alpha}, \vec{\gamma})^N = C(N\vec{\alpha}, \vec{\gamma}) = C(0, \vec{\gamma}) = C(-\vec{\gamma}, 0) = 1 ,$$

if N is the order of $\vec{\alpha}$. This shows that C must be a phase after all, and is in fact an N^{th} root of unity. The classification of all allowed phase choices is then exactly as in [9], and will be explained below. This concludes the proof of completeness for odd orders.

Now we still have to deal with even N . Let us again start with a single factor. In \mathbf{Z}_N , N even one cannot (in general) 'take the square root' of J . If J has integer spin this is still not a problem, because one can work with the 'chirally' twisted sectors $\mathcal{X}_a \mathcal{X}_{J^2 a}^*$, and everything goes through just as before. What happens if J has some other spin? It seems natural to define

$$P[m, n] = \sum_a e^{2\pi i m Q(a)} \mathcal{X}_a \mathcal{X}_{J^n a}^* \tag{2.2}$$

Note that choosing a or $J^n a$ as the argument of Q yields a different result (except if J has half-integer spin). However, the real problem is that P transforms with a phase that depends on n, m and a, b, c, d in a very complicated, way. In fact, we do not even know a general formula for that phase, except when the current has half-integer spin. Then we find

$$P[m, n] \left(\frac{a\tau + b}{c\tau + d} \right) = (-1)^{am+bn+m} (-1)^{cm+dn+n} P[am + bn, cm + dn](\tau) .$$

Keeping these phases in a sum over a modular orbit, we find for example if $N = 2$ that

$$P[0, 0] \pm (P[1, 0] + P[0, 1] - P[1, 1])$$

is modular invariant. To keep the identity the \pm sign must be taken to be $+$, and

this choice gives the expected automorphism invariant generated by a spin- $\frac{1}{2}$ current. Formally this is analogous to the prescription given in [9] for orbifolding free fermions, but half-integer spin simple currents occur in many CFT's that have nothing to do with free fermions.

Beyond this example the procedure of [9] becomes really inadequate for our purpose. For example, consider $SU(4)$ level 2. This has four simple currents of spin 0, twice $\frac{3}{4}$ and 1. The spin-1 invariant is easy to get, but there is an extra automorphism generated by the spin- $\frac{3}{4}$ current J . One might think that this can be gotten by using the operator JJ^c , but that is not true. This just gives the same spin-1 invariant. The 'correct' answer turns out to be to modify (2.2) by using in the exponential not $Q(a)$ nor $Q(J^n a)$ but the average:

$$P[m, n] = \sum_a e^{2\pi i m \frac{1}{2}(Q(a)+Q(J^n a))} \mathcal{X}_a \mathcal{X}_{J^n a}^* .$$

This expression is ill-defined as it stands, since charges are defined modulo 1. By writing the exponent as $i\pi(2Q(a) + Q(J^n))$ we see that in particular the second term requires more care. It turns out that under S the transformation is as for odd N , for any valid definition of Q . However, the T -transformation is sensitive to the precise definition of $Q(J)$. If we define $Q(J)$ with the same matrix R used also in the definition for the conformal weight (see (1.1)) one gets the following transformation under T :

$$P[m, n](\tau + 1) = e^{2\pi i \frac{nr}{2}} P[m + n, n](\tau) . \tag{2.3}$$

The extra phase disappears if r is even, which indeed is a necessary condition for the existence of the modular invariant, and is automatic for the effective center. Without the phase, $P[m, n]$ transforms exactly as before, and by summing over a modular orbit one gets the desired invariant, which is precisely (1.2).

Incidentally, one may ask what happens if, for r odd, one simply keeps the phase and sums over the modular transformations. Then the sum will have to extend over twice the modular domain, because $P[m, n] \neq P[m, n + N]$ for odd m . The result is that the terms with improper periodicity cancel out, and one finally gets some invariant corresponding to a smaller subgroup.

What is somewhat disturbing about (2.3) is that the phase factor does not seem to correspond to any symmetry operator g acting on the states of the theory, like the 'g'

of the orbifold approach. An operator that produces such eigenvalues would be $\sqrt{JJ^c}$, which looks rather unpleasant. Furthermore the 'twisted' sectors are not created with this operator, but with J acting on the right-moving sector. Nevertheless the proof of modular invariance at one loop is completely rigorous. Note that these arguments, as well as the following ones, hold for odd as well as even orders.

Now let us consider the general case. Clearly we would like to define objects like

$$P[\vec{\alpha}, \vec{\beta}] = \sum_a e^{\pi i \vec{\alpha} \cdot (\vec{Q}(a) + \vec{Q}([\vec{\beta}]_a))} \mathcal{X}_a \mathcal{X}_{[\vec{\beta}]_a}^* .$$

It is easy to show that these functions transform correctly under S , and under T they transform with an extra phase $\exp(\pi i \sum_i \beta_i r_{ii})$, that is equal to 1 if r_{ii} is even, which is always true for the effective center. Here and in the following all charges of currents are defined using the matrix R of (1.1), whose diagonal elements are defined modulo 2. As before, this is imposed by T -invariance. One loop modular transformations invariance do not impose any particular choice for the mod 2 ambiguity in the off-diagonal elements of R , as long as R is symmetric modulo 2. Note that the conformal weights also do not fix this ambiguity. We simply make an arbitrary choice to fix this ambiguity in R (which amounts to a sign-ambiguity for P). For odd N_i it is convenient to choose all r_{ij} even.

First we will show, as before, that any modular invariant partition function can be written in this basis. Any simple current invariant has the form

$$\sum_{a, \vec{\beta}} M(a, \vec{\beta}) \mathcal{X}_a \mathcal{X}_{[\vec{\beta}]_a}^* .$$

As before we use the result of [6] that M depends on a only via the charge \vec{Q} of a . Then we can write this as

$$\sum_{\vec{q}} \sum_{a(\vec{q})} M(\vec{q}, \vec{\beta}) \mathcal{X}_{a(\vec{q})} \mathcal{X}_{[\vec{\beta}]_{a(\vec{q})}}^* .$$

Now we Fourier-transform M with respect to \vec{q} :

$$M(\vec{q}, \vec{\beta}) = \sum_{\vec{\alpha}} \tilde{C}(\vec{\alpha}, \vec{\beta}) e^{2\pi i \vec{\alpha} \cdot \vec{q}} .$$

This will not yield exactly the functions $P(\vec{\alpha}, \vec{\beta})$ we are trying to get, so what we have

to do is redefine the coefficients as follows

$$\tilde{C}(\vec{\alpha}, \vec{\beta}) = e^{\pi i \vec{\alpha} \cdot R \cdot \vec{\beta}} C(\vec{\alpha}, \vec{\beta}) .$$

Substituting this we have indeed written the modular invariant as

$$\sum_{\vec{\alpha}, \vec{\beta}} C(\vec{\alpha}, \vec{\beta}) P[\vec{\alpha}, \vec{\beta}] . \tag{2.4}$$

This shows that we can indeed write every modular invariant in this basis. Note that the ambiguities in the definition of the off-diagonal elements of R modulo 2 cancel between C and P .

Now we have to answer the question: for which choices of C do we get a modular invariant. First of all, suppose one knows one valid choice. Then one can factor it out of all the C 's, and the remainder should then satisfy the same equations as the discrete torsions introduced earlier. Now previously there was always a trivial solution, namely $C = 1$. This may appear to be true here as well, but on closer inspection it is not.

Eq. (2.4) is summed over a set of vectors belonging to a domain that covers all inequivalent vectors exactly once. It is manifestly modular invariant under all transformations that map these vectors within this domain. If P has the proper periodicity it is then modular invariant. However, P does not always have the right periodicity. The factor that may violate the periodicity is

$$e^{i\pi \vec{\alpha} \cdot \vec{Q}(\vec{\beta})} = e^{i\pi \vec{\alpha} \cdot R \cdot \vec{\beta}} .$$

In general, $R_{ij} = \frac{r_{ij}}{N_i}$. Thus if we shift $\vec{\alpha}$ by a period N_k for some k we get a phase $\exp(i\pi r_{kl} \beta_l)$. If all the matrix elements of r are equal to an even integer this phase is equal to 1. This property of r can always be arranged to hold for odd N_i . Furthermore it always holds for the diagonal elements if we restrict ourselves to the effective center, but it need not hold for the off-diagonal ones.

Clearly the choice $C = 1$ will not do in that case. We need some choice of C_0 that respects modular invariance within the domain described above, but also has the wrong periodicity to compensate the wrong periodicity of P . It doesn't matter how we get

such a C_0 , and it also doesn't matter which choice we make. If someone else makes a different choice, it follows from the above arguments that his choice can be obtained from ours by multiplication with a discrete torsion factor. Since P transforms 'correctly' within the domain, C_0 must transform like a discrete torsion factor within the domain. An obvious choice for C_0 is

$$C_0[\vec{\alpha}, \vec{\beta}] = e^{i\pi\vec{\alpha}\cdot E\cdot\vec{\beta}} , \quad (2.5)$$

where E is an antisymmetric matrix satisfying $E_{ij} = R_{ij}$ for $j > i$. This does indeed satisfy all conditions of a discrete torsion, and has the wrong periodicity whenever R does.

Apart from this subtlety, the rest of the argument goes exactly as for odd order. In both cases, the general discrete torsion can be written as

$$C(\vec{\alpha}, \vec{\beta}) = e^{2i\pi\vec{\alpha}\cdot e\cdot\vec{\beta}} ,$$

where e is an antisymmetric matrix with matrix elements

$$e_{ij} = \frac{\epsilon_{ij}}{N_i} = -\frac{\epsilon_{ji}}{N_j} , \quad \text{with } \epsilon_{ij} \in \mathbf{Z} . \quad (2.6)$$

That this is the most general form follows from [9], and can be seen easily by defining $C = e^\Gamma$. Then the equations for C yield

$$\Gamma(\vec{\alpha} + \vec{\beta}, \vec{\gamma}) = \Gamma(\vec{\alpha}, \vec{\gamma}) + \Gamma(\vec{\beta}, \vec{\gamma})$$

and

$$\Gamma(\vec{\alpha}, \vec{\beta}) = -\Gamma(\vec{\beta}, \vec{\alpha}) ,$$

so that Γ is linear in both its arguments. Furthermore

$$\Gamma(\vec{\alpha}, \vec{\alpha}) = 0 .$$

Thus Γ is a bilinear antisymmetric object, and must be of the form $\vec{\alpha} \cdot e \cdot \vec{\beta}$ for some e . The matrix elements of e are then restricted to the form (2.6) by requiring the correct periodicity.

Now we can absorb C into P and write the most general invariant as follows

$$\sum_a \sum_{\vec{\alpha}, \vec{\beta}} e^{2\pi i \vec{\alpha} \cdot \vec{Q}(a)} e^{2\pi i \vec{\alpha} \cdot X \cdot \vec{\beta}} \chi_a \chi_{[\vec{\beta}]a}^* , \quad (2.7)$$

where X is a matrix satisfying $X + X^T = R$,

$$X_{ij} = \frac{\chi_{ij}}{N_i}, \quad \chi_{ij} \in \mathbf{Z} . \quad (2.8)$$

Note that the matrix elements of X are quantized precisely like those of \tilde{R} : since X is defined modulo integers, χ_{ij} must be defined modulo N_i ; furthermore, in order to satisfy $X + X^T = R$, χ_{ij} must be proportional to $N_i/\text{GCD}(N_i, N_j)$, and hence $\chi_{ij}N_j = 0 \pmod{N_i}$. In our previous notation $X = e + \frac{1}{2}R$ or $e + \frac{1}{2}(R + E)$, where the second expressions is used if R is not divisible by 2. Note that the rôle of the discrete torsions is simply to provide an antisymmetric part to R . The invariant (2.7) is not yet normalized, but it is easy to see that one must simply divide by the order of the group.

This result can be simplified further by observing that the sum over $\vec{\alpha}$ is just a δ -function. This yields the final, and undoubtedly most simple and elegant formula for the modular invariant partition function. Before presenting it, let us summarize the main result of this paper.

Suppose one has a conformal field theory with simple currents generating a center \mathcal{C} . Then the complete set of simple current invariants of that theory can be obtained by the following procedure

1. Choose any subgroup \mathcal{H} of \mathcal{C} .
2. Choose a basis of currents J_1, \dots, J_k that generate \mathcal{H} .
3. Compute the current-current monodromies R_{ij} in that basis.
4. Choose any properly quantized matrix X (see (2.8)) whose symmetric part is $\frac{1}{2}R \pmod{1}$ (in other words $X + X^T = R$). The modular invariant partition function

* Note that X is defined modulo 1 and that R is defined modulo 2 on the diagonal and modulo 1 elsewhere. The equations are defined with exactly the same periodicities as R . In the following these periodicities will be omitted from the equations.

corresponding to this choice is then given by a matrix whose only non-zero elements are

$$M_{a, [\vec{\beta}]a} = \text{Mult}(a) \prod_i \delta^1(Q_i(a) + X_{ij}\beta_j) , \quad (2.9)$$

where δ^1 is equal to 1 if its argument is an integer, and vanishes otherwise. The factor $\text{Mult}(a)$ appears because a may be a fixed point of some currents. In that case the β -sum in (2.7) includes all terms involving a more than once, and $\text{Mult}(a)$ is the number of times this happens. This is the generalization of (1.2) to more than one factor.

As we mentioned earlier, this is the complete set of solutions to a different set of conditions than those considered in [6] and [5] (and in most other papers on modular invariance of conformal field theories other than those built out of free bosons and fermions). Usually one tries to determine all positive and properly normalized matrices M that commute with S and T , a given set of representations of the (one-loop) modular group. In our case it turned out to be convenient to replace the positivity condition by a higher genus condition. This goes a little bit against the original spirit, since it requires more information than just S and T . Indeed, we do not even know how to write down explicitly the complete higher genus modular invariance conditions for a generic CFT of which *a priori* we only know S and T . In the special case of simple current invariants the orbifold analogy strongly suggested a higher genus transformation rule which turned out to give a very effective shortcut in the completeness proof (an interesting question is whether a similar higher genus approach could be applied successfully to the long-standing classification problem of exceptional invariants). Since the resulting invariants are manifestly positive, they form at least a subset of the solutions to the original genus-1 problem. Since in all known cases we find in fact *all* solutions, it is natural to conjecture that this will be true in general. If this conjecture turns out to be wrong, the exceptions most likely do not correspond to well-defined CFT's, unless their higher genus behavior is unexpectedly subtle.

Finally, note that for some subgroups \mathcal{H} discrete torsion is *required* to get any invariant at all. Consider for example $SO(8)$ level 1. This has a center $\mathbf{Z}_2 \times \mathbf{Z}_2$, and the

matrix R is

$$\frac{1}{2} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix},$$

so that indeed the periodicity problem occurs here. There are in total six invariants, corresponding to the six permutations of the conjugacy classes (v) , (s) , and (c) . The trivial permutation (the diagonal invariant) belongs to the identity subgroup. There are three \mathbf{Z}_2 subgroups, which correspond to the three permutations of order 2. And finally, there are two invariants corresponding to the $\mathbf{Z}_2 \times \mathbf{Z}_2$ subgroup, namely the two cyclic permutations. Clearly these two are completely equivalent, and there is no sense in which one of them has discrete torsion, and the other does not. Indeed, the difference between them is simply that in one case one uses C_0 , as in (2.5), and in the other case the complex conjugate. In both cases there is discrete torsion.

3. Chiral algebras and Automorphisms

We have now classified all simple current invariants for any center. In this chapter we will investigate their properties more closely, to determine the extensions of the chiral algebra and the automorphisms they imply. This will also clarify the relation with [5] and [6] where the various kinds of invariants were considered separately. Finally, we will study products of invariants of the form (2.9) and explicitly prove closure under multiplication.

3.1. EXTENDED CHIRAL ALGEBRAS

From (2.9) one can immediately read off the extensions of the right algebra, *i.e.* the nonvanishing matrix elements $M_{0, [\vec{\alpha}]}$. The condition is simply that $X\vec{\alpha} = 0 \pmod{1}$, or in other words that $\vec{\alpha}$ is in the kernel of X . To see how the left algebra is extended we need the following reflected version of (2.9):

$$M_{[\vec{\beta}]a,a} = \text{Mult}(a) \prod_i \delta^1(Q_i(a) + X_{ij}^T \beta_j). \quad (3.1)$$

Clearly the currents in the left algebra form precisely the kernel of X^T .

If a current $\vec{\alpha}$ appears in the left as well as the right algebra, then $X\vec{\alpha} = X^T\vec{\alpha} = 0$, so that $R\vec{\alpha} = 0$. This implies that it is local with respect to all currents in the subgroup \mathcal{H} . Obviously the converse is also true: if a current is local with respect to all other currents in a subgroup, it is either in both the left and the right algebra or in neither, in all invariants belonging to that subgroup (and subgroups thereof). Hence if a current is local with respect to all simple currents in the full center \mathcal{C} it can not appear 'heterotically' in any simple current invariant. This was proved in a different way in [6]. Furthermore it is true that if a current is non-local with respect to at least one other currents in a subgroup, it cannot appear simultaneously in the left and right algebras in any of the invariants corresponding to that subgroup (and all subgroups containing it).

It is of course interesting to see if we can get any general restrictions on the possible combinations of left and right chiral algebras. We begin with an instructive example: Consider a center $\mathcal{C} = \mathbf{Z}_9 \times \mathbf{Z}_9$ with monodromy matrix $R_{12} = 1/9$, $R_{11} = R_{22} = 0$ and its diagonal subgroup $\mathcal{H} = \mathbf{Z}_9 \times \mathbf{Z}_3$. Choosing the discrete torsions such that

$$X = \begin{pmatrix} 0 & 0 \\ \frac{1}{3} & 0 \end{pmatrix}$$

we have $\mathcal{A}_L = \mathbf{Z}_9$ and $\mathcal{A}_R = \mathbf{Z}_3 \times \mathbf{Z}_3$, so that the chiral algebras obviously need not be isomorphic. Furthermore, in this case $\mathcal{C}/\mathcal{A}_L = \mathbf{Z}_9$ and $\mathcal{C}/\mathcal{A}_R = \mathbf{Z}_3 \times \mathbf{Z}_3$ are not isomorphic either, but $\mathcal{H}/\mathcal{A}_L = \mathbf{Z}_3$ and $\mathcal{H}/\mathcal{A}_R = \mathbf{Z}_3$ are. The latter isomorphism is in fact a general feature.

To see this note that we are dealing with two kinds of groups: A current subgroup \mathcal{H} is a set of (equivalence classes of) currents that closes under fusion. The corresponding charge subgroup \mathcal{H}^* is the additive group of charges modulo 1 that are allowed by \mathcal{H} . These two groups are of course isomorphic. The matrix X maps $\mathcal{H}/\mathcal{A}_R$ linearly onto a subgroup of \mathcal{H}^* , whereby the group structure is obviously preserved. Now if $q = X\beta$ is in the image of X , then $\lambda q = \lambda X\beta = 0$ for all $\lambda \in \mathcal{A}_L$. Hence $\text{Im } X$ is a subgroup of $(\mathcal{H}/\mathcal{A}_L)^*$, and $\mathcal{H}/\mathcal{A}_R$ must be isomorphic to a subgroup of $\mathcal{H}/\mathcal{A}_L$. Replacing X by X^T we also have the opposite inclusion, and thus the two quotients must be isomorphic.

We can also reverse the argument: Given a subgroup \mathcal{H} of the center and two subgroups $\mathcal{A}_L, \mathcal{A}_R$ of \mathcal{H} with isomorphic quotients, we can choose any isomorphism between $\mathcal{H}/\mathcal{A}_R$ and $(\mathcal{H}/\mathcal{A}_L)^*$ to define a unique matrix X that is appropriately quantized and

has \mathcal{A}_R and \mathcal{A}_L as its kernels. In practice, however, this is of little use since one always starts with a definite monodromy matrix R . A particular extension thus can occur if and only if $X + X^T$ equals R for at least one of the isomorphisms.

Given two subgroups \mathcal{A}_L and \mathcal{A}_R of equal order, the requirement that there is any group \mathcal{H} such that $\mathcal{H}/\mathcal{A}_L$ and $\mathcal{H}/\mathcal{A}_R$ are isomorphic is, of course, rather trivial. But as we start with a given center \mathcal{C} we get, in general, at least a restriction on the subgroups that can possibly produce a given heterotic combination. For some centers we even get more: Consider, for example, $\mathcal{C} = \mathbf{Z}_{p^A} \times \mathbf{Z}_p \times \mathbf{Z}_p$, where p is a prime number. It is obvious that we can never get a modular invariant with $\mathcal{A}_L = \mathbf{Z}_{p^2}$, $\mathcal{A}_R = \mathbf{Z}_p \times \mathbf{Z}_p$ and $\mathcal{A}_L \cap \mathcal{A}_R = 1$, because the smallest group that contains \mathcal{A}_L and \mathcal{A}_R with isomorphic quotients is $\mathbf{Z}_{p^2} \times \mathbf{Z}_{p^2} \times \mathbf{Z}_p$, which is not a subgroups of \mathcal{C} .

Unfortunately we do not know a simple general rule that covers all possible left-right combinations of algebras, like the rule formulated in [6] for $(\mathbf{Z}_p)^k$. It seems that the general problem is rather complicated. The rules explained above are necessary, but not sufficient, and are in any case of some help. We emphasize, however, that although there is no simple rule, all possibilities can be enumerated easily by generating all matrices X .

3.2. AUTOMORPHISMS

If the kernels of X are trivial, it corresponds to an automorphism. In [5] automorphisms were defined by means of an integral matrix μ_{ij} , defined modulo N_j and satisfying $N_i \mu_{ij} = 0 \pmod{N_j}$. (Note that the transpose of μ has the same quantization as the matrices r and χ .) Ordinary matrix multiplication closes on such a set of matrices: if μ_1 and μ_2 satisfy the quantization rule, so does $\mu_1 \mu_2$. We can define an identity matrix: $\delta_{ij} = 1 \pmod{N_j}$ if $i = j$ and $\delta_{ij} = 0 \pmod{N_j}$ if $i \neq j$ (note that the definition of this element is *not* symmetric in i and j). Furthermore we can define an inverse for a subset of all matrices μ . The kernel of μ is defined as the set of vectors $\vec{\beta}$ satisfying $\sum_i \beta_i \mu_{ij} = 0 \pmod{N_j}$. The matrix μ_{ij} specifies for each basic charge i by how much the automorphism moves it in the direction j . The kernel corresponds thus to those charges that are not moved at all. This definition of the kernel is identical to the one given above for X provided that one writes it in terms of the matrix $\frac{1}{N} \mu^T$, which has the same quantization rule as X .

It is easy to show that the left and right inverse are the same, that an inverse exists if and only if the kernel is trivial, and that this inverse is unique modulo the usual periodicities. The matrix μ^T belongs to a similar, but different set on which the ‘mirror image’ of all these properties holds. In particular, if μ is invertible so is μ^T , and its inverse is the transpose of that of μ .

Not all such matrices μ define an automorphism, but only those that satisfy the equation [6]

$$\mu \frac{1}{N} + \frac{1}{N} \mu^T + \mu R \mu^T = 0 \pmod{1} .$$

If μ is invertible we can multiply from the left with μ^{-1} and from the right with μ^{T-1} to get

$$\frac{1}{N} \mu^{T-1} + \mu^{-1} \frac{1}{N} + R = 0 \pmod{1} .$$

Comparing this with the equation for X we see that it is satisfied by

$$\frac{1}{N} \mu^{T-1} = -X = -\frac{1}{N} \chi ,$$

or $\mu = -\chi^{T-1}$. This can also be derived directly from (2.9) and the definition of μ . Note that χ is indeed always invertible if it represents an automorphisms.

If μ were always invertible this would establish a one-to-one relationship between the two descriptions. However, if μ is not invertible, it has a non-trivial kernel, and hence there are charges on which the automorphism acts trivially. These charges can always be removed by restricting to a subgroup. By choosing a small enough subgroup, one can always make μ invertible. This reflects a difference in philosophy between the orbifold method used here and the approach of [5] and [6]: The former works always within subgroups of the center, whereas the latter constructs all invariants directly within the full center. Taking this into account, we find thus an exact one-to-one mapping between the pure automorphisms in both formalisms. Since for automorphisms both formalisms yield the complete set of solutions to their respective conditions for any center, this provides additional evidence for our conjecture that those conditions are in fact equivalent.

3.3. PRODUCTS OF INVARIANTS

If one multiplies two matrices M_1 and M_2 that define a modular invariant, the result is obviously modular invariant as well. Furthermore the product is positive. If M_1 and M_2 are both simple current invariants, so is the product. In general, the matrix element M_{00} of the product matrix is not equal to 1, but at least for simple current invariants this is always just an overall factor, which can be divided out. Hence the complete set of positive simple current invariants must close under matrix multiplication.

The physical meaning of matrix multiplication in terms of conformal field theory is not clear in general, and indeed it may well happen that for exceptional invariants the product does not correspond to a meaningful CFT (products of simple current invariants can usually be interpreted as consecutive orbifold twists). It is also not obvious that higher loop conditions are automatically satisfied for such a product, or even how to formulate them. As we discussed at the end of section 2, positivity and higher genus invariance are probably equivalent in the present context. If that is true, our set of invariants should close under matrix multiplication. Checking closure is in any case a good test of this conjecture.

This means that there must be an associative product operation for pairs $(\mathcal{H}, X_{\mathcal{H}})$, where $R_{\mathcal{H}} = X_{\mathcal{H}} + X_{\mathcal{H}}^T$ has to be the monodromy matrix for the subgroup \mathcal{H} of the center. Let us first consider the simplest case of two invariants $M(\mathcal{H}, X_{\mathcal{H}})$ and $M(\mathcal{H}, Y_{\mathcal{H}})$ corresponding to the same subgroup. We then have[★]

$$M(X)_{[\vec{\beta}]a, [\vec{\gamma}]a} = \delta^1(\vec{Q}(a) + X^T \vec{\beta} + X \vec{\gamma}),$$

where $\delta^1(\vec{q})$ is 1 if all components of the vector \vec{q} are integer. Thus

$$\begin{aligned} (M(X)M(Y))_{a, [\vec{\gamma}]a} &= \sum_{\vec{\beta}} \delta^1(\vec{Q}(a) + X \vec{\beta}) \delta^1(\vec{Q}(a) + (R - Y) \vec{\beta} + Y \vec{\gamma}) \\ &= \sum_{\vec{\beta}} \delta^1(\vec{Q}(a) + X \vec{\beta}) \delta^1(Y \vec{\gamma} - (X + Y - R) \vec{\beta}). \end{aligned} \quad (3.2)$$

If the antisymmetric matrix $\Delta = X + Y - R$ in eq. (3.2) is invertible we can solve for $\vec{\beta}$

★ For notational simplicity we assume that a is not a fixed point.

and obtain the product invariant $M(X * Y) = M(X)M(Y)$ with

$$X * Y = X(X + Y - R)^{-1}Y. \quad (3.3)$$

There are two reasons why Δ may not be invertible. First of all there may be vectors β that are in the kernel of Δ and also in the kernel of X . Clearly the summand in (3.2) is totally independent of such vectors, and performing the sum just gives an overall factor independent of \vec{Q} and $\vec{\gamma}$. This factor is

$$N(X, Y) = |\text{Ker } X \cap \text{Ker } \Delta| = |\text{Ker } X \cap \text{Ker } Y^T| = |\mathcal{A}_R(X) \cap \mathcal{A}_L(Y)|.$$

A trivial example of this situation is to multiply two identical left-right symmetric pure chiral algebra extensions ($X = Y = R = 0$ within \mathcal{H}).

In addition it may happen that the product $M(X)M(Y)$ cannot be written in the form $M(Z)$ for any Z with the same subgroup \mathcal{H} , but only for a subgroup of \mathcal{H} . This is the case exactly if the last δ -function in (3.2) constrains the possible values of $\vec{\gamma}$, i.e. if the image of Δ does not contain the image of Y . This can be understood as a partial cancellations of the automorphism actions of X and Y . The simplest example is $Y = X^T$, with both X and Y invertible, so that X and Y define mutually inverse automorphisms. Note that the second δ -function implies a charge independent restriction on $\vec{\gamma}$, which can certainly not be of the form (2.9). The reduced subgroup \mathcal{H}' is thus expected to be the set of vectors $\vec{\gamma}$ for which the equation $Y\vec{\gamma} = \Delta\vec{\beta}$ has at least one solution $\vec{\beta}$. If there are more solutions they differ by vectors in $\text{Ker } \Delta$. We can now perform the sum over $\vec{\beta} \in \text{Ker } \Delta \cap \text{Ker } X$ to obtain

$$N(X, Y) \sum_{\vec{\beta} \bmod \text{Ker } \Delta} \sum_{\vec{\beta}_0 \in \text{Ker } \Delta \bmod \text{Ker } X} \delta^1(\vec{Q}(a) + X\Delta^{-1}Y\vec{\gamma} + X\vec{\beta}_0)\delta^1(Y\vec{\gamma} - \Delta\vec{\beta}), \quad (3.4)$$

where $\Delta^{-1}Y\vec{\gamma}$ is a formal notation for one representative of the solutions to $Y\vec{\gamma} = \Delta\vec{\beta}$. Of course this set of representatives can be chosen such that it depends linearly on $\vec{\gamma}$. The second δ -function restricts \mathcal{H} to \mathcal{H}' for the currents $\vec{\gamma}$. For a fixed $\vec{\gamma}$, the constraint of the first δ -function in (3.4) has at most one solution $\vec{\beta}_0$, since we have already summed over $\text{Ker } \Delta \cap \text{Ker } X$. The sum over $\vec{\beta}_0$ effectively reduces the number of δ -restrictions. Hence it must be possible to write the first δ -function as $\delta^1(P(\vec{Q}(a) + Z\vec{\gamma}))$ for some linear map Z , where P projects to a subgroup $\mathcal{H}'' = \text{Im } P$.

It is easy to see that $\mathcal{H}'' \supset \mathcal{H}'$: Consider a vector $\vec{v} \in \mathcal{H}'$. By definition of \mathcal{H}' there exists some vector $\vec{\beta}'$ with $\vec{v}Y^T = \vec{\beta}'\Delta^T$, therefore any vector $\vec{v} \in \mathcal{H}'$ is orthogonal to the term $X\vec{\beta}_0$ in (3.4),

$$\vec{v} \cdot X\vec{\beta}_0 = \vec{v} \cdot Y^T\vec{\beta}_0 = \vec{\beta}' \cdot \Delta^T\vec{\beta}_0 = -\vec{\beta}' \cdot \Delta\vec{\beta}_0 = 0 ,$$

where we used $\vec{\beta}_0 \in \text{Ker } \Delta$ and $\Delta = X - Y^T = -\Delta^T$. It follows that the projection of the δ -function argument on \mathcal{H}' is consistent with the sum over $\vec{\beta}_0$, so that indeed \mathcal{H}'' must contain \mathcal{H}' .

It then only remains to show that the dimensions of \mathcal{H}' and \mathcal{H}'' are equal. This can be concluded from the one-loop modular invariance of the product invariant (3.4), as we show in the Appendix, or from a direct computation: The dimension of \mathcal{H}/\mathcal{H}' is given by the number of restrictions on $\vec{\gamma}$ that come from the second δ -function, which is $|\text{Im } Y|/|\text{Im } Y \cap \text{Im } \Delta|$. The number $|\mathcal{H}/\mathcal{H}''|$ of different charges that are allowed for a given $\vec{\gamma}$, on the other hand, is equal to $|\text{Ker } \Delta|/|\text{Ker } \Delta \cap \text{Ker } Y^T|$, *i.e.* the number of vectors $\vec{\beta}_0 \in \text{Ker } \Delta \text{ mod Ker } X$. Now we already know from section 3.1 that $\text{Im } Y = (\mathcal{H}/\text{Ker } Y^T)^*$ for any matrix Y satisfying the quantization conditions, and therefore also $\text{Im } Y \cap \text{Im } \Delta = (\mathcal{H}/\langle \text{Ker } Y^T, \text{Ker } \Delta \rangle)^*$, where $\langle A, B \rangle$ denotes the span of A and B . Altogether we find

$$\frac{|\mathcal{H}|}{|\mathcal{H}'|} = \frac{|\text{Im } Y|}{|\text{Im } Y \cap \text{Im } \Delta|} = \frac{|\mathcal{H}|/|\text{Ker } Y^T|}{|\mathcal{H}|/|\langle \text{Ker } Y^T, \text{Ker } \Delta \rangle|} = \frac{|\text{Ker } \Delta|}{|\text{Ker } Y^T \cap \text{Ker } \Delta|} = \frac{|\mathcal{H}|}{|\mathcal{H}''|}.$$

Hence we have show now that $\mathcal{H}' = \mathcal{H}''$, and therefore

$$M(\mathcal{H}, X)M(\mathcal{H}, Y) = |\mathcal{A}_R(X) \cap \mathcal{A}_L(Y)| M(\mathcal{H}', Z_{\mathcal{H}'}) \quad (3.5)$$

for some matrix $Z_{\mathcal{H}'}$. Roughly speaking, this matrix is nothing but $X*Y$ defined earlier, but with a choice of representatives for the inverse map Δ^{-1} plus a projection to \mathcal{H}' . Using once more that the product is in any case one-loop modular invariant we can show (see Appendix A) that Z satisfies $Z + Z^T = R$ in \mathcal{H}' . The subgroup \mathcal{H}' can formally be written as $\mathcal{H}' = \text{Im}(Y^{-1}\Delta) \subseteq \mathcal{H}$, where Y^{-1} is a one-to-one map from $\text{Im } Y$ to $\mathcal{H} \text{ mod Ker } Y$ (in general it is thus not a well-defined map from \mathcal{H} to \mathcal{H} , but the image of $Y^{-1}\Delta$ is nevertheless a well-defined subset of \mathcal{H}). With a similar notation, the extended chiral algebras are given by $\mathcal{A}'_R = Y^{-1}Y^T\mathcal{A}_R(X)$, where of course $\mathcal{A}_R(Y) \subseteq \mathcal{A}'_R \subseteq \mathcal{H}'$, and by $\mathcal{A}'_L = (X^T)^{-1}X\mathcal{A}_L(Y) \supseteq \mathcal{A}_L(X)$.

The formula (3.5) for the overall multiplicity in the product invariant is of course correct also in the general case of two different effective subgroups \mathcal{F} and \mathcal{G} . In that case, the effective subgroup \mathcal{H}' of the product invariant is obviously a subgroup of the group \mathcal{H} that is spanned by \mathcal{F} and \mathcal{G} . It is thus convenient to refer to a basis of \mathcal{H} , in terms of which bases of \mathcal{F} , \mathcal{G} and $\mathcal{I} = \mathcal{F} \cap \mathcal{G}$ can be defined by matrices F , G and $I = I_F F = I_G G$ as explained in Appendix A. Any vector $\vec{\gamma} \in \mathcal{H}$ can be written as

$$\vec{\gamma} = F^T \vec{\gamma}_F + G^T \vec{\gamma}_G .$$

Of course if \mathcal{F} and \mathcal{G} have a non-trivial overlap there is more than one way to make this decomposition. Now consider a product $\sum_{\vec{\beta}} M_{a, [\vec{\beta}]a} M_{[\vec{\beta}]a, [\vec{\gamma}]a}$ where the first matrix is defined in \mathcal{F} and the second in \mathcal{G} . The sum over $\vec{\beta}$ is *a priori* over all of \mathcal{H} , but is restricted to the subset $\vec{\beta}_G = 0 \bmod \mathcal{I}$ by the first matrix and $\vec{\beta}_F = \vec{\gamma}_F \bmod \mathcal{I}$ by the second one, *i.e.* $\vec{\beta} = F^T \vec{\gamma}_F + I^T \vec{\beta}_I$ because the matrix elements can both be non-vanishing only if $[\vec{\beta}] \in \mathcal{F}$ and $[\vec{\gamma} - \vec{\beta}] \in \mathcal{G}$. Hence we are left with a sum over \mathcal{I} only. Using again the notation of Appendix A we get now for the product

$$\begin{aligned} & \sum_{\vec{\beta}_I \in \mathcal{I}} \delta^1 \left(F \vec{Q}(a) + X_F (\vec{\gamma}_F + I_F^T \vec{\beta}_I) \right) \\ & \delta^1 \left(G \vec{Q}(a) + G R (F^T \vec{\gamma}_F + I^T \vec{\beta}_I) + Y_G (\vec{\gamma}_G - I_G^T \vec{\beta}_I) \right) , \end{aligned} \quad (3.6)$$

where \vec{Q} and R are defined with respect to the basis of \mathcal{H} , whereas X_F and Y_G are given in the bases of \mathcal{F} and \mathcal{G} , respectively. Equating the \mathcal{I} -charges $IQ(a)$ as constrained by the two δ -functions, we get a factor

$$\delta^1 \left(I_F X^T \vec{\gamma}_F + I_G Y \vec{\gamma}_G - (X_I + Y_I - R_I) \vec{\beta}_I \right) , \quad (3.7)$$

where $X_I = I_F X I_F^T$, $Y_I = I_G Y I_G^T$ and $R_I = I R I^T$. This is the analog of the last δ -function in (3.2) and again determines the subgroup \mathcal{H}' of allowed values $\vec{\gamma}$ (the ambiguity in the definition of $\vec{\gamma}_F$ and $\vec{\gamma}_G$ just amounts to a shift in $\vec{\beta}_I$ and thus is irrelevant, as it should be). Invertibility of $\Delta_I = X_I + Y_I - R_I$ is again a sufficient condition for having the full effective subgroup $\mathcal{H}' = \mathcal{H}$. If Δ_I is not invertible (3.7) determines the subgroup \mathcal{H}' on which the product operates non-trivially exactly as before. It consists of all the $\vec{\gamma}$'s for which the δ -restriction has a solution, but it seems hard to give a more explicit description.

To show that the product is again of the form (2.9) we can now proceed just as in the previous case. First we sum over those $\vec{\beta}_I$ on which the summand doesn't depend at all, namely $\vec{\beta}_I \in \text{Ker } X_F I_F^T \cap \text{Ker } Y_G^T I_G^T = \text{Ker } X \cap \text{Ker } Y^T$. Then we solve (formally at least) (3.7) which leaves us with a sum over $\text{Ker } \Delta_I \cap (\text{Ker } X \cap \text{Ker } Y^T)$. For each charge, there can be at most one term in that sum that contributes. Just as before, the sum over $\vec{\beta}_I \in \Delta_I$ restricts the first δ -function to a group $\mathcal{H}'' \supset \mathcal{H}'$. To see this consider $\vec{v} \cdot \vec{Q} = \vec{v}_F \cdot F\vec{Q} + \vec{v}_G \cdot G\vec{Q}$ for all $\vec{v} \in \mathcal{H}'$. Using (3.6) one can express this in terms of $\vec{\gamma}$ and $\vec{\beta}$, and then using (3.7) one can show that the $\vec{\beta}$ -dependence cancels. The rest is exactly as before, because we now have $|\mathcal{H}/\mathcal{H}'| = |\text{Im } Y_I|/|\text{Im } Y_I \cap \text{Im } \Delta_I|$, which is equal to $|\mathcal{H}/\mathcal{H}''| = |\text{Ker } \Delta_I|/|\text{Ker } \Delta_I \cap \text{Ker } Y_I^T|$.

Thus, although we could derive an explicit product formula only in case of equal subgroups, we have been able to give a general proof of closure under matrix multiplication.

4. Discussion

The central result of our investigation is that all simple current modular invariants for arbitrary centers are of the form

$$M_{a, [\vec{\beta}]a} = \text{Mult}(a) \prod_i \delta^1(Q_i(a) + X_{ij}\beta_j) \quad (4.1) ,$$

where the simple currents $[\vec{\beta}]$ are in a subgroup \mathcal{H} of the (effective) center and where $R_{\mathcal{H}} = X + X^T$ is the monodromy matrix for that subgroup with $X_{ij}N_j$ integer. The number of these invariants is given by the sum of the number of properly quantized antisymmetric matrices, which is just the usual number of allowed discrete torsions, over all subgroups of the center. For symmetry groups that are generated by monodromy charges of simple currents this shows that, in general, the usual orbifold construction indeed gives all simple current invariants. There is, however, a complication if there are odd (off-diagonal) entries in the monodromy matrix (for group factors of even order). Then discrete torsion is a necessity rather than a possibility, and there is no (genuine) orbifold with a modular invariant partition function.

Our procedure can be summarized as follows

1. We start with an orbifold inspired ansatz for a basis of invariants.
2. We use the fact that (according to [6]) $M_{a, [\vec{\beta}]a}$ can only depend on the charge of a (provided S satisfies the regularity conditions of [6]).
3. Fourier analysis of any such invariant shows that it can be expressed as a complex linear combination of the basis of point 1.
4. Modular invariance and factorization restricts the complex coefficients to certain phases. Modular invariance is used at one and two loops. The two-loop transformation is an assumption, since for a general CFT's we do not know explicit formulas for the two-loop characters and their transformations.
5. The resulting invariants are positive. In all cases where a comparison is possible, they coincide with the already classified simple current invariants. In all cases where orbifolding works straightforwardly, they coincide with the set of orbifold invariants with discrete torsion. In all other cases, it is in any case true that for any subgroup \mathcal{H} of the center, after a redefinition of the phases the remaining freedom is exactly like the one for discrete torsions of orbifolds

The above formula is extremely useful in discussing some general features of simple current invariants, which we did in section 3. Obviously, the kernels of X^T and X define the left and right extensions of the chiral algebras. Some conditions on monodromies for heterotic combinations are immediate from $X + X^T = R$. We also showed that the quotients of the effective subgroup by the left/right algebra extensions are isomorphic. The chiral algebra is not enlarged if and only if X is invertible. Then X defines an automorphism and we recover the results of a classification of that type of invariants [5]. Finally we considered multiplication of our modular invariants, which are functions of subgroups of the center and matrices X . Although we did not present an explicit formula for arbitrary products we did prove that our set of solutions closes under matrix multiplication. This is an important check of the two-loop assumption mentioned above.

Our results are also very useful from a practical point of view. The structure of the final result is very simple so that one can easily search for particular features, or implement a fast code for systematic constructions of all different invariants. Redundancies still could arise in case of permutation symmetries of tensor products of identical conformal field theories, but these can now straightforwardly be eliminated with the

same methods as for orbifolds [4]. Along these lines, however, an important problem remains to extend our classification to cases where exceptional invariants are known and, for example, find a characterization of all invariants that can be written as products of simple current invariants and the exceptional one(s).

APPENDIX A

In this appendix we show how to write a modular invariant defined with respect to a subgroup \mathcal{H} in terms of charges and currents of a group $\mathcal{G} \supset \mathcal{H}$.

Consider a set of N currents J_i generating a group \mathcal{G} and another set of M currents J_a generating a subgroup \mathcal{H} . Then there is a $M \times N$ integral matrix H such that

$$J_a = \prod_i J_i^{H_{ai}} \quad (\text{A.1})$$

The charges Q_a with respect to J_a are related to the \mathcal{G} -charges Q_i as

$$Q_a = \sum_i H_{ai} Q_i. \quad (\text{A.2})$$

Currents in \mathcal{H} are of the form $[\vec{\alpha}] = \prod_a J_a^{\alpha_a}$. In the original basis one therefore has $\alpha_i = \sum_a \alpha_a H_{ai}$. Denoting charges Q_a as \vec{Q}_H and similarly for Q_i , α_a and α_i we can summarize this as

$$\begin{aligned} \vec{Q}_H &= H \vec{Q}_G \\ \vec{\alpha}_G &= H^T \vec{\alpha}_H. \end{aligned} \quad (\text{A.3})$$

The R -matrix for the currents of \mathcal{H} is given by $R_H = H R H^T$. An invariant of the form (2.9) defined within \mathcal{H} and with a matrix X_H can be written in the basis of G in the following way:

$$M_{a, [\vec{\beta}]_a} = \delta^1(H \vec{Q} + X_H H^T \vec{\beta}) \delta^1(U \vec{\beta}). \quad (\text{A.4})$$

Here \vec{Q} and $\vec{\beta}$ are defined with respect to the \mathcal{G} -basis (note that in general there does not exist a representation $X_H = H X H^T$ with a matrix X satisfying the quantization conditions of \mathcal{G}). As usual in the δ -functions a product over all components of the \mathcal{G} basis is implicit. The matrix U is a rational matrix, quantized like X , with the property that $\text{Ker } U = \mathcal{H}$. It ensures that M has no matrix elements related to currents that are not in \mathcal{H} .

If we restrict $\vec{\beta}$ to \mathcal{H} as required by the second δ -function we may write the argument of the first δ as $H\vec{Q} + X_H\vec{\beta}_H = Q_H + X_H\vec{\beta}_H$, which indeed is the correct expression within \mathcal{H} .

It is undoubtedly possible to prove that any partition function of the form (A.4) is one-loop modular invariant provided that $\text{Im } H = \text{Ker } U \equiv \mathcal{H}$ and that $X_H + X_H^T = R_H$ (*i.e.* no higher loop considerations are required here). We only need this fact in cases where it is already known that $\text{Ker } U \subset \text{Im } H$. To prove it, we make use of a result of [6] that for any simple current invariant the sum over a row (or column) of M_{ab} is equal to zero if the corresponding field is non-local with respect to the left (or right) chiral algebra, and otherwise it is equal to the number of currents in the chiral algebra. This implies that if we view $M_{a, [\vec{\alpha}]a}$ as a function of the charge rather than a , then the sum over all currents and all allowed charges is always equal to $|\mathcal{H}|$ for any group \mathcal{H} that contains all currents $\vec{\alpha}$ for which some $M_{a, [\vec{\alpha}]a} \neq 0$. [To see this note that it is manifestly true if there is no chiral algebra; if there is a chiral algebra with N currents, then the number of allowed charges is reduced by a factor N , but each of the surviving charges now contributes N times to the sum].

Now the first factor in (A.4) satisfies this sumrule for any X_H . If $\text{Ker } U$ is smaller than $\text{Im } H$ this means that the second δ imposes extra restrictions on \mathcal{H} -currents, so that some matrix elements are put to zero. But then clearly the sum rule is not satisfied for \mathcal{H} and the result cannot be modular invariant.

If one makes the ansatz (A.4) with $\text{Im } H = \text{Ker } U$, the restriction on X follows already from T -invariance alone. If $M_{a, [\vec{\beta}]a}$ is non-zero, it must be true that $h(a) = h([\vec{\beta}]a) \text{ mod } 1$. This yields the condition $h(\vec{\beta}) = \vec{\beta} \cdot \vec{Q}(a) \text{ mod } 1$, which within \mathcal{H} (and with \mathcal{H} -indices omitted) can be written as

$$-\frac{1}{2}\vec{\beta} \cdot R \cdot \vec{\beta} = -\vec{\beta} \cdot X \cdot \vec{\beta} \text{ mod } 1 . \tag{A.5}$$

This must be true for any $\vec{\beta}$ since there is always a charge $-X\vec{\beta}$ for which $M_{a, [\vec{\beta}]a}$ is non-zero. It is then clear that (A.5) is equivalent to $X + X^T = R$, with the usual periodicities.

REFERENCES

- [1] A.N. Schellekens and S. Yankielowicz, Nucl. Phys. B327 (1989) 673.
- [2] J. Fuchs, A. Klemm, C. Scheich and M. Schmidt, Phys. Lett. B232 (1989) 317; Ann. Phys. 204 (1990) 1.
- [3] A.N. Schellekens and S. Yankielowicz, Nucl. Phys. B330 (1990) 103; Tables Supplement CERN-TH.5440T/89 (1989) (unpublished).
- [4] M. Kreuzer and H. Skarke, ADE string vacua with discrete torsion, CERN-TH.6931/93 preprint
- [5] B. Gato-Rivera and A.N. Schellekens, Nucl. Phys. B353 (1991) 519.
- [6] B. Gato-Rivera and A.N. Schellekens, Commun. Math. Phys. 145 (1992) 85
- [7] A.N. Schellekens and S. Yankielowicz, Int. J. Mod. Phys. A5 (1990) 2903.
- [8] A.N. Schellekens and S. Yankielowicz, Phys. Lett. B227 (1989) 387.
- [9] C. Vafa, Nucl. Phys. B273 (1986) 592.
- [10] A.N. Schellekens, Phys. Lett. B244 (1990) 255.
- [11] K. Intriligator, Nucl. Phys. B332 (1990) 541.