

Relativistic Dynamics of Spin in Strong External Fields *

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Abstract

The dynamics of relativistic spinning particles in strong external electromagnetic or gravitational fields is discussed. Spin-orbit coupling is shown to affect such relativistic phenomena as time-dilation and perihelion shift. Possible applications include muon decay in a magnetic field and the dynamics of neutron stars in binary systems.

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1. Introduction

The classical mechanics of spinning non-relativistic particles in external fields predicts several interesting effects as a result of spin-orbit and/or spin-spin coupling, which cause their behaviour to deviate from that of scalar point particles under similar conditions [1]-[9]. One such effect is that relativistic time dilation can have a *dynamical* component [3, 4], in addition to the universal kinematic time dilation which disappears in any inertial frame with respect to which the particle is at rest. Such a non-universal dynamical effect can arise both in special and in general relativity [5, 6].

It is the purpose of this review to present some simple examples of these dynamical phenomena, derived for spinning particles which satisfy equations of motion of the type proposed by Frenkel [2] and Bargmann, Michel and Telegdi [7] for motion in an electro-magnetic field, or a generalisation of these equations to motion in a gravitational field [8, 4, 6, 9]. However, in order to keep the analysis simple I have generally neglected corrections to the motion of spins compensating for a non-canonical value of the gyromagnetic factor, assuming where necessary that $g = 2$ as for an electron or other Dirac particle. Also, I will restrict the analysis below to effects which depend only linearly on the spin, arising essentially from spin-orbit coupling, whilst non-linear spin effects such as associated with spin-spin coupling are disregarded. In view of the smallness of spin-effects this approximation seems well-justified and it is not to be expected that these restrictions alter the conclusions in a significant way, at least not qualitatively. After all, in atomic physics hyperfine splitting is generally small compared to fine-splitting, which in turn tends to be small compared to the transition energies between states of different principal quantum numbers. And at the other extreme, the ratio of intrinsic to total angular momentum of a rapidly spinning neutron star orbiting a heavy companion is typically of the order of 10^{-3} or smaller. Finally, since the effects predicted are not special to classical particles, but have quantum mechanical counterparts, one may feel quite confident that the considerations presented below are of fairly general physical interest.

2. Charged point particle in a magnetic field

To show how relativistic effects of spin in an external field may arise, I consider first the motion of a spinless charged point particle in a constant magnetic field, neglecting the back reaction of the motion of the charge on the field. The Lorentz force law provides the classical equation of motion for the particle:

$$\frac{d}{dt} \frac{m\vec{v}}{\sqrt{1 - \vec{v}^2/c^2}} = \frac{q}{c} \vec{v} \times \vec{B}. \quad (1)$$

Taking the direction of the field as the z -axis, the motion can be decomposed into a linear motion with constant velocity v_z parallel to the field, and a circular motion with frequency

$$\omega = \frac{qB}{mc} \sqrt{1 - \vec{v}^2/c^2}, \quad (2)$$

(the cyclotron frequency) perpendicular to the field. Thus the solution of eq.(1) is

$$\vec{v} = (\omega r \sin \omega t, \omega r \cos \omega t, v_z). \quad (3)$$

Elimination of ω then gives

$$\begin{aligned} 1 - \frac{\vec{v}^2}{c^2} &= 1 - \frac{v_z^2}{c^2} - \frac{\omega^2 r^2}{c^2} \\ &= \frac{1 - v_z^2/c^2}{1 + (qBr/mc^2)^2}. \end{aligned} \quad (4)$$

To interpret this formula we associate an effective magnetic moment with the circulating charge, as induced by the external field:

$$\vec{\mu} = \frac{q}{2mc} \vec{r} \times \frac{m\vec{v}}{\sqrt{1 - \vec{v}^2/c^2}}. \quad (5)$$

In agreement with Lenz's law, it is directed opposite to the field and its component in this direction has magnitude

$$\frac{\vec{\mu} \cdot \vec{B}}{mc^2} = -\frac{1}{2} \left(\frac{qBr}{mc^2} \right)^2. \quad (6)$$

Combining eqs.(4) and (6) one finally obtains

$$\frac{1}{1 - \vec{v}^2/c^2} = \frac{1 - 2\vec{\mu} \cdot \vec{B}/mc^2}{1 - v_z^2/c^2}. \quad (7)$$

As a result the relativistic expression for the energy can be written as

$$E = mc^2 \sqrt{\frac{1 - 2\vec{\mu} \cdot \vec{B}/mc^2}{1 - v_z^2/c^2}}, \quad (8)$$

or equivalently

$$E^2 = m^2 c^4 + p_z^2 c^2 - 2\vec{\mu} \cdot \vec{B} mc^2. \quad (9)$$

If the particle is a pion, for example, its mean life time is changed by the relativistic time dilation factor

$$\Delta t = \Delta \tau \sqrt{\frac{1 - 2\vec{\mu} \cdot \vec{B}/mc^2}{1 - v_z^2/c^2}}. \quad (10)$$

Here τ denotes the proper time, and t the laboratory time measured in the observers rest frame. To an observer at a distance $R \gg r$, for whom the circular motion is not resolved, it seems that the pion moves linearly parallel to the field with a magnetic moment $\mu_z = -q^2 r^2 B / 2mc^2$. For this observer the mean life time depends both on the velocity (along the field direction) *and* on the energy $\vec{\mu} \cdot \vec{B}$ associated with the

magnetic moment. In particular, the time dilation does not vanish in the limit in which the translational velocity v_z goes to zero. In the following similar results will be derived for particles with an intrinsic magnetic moment.

3. Classical dipoles

Consider a charged particle with an intrinsic electric dipole moment \vec{d} and a magnetic dipole moment $\vec{\mu}$. Like the electric and magnetic fields themselves they can be assembled into a covariant anti-symmetric tensor, which we associate with the spin (for $g = 2$), as follows:

$$\frac{q}{mc} S^{\mu\nu} = \begin{pmatrix} 0 & -cd_x & -cd_y & -cd_z \\ cd_x & 0 & \mu_z & -\mu_y \\ cd_y & -\mu_z & 0 & \mu_x \\ cd_z & \mu_z & -\mu_x & 0 \end{pmatrix}. \quad (11)$$

Equivalently, we may associate the electric and magnetic dipole moments with the Lorentz four-vectors

$$\begin{aligned} D_\mu &= \frac{q}{mc^3} S_{\mu\nu} \dot{x}^\nu, \\ M_\mu &= \frac{q}{2mc^2} \varepsilon_{\mu\nu\kappa\lambda} \dot{x}^\nu S^{\kappa\lambda}, \end{aligned} \quad (12)$$

where the overdot denotes a proper-time derivative. Clearly, in the rest frame these four-vectors reduce to the electric and magnetic three-vectors $(\vec{d}, \vec{\mu})$. However, in a general inertial frame the electric and magnetic dipole moments mix, and a particle with vanishing electric dipole moment in the rest frame nevertheless develops one with respect to a moving observer.

The energy of the dipoles in an external field can now be written as

$$\vec{d} \cdot \vec{E} + \vec{\mu} \cdot \vec{B} = \frac{q}{2mc} S^{\mu\nu} F_{\mu\nu}. \quad (13)$$

From the right-hand side of this equation we infer, that this energy is a relativistic invariant.

We now propose the following equations of motion for a classical particle with magnetic and/or electric dipole moments:

$$\begin{aligned} m\ddot{x}^\mu &= qF^\mu{}_\nu \dot{x}^\nu + \frac{q}{2mc} S^{\kappa\lambda} \partial^\mu F_{\kappa\lambda}, \\ \dot{S}^{\mu\nu} &= \frac{q}{m} [F, S]^{\mu\nu}. \end{aligned} \quad (14)$$

The right-hand side of the last equation is the ordinary commutator of the two tensors F and S considered as 4×4 matrices. These equations have the following physical interpretation: the first term on the right-hand side of the first equation represents the usual Lorentz force; the second term is a direct covariant extension of the Stern-Gerlach force, which results from the coupling of the dipole moment to the gradient of the field. The last equation is the expression of Frenkel [2] and Bargmann,

Michel and Telegdi [7] for the precession of a dipole moment in an external field in the limit $g = 2$.

The first integral of motion obtained from these equations leads to the following relativistic energy-momentum relation:

$$(p_\mu - qA_\mu)^2 - \frac{q}{c} S^{\mu\nu} F_{\mu\nu} + m^2 c^2 = 0. \quad (15)$$

This replaces the usual mass-shell condition and reduces in the non-relativistic limit to

$$E = mc^2 + \frac{1}{2m} (\vec{p} - q\vec{A})^2 + q\phi - (\vec{d} \cdot \vec{E} + \vec{\mu} \cdot \vec{B}) + \dots \quad (16)$$

Here $q\phi$ is the electrostatic energy of the charge. The full solution of the energy-momentum relation and the equations of motion is [4, 5]

$$\begin{aligned} \vec{p} - q\vec{A} &= m\vec{v} \sqrt{\frac{1 - 2(\vec{d} \cdot \vec{E} + \vec{\mu} \cdot \vec{B})/mc^2}{1 - \vec{v}^2/c^2}}, \\ E - q\phi &= mc^2 \sqrt{\frac{1 - 2(\vec{d} \cdot \vec{E} + \vec{\mu} \cdot \vec{B})/mc^2}{1 - \vec{v}^2/c^2}}, \end{aligned} \quad (17)$$

For the relativistic time dilation we now obtain the result

$$\Delta t = \left(\frac{E - q\phi}{mc^2} \right) \Delta\tau = \Delta\tau \sqrt{\frac{1 - 2(\vec{d} \cdot \vec{E} + \vec{\mu} \cdot \vec{B})/mc^2}{1 - \vec{v}^2/c^2}}. \quad (18)$$

This agrees completely with our previous result for the motion of a point particle in a magnetic field, if we replace the induced magnetic moment by the intrinsic dipole moments. Note that again a time dilation factor remains associated with the energy of the dipole in an external field even in the limit of zero translational velocity.

4. Comparison with the Dirac equation

Now compare our classical equations with the quantum theory of spinning particles, as represented by the Dirac equation for a particle in an external field:

$$\left(\gamma \cdot \mathcal{D} + \frac{mc}{\hbar} \right) \Psi = 0, \quad (19)$$

where the external field is taken to be electro-magnetic and the interaction with the Dirac particle is described by minimal coupling:

$$\mathcal{D}_\mu = \partial_\mu - \frac{i}{\hbar} qA_\mu. \quad (20)$$

Multiplying the Dirac equation by the operator $(-\gamma \cdot \mathcal{D} + mc/\hbar)$ and working out the algebra of Dirac matrices and covariant derivatives, we get a generalised Klein-Gordon equation

$$\left(-\mathcal{D}_\mu^2 + \frac{i}{\hbar} q\sigma^{\mu\nu} F_{\mu\nu} + \frac{m^2 c^2}{\hbar^2}\right) \Psi = 0. \quad (21)$$

This corresponds exactly to the classical equation (15) if make the operator correspondence

$$p_\mu \rightarrow -i\hbar\partial_\mu, \quad S^{\mu\nu} \rightarrow -i\hbar\sigma^{\mu\nu}. \quad (22)$$

Therefore our classical equations of motion (14) may indeed be taken to represent the classical limit of the Dirac equation, provided we take the particle to have vanishing electric dipole moment in its rest frame far away from external fields.

At this point it is however necessary to issue a warning: although it is perfectly permissible to impose the vanishing of the intrinsic electric dipole moment as an initial condition, its preservation in time is violated in non-homogeneous fields by terms quadratic in the spin, unless special conditions on the spin variables are imposed [4]. However, as stated in the introduction, all such effects quadratic in the spin are neglected here because of their numerical smallness.

5. Muon decay

As an application of our results so far we consider β -decay of a muon in an external field:

$$\mu \rightarrow e\bar{\nu}_e\nu_\mu. \quad (23)$$

The mean life time of the free muon τ_μ is accurately known to be 2.19703×10^{-6} sec. Given a muon mass of 105.7 MeV, the change in the life time we predict by placing the particle in an external magnetic field of magnitude B is

$$\frac{\delta\tau}{\tau_\mu} = -\frac{\vec{\mu} \cdot \vec{B}}{mc^2} = 0.28 \times 10^{-14} \times B, \quad (24)$$

with B measured in Tesla. In order to obtain a significant variation in the mean life time we therefore need fields of at least 5×10^9 T. This corresponds to a magnetic energy of

$$\vec{\mu} \cdot \vec{B} \geq 1 \text{ keV}. \quad (25)$$

Such magnetic interaction energies are not unrealistic; for example, the hyperfine splitting of muon energy levels in the field of high- Z atomic nuclei like Nb or Bi are of the order of 3 – 5 keV. Practical difficulties in measuring the corresponding change in life time are the result of competing effects like muon-capture by the nucleus, and of the short life-time of the higher state in the hyperfine doublet: the rate of transition to the groundstate, an electromagnetic process, is much higher than the rate of muon decay, which is a weak interaction process. A reliable measurement would require long-lived high population density in the excited state of the doublet, so as to allow accurate comparison of the relative life times between the two states.

Another place where very strong magnetic fields are encountered is near neutron stars. It would be very interesting if it were possible to extract information about unstable particle decay from such distant sources.

6. Gravitational interactions

Next I turn to the case of a spinning particle moving in a gravitational background field. Again we can make an inspired guess for the classical equations of motion based upon correspondence with the Dirac equation. The simplest conjecture [4, 6], which disregards terms quadratic in the spin [9], is

$$\begin{aligned} m \frac{D^2 x^\mu}{D\tau^2} &= \frac{1}{2} S^{\kappa\lambda} R_{\kappa\lambda}{}^{\mu}{}_{\nu} \dot{x}^\nu, \\ \frac{DS^{\mu\nu}}{D\tau} &= 0. \end{aligned} \tag{26}$$

Here $R_{\mu\nu\kappa}{}^\lambda$ is the Riemann curvature tensor. Note that the form of the first equation is formally very similar to the Lorentz force law, with the electromagnetic field strength replaced by the space-time curvature and the scalar charge q replaced by the tensorial coupling parameter $S^{\mu\nu}$, which is taken here to be covariantly constant. No additional Stern-Gerlach term has been introduced since it would be quadratic in the spin.

As a first consequence one obtains as a first integral of motion the mass-shell condition

$$m^2 g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu + m^2 c^2 = 0. \tag{27}$$

As shown below, dynamical spin-dependent effects are now caused by spin dependence of the solution $\vec{x}(t)$ for the orbit of the particle.

I will confine the discussion here to the simplest relevant physical situation where these equations might apply: the motion of a spinning particle in the field of a large spherically symmetric mass, like a star or a black hole [6]. In this case the space-time geometry is described by the Schwarzschild solution of the Einstein equations:

$$ds^2 = - \left(1 - \frac{\alpha}{r}\right) dt^2 + \frac{1}{\left(1 - \frac{\alpha}{r}\right)} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2, \tag{28}$$

with $\alpha = 2GM$, M being the total central mass and G Newton's gravitational constant.

The solution of the equation of motion in the absence of spin is well-known: orbital angular momentum is conserved, the motion is planar and, for bound states, the orbit is to high accuracy an ellipse, with a precession of the perihelion of magnitude

$$\Delta\phi = \frac{6\pi GM}{k} \left(1 + (18 + e^2) \frac{GM}{4k} + \dots\right). \tag{29}$$

In this equation e is the eccentricity of the ellipse, and k the semi-latus rectum [10].

The corrections to first order in the spin now amount to the following: instead of conservation of orbital angular momentum \vec{L} , we now only have conservation of *total* angular momentum $\vec{J} = \vec{L} + \vec{S}$. If \vec{L} and \vec{S} are not parallel, they will both precess around \vec{J} in such a way as to keep \vec{J} constant. Hence in general the motion is no longer planar. Of course, if $|S| \ll |L|$ the motion is still approximately planar. If \vec{S} and \vec{L} are parallel, this is an exact result. The orbit will again be an approximate ellipse, but the rate of precession of the perihelion is changed by spin-orbit coupling. One finds [6]

$$\Delta\phi = \frac{6\pi GM}{k} \left(1 + \Delta + (18 + e^2) \frac{GM}{4k} + \dots \right). \quad (30)$$

where, taking \vec{L} in the z -direction, $\Delta = S_z/L$.

Another constant of motion in a static space-time geometry, like the Schwarzschild metric, is the space-time energy (the time component of the canonical energy-momentum 4-vector). For a spinning particle it is of the form

$$E = mc^2 \left(1 - \frac{\alpha}{r} \right) \frac{dt}{d\tau} - \frac{\alpha}{2r^2} S^{rt}. \quad (31)$$

Now for motion in a plane we have separate conservation of orbital and spin angular momentum; hence

$$L_z = mr^2 \dot{\phi} \quad (32)$$

is a constant of motion. In addition, absence of the electric dipole components of the spin gives

$$S^{\mu\nu} \dot{x}_\nu = 0 \rightarrow S^{rt} = \frac{\vec{L} \cdot \vec{S}}{Er}. \quad (33)$$

Combining these results we arrive at the following formula for time-dilation in a Schwarzschild geometry

$$dt = \frac{d\tau}{\left(1 - \frac{\alpha}{r} \right)} \frac{E}{mc^2} \left(1 + \frac{\alpha \vec{L} \cdot \vec{S}}{2E^2 r^3} \right). \quad (34)$$

Thus there is the usual universal gravitational redshift, given by the first term inside the parentheses, and in addition a dynamical non-universal redshift proportional to the spin-orbit coupling.

These examples show that the phenomena we encountered for the motion of spinning particles in an electro-magnetic field generalize to the case of gravitational fields.

7. The binary pulsar

I can not predict precisely to what extent our results are valid for general spinning particles, which do not have the canonical value of the gyromagnetic factor. However, assuming that the above results are valid at least qualitatively in the general

case, we may estimate the order of magnitude of the various effects for a system like the binary pulsar PSR 1913 + 16. This is a binary system including a rapidly spinning neutron star, the radio signal of which has been studied for many years, giving very precise information about its orbital parameters. The rapid spin of the neutron stars make them the only known macroscopic objects whose spin has to be treated relativistically.

For a spherical mass m with radius R the moment of inertia is $I = 2/5 mR^2$, and the intrinsic angular momentum

$$S = I\omega = \frac{2}{5} mR^2\omega. \quad (35)$$

With an orbital angular momentum given by eq.(32) the ratio of the two quantities becomes

$$\Delta = \frac{S}{L} = \frac{2}{5} \frac{R^2\omega}{r^2\dot{\phi}}. \quad (36)$$

I have used here a non-relativistic expression for S , but this should be allowed on order to make an order-of-magnitude estimate as it provides an upper limit.

Now for PSR 1913 + 16 the respective quantities are approximately

$$\begin{aligned} R &= 2.5 \times 10^4 \text{ m}, & \omega &= 106.4 \text{ rad sec}^{-1}, \\ r &= 6.2 \times 10^8 \text{ m}, & \dot{\phi} &= 2.252 \times 10^{-4} \text{ rad sec}^{-1}. \end{aligned} \quad (37)$$

As a result we find $\Delta = S/L \approx 0.35 \times 10^{-3}$.

Finally we substitute this estimate into the equations for the perihelion shift and the redshift formula. For the binary pulsar, the next-to-leading order result for the perihelion shift of a spinless mass point is determined by the quantity

$$18 \frac{GM}{4k} \approx 0.23 \times 10^{-4}, \quad (38)$$

cf. eq.(29). This is to be compared with Δ in computing the precession of the perihelion as in eq.(30). Clearly Δ is an order of magnitude *larger* than the next-to-leading order general relativistic correction. Given the accuracy of the measurement of the orbit of the binary pulsar, this effect seems within the limits of observation.

For the redshift effect, we have to compute the spin-orbit term

$$\frac{\alpha \vec{L} \cdot \vec{S}}{2E^2 r^3} \approx 1.4 \times 10^{-12} \Delta. \quad (39)$$

Clearly, for the value of Δ in the order of 10^{-3} as quoted above this is completely negligible compared to unity. Thus the spin-dependent non-universal redshift seems unobservable, at least for this system.

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