

## Seventy Relatives of the Monster Module<sup>\*</sup>

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### Astract

Recent work on the classification of conformal field theories with one primary field (the identity operator) is reviewed. The classification of such theories is an essential step in the program of classification of all rational conformal field theories, but appears impossible in general. The last managable case, central charge 24, is considered here. We found a total of 71 such theories (which have not all been constructed yet), including the monster module. The complete list of modular invariant partition functions has already appeared elsewhere [1]. This paper contains an easily readable account of the method, as well as a few examples and some comments.

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## 1. Introduction

The simplest conformal field theories, from the point of view of the modular group or the fusion rules, are those with just one primary field with respect to some integer spin chiral algebra. It is elementary to show that unitary conformal field theories of this kind must have a central charge that is a multiple of 8. They transform according to a one-dimensional representation of the modular group with  $S = 1$  and  $T$  a cubic root of unity. Furthermore, if the central charge is a multiple of 24 the single character is modular invariant by itself, and can be written as a polynomial in the absolute modular invariant  $j$ ,

$$j = \frac{1}{q} + 744 + 196884q + 21493760q^2 + \dots ,$$

with a leading term  $q^{-n}$  if the central charge  $c = 24n$ ; here  $q = e^{2\pi i\tau}$ . Since the character  $\mathcal{X}$  is modular invariant by itself one may consider, instead of the usual “diagonal” CFT with partition function  $\mathcal{X}\mathcal{X}^*$ , a purely chiral conformal field theory with partition function  $\mathcal{X}$ . Such a theory will be called a *meromorphic* conformal field theory, and denoted MCFT.

The classification of these theories is an essential part of the programme of classification of rational conformal field theories, initiated a few years ago. Indeed, one can argue that the entire RCFT classification problem can be embedded in that of classification of MCFT’s, provided that one can show that any RCFT has a *complement*. This is a RCFT with the same number of primary fields and complex conjugate  $S$  and  $T$  matrices. (A complement can easily be constructed for all WZW-models and for all coset theories without field identification fixed points). Then any diagonal RCFT with a modular invariant  $\sum_i \mathcal{X}_i(\mathcal{X}_i)^*$  can be mapped to a meromorphic one with partition function  $\sum_i \mathcal{X}_i\mathcal{X}_i^C$ , where ‘ $C$ ’ denotes the complement.

In any case it is clear that the RCFT classification problem is not solved as long as we cannot even classify the theories with just one primary field. This is bad news, since for  $c \geq 32$  the number of such theories grows so fast with the central charge that listing them is simply impossible. Indeed, for  $c = 32$  the number of such theories is known to be larger than  $8 \times 10^7$ . The problem looks substantially easier for  $c \leq 24$ , and with some (though probably unfounded) optimism one may hope that the information contained in the  $c \geq 32$  theories will never really be needed in practice.

The fact that enumeration is impossible for  $c \geq 32$  may dampen one's enthusiasm for attempting an enumeration for  $c \leq 24$ . Nevertheless, there are indications that the  $c \leq 24$  theories (and in particular those with  $c = 24$ ) are of some intrinsic interest. In physics,  $c = 24$  is special because of the bosonic string, whose transverse dimension is 24; in mathematics the number 24 plays a special role in many contexts, such as the theory of sphere packings or the Monster group (the largest of the sporadic simple finite groups), for which a meromorphic  $c = 24$  theory provides a “natural”  $q$ -graded representation, the “monster module” [2]. One may hope that a list of the CFT-relatives of the monster module places this object in a new and interesting context. These admittedly rather vague motivations will probably turn out to be the most important ones for attempting to classify the meromorphic  $c = 24$  CFT's. A somewhat more practical motivation is that a listing of such theories has enabled us to complete another classification problem, that of ten-dimensional heterotic strings [3]. Yet another unsolved problem about which we have learned a few interesting new facts (without solving it, though) is that of the classification of Kac-Moody modular invariants. Several new non-diagonal invariants of simple Kac-Moody algebras were found that are ‘highly exceptional’: they are not simple current invariants or conformal embeddings, nor are they related to such invariants by rank-level duality.

A large class of MCFT's can be constructed by taking  $8n$  free bosons with momenta quantized on an even self-dual lattice. This gives 1, 2 and 24 [4,5] distinct theories for  $c = 8, 16$  and 24 respectively (and more than  $8 \times 10^7$  for  $c = 32$ ). This class can be enlarged by a  $\mathbf{Z}_2$  orbifold twist, using the symmetry that sends every boson  $X$  to  $-X$  [6,7]. This gives back the same  $E_{8,1}$  theory for  $c = 8$ , and maps the two  $c = 16$  MCFT's ( $(E_{8,1})^2$  and  $D_{16,1}([0] + [s])$ ) to one another (the argument denotes the conjugacy classes that appear). The result is more interesting when this twist is applied to the Leech lattice and the 23 Niemeier lattices: The former gives a new MCFT, the monster module, while from the latter one gets other Niemeier lattices in 9 cases, and new MCFT's in the 14 remaining cases [7]. Altogether this gives us thus 1, 2 and 39 MCFT's for  $c = 8, 16$  and 24.

Clearly there are other orbifold twists one might consider, but it becomes rather difficult to prove the consistency of the resulting theories. More importantly, even an exhaustive classification of all orbifolds of known theories is not sufficient to show that the result is complete. The same is true for other kinds of constructions. For example,

one could study all tensor products of Kac-Moody algebras with total central charge  $8n$ , and determine their meromorphic modular invariants. Even though this is a finite problem, there is no guarantee that the answer will be complete, since in general only part of the central charge will be saturated by (non-abelian) Kac-Moody algebras. As soon as one allows rational  $U(1)$  factor the problem is not finite anymore, and it gets still worse if one adds factors without spin-1 currents (e.g. coset theories). In any case, it was already known for some time that the number of MCFT's with  $c = 24$  is larger than the 39 mentioned so far: two additional candidates were presented in [8], one of which can certainly be constructed explicitly.

While explicit constructions approach the set of solutions from below it is possible in some cases to limit the set of solutions from above, i.e. to derive necessary rather than sufficient conditions for the existence of solutions. An example is the set of  $c = 8$  and  $c = 16$  solutions. Any such theory can be used to build a supersymmetric heterotic string theory in 10 dimensions. It can be shown in general ([9], see also [10]) that modular invariance of such a theory implies that all gauge and gravitational anomalies of the resulting field theory must factorize à la Green-Schwarz [11]. But all possibilities for such anomaly cancellations are known [11] [12], and this immediately reduces the  $c = 8$  and  $c = 16$  theories to  $(E_{8,1})^2$  and  $D_{16,1}$ . There cannot exist more such theories, and since both can be constructed using self-dual lattices, there are no fewer either.

It turns out that a similar argument can be applied, with a considerably larger effort, to the  $c = 24$  theories [1]. Beyond  $c = 24$  the nature of the problem changes drastically, and these methods become useless, not just in practice but even in principle. The basic idea is to write down a character valued partition function for a given  $c = 24$  theory analogous to similar functions introduced in [9] for the chiral sector of heterotic strings. This function generalizes the ordinary one-loop partition function

$$P(q) = \sum_{n=1}^{\infty} d_n q^n ,$$

by replacing the multiplicities  $d_n$  by Chern-characters of the representation at each level. Thus we get

$$P(q, F) = \sum_{n=1}^{\infty} \text{Tr} e^F q^n .$$

Here  $F$  is some representation matrix of a semi-simple Lie-algebra, in the representation

of the  $n^{\text{th}}$  level.

To write down such a partition function we must have a Lie algebra that organizes the levels according to its representations. This happens if the theory has a set of spin-1 currents, which necessarily close into a Kac-Moody algebra, plus possibly some  $U(1)$ -currents [13]. Note that at this point we are certainly not assuming that these algebras saturate the central charge.

In general, the Kac-Moody algebra consists of several simple factors, and the partition function can be expressed in terms of the characters  $\mathcal{X}_{i_\ell}^\ell$  of the  $\ell^{\text{th}}$  factor and an unknown function without spin-1 contributions:

$$P(q, \vec{F}_1, \dots, \vec{F}_L) = \sum_{i_1, \dots, i_L} \mathcal{X}_{i_1}^1(q, \vec{F}_1) \dots \mathcal{X}_{i_L}^L(q, \vec{F}_L) \mathcal{X}_{i_1, \dots, i_L}(q) .$$

Here  $\vec{F}_\ell$  denotes the decomposition of  $F$  with respect to a basis of Lie-algebra generators  $J_0^a$  in each of the simple factors:  $F = \sum_a F^a J_0^a = \vec{F} \cdot \vec{J}_0$ . Now we wish to make use of the modular transformation properties of the theory. For  $c = 8n$   $P$  transforms with  $S = 1$  and  $T = e^{-2\pi i n/3}$ . It is convenient to multiply  $P$  with  $\eta(q)^{8n}$  to remove the phase in the  $T$  transformation. Then the function  $\hat{P}(q, 0, \dots, 0) = [\eta(q)]^{8n} P(q, 0, \dots, 0)$  transforms as a modular function of weight  $4n$ . Furthermore we know the transformation properties of the Kac-Moody characters [14]

$$\begin{aligned} \tau \rightarrow \tau + 1 & : \quad \mathcal{X}_i(\tau + 1, \vec{F}) = e^{2\pi i(h_i - c/24)} \mathcal{X}_i(\tau, \vec{F}) \\ \tau \rightarrow -\frac{1}{\tau} & : \quad \mathcal{X}_i\left(-\frac{1}{\tau}, \frac{\vec{F}}{\tau}\right) = e^{-\frac{i}{8\pi\tau} \frac{k}{g} \text{Tr}_{\text{adj}} F^2} S_{ij} \mathcal{X}_j(\tau, \vec{F}) , \end{aligned} \tag{1.1}$$

where

$$\mathcal{X}_i(\tau, \vec{F}) = \text{Tr}_i e^{\vec{F} \cdot \vec{J}_0} e^{2\pi i \tau (L_0 - c/24)} , \tag{1.2}$$

with the trace evaluated over the positive norm states of the representation “ $i$ ”. In (1.1)  $g$  is the dual Coxeter number of the Kac-Moody algebra, and we have traded  $q$  for  $\tau$ , with  $q = e^{2\pi i \tau}$ . The trace in (1.1) is evaluated in the adjoint representation.\*

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\* Conventions:  $J_0^a$  is Hermitian,  $f_{abc} f_{abe} = 2g \delta_{ce}$ . For  $U(1)$  factors the adjoint representation is not suitable, but one can use any non-trivial representation, provided that  $k/g$  is replaced by some normalization  $N$ . This will be implicitly assumed in the following.

Using (1.1) and the fact that the  $\hat{P}$  must be a modular function for  $\vec{F} = 0$ , we can derive how it must transform when  $\vec{F} \neq 0$ . One finds

$$\hat{P}\left(\frac{a\tau + b}{c\tau + d}, \frac{\vec{F}}{c\tau + d}\right) = \exp\left[\frac{-ic}{8\pi(c\tau + d)}\mathcal{F}^2\right] (c\tau + d)^{4n} \hat{P}(\tau, \vec{F}), \quad (1.3)$$

where we have defined

$$\mathcal{F}^2 = \sum_{\ell} \frac{k_{\ell}}{g_{\ell}} \text{Tr}_{\text{adj}} F_{\ell}^2.$$

To analyse the consequences of these transformation properties we need the Eisenstein functions, for convenience normalized as follows

$$\begin{aligned} E_2(q) &= 1 - 24 \sum_{n=1}^{\infty} \frac{nq^n}{1 - q^n}, \\ E_4(q) &= 1 + 240 \sum_{n=1}^{\infty} \frac{n^3 q^n}{1 - q^n}, \\ E_6(q) &= 1 - 504 \sum_{n=1}^{\infty} \frac{n^5 q^n}{1 - q^n}, \end{aligned}$$

The last two are entire modular functions of weight 4 and 6 respectively, whereas  $E_2$  has an anomalous term in its modular transformation

$$E_2\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^2 E_2(\tau) - \frac{6i}{\pi} c(c\tau + d).$$

The anomalous term in the  $E_2$  transformation can be used to cancel the exponential prefactor in (1.3). Indeed, if we define

$$\tilde{P}(q, \vec{F}) = e^{-1/48 E_2(q) \mathcal{F}^2} \hat{P}(q, \vec{F})$$

we find

$$\tilde{P}\left(\frac{a\tau + b}{c\tau + d}, \frac{\vec{F}}{c\tau + d}\right) = (c\tau + d)^{4n} \tilde{P}(\tau, \vec{F}). \quad (1.4)$$

Expanding  $\tilde{P}$  in powers of  $F$  one finds that the expansion coefficients of terms of order  $m$  must be modular functions of weight  $4n + m$ . Furthermore they do not have poles

at  $\tau = i\infty$  because we have taken out the required number of  $\eta$ -functions. It will be necessary to *assume* that they do not have poles elsewhere in the upper half-plane. This is automatically true for any conformal field theory whose chiral algebra is generated by a finite number of currents [15]. Since all known unitary RCFT's have that property, this is probably a very mild assumption. Basic theorems on modular functions can then be invoked to show that all coefficient functions must be polynomials in  $E_4$  and  $E_6$ .

We define the functions  $\mathcal{E}_n$  as polynomials in  $E_4$  and  $E_6$  with total weight  $n$ . These functions have one or more free parameters:  $\mathcal{E}_{12k+l}$  depends on  $k+1$  parameters for  $l = 0, 4, 6, 8$  and  $10$ , and  $k$  for  $l = 2$ .

The characters out of which  $P$  was built can be expanded in traces over some fixed representation (called the *reference* representation in the following). Furthermore all traces can be expressed in terms of a number (equal to the rank) of basic traces  $\text{Tr } F^s$ , where  $s$  is equal to the order of one of the fundamental Casimir operators of the Lie algebra. The reference representation must be chosen so that for all  $s$  these basic traces are non-trivial and cannot be expressed in terms of lower-order traces. In the following all traces will be over the reference representation unless a different one is explicitly indicated.

Thus we arrive at the following expression for the character-valued partition function

$$P(q, F_1, \dots, F_L) = e^{\frac{1}{48}E_2(q)\mathcal{F}^2} (\eta(q))^{-8n} \sum_{m=0}^{\infty} \sum_i \mathcal{E}_{4n+m}(i) \mathcal{T}_i^m. \quad (1.5)$$

Here  $\mathcal{T}_i^m$  denotes a trace of total order  $m$ , and  $i$  labels the various combinations of traces of that order.

Level zero Now we feed in some facts about the representations at the zeroth level to determine some of the parameters in the coefficient functions  $\mathcal{E}$ . Since the ground state is a singlet representation of the theory, it does not contribute to any of the higher traces. This allows us to rewrite the partition function in the following way

$$\begin{aligned}
P(q, F) = & \exp\left(\frac{1}{48}E_2(q)\mathcal{F}^2\right) \eta(q)^{-8n} \\
& \times \left\{ \mathcal{E}_{4n}(0) + (E_4(q))^n \left[ \cosh\left(\frac{1}{48}\sqrt{E_4(q)}\mathcal{F}^2\right) - 1 \right] \right. \\
& - (E_4(q))^{n-3/2} E_6(q) \left[ \sinh\left(\frac{1}{48}\sqrt{E_4(q)}\mathcal{F}^2\right) \right] \\
& \left. + \sum_{m=2}^{\infty} \sum_i \Delta \mathcal{E}_{4n+m-12}(i) \mathcal{T}_i^m \right\}, \tag{1.6}
\end{aligned}$$

where  $\mathcal{E}_{4n}(0)$  has a leading term equal to 1. The cosh and sinh terms, when expanded in  $F$ , produce coefficient functions that are polynomials in  $E_4$  and  $E_6$  of the correct weight. Their rôle is to cancel for the leading term in  $q$  the contribution of the exponential pre-factor. We can take out a factor  $\Delta = \eta^{24}$  from the remaining coefficient functions, because we know that they must be proportional to  $q$ . This leaves  $\mathcal{E}_{4n+m}/\Delta$ , which is an entire modular function of weight  $4n + m - 12$  (since  $\Delta$  has no zeroes). Note that this shifts the weight of the unknown functions  $\mathcal{E}$  by  $-12$ , effectively removing one free parameter for each coefficient function. The functions  $\mathcal{E}_l$  exist only for  $l = 0$  and  $l \geq 4$ ,  $l$  even. For all other values that occur in the sum they must be interpreted as 0.

Level One Now consider the first excited level. Expanding (1.6) to second order in  $F$  one gets

$$\mathcal{N} + \left(15 - \frac{31}{6}n + \frac{\mathcal{N}}{48}\right) \mathcal{F}^2 + \sum_{\ell} \alpha_{\ell} \text{Tr}_{\text{adj}} F_{\ell}^2. \tag{1.7}$$

Here  $\alpha_{\ell}$  is the leading coefficient of  $\mathcal{E}_{4n-10}(\ell)$  (times a factor for the conversion from reference to adjoint representation). This term vanishes if  $n \leq 3$ . Since by construction the first excited level (the spin-1 currents) consists entirely of adjoint representation of the Kac-Moody algebras, the result should be equal to the Chern-character  $\text{Tr} e^{F \cdot \Lambda}$ , where  $\Lambda$  is the adjoint representation matrix. Upon expansion this yields, for non-Abelian algebras

$$\sum_{\ell} \left( \dim_{\ell} + \frac{1}{2} \text{Tr}_{\text{adj}} F_{\ell}^2 \right). \tag{1.8}$$

For  $U(1)$  factors there is no  $F^2$  contribution in (1.8), and any non-trivial representation can be used for the other traces. Comparing (1.7) and (1.8) we get, for non-Abelian

algebras

$$\sum_{\ell} \dim_{\ell} = \mathcal{N} \quad \text{and} \quad \left(30 - \frac{31}{3}n + \frac{\mathcal{N}}{24}\right) \frac{k_{\ell}}{g_{\ell}} + \alpha_{\ell} = 1 \quad (1.9)$$

For  $n > 3$  (i.e.  $c \geq 32$ ) the second equation simply determines the coefficients  $\alpha_{\ell}$ , and one does not learn anything about the possible Kac-Moody algebras. However, for  $n \leq 3$  these coefficients are absent, and we get

$$\frac{g_{\ell}}{k_{\ell}} = 30 - \frac{31}{3}n + \frac{\mathcal{N}}{24}, \quad (1.10)$$

which is independent of  $\ell$ . For  $U(1)$  factors the right-hand side of the second equation in (1.9) is zero instead of one, and  $k_{\ell}/g_{\ell}$  is replaced by the non-vanishing normalization constant  $N_{\ell}$ . Hence in this case we find (if  $n \leq 3$ )

$$\mathcal{N} = 248n - 720. \quad (1.11)$$

This makes sense only if  $n = 3$ . Then one finds that  $\mathcal{N} = 24$ , and substituting this into (1.10) we conclude that any non-Abelian factor that might still be present must have vanishing dual Coxeter number. Since this is not possible, all 24 spin-1 currents must generate  $U(1)$ 's. This saturates the central charge, and hence the entire theory can be written in terms of free bosons with momenta on a Niemeier lattice. The only such lattice with 24 spin-1 currents is the Leech lattice. Therefore this is the only meromorphic  $c = 24$  theory in which Abelian factors appear.

Hence we may ignore  $U(1)$ 's from here on, and focus on non-Abelian factors. It is instructive to compute the total Kac-Moody central charge:

$$\begin{aligned} c_{\text{tot}} &= \sum_{\ell} \frac{k_{\ell} \dim_{\ell}}{k_{\ell} + g_{\ell}} \\ &= 24 \frac{\mathcal{N}}{248(3 - n) + \mathcal{N}}, \end{aligned}$$

which is valid only if  $n \leq 3$ . For  $n = 3$  we see that the result is always equal to 24, which implies that the Kac-Moody system “covers” the entire theory, and that the unknown part of the theory defined above is necessarily trivial. Our results so far can be summarized as follows

Let  $\mathcal{C}$  be a modular invariant meromorphic  $c = 24$  theory whose chiral algebra is finitely generated and contains  $\mathcal{N}$  spin-1 currents, with  $\mathcal{N} \neq 0$ . Then either  $\mathcal{N} = 24$ , and  $\mathcal{C}$  is the conformal field theory of the Leech lattice, or  $\mathcal{N} > 24$ , and the spin-1 currents form a Kac-Moody algebra with total central charge 24. The values of  $g/k$  for each simple factor of this algebra are equal to one another, and given by  $\mathcal{N}/24 - 1$ .

For the special case of simply laced, level-1 Kac-Moody algebras (yielding even self-dual lattices) this result has been proved by Venkov [16], who also observed that all the solutions to these conditions correspond precisely to the Niemeier lattices. Interestingly, Niemeier was able to classify all lattices without knowing this fact.

This is all that can be learned from the trace identities at the first level. The identities for higher-order traces involve always unknown coefficients analogous to  $\alpha_\ell$  above. These coefficients can be determined and then used to compute traces over the second excitation level.

Level two At the second level we do not know in advance which representations will appear, but at least we know which representations are allowed to appear, namely all combinations of Kac-Moody representations with total spin 2. For all types of traces of total order 0, 2, 4, 6, 8, 10 and 14 we can compute the total value of that trace. This must be matched by some combination of the spin-2 fields. By allowing arbitrary positive integer coefficients for the multiplicity of each spin-2 field we get thus a set of equations for those multiplicities (of course descendants of the spin-0 and spin-1 states must be taken into account as well).

To write down these higher-order trace identities we first need some definitions. The indices  $J_{m_1, \dots, m_r}(R)$  of a representation  $R$  of a simple Lie algebra are defined as

$$\mathrm{Tr}_R F^m = \sum J_{m_1, \dots, m_r}(R) \prod_{i=1}^r \mathrm{Tr}(F^{s_i})^{m_i} ,$$

where the traces on the right-hand side are over the reference representation, and  $\sum_i m_i s_i = m$ . Here  $r$  is the rank of the Lie algebra, and the sum is over all combinations of basic traces with the correct total order  $m$ . Note that with this definition the indices depend on the reference representation. For our purposes it will be sufficient to consider the coefficients  $J_{m, 0, \dots, 0}$ , i.e. the coefficient of  $(\mathrm{Tr} F^2)^m$ . In a tensor product of

$L$  Kac-Moody algebras we will denote the coefficient of  $(\text{Tr}(F_1)^2)^{n_1} \times \dots \times (\text{Tr}(F_L)^2)^{n_L}$  for a representation  $R = (R_1, \dots, R_\ell)$  as  $K_R(n_1, \dots, n_L)$ . Thus

$$K_R(n_1, \dots, n_L) = \prod_{\ell=1}^L J_{n_\ell, 0, \dots, 0}(R_\ell) .$$

The second-level trace identities can now be derived from (1.6). After a rather lengthy computation we get

$$\begin{aligned} \sum_R K_R(n_1, \dots, n_L) = & \left[ \prod_{\ell=1, n_\ell \neq 0}^L \frac{(2n_\ell - 1)!}{2^{n_\ell - 1} (n_\ell - 1)!} \left( \frac{k_\ell}{2N_\ell} \right)^{n_\ell} \right] \\ & \times \left[ C_P - \sum_{\ell=1}^L \sum_{k=1}^{n_\ell} \frac{2^{k+1} n_\ell!}{(n_\ell - k)! (P + k - 1)! B_{2k}} \left( \frac{2N_\ell}{k_\ell} \right)^k C_{k, \ell} \right] , \end{aligned} \tag{1.12}$$

which is valid if the total order,  $P = \sum_\ell n_\ell$ , is smaller than or equal to 5. The identity is valid for any (non-trivial) choice of reference representation. The dependence on this choice enters via the exponential ‘‘anomaly’’ factor in (1.6), and manifests itself through the normalization constants  $N_\ell$ . They are defined by the quadratic trace of the reference representation matrices  $\Lambda_\ell$  in the  $\ell^{\text{th}}$  group

$$\text{Tr } \Lambda_\ell^a \Lambda_\ell^b = 2N_\ell \delta^{ab} .$$

If one chooses the adjoint representation one must set  $N_\ell = g_\ell$  (the adjoint is a valid choice as long as only quadratic traces appear). The coefficients  $C_{k, \ell}$  are the indices of the adjoint representation in the  $\ell^{\text{th}}$  factor, e.g.  $C_{k, 1} = K_{\text{adj}}(k, 0, \dots, 0)$ , with respect to the reference representation. The coefficients  $C_L$  in (1.12) are respectively equal to 196884, 32760, 5040, 720, 96, and 12 for  $P = 0, 1, 2, 3, 4$ , and 5, where  $P$  is the total order of the trace,  $P = \sum_\ell n_\ell$ . Finally,  $B_{2k}$  are the Bernoulli numbers. There is an additional identity for traces of order 14, which is a bit more complicated because the unknown parameters of the 12<sup>th</sup> order traces must be cancelled.

There are many other higher trace identities one could write down. Unfortunately traces of higher than second order are decomposable for the algebra  $SU(2)$ , so that in that case only combinations of second order traces can be used. Most of the accidental solutions to the first level trace identities are precisely due to  $SU(2)$ 's. For this reason it was not worthwhile to consider higher trace identities.

Solution methods Although all solutions to the first level identities correspond to Niemeier lattices if one considers only simply laced algebras, this is not true in general. Our first priority is therefore to rule out as many of the 221 Kac-Moody combinations as possible. To do so, we would like to solve the second level equations for each of the 221 Kac-Moody combinations, or prove that no solution exists. Unfortunately, the number of spin-2 fields is often much too large. This problem can be solved by symmetrizing the equations over identical factors in the tensor product, which usually reduces the number of variables much more than the number of equations. Obviously, the existence of a solution to the symmetrized equation is a necessary, but not a sufficient condition for the existence of a solution to the full set of equations. The number of variables is typically about 50, and at most 288. The number of equations is usually larger than the number of variables, although not all *a priori* distinct equations are independent. For  $\mathcal{N} < 36$  one finds, however, often fewer equations than variables. One of the worst cases has 63 equations for 248 variables.

There are several ways of dealing with these equations. The simplest procedure is to compute the greatest common divisor of the coefficients. Consider for example the combination  $(B_{4,1})^4 A_{6,1}$  with  $\mathcal{N} = 192$ . The right hand side of the zeroth order trace identity is computed by subtracting the descendant contributions from 196884, which yields:

$$\begin{aligned} & 196884 - (4 \times 36 + 48) \\ & \quad - (4 \times 36 \times 48 + 6 \times 36^2) \\ & \quad - (4 \times (\tfrac{1}{2}[36 \times 37] - 495) + (\tfrac{1}{2}[48 \times 49] - 735)) = 180879 \end{aligned}$$

In this case there is just one spin-2 field available, as one may easily check, namely a combination of the four vector representations of  $B_{4,1}$ . This has a ground state dimension  $9^4 = 6561$ , which is not a divisor of 180879. Hence for this combination there is no solution to the trace identity, and therefore no conformal field theory can exist. There are three other  $\mathcal{N} = 192$  combinations, for which solutions *do* exist. They correspond to the Niemeier lattice  $(A_{6,1})^4$ , the  $\mathbf{Z}_2$ -twisted Niemeier lattice  $(B_{4,1})^2 D_{8,2}$  (derived from  $D_{9,1} A_{15,1}$ ) and a new theory  $B_{4,1}(C_{6,1})^2$ .

Many combinations with large values of  $\mathcal{N}$  can be ruled out by this sort of argument. It occurs rather frequently that for one or more kinds of traces all allowed spin-2 fields have a common factor, which does not divide the right hand side. This is the easiest

way to rule out ‘fake’ solutions (i.e. solutions to the level-one conditions without a corresponding modular invariant partition function).

If this common divisor method does not yield inconsistencies, ruling out fake solutions becomes more difficult. Very often the following method works. We know that all unknowns must be positive integers. Furthermore they are bounded from above since the total number of states at the second level must be 196884. Using a set of bounds on all variables, one can compute new bounds from the equations, or from suitably chosen linear combinations of the equations. Very often one ends up with an inconsistency for the bounds on some variable.

If this does not work, one can make use of the knowledge that the unknowns should be integers. To do so, one can use standard methods for solving linear equations. If the number of variables is larger than the number of equations, the best one can do is try all allowed integer values (within the boundaries) for the remaining variables. This might easily have failed, since the number of possibilities grows exponentially with the number of undetermined variables, but luckily in the few cases where it was necessary, it was possible. The final result is that only 69 of the 221 combinations remain.

Of course all these computations were done with a computer. This has two disadvantages. First of all, it is not possible to present details of the elimination process as we did for the example above. Thus there is no “presentable” proof. Secondly, one has to worry about programming errors and accuracy. Most errors of the former kind would almost certainly affect one of the known solutions, and therefore such errors are not very likely. Accuracy becomes an important issue especially in those cases where it was necessary to solve the linear equations. The computations were done in FORTRAN using extended precision floating point arithmetic. Integer arithmetic is exact, but the integers become rapidly extremely large, and easily exceed the maximum value (i.e.  $\approx 10^9$ ) with disastrous consequences. The main worry with floating point arithmetic is loss of accuracy. However, the 32-digit accuracy that was used should be more than sufficient. Nevertheless, it might be worthwhile to repeat the computations with an algebraic program, to solve the linear equations exactly.

### Modular Invariant Partition Functions

For the remaining 69 combinations we expect a conformal field theory to exist, since that is the only way to make sense of the fact that many equations can be satisfied with

simple integers. Therefore we expect that there must exist a modular invariant partition function. The level-2 solution gives us partial information about that function, but we still have to “unsymmetrize” it (if there are several identical factors) and determine the higher spin content. This task seems hopeless at first, but the problem is simplified drastically by simple currents. If among the known spin-2 fields that appear there are one or more simple currents, then we know that

- Fields with fractional charges with respect to those simple currents cannot appear (this simply follows from the requirement of locality of the operator algebra).
- The multiplicity of all fields is constant on the simple current orbits. This can be proved using the form of the matrix  $S$  due to simple currents.

Each of these two points allows us to reduce the number of primary fields that we need to consider by a factor  $N$ , where  $N$  is the order of the simple current (the reduction is slightly less when there are fixed points).

Here again a bit of luck was needed to make the problem manageable. In some cases, the spin-2 fields do not include any simple current. Fortunately it was possible to investigate those without any reduction in the number of fields. In other cases the reduction of the number of fields was large enough to make the problem amenable to computer calculation. After taking into account all known simple currents, the effective number of integer spin fields was less than 250, except for one case ( $(A_{1,4})^{12}$ ) with 1147 integer spin fields, which required special treatment. In these calculations issues related to computer accuracy are far less important, since problems of this kind are far more likely to eliminate valid solutions than to generate invalid ones.

For each of the 69 remaining cases we found precisely one meromorphic modular invariant (a few of the Niemeier lattices were not investigated, because they have in any case already been classified completely). The complete list appears in [1], and will not be repeated here. Thus, if the monster module is indeed unique, and if there is precisely one MCFT per modular invariant, then the total number of such theories is 71.

Some of the Kac-Moody combinations that were ruled out by the level-two trace identities had a small enough number of integer spin fields to be checked explicitly for meromorphic modular invariants. Still others could be checked under the additional assumption that some integer spin simple currents appear in the chiral algebra. Several

meromorphic invariants were indeed found, but they all involve spin-1 currents. This means that they have a larger Kac-Moody sub-algebra, and are embedded conformally in some other meromorphic theory. For example, all theories with  $\mathcal{N} = 48$  can be embedded in  $D_{24}$  (using the embedding  $SO(\dim(H)) \supset H$ , with the vector representation branching to the adjoint of  $H$ ). These embeddings do not appear as a solution to the trace identities, since they violate the assumption that the spin-1 currents are all absorbed into adjoint representations. The fact that they are found as modular invariant partition functions whenever expected, and that no other invariants are found is an important check on the calculations. Unfortunately this check is not available in all cases, because the number of integer spin currents is simply too large.

### Construction

Explicit constructions exist for 39 of the 71 theories, and most likely for two additional ones. The 24 lattice theories and the 15  $\mathbf{Z}_2$ -twisted theories have been constructed [7]. The  $E_{8,2} \times B_{8,1}$  theory can be obtained from a (different)  $\mathbf{Z}_2$ -orbifold twist [17] or can be built out of free fermions [18]. Since free fermion theories (even with real boundary conditions) can be formulated on arbitrary genus Riemann surfaces, there should be no difficulty in writing down a multi-loop partition function (see [19-21]). By factorization, that implies the existence of all correlation functions on arbitrary Riemann surfaces, which is tantamount to existence of the theory.

An investigation of other orbifold twists of meromorphic CFT's was presented by P. Montague [22]. Unfortunately, the consistency conditions for such orbifold are apparently difficult to verify. Many candidate theories appear that do not correspond to an allowed Kac-Moody combination, and that therefore must be inconsistent. In [22] four theories are found that appear on the list of 221 combinations satisfying the level-1 conditions, but only one of those ( $E_{6,2}A_{5,1}C_{5,1}$ ) survives the level-2 conditions. The latter is obtained by starting with the Niemeier lattice  $(E_{6,1})^4$ , with a  $\mathbf{Z}_2$  twist consisting of an interchange of two  $E_6$  factors and two different involutions on the other two  $E_6$ 's. If the consistency of this procedure can be proved, we would have a construction of one more theory on the list. (The other three candidates were all obtained by applying an extra twist to some of the 15 theories of [7].)

This still leaves 30 theories to be constructed. Perhaps they can all be obtained using orbifold twists on some other theory, but there is at present no evidence to support this.

It might be worthwhile to explore how many of them can be constructed using free fermions. This set probably includes the Niemeier lattices (most of them can indeed be constructed out of free fermions with complex boundary conditions, but I have not checked this for all of them), then most likely also the 15 twisted lattice theories, as well as the  $E_{8,2}B_{8,1}$  theory. Even though combinations like  $F_{4,6} \times A_{2,2}$  consist of factors that cannot separately be constructed out of free fermions (the central charges are  $\frac{104}{5}$  and  $\frac{16}{5}$ ), perhaps the combination *can* be obtained in this way.

It is possible to apply the methods of [23][19] to construct systematically *all* possible meromorphic free fermionic theories with  $c = 24$ . However, the still preliminary results are rather disappointing. So far all allowed fermion boundary conditions have been constructed, but not all the allowed phases have taken into account. For complex boundary conditions I do seem to get all the Niemeier lattices (for example the not straightforwardly fermionic  $A_{24}$  theory has appeared). However, there are relatively few choices of boundary conditions for real, unpaired fermions. Among the resulting spectra I do find a surprisingly simple realization of the monster module (which is probably already known to mathematicians, although the precise correspondence is hard to establish), as well as the  $E_{8,2} \times B_{8,1}$  theory and some twisted Niemeier lattices. Nothing genuinely new has emerged so far, however. Some more details may appear in a future publication.

### The number 71

One may hope that the complete list of 71 theories displays some interesting underlying structure, just as the list of  $SU(2)$  modular invariants revealed a relation to ADE-Dynkin diagrams. Indeed, the list of 24 even self-dual lattices might make more sense when it is embedded in the list of 71 meromorphic CFT's. Clearly what is missing is some organizing principle that makes our result something more than an uncorrelated list of modular invariants.

So far not much has emerged. There are, however, two rather intriguing, though highly speculative observations concerning the total number of (candidate) MCFT's, 71. First of all this number is equal to the largest prime factor in the number of elements of the monster group,

$$2^{46} \cdot 3^{30} \cdot 5^9 \cdot 7^6 \cdot 11^3 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23 \cdot 31 \cdot 41 \cdot 47 \cdot 59 \cdot 71 .$$

It is totally mysterious what this might mean, if anything.

A second observation concerns the equation

$$\sum_{i=0}^N i^2 = M^2 ,$$

with  $N$  and  $M$  positive integers. The only solutions I know to this equation are two trivial ones,  $(N, M) = (0, 0); (1, 1)$ , and a unique non-trivial solution,  $(24, 70)$ . The latter solution does indeed play a rôle in this context: the Leech lattice can be characterized as a subset of the lattice  $\Gamma_{25,1}$  (which can be described as  $D_{26}$ , but with a Lorentzian signature, and with a spinor conjugacy class added), consisting of a set of points orthogonal to a lightlike vector  $\lambda$  modulo that vector. One choice for  $\lambda$  is the vector  $(0, 1, 2, 3, \dots, 24; 70)$ . This suggests that there might exist some ‘natural’ correspondence between the integers from 1 to 71 and the 71 MCFT’s, with 70 assigned to the Leech lattice and perhaps 71 to the monster module. The problem with this idea is that the light-like vectors specifying the Niemeier lattices within  $\Gamma_{25,1}$  are by no means unique. One would have to find some natural choice among the infinity of possibilities. Furthermore one has to find a CFT-generalization of the concept of a light-like vector and a lattice ‘orthogonal’ to it.

Clearly, if any of these observations is more than just a numerological coincidence, this would be an extremely exciting development.

## REFERENCES

- [1] A.N. Schellekens, *Meromorphic  $c=24$  Conformal Field Theories*, CERN preprint, CERN-TH.6478/92, to appear in Commun. Math. Phys.
- [2] I. Frenkel, J. Lepowsky and J. Meurman, Proc. Natl. Acad. Sci. U.S.A. 81, 32566 (1984).
- [3] A.N. Schellekens, Phys. Lett. B277 (1992) 277.
- [4] J. Leech, Canadian Journal of Mathematics 19 (1967) 251.
- [5] H. Niemeier, Journal of Number Theory 5 (1973) 142.
- [6] L. Dixon, P. Ginsparg and J. Harvey, Commun. Math. Phys. 119 (1988) 221.
- [7] L. Dolan, P. Goddard and P. Montague, Phys. Lett. B236 (1990) 165.

- [8] A.N. Schellekens and S. Yankielowicz, Phys. Lett. B226 (1989) 285.
- [9] A.N. Schellekens and N.P. Warner, Phys. Lett. B177 (1986) 317; Phys. Lett. B181 (1986) 339; Nucl. Phys. B287 (1987) 317.
- [10] It was pointed out by Vafa, Ginsparg and Moore that the conclusions of [9] can be generalized immediately to arbitrary Kac-Moody algebras using the results of Kac and Peterson on the transformation properties of Kac-Moody characters; C. Vafa, private communication; P. Ginsparg *Informal String Lectures*, in the Proceedings of the U.K. Institute for Theoretical High Energy Physics, Cambridge, 16 Aug. - 5 Sept. 1987.
- [11] M. Green and J. Schwarz, Phys. Lett. 149B (1984) 117.
- [12] J. Thierry-Mieg, Phys. Lett. 171B (1986) 163.
- [13] A. Zamolodchikov, Theor. Math. Phys. 65 (1986) 1205.
- [14] V.G. Kac and D.H. Peterson, Adv. in Math. 53 (1984) 125.
- [15] E. Kiritsis, Nucl. Phys. B324 (1989) 475.
- [16] B.B Venkov, *The Classification of Integral Even Unimodular Quadratic Forms*, Proc. of the Steklov Institute of Mathematics 4 (1980) 63.
- [17] L. Dixon and J. Harvey, Nucl. Phys. B274 (1986) 93.
- [18] H. Kawai, D. Lewellen and S. Tye, Phys. Rev. D34 (1986) 3794.
- [19] I. Antoniadis, C. Bachas and C. Kounnas, Nucl. Phys. B289 (1987) 87.
- [20] H. Kawai, D. Lewellen, J.A. Schwartz and S.-H.H. Tye Nucl. Phys. B299 (1988) 431.
- [21] R. Bluhm, L. Dolan and P. Goddard, Nucl. Phys. B309 (1988) 330.
- [22] P. Montague, *Discussion of Self-Dual  $c=24$  Conformal Field Theories*, Cambridge preprint, hep-th/920567.
- [23] H. Kawai, D. Lewellen and S. Tye, Nucl. Phys. B288 (1987) 1.