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SUSY in the sky

G.W. Gibbons[1]
DAMTP, Cambridge U.K.

R.H. Rietdijk[2]
Physics Dept., Univ. of Durham U.K.

J.W. van Holten[3]
NIKHEF-H, Amsterdam NL

Abstract

Spinning particles in curved space-time can have fermionic symmetries generated by the square root of bosonic constants of motion other than the Hamiltonian. We present a general analysis of the conditions under which such new supersymmetries appear, and discuss the Poisson-Dirac algebra of the resulting set of charges, including the conditions of closure of the new algebra. An example of a new non-trivial supersymmetry is found in black-hole solutions of the Kerr-Newman type and corresponds to the Killing-Yano tensor, which plays an important role in solving the Dirac equation in these black-hole metrics.

[1]e-mail: gwg1@amtp.cam.ac.uk
[2]e-mail: r.h.rietdijk@durham.ac.uk
[3]e-mail: t32@nikhef.nl

1 Introduction

In this paper we investigate the symmetries of classical space-times in terms of the motion of pseudo-classical spinning point particles in a curved Lorentzian manifold. Pseudo-classical spinning point particles are described by the $d = 1$ supersymmetric extension of the simple (spinless) relativistic point particle, as developed in [1]-[6]. The general relations between space-time symmetries and the motion of spinning point particles has been analysed in detail in [7]-[9]. These methods may be applied to any space-time, but since undoubtedly the most important solution of the Einstein vacuum equations in 4 dimensions is the Kerr solution, which represents the gravitational field of an isolated rotating black hole, the detailed applications considered in the later sections of this paper are taylored more in particular to that case and, almost equally interesting from the theoretical point of view, to the Kerr-Newman solutions of the combined Einstein-Maxwell equations. With the exception of the charged multi-black-hole metrics found by Papapetrou and Majumdar, these are believed to be the unique stationary and asymptotically flat solutions of the Einstein-Maxwell equations which are regular outside the (equally regular) event horizon. If the cosmic censorship hypothesis holds, then it seems likely that these metrics represent the only possible final exterior equilibrium states of gravitational collapse.

The Kerr-Newman solutions are invariant under two continuous symmetries: time translations and rotations about an axis of symmetry, which are generated by Killing fields K^μ and M^μ, respectively. These symmetries give rise to two constants of motion: energy E and angular momentum J, for particles orbiting in these backgrounds. Both constants of motion are linear in the particle's 4-momentum p_μ:

$$E = -K^\mu p_\mu, \tag{1}$$

and

$$J = M^\mu p_\mu. \tag{2}$$

It came, therefore, as a considerable surprise when Carter succeeded in showing that because of the existence of a further constant of motion, quadratic in the 4-momentum, the equations for the geodesics and the orbits of charged particles constituted a completely integrable system in the sense of Liouville [10]. Carter's constant of motion has the form

$$Z = \frac{1}{2} K^{\mu\nu} p_\mu p_\nu, \tag{3}$$

which commutes with the covariant Hamiltonian

$$H = \frac{1}{2} g^{\mu\nu} p_\mu p_\nu \tag{4}$$

in the sense of Poisson brackets. This is guaranteed because $K^{\mu\nu}$ is a symmetric second-rank contravariant tensor field satisfying a generalised Killing equation [7, 8], a class known as Stackel-Killing tensors. The four constants of motion (E, J, Z, H) now form a mutually Poisson-commuting set of functions on the cotangent bundle, two of which are linear and two of which are quadratic in the momenta.

Historically, it was shown by Carter [11] that the Klein-Gordon equation with minimal coupling to the electromagnetic field is soluble by separation of variables in these backgrounds. Also various other field equations were shown to separate. Most significantly for our present purpose, Chandrasekhar achieved separation of the Dirac-equation [12].

Carter was able to reinterpret the separability of the Klein-Gordon equation for charged particles in terms of a second order differential operator constructed from the Stackel-Killing tensor $K^{\mu\nu}$ similar to the construction of the Laplace-Beltrami operator from the contravariant components of the metric $g^{\mu\nu}$: one considers the operator $\hat{K}$ operating on a scalar wave function Ψ by

$$\hat{K}\,\Psi = D_\mu K^{\mu\nu} D_\nu \Psi. \tag{5}$$

Here D_μ is a covariant derivative including minimal coupling to the electromagnetic field. This operator $\hat{K}$ now commutes with the covariant Laplacian

$$\Delta = D_\mu g^{\mu\nu} D_\nu. \tag{6}$$

It also commutes with the Lie derivatives with respect to the Killing vector fields (K^μ, M^μ), acting on a scalar wave function by linear differential operators:

$$K^\mu \partial_\mu \Psi, \qquad M^\mu \partial_\mu \Psi. \tag{7}$$

Thus Carter established a quantum version of his earlier classical results.

In subsequent work with McLenaghan [13], Carter constructed a linear differential operator which *commutes* with the Dirac operator in the Kerr-Newman background. The construction of this operator depended upon another remarkable fact, discovered by Penrose and Floyd, that the Stackel-Killing tensor $K^{\mu\nu}$ has a certain square root, which defines a Killing-Yano 2-form $f_{\mu\nu} = -f_{\nu\mu}$ such that

$$K^\mu_{\ \nu} \;=\; f^\mu_{\ \lambda} f^\lambda_{\ \nu}. \tag{8}$$

Here indices are raised and lowered with the space-time metric $g_{\mu\nu}$ and its inverse.

These remarkable discoveries were at the same time both useful and mysterious. On the one hand they made possible a whole range of calculations, both classical and quantum mechanical, which can and are being applied to various physical processes near black holes in our universe. On the other hand they raise a number of theoretical questions, including what is the *classical* counter part of Carter and McLenaghan's work on the Dirac equation and what, if any, is the relation between *supersymmetry* and the mysterious Killing-Yano type square root of the Stackel-Killing tensor. It is the purpose of this paper to address these two questions and to show how supersymmetric particle mechanics involving classically anti-commuting Grassmann variables can be used to understand this aspect of black holes. Furthermore it is shown how this construction fits into the more general framework for finding constants of motion which involve higher-rank tensors and p-forms, as outlined in refs.[7, 8]. The passage to quantum mechanics is then made as discussed for example in [1]-[6], and no special problems are expected to occur.

The conclusion of our analysis is, that the existence of the Killing-Yano tensor discovered by Penrose and Floyd and its properties may be understood in a *systematic* way as a particularly interesting example of a more general phenomenon, the appearance of an additional supersymmetry in the usual $N = 1$ supersymmetric extension of point particle mechanics in curved space-time. It was discovered by Zumino [15], that demanding the system to admit an additional supersymmetry of the usual type, generating an $N = 2$ super-Poincaré algebra, restricts the background metric (which he took to have

Euclidean signature) to correspond to a Kähler manifold. This implies that the metric $g_{\mu\nu}$ should admit a covariantly constant 2-form $f_{\mu\nu} = -f_{\nu\mu}$:

$$D_\lambda \, f_{\mu\nu} \;=\; 0. \tag{9}$$

Quite aside from the problem of the space-time signature (which could possibly be changed to Kleinian, $++--$) this condition is far too restrictive to be useful in any direct astrophysical application. However, it is possible to make the weaker demand that an extra supersymmetry exists but not to prejudge the algebra it is to satisfy. This weaker condition *is* compatible with a Lorentzian signature and gives rise to the correspondingly weaker condition

$$D_\mu \, f_{\nu\lambda} \,+\, D_\nu \, f_{\mu\lambda} \;=\; 0, \tag{10}$$

where the covariant tensor field $f_{\mu\nu}$ need not necessarily be anti-symmetric. In the case that $f_{\mu\nu}$ is anti-symmetric, we are led precisely to the condition for the existence of a second-rank Killing-Yano tensor of the type found by Penrose and Floyd, and exploited so succesfully by Carter and McLenaghan. Moreover it is easy to see that the expression given by them for a first-order differential operator which *commutes* with the Dirac operator $\gamma^\mu D_\mu$, to wit:

$$i\gamma_5\gamma^\mu \left(f_\mu{}^\nu D_\nu \,-\, \frac{1}{6}\,\gamma^\nu\gamma^\lambda D_\lambda f_{\mu\nu} \right), \tag{11}$$

anti-commutes with the Dirac operator after multiplication by γ_5, which is more natural from the point of view of supersymmetry.

As is well-known, in the quantum theory of the spinning point particle the first Poincaré-type supercharge is represented by a multiple of γ_5 times the Dirac operator acting on spinor fields on the space-time manifold. In the special case one is dealing with a Kähler metric, for which $D_\lambda f_{\mu\nu} = 0$, the second Poincaré supercharge is then given by a multiple of

$$i\gamma_5\gamma^\nu \, f_\nu{}^\mu \, D_\mu. \tag{12}$$

This operator is constructed from the Levi-Civita connection on spinor fields in terms of the gamma matrices $i f_\nu{}^\mu \gamma^\nu$, which have been rotated with respect to those appearing in the Dirac operator using a complex structure $f^\mu{}_\nu$, provided it is normalised to satisfy

$$f^\mu{}_\lambda f^\lambda{}_\nu \;=\; -\delta^\mu_\nu. \tag{13}$$

The expression of Carter and McLenaghan is a natural generalisation of eq. (12) and is the quantum version of the generalised second supersymmetry we will show to exist in the Kerr-Newman background.

At this point we should remark, that space-time supersymmetry has previously been applied to charged black holes in the context of $N = 2$ supergravity theory. The application of world-line supersymmetry in this paper seems at first sight to be unrelated to that work. For example, our results concerning a 'hidden' supersymmetry related to the motion of spinning point particles are applicable to *all* members of the Kerr-Newman family of black-hole solutions, while the Killing spinors giving rise to symmetries of the solutions of supergravity field equations arise only in the case of extreme solutions (or indeed naked singularities) whose mass and charge in suitable units are equal.

It seems therefore that our scheme enables us for the first time to succesfully apply supersymmetry to a problem of genuine astrophysical interest. Indeed it is widely believed that our universe contains large numbers of macroscopic rotating black holes with masses up to perhaps 10^6 solar masses, all described to good approximation by the Kerr metric. Hence the title of our paper. Of course we do not claim this supersymmetry to be connected with any hypothetical supersymmetry acting at elementary particle scales.

Another remark we should like to make is that from the point of view of Hamiltonian mechanics the two symmetric contravariant tensor fields $g^{\mu\nu}$ and $K^{\mu\nu}$ appear on a completely symmetrical footing. For example, the Poisson commutativity of their associated constants of motion may be expressed on the space-time manifold entirely in terms of the vanishing of their Schouten bracket, which is an (anti-)symmetrical expression in terms of their partial derivatives not containing any covariant (i.e. Levi-Civita) derivatives: it is covariant as it stands. One might therefore be tempted to conjecture, at least for the Kerr solution, that this symmetry might run even deeper. This however does not seem to be the case, at least for the Kerr solution, because quite independently of any global questions, the two tensors differ in their purely local properties. Consider the symmetric covariant tensor fields given by their inverses. In the case of $g_{\mu\nu}$ it satisfies the vacuum Einstein equations. In the case of $(K^{-1})_{\mu\nu}$, substitution of this symmetric tensor as a metric into the Einstein tensor does *not* lead to a solution of the vacuum Einstein equations. In fact one may also compute the Ricci tensor of $g_{\mu\kappa} g_{\nu\lambda} K^{\kappa\lambda}$. It does not vanish either. Nevertheless, the study of these 'linked geometries' is an interesting one which will be the subject of a separate publication.

The plan of this paper is as follows. In sect. 2 we review the formalism of pseudo-classical spinning point particles in an arbitrary background spacetime, using anticommuting Grassmann variables to describe the spin degrees of freedom. We should point out here that the equations of motion apply in practice as a suitable semi-classical approximation to the dynamics of a massive spin-1/2 particle such as the electron. The question of what is the correct equation of motion for an extended macroscopic object with angular momentum, such as a spinning neutron star, is a separate one not addressed in this paper, although the qualitative features will surely be similar. In particular, the Killing tensors of Stackel and Yano type are also useful in dealing with such macroscopic spinning bodies. Another related question is what is the correct equation of motion for a classical massless spin-1 particle such as a photon. It is well-known that one may take a suitable WKB-approximation to the classical Maxwell equations and discover that the appropriate equation for the classical rays is that they move along null geodesics, and that the polarisation vector is parallel transported along the null geodesic. Again both types of Killing tensors are useful to deal with these equations and have been used for that purpose. For example, the rotation of the plane of polarisation of radio waves passing near a spinning black hole has been calculated using the Penrose-Floyd Yano-Killing tensor [16, 17]. Thus our results are also relevant to that case though the equations of motion themselves are not the same.

In sect. 3 we review the general relation between symmetries, supersymmetries and constants of motion for these equations.

In sec. 4 we take up the question of the existence of extra supersymmetries and their algebras. Supersymmetries are shown to depend on the existence of a second rank tensor field $f_{\mu\nu}$ which we refer to as *f-symbols*.

The general properties of f-symbols are investigated in sect. 5 and their relation to Killing-Yano tensors is pointed out. The results of this section are rather general. They may, for instance, readily be applied to the special case of hyper-Kähler four-manifolds, though we shall not do so in this paper. Hyper-Kähler manifolds admit three covariantly constant 2-forms and therefore exhibit $N = 4$ supersymmetry. This would be relevant to Kaluza-Klein monopoles, described by the Taub-NUT metrics, or the moduli space of two BPS Yang-Mills monopoles, as given by the Atiyah-Hitchin metric.

Finally, in sect. 6 we turn to the main (astrophysically relevant) application of this paper and exhibit the exact form of the constants of motion in the Kerr-Newman geometry.

2 Spinning Particles in Curved Space-Time

As is well-known the pseudo-classical limit of the Dirac theory of a spin-1/2 fermion in curved space-time is described by the supersymmetric extension of the ordinary relativistic point-particle [1]-[6]. The configuration space of this theory is spanned by the real position variables $x^\mu(\tau)$ and the Grassmann-valued spin variables $\psi^a(\tau)$, where μ and a both run from $1, ..., d$, with d the dimension of space-time. The index μ labels the space-time co-ordinates and components of vectors in space-time (world vectors), and a labels the components of vectors in tangent space (local Lorentz vectors), among them the anti-commuting spin-co-ordinates. These types of vectors can be converted into each other by the vielbein $e_\mu^{\;\,a}(x)$ and its inverse $e^\mu_{\;\,a}(x)$; for example it is sometimes convenient to introduce the object

$$\psi^\mu(x) \,=\, e^\mu_{\;\,a}(x)\,\psi^a, \tag{14}$$

transforming under general co-ordinate and local Lorentz transformations as a world vector rather than a local Lorentz vector. The world-line parameter τ is the invariant proper time,

$$c^2 d\tau^2 \,=\, -g_{\mu\nu}(x)\,dx^\mu dx^\nu, \tag{15}$$

in our conventions; by choosing units such that $c = 1$ this constant no longer explicitly appears in our equations.

The equations of motion of the pseudo-classical Dirac particle can be derived from the Lagrangian

$$L = \frac{1}{2}\,g_{\mu\nu}\,\dot{x}^\mu\dot{x}^\nu + \frac{i}{2}\,\eta_{ab}\,\psi^a\frac{D\psi^b}{D\tau}, \tag{16}$$

where η_{ab} is the flat-space (Minkowski) metric. Here and in the following the overdot denotes an ordinary proper-time derivative $d/d\tau$, whilst the covariant derivative of the spin variable, transforming as a local Lorentz vector, is

$$\frac{D\psi^a}{D\tau} \,=\, \dot{\psi}^a \,-\, \dot{x}^\mu\,\omega_{\mu\;\,b}^{\;\,a}\,\psi^b, \tag{17}$$

with $\omega_\mu{}^a{}_b$ the spin connection. In order to fix the dynamics completely one has to add the condition expressed by eq.(15), which is equivalent to the mass-shell condition, plus others necessary to select the physical solutions of the equations of motion. For example, the restriction that spin be space-like reads

$$\mathcal{Q} \equiv e_{\mu a}\, \dot{x}^\mu\, \psi^a \;=\; 0, \tag{18}$$

implying that ψ has no time-component in the rest frame. These supplementary conditions have to be compatible with the equations of motion derived from the Lagrangian L [8, 9]; however, in our formulation of spinning particle dynamics they are only to be imposed *after* solving these equations[4].

The configuration space of spinning particles spanned by (x^μ, ψ^a) is sometimes refered to as spinning space. The solutions of the Euler-Lagrange equations derived from the Lagrangian L may then be considered as generalizations of the concept of geodesics to spinning space. In this formulation the supplementary conditions then select those geodesics which correspond to the world lines of the physical spinning particles.

The variation of the Lagrangian under arbitrary variations $(\delta x^\mu, \delta\psi^a)$ is

$$\begin{aligned} \delta L &= \delta x^\mu \left(-g_{\mu\nu} \frac{D^2 x^\nu}{D\tau^2} - \frac{i}{2} \psi^a \psi^b R_{ab\mu\nu} \dot{x}^\nu \right) \\ &+ \Delta\psi^a\, \eta_{ab}\, \frac{D\psi^b}{D\tau} + \text{total derivative} \end{aligned} \tag{19}$$

For notational convenience we have introduced the covariantized variation of ψ^a [7]

$$\Delta\psi^a \;=\; \delta\psi^a - \delta x^\mu\, \omega_\mu{}^a{}_b\, \psi^b. \tag{20}$$

The equations of motion can be read off immediately from eq.(19):

[4] Actually, we work in a gauge-fixed formulation of the super-reparametrization invariant theory, hence we have to impose the gauge conditions as separate restrictions on the dynamics.

$$\begin{aligned} \frac{D^2x^\mu}{D\tau^2} &= \ddot{x}^\mu - \Gamma_{\lambda\nu}^{\;\;\;\mu}\,\dot{x}^\lambda\dot{x}^\nu \,=\, -\frac{i}{2}\,\psi^a\psi^b\,R_{ab\;\;\nu}^{\;\;\;\mu}\,\dot{x}^\nu \\ \frac{D\psi^a}{D\tau} &= 0. \end{aligned} \tag{21}$$

The canonical momentum conjugate to x^μ is

$$p_\mu \,=\, \frac{\partial L}{\partial \dot{x}^\mu} \,=\, g_{\mu\nu}\dot{x}^\nu - \frac{i}{2}\,\omega_{\mu ab}\,\psi^a\psi^b, \tag{22}$$

whilst the canonical momentum conjugate to ψ^a is proportional to ψ^a itself:

$$\pi_a \,=\, \frac{\partial L}{\partial \dot{\psi}^a} \,=\, -\frac{i}{2}\,\psi_a. \tag{23}$$

This implies a second-class constraint. Eliminating the constraint by Dirac's procedure one obtains the canonical Poisson-Dirac brackets

$$\{x^\mu, p_\nu\} \,=\, \delta^\mu_\nu, \qquad \left\{\psi^a, \psi^b\right\} \,=\, -i\eta^{ab}. \tag{24}$$

The Poisson-Dirac brackets for general functions F and G of the canonical phase-space variables (x, p, ψ) accordingly read

$$\{F, G\} \,=\, \frac{\partial F}{\partial x^\mu}\frac{\partial G}{\partial p_\mu} - \frac{\partial F}{\partial p_\mu}\frac{\partial G}{\partial x^\mu} + i\,(-1)^{a_F}\,\frac{\partial F}{\partial \psi^a}\frac{\partial G}{\partial \psi_a}. \tag{25}$$

Here a_F is the Grassmann parity of F: $a_F = (0, 1)$ for $F =$ (even, odd). The canonical Hamiltonian of the theory is given by

$$H \,=\, \frac{1}{2}\,g^{\mu\nu}\,\left(p_\mu + \omega_\mu\right)\left(p_\nu + \omega_\nu\right), \tag{26}$$

where we define $\omega_\mu = i/2\,\omega_{\mu ab}\psi^a\psi^b$. Indeed, the time-evolution of any function $F(x, p, \psi)$ is generated by its Poisson-Dirac bracket with this Hamiltonian:

$$\frac{dF}{d\tau} \,=\, \{F, H\}\,. \tag{27}$$

Eqs. (24)-(27) summarize the canonical structure of the theory. In this formulation the fundamental brackets (24) take their simplest form. The disadvantage of the canonical formulation is, that one looses manifest covariance.

For this reason it is often convenient to describe the theory in terms of a set of covariant phase-space variables, defined by x^μ, ψ^a and the covariant momentum

$$\Pi_\mu \equiv p_\mu + \omega_\mu = g_{\mu\nu} \dot{x}^\nu. \tag{28}$$

The Poisson-Dirac brackets for functions of the covariant phase-space variables (x, Π, ψ) then read

$$\{F, G\} = \mathcal{D}_\mu F \frac{\partial G}{\partial \Pi_\mu} - \frac{\partial F}{\partial \Pi_\mu} \mathcal{D}_\mu G - R_{\mu\nu} \frac{\partial F}{\partial \Pi_\mu} \frac{\partial G}{\partial \Pi_\nu} + i (-1)^{a_F} \frac{\partial F}{\partial \psi^a} \frac{\partial G}{\partial \psi_a}, \tag{29}$$

where the notation used is

$$\begin{aligned} \mathcal{D}_\mu F &= \partial_\mu F + \Gamma_{\mu\nu}{}^\lambda \Pi_\lambda \frac{\partial F}{\partial \Pi_\nu} + \omega_\mu{}^a{}_b \psi^b \frac{\partial F}{\partial \psi^a}, \\ R_{\mu\nu} &= \frac{i}{2} \psi^a \psi^b R_{ab\mu\nu}. \end{aligned} \tag{30}$$

Note that in particular

$$\{\Pi_\mu, \Pi_\nu\} = -R_{\mu\nu}, \tag{31}$$

which is the classical analogue of the Ricci-identity (in the absence of torsion). In terms of the new covariant phase-space variables the Hamiltonian becomes

$$H = \frac{1}{2} g^{\mu\nu} \Pi_\mu \Pi_\nu. \tag{32}$$

The dynamical equation (27) remains of course unchanged. The constraints (15) and (18) become

$$2H = g^{\mu\nu} \Pi_\mu \Pi_\nu = -1, \qquad \mathcal{Q} = \Pi \cdot \psi = 0. \tag{33}$$

Again these are to be imposed only after solving the theory, since they are not compatible with the Poisson-Dirac brackets in general. However, one easily establishes that

$$\{\mathcal{Q}, H\} = 0. \tag{34}$$

This establishes the conservation of $\mathcal{Q}$, whilst the Hamiltonian itself is trivially conserved. Therefore the values of H and $\mathcal{Q}$ as chosen in (33) are preserved in time, and the physical conditions we impose are consistent with the equations of motion[5].

3 Symmetries and Constants of Motion

The theory described by the Lagrangian (16), or equivalently the Hamiltonian (32), admits a number of symmetries which are very useful in obtaining explicit solutions of the equations of motion [9], in particular because of the connection with constants of motion via Noether's theorem. The symmetries of a spinning-particle model can be divided into two classes:
– *generic* symmetries, which exist for any space-time metric $g_{\mu\nu}(x)$;
– *non-generic* symmetries, which depend on the explicit form of the metric.
In refs. [7, 8] it was shown that the theory defined by the Lagrangian (16) possesses 4 generic symmetries: proper-time translations, generated by the Hamiltonian H; supersymmetry generated by the supercharge $\mathcal{Q}$, eq.(33); and furthermore chiral symmetry, generated by the chiral charge

$$\Gamma_\star = -\frac{i^{[\frac{d}{2}]}}{d!}\, \varepsilon_{a_1 \dots a_d}\, \psi^{a_1} \dots \psi^{a_d}, \tag{35}$$

and dual supersymmetry, generated by the dual supercharge

$$\mathcal{Q}^\star = i\, \{\mathcal{Q}, \Gamma_\star\} = \frac{-i^{[\frac{d}{2}]}}{(d-1)!}\, \varepsilon_{a_1 \dots a_d}\, e^{\mu a_1}\, \Pi_\mu\, \psi^{a_2} \dots \psi^{a_d}. \tag{36}$$

It is straightforward to check, that all these quantities have vanishing Poisson-Dirac brackets with the Hamiltonian, and therefore are constants of motion.

To obtain all symmetries, including the non-generic ones, one has to find all functions $\mathcal{J}(x, \Pi, \psi)$ which commute with the Hamiltonian in the sense of Poisson-Dirac brackets:

$$\{H, \mathcal{J}\} = 0. \tag{37}$$

The covariant form (29) of the brackets immediately gives

[5] A more detailed discussion of this point can be found in the second reference [8].

$$\Pi^\mu \left(\mathcal{D}_\mu \mathcal{J} + R_{\mu\nu} \frac{\partial \mathcal{J}}{\partial \Pi_\nu} \right) = 0. \tag{38}$$

Note that if $\mathcal{J}$ depends on the covariant momentum only via the Hamiltonian: $\mathcal{J}(x,\Pi,\psi) = \mathcal{J}(x,H,\psi)$, then the second term vanishes identically and the equation simplifies to

$$\Pi \cdot \mathcal{D}\, \mathcal{J} = 0. \tag{39}$$

In all other cases we require the full eq.(38). If we expand $\mathcal{J}$ in a power series in the covariant momentum

$$\mathcal{J} = \sum_{n=0}^{\infty} \frac{1}{n!}\, J^{(n)\,\mu_1 \ldots \mu_n}(x,\psi)\, \Pi_{\mu_1} \ldots \Pi_{\mu_n}, \tag{40}$$

then eq.(38) is satisfied for arbitrary Π_μ if and only if the components of $\mathcal{J}$ satisfy

$$D_{(\mu_{n+1}}\, J^{(n)}_{\mu_1 \ldots \mu_n)} + \omega_{(\mu_{n+1}\;\;b}^{\;\;\;\;\;\;\;a}\, \psi^b\, \frac{\partial J^{(n)}_{\mu_1 \ldots \mu_n)}}{\partial \psi^a} = R_{\nu(\mu_{n+1}}\, J^{(n+1)\;\;\nu}_{\mu_1 \ldots \mu_n)}, \tag{41}$$

where the parentheses denote full symmetrization over the indices enclosed. Eqs.(41) are the generalizations of the Killing equations to spinning space first obtained in [7]. An important aspect of these equations is, that with the exception of the case described by eq.(39), they couple spinning-space Killing tensors $J^{(n)}$ of different rank, unlike the case of Killing tensors in ordinary space.

We also observe, that any constant of motion $\mathcal{J}$ satisfies

$$\begin{aligned} \{\mathcal{Q}, \mathcal{J}\} &= -\psi^\mu \left(\mathcal{D}_\mu \mathcal{J} + R_{\mu\nu} \frac{\partial \mathcal{J}}{\partial \Pi_\nu} \right) - i\, e^{\mu a}\, \Pi_\mu \frac{\partial \mathcal{J}}{\partial \psi^a} \\ &= -\left(\psi \cdot \mathcal{D}\, \mathcal{J} + i\, \Pi \cdot \frac{\partial \mathcal{J}}{\partial \psi} \right), \end{aligned} \tag{42}$$

where the last line follows from the observation that the curvature term contains three contractions with the anti-commuting spin variables, in combination with the Bianchi identity $R_{[\mu\nu\lambda]\kappa} = 0$. Taking in particular $\mathcal{J} = \mathcal{Q}$ one finds

$$\{\mathcal{Q}, \mathcal{Q}\} = -2iH, \tag{43}$$

the conventional supersymmetry algebra. From this result and the Jacobi-identity for 2 $\mathcal{Q}$'s and any constant of motion $\mathcal{J}$ it follows, that

$$\Theta \equiv \{\mathcal{Q}, \mathcal{J}\} \tag{44}$$

is a superinvariant and hence a constant of motion as well:

$$\{\mathcal{Q}, \Theta\} = 0, \qquad \{H, \Theta\} = 0. \tag{45}$$

Thus we infer that constants of motion generally come in supermultiplets $(\mathcal{J}, \Theta)$, of which the prime example is the multiplet $(\mathcal{Q}, H)$ itself. The only exceptions to this result are the constants of motion whose bracket with the supercharge vanishes ($\Theta = 0$), but which are not themselves obtained from the bracket of $\mathcal{Q}$ with another constant of motion.

From eq.(42) it follows that a superinvariant is a solution of the equation

$$\{\mathcal{Q}, \mathcal{J}\} = -\left(\psi \cdot \mathcal{D}\,\mathcal{J} + i\,\Pi \cdot \frac{\partial \mathcal{J}}{\partial \psi} \right) = 0. \tag{46}$$

Let us write for $\mathcal{J}$ the series expansion

$$\mathcal{J}(x, \Pi, \psi) = \sum_{m,n=0}^{\infty} \frac{i^{\left[\frac{m}{2}\right]}}{m!n!}\, \psi^{a_1} ... \psi^{a_m}\, f^{(m,n)\,\mu_1 ... \mu_n}_{a_1 ... a_m}(x)\, \Pi_{\mu_1} ... \Pi_{\mu_n}, \tag{47}$$

where $f^{(n,m)}$ is completely symmetric in the n upper indices $\{\mu_k\}$ and completely anti-symmetric in the m lower indices $\{a_i\}$; one then obtains the component equation

$$n\, f^{(m+1,n-1)\,(\mu_1 ... \mu_{n-1}}_{a_0 a_1 ... a_m}\; e^{\mu_n)\, a_0} = m\, D_{[a_1}\, f^{(m-1,n)\,\mu_1 ... \mu_n}_{a_2 ... a_m]}, \tag{48}$$

where $D_a = e^{\mu}_{\;\;a} D_\mu$ are ordinary covariant derivatives, and indices in parentheses are to be symmetrized completely, whilst those in square brackets are to be anti-symmetrized, all with unit weight. Note in particular for $m = 0$:

$$f^{(1,n)\,(\mu_1 ... \mu_n}_{a}\; e^{\mu_{n+1})\, a} = 0. \tag{49}$$

In a certain sense these equations represent a square root of the generalized Killing equations, although they only provide sufficient, not necessary conditions for obtaining solutions. However, of each supermultiplet (singlet or non-singlet) at least *one* component is a solution of equation (46). Having found Θ we can then try to reconstruct the corresponding $\mathcal{J}$ by solving (44).

The content of eqs.(48) is twofold. On the one hand they partly solve $f^{(m+1,n-1)}$ in terms of $f^{(m-1,n)}$. Only that part of $f^{(m+1,n-1)}$ is solved which is symmetrized in one flat index and all $(n-1)$ curved indices. On the other hand eqs.(48) do not automatically imply that $f^{(m+1,n-1)}$ is completely anti-symmetric in the first $(m+1)$ indices. Imposing that condition on eqs.(48) one finds a new set of equations which are precisely the generalised Killing equations for that part of $f^{(m+1,n-1)}$ which was *not* given in terms of $f^{(m-1,n)}$, and which should still be solved for. This is the part of $f^{(m+1,n-1)}$ which is anti-symmetized in one curved index and all $(m+1)$ flat indices.

Hence eqs.(48) clearly have advantages over the generalized Killing equations (41). In order to find the constant of motion corresponding to a Killing tensor of rank n:

$$\mathcal{D}_{(\mu_{n+1}}\, \mathcal{J}^{(n)}_{\mu_1 \dots \mu_n)} \,=\, 0, \tag{50}$$

one has to solve the complicated hierarchy of partial differential equations (41) for $(\mathcal{J}^{(n-1)}, ..., \mathcal{J}^{(0)})$ and add the terms, as in expression (40). However, if we have a solution $f^{(m,n)\,\mu_1 \dots \mu_n}_{a_1 \dots a_m}$ of the equation

$$f^{(m,n)\,(\mu_1 \dots \mu_n}_{a_1 \dots a_m}\; e^{\mu_{n+1})a_1} \,=\, 0, \tag{51}$$

then we generate at least part of the components $f^{(m+2\alpha,n-\alpha)\,\mu_1 \dots \mu_{n-\alpha}}_{a_1 \dots a_{m+2\alpha}}$ for $\alpha = 1, ..., n$ by mere differentiation. Eq.(47) then gives us the corresponding constant of motion. In section 5 we consider an example in which these advantages become clear.

Finally we observe, that eqs.(44),(45) imply that the Poisson-Dirac bracket with $\mathcal{Q}$ defines a nilpotent operation in the space of constants of motion. Hence the supersinglets span the cohomology of the supercharge, whilst the supermultiplets $(\mathcal{J}, \Theta)$ form pairs of $\mathcal{Q}$-exact and $\mathcal{Q}$-coexact forms. The solutions of eq.(46) then correspond to the $\mathcal{Q}$-closed forms.

4 New Supersymmetries

The constants of motion generate infinitesimal transformations of the coordinates leaving the equations of motion invariant:

$$\delta x^{\mu} \,=\, \delta\alpha\, \left\{ x^{\mu}, \mathcal{J} \right\}, \qquad\qquad \delta\psi^{a} \,=\, \delta\alpha\, \left\{ \psi^{a}, \mathcal{J} \right\}, \tag{52}$$

with $\delta\alpha$ the infinitesimal parameter of the transformation. In particular, the action as defined by L, eq.(16), is invariant under the generic symmetries, such as supersymmetry:

$$\begin{aligned} \delta x^{\mu} &= i\epsilon \left\{ \mathcal{Q}, x^{\mu} \right\} \,=\, -i\epsilon\, e^{\mu}_{\;\;a}\, \psi^{a}, \\ \delta\psi^{a} &= i\epsilon \left\{ \mathcal{Q}, \psi^{a} \right\} \,=\, \epsilon\, e_{\mu}^{\;\;a}\, \dot{x}^{\mu} \,+\, \delta x^{\mu}\, \omega_{\mu\;\;b}^{\;\;a} \psi^{b}, \end{aligned} \tag{53}$$

where the infinitesimal parameter ϵ of the transformation is Grassmann-odd.

We now ask whether the theory might admit other (non-generic) supersymmetries of the type

$$\delta x^{\mu} \,=\, -i\epsilon\, f^{\mu}_{\;\;a}\, \psi^{a} \,\equiv\, -i\epsilon\, J^{(1)\mu}. \tag{54}$$

Such a transformation is generated by a phase-space function $\mathcal{Q}_f$

$$\mathcal{Q}_f \,=\, J^{(1)\mu}\, \Pi_{\mu} \,+\, J^{(0)}, \tag{55}$$

where $J^{(0,1)}(x,\psi)$ are independent of Π. Inserting this Ansatz into the generalized Killing equations (41), one finds

$$J^{(0)}(x,\psi) \,=\, \frac{i}{3!}\, c_{abc}(x)\, \psi^{a}\psi^{b}\psi^{c}, \tag{56}$$

with the tensors $f^{\mu}_{\;\;a}$ and c_{abc} satisfying the conditions

$$D_{\mu}\, f_{\nu a} \,+\, D_{\nu}\, f_{\mu a} \,=\, 0, \tag{57}$$

and

$$D_{\mu}\, c_{abc} \,=\, -\left(R_{\mu\nu ab}\, f^{\nu}_{\;\;c} \,+\, R_{\mu\nu bc}\, f^{\nu}_{\;\;a} \,+\, R_{\mu\nu ca}\, f^{\nu}_{\;\;b} \right). \tag{58}$$

If we have N symmetries of this kind, specified by N sets of tensors $(f^{\mu}_{i\;a}, c_{i\,abc})$, $i = 1, ..., N$, then the corresponding generators are

$$\mathcal{Q}_i \,=\, f^{\mu}_{i\,a}\,\Pi_\mu\,\psi^a \,+\, \frac{i}{3!}\,c_{i\,abc}\,\psi^a\psi^b\psi^c. \tag{59}$$

Observe, that the supercharge (33) is precisely of this form with $f^{\mu}{}_{a} = e^{\mu}{}_{a}$ and $c_{abc} = 0$. For notational convenience we sometimes refer to these quantities defining the standard supersymmetry by assigning them the index $i = 0$: $\mathcal{Q} = \mathcal{Q}_0$, $e^{\mu}{}_{a} = f^{\mu}_{0\,a}$, etc.

From the general result (29) one now obtains the following algebra of Poisson-Dirac brackets of the conserved charges $\mathcal{Q}_i$:

$$\{\mathcal{Q}_i, \mathcal{Q}_j\} \,=\, -\,2i\,Z_{ij}, \tag{60}$$

with

$$Z_{ij} \,=\, \frac{1}{2}\,K^{\mu\nu}_{ij}\,\Pi_\mu\Pi_\nu \,+\, I^{\mu}_{ij}\,\Pi_\mu \,+\, G_{ij}, \tag{61}$$

and

$$\begin{aligned}
K^{\mu\nu}_{ij} &= \frac{1}{2}\left(f^{\mu}_{i\,a}f^{\nu a}_{j} + f^{\nu}_{i\,a}f^{\mu a}_{j}\right), \\
I^{\mu}_{ij} &= \frac{i}{2}\,\psi^a\psi^b\,I^{\mu}_{ij\,ab} \\
&= \frac{i}{2}\,\psi^a\psi^b\,\left(f^{\nu}_{i\,b}D_\nu f^{\mu}_{j\,a} + f^{\nu}_{j\,b}D_\nu f^{\mu}_{i\,a} + \frac{1}{2}\,f^{\mu c}_{i}c_{j\,abc} + \frac{1}{2}\,f^{\mu c}_{j}c_{i\,abc}\right), \qquad (62) \\
G_{ij} &= -\frac{1}{4}\,\psi^a\psi^b\psi^c\psi^d\,G_{ij\,abcd} \\
&= -\frac{1}{4}\,\psi^a\psi^b\psi^c\psi^d\,\left(R_{\mu\nu ab}f^{\mu}_{i\,c}f^{\nu}_{j\,d} + \frac{1}{2}c_{i\,ab}{}^{e}c_{j\,cde}\right).
\end{aligned}$$

An explicit calculation shows, that $K_{ij\,\mu\nu}$ is a symmetric Killing tensor of 2nd rank :

$$D_{(\lambda}\,K_{ij\,(\mu\nu)} \,=\, 0, \tag{63}$$

whilst I^{μ}_{ij} is the corresponding Killing vector:

$$\mathcal{D}_{(\mu}\, I_{ij\;\nu)} \;=\; \frac{i}{2}\,\psi^a\psi^b\, D_{(\mu}\, I_{ij\;\nu)ab} \;=\; \frac{i}{2}\,\psi^a\psi^b\, R_{ab\lambda(\mu}\, K_{ij\;\nu)}^{\;\;\lambda}, \tag{64}$$

and G_{ij} the corresponding Killing scalar:

$$\mathcal{D}_\mu\, G_{ij} \;=\; -\frac{1}{4}\,\psi^a\psi^b\psi^c\psi^d\, D_\mu\, G_{ij\,abcd} \;=\; \frac{i}{2}\,\psi^a\psi^b\, R_{ab\lambda\mu}\, I_{ij}^{\lambda}. \tag{65}$$

Since the Grassmann-even phase-space functions Z_{ij} satisfy the generalized Killing equations, their bracket with the Hamiltonian vanishes and they are constants of motion:

$$\frac{dZ_{ij}}{d\tau} \;=\; 0. \tag{66}$$

Note that for $i = j = 0$ we reobtain the usual supersymmetry algebra:

$$\{\mathcal{Q}, \mathcal{Q}\} \;=\; -\,2i\, H, \tag{67}$$

where H is the Hamiltonian. The Z_{ij} with i or j not equal to zero correspond to new bosonic symmetries, unless $K_{ij}^{\mu\nu} \;=\; \lambda_{(ij)}\, g^{\mu\nu}$, with $\lambda_{(ij)}$ a constant (maybe zero). In that case the corresponding Killing vector I_{ij}^{μ} and Killing scalar G_{ij} vanish identically; moreover if $\lambda_{(ij)} \neq 0$, the corresponding supercharges close on the Hamiltonian, which proves the existence of a second supersymmetry of the standard type; we then have an N-extended supersymmetry, with $N \geq 2$. On the other hand, if there exists a second independent Killing tensor $K^{\mu\nu}$ not proportional to $g^{\mu\nu}$, then we obtain a genuine new type of supersymmetry.

Now, as proven in sect. 4, the bracket of a supersymmetry $\mathcal{Q}_i$ with the original supercharge $\mathcal{Q}$ vanishes, and hence $\mathcal{Q}_i$ is superinvariant, if and only if

$$K_{0i}^{\mu\nu} \;=\; f^{\mu}_{\;\;a}\, e^{\nu a} \,+\, f^{\nu}_{\;\;a}\, e^{\mu a} \;=\; 0. \tag{68}$$

In the language of $\mathcal{Q}$-cohomology, $\mathcal{Q}_i$ is $\mathcal{Q}$-closed; according to the discussion at the end of sect. 2 we can then construct the full constant of motion Z_{ij} directly by repeated differentiation of $f^{\mu}_{\;\;a}$.

Finally, since the Z_{ij} are symmetric in (ij) by construction we can diagonalize them and thus obtain an algebra[6]

[6]No summation over repeated indices is implied on the right-hand side.

$$\{\mathcal{Q}_i, \mathcal{Q}_j\} \,=\, -\,2i\,\delta_{ij}\,Z_i, \tag{69}$$

with $N+1$ conserved bosonic charges Z_i. If condition (68) is satisfied for all $\mathcal{Q}_i$, the first of these diagonal charges (with $i=0$) is the Hamiltonian: $Z_0 = H$.

5 Properties of the f-symbols

In order to study the properties of the new supersymmetries, we now turn our attention to the quantities $f^{\mu}_{\;\;a}$. It is convenient to introduce the 2nd rank tensor

$$f_{\mu\nu} \,=\, f_{\mu a}\, e_{\nu}^{\;\;a}, \tag{70}$$

which will be refered to as the f-symbol. The defining relation (57) implies

$$D_\nu\, f_{\lambda\mu} \,+\, D_\lambda\, f_{\nu\mu} \,=\, 0. \tag{71}$$

It follows that the f-symbol is divergence-less on its first index

$$D_\nu\, f^{\nu}_{\;\;\mu} \,=\, 0. \tag{72}$$

By contracting of eq.(71) one finds

$$D_\nu\, f_{\mu}^{\;\;\nu} \,=\, -\partial_\mu\, f_{\nu}^{\;\;\nu}. \tag{73}$$

Hence the divergence on the second index vanishes if and only if the trace of the f-symbol is constant:

$$D_\nu\, f_{\mu}^{\;\;\nu} \,=\, 0 \quad\Leftrightarrow\quad f_{\mu}^{\;\;\mu} \,=\, const. \tag{74}$$

Now observe, that the metric tensor $g_{\mu\nu}$ is a trivial solution of eq.(71); therefore if the trace is constant, it maybe subtracted from the f-symbol without spoiling condition (71). It follows, that in this case one may without loss of generality always take the constant equal to zero and hence f to be traceless.

The symmetric part of the ith f-symbol is the tensor

$$S_{\mu\nu} \,\equiv\, K_{i0\,\mu\nu} \,=\, \frac{1}{2}\,\left(f_{\mu\nu} + f_{\nu\mu}\right), \tag{75}$$

defined in the first eq.(62) with $f_{0\,a}^{\mu} = e^{\mu}_{\;\;a}$. As was discussed there, it satisfies the generalized Killing equation

$$D_{(\mu}\, S_{\nu\lambda)} \,=\, 0. \tag{76}$$

We can also construct the anti-symmetric part

$$B_{\mu\nu} \,=\, -B_{\nu\mu} \,=\, \frac{1}{2}\,\left(f_{\mu\nu} - f_{\nu\mu}\right). \tag{77}$$

It obeys the condition

$$D_{\nu}\, B_{\lambda\mu} \,+\, D_{\lambda}\, B_{\nu\mu} \,=\, D_{\mu}\, S_{\nu\lambda}. \tag{78}$$

It follows, that if the symmetric part does not vanish and is not covariantly constant, then the anti-symmetric part $B_{\mu\nu}$ by itself is *not* a solution of eq.(71). But by the same token the anti-symmetric part of f can not vanish either, hence f can be completely symmetric only if it is covariantly constant.

It is of considerable interest to study the case in which the f-symbol is completely anti-symmetric: $f_{\mu\nu} \,=\, B_{\mu\nu}$. This is precisely the condition (68) for the supercharge $\mathcal{Q}_f$ to anti-commute with ordinary supersymmetry in the sense of Poisson-Dirac brackets. In this case also eq.(74) is satisfied automatically.

For anti-symmetric $f_{\mu\nu}$ it is possible to say much more about the explicit form of the quantities that were introduced above. First of all, if the symmetric part of a certain $f_{i\,\mu\nu}$ vanishes:

$$S_i^{\mu\nu} \,=\, K_{i0}^{\mu\nu} = 0, \tag{79}$$

then the corresponding Killing vector I_{i0}^{μ} and the Killing scalar G_{i0} vanish as well. Hence for this particular value of i the complete $Z_{i0} = 0$, and therefore

$$\{\mathcal{Q}_i, \mathcal{Q}\} \,=\, 0, \tag{80}$$

showing that $\mathcal{Q}_i$ is superinvariant. To prove these assertions, we first note that for anti-symmetric $f^{\mu\nu}$ eq.(71) becomes

$$D_{\nu}\, B_{\lambda\mu} \,=\, -D_{\lambda}\, B_{\nu\mu}. \tag{81}$$

Together with the anti-symmetry of $B_{\mu\nu}$ it follows that the gradient is completely anti-symmetric:

$$D_\mu\, B_{\nu\lambda} \,=\, D_{[\mu}\, B_{\nu\lambda]} \,\equiv H_{\mu\nu\lambda}. \tag{82}$$

By taking the second covariant derivative of $f_{\mu\nu}$, commuting the derivatives and using eq.(71) we derive the identity

$$D_\mu\, D_\nu\, f_{\lambda\kappa} \,=\, R_{\nu\lambda\mu}{}^{\sigma}\, f_{\sigma\kappa} \,+\, \frac{1}{2}\, \left(R_{\nu\lambda\kappa}{}^{\sigma}\, f_{\mu\sigma} \,+\, R_{\mu\lambda\kappa}{}^{\sigma}\, f_{\nu\sigma} \,-\, R_{\mu\nu\kappa}{}^{\sigma}\, f_{\lambda\sigma} \right). \tag{83}$$

For the special case of anti-symmetric $f_{\mu\nu}$ this implies

$$D_\mu\, H_{\nu\lambda\kappa} \,=\, \frac{1}{2}\, \left(R_{\nu\lambda\mu}{}^{\sigma}\, f_{\sigma\kappa} \,+\, R_{\lambda\kappa\mu}{}^{\sigma}\, f_{\sigma\nu} \,+\, R_{\kappa\nu\mu}{}^{\sigma}\, f_{\sigma\lambda} \right). \tag{84}$$

Comparison with eq.(58) shows, that

$$-\,\frac{1}{2}\, c_{abc} \,=\, H_{abc} \,=\, e^{\mu}{}_{a} e^{\nu}{}_{b} e^{\lambda}{}_{c}\, H_{\mu\nu\lambda}, \tag{85}$$

modulo a covariantly constant term. This result is an instance of eq.(48) with $n = 1$, $m = 2$. Note that we can always choose the covariantly constant term to vanish, since in order to construct a constant of motion we only need a particular solution of eq.(58).

As a side-remark we note, that if a covariantly constant three-index tensor c_{abc} exists, then it always provides us with another symmetry, corresponding to the Killing vector

$$I_\mu \,=\, \frac{i}{2}\, \psi^a \psi^b\, e_{\mu}{}^{c}\, c_{abc}. \tag{86}$$

More precisely, if $D_\mu\, c_{abc} \,=\, 0$ then

$$\mathcal{D}_\mu\, I_\nu \,=\, 0, \tag{87}$$

and the generalized Killing equation is automatically satisfied for I_μ. In this case we are free to add the term with c_{abc} to the supercharge, but it is not required, since both terms are conserved separately.

Returning to eq.(80), we first observe that according to eq.(79) $K_{0i}^{\mu\nu} = 0$. Moreover, since $c_{0\,abc} = 0$ identically, the r.h.s. of the second of eqs.(62) now becomes

$$I_{i0\,\mu\nu\lambda} \equiv I_{i0\,\mu ab}\, e_{\nu}^{\;\;a}\, e_{\lambda}^{\;\;b} = D_{\lambda}\, B_{i\,\mu\nu} + \frac{1}{2}\, c_{i\,\mu\nu\lambda} = 0. \tag{88}$$

The last equality follows from eq.(85). Finally, the Killing scalar G_{i0} vanishes because of the cyclic Bianchi identity for the Riemann tensor $R_{\mu\nu\lambda\kappa}$ and the vanishing of at least one of the three-index tensors: $c_{0\,abc} = 0$. This proves eq.(80).

Anti-symmetric f-symbols and their corresponding Killing-tensors have been studied extensively in refs.[10, 13] in the related context of finding solutions of the Dirac-equation in non-trivial curved space-time. They correspond precisely to the Killing-Yano and Stackel-Killing tensors described in these papers. The analysis presented here shows, that they belong to a larger class of possible structures which generate generalized supersymmetry algebras.

6 The Kerr-Newman metric

In this section we apply the results obtained previously to show that a new kind of supersymmetry exists in spinning Kerr-Newman space.

The gravitational and electromagnetic field of a rotating particle with mass M and charge Q are described by the Kerr-Newmann metric, which reads

$$\begin{aligned} ds^2 &= -\frac{\Delta}{\rho^2}\left[dt - a\sin^2\theta\, d\varphi\right]^2 + \frac{\sin^2\theta}{\rho^2}\left[(r^2+a^2)d\varphi - a dt\right]^2 + \\ &\quad + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2, \end{aligned} \tag{89}$$

and the electromagnetic field tensor

$$\begin{aligned} F &= \frac{Q}{\rho^4}(r^2 - a^2\cos^2\theta)dr \wedge \left[dt - a\sin^2\theta\, d\varphi\right] + \\ &+ \frac{2Qar\cos\theta\sin\theta}{\rho^4} d\theta \wedge \left[-a dt + (r^2+a^2)d\varphi\right]. \end{aligned} \tag{90}$$

Here

$$\begin{aligned}\Delta &= r^2 + a^2 - 2Mr + Q^2, \\ \rho^2 &= r^2 + a^2 \cos^2\theta,\end{aligned} \tag{91}$$

and the total angular momentum is $J = Ma$. The expressions for ds^2 and F only describe the fields *outside* the horizon, which is located at

$$r = M + \sqrt{M^2 - Q^2 - a^2}. \tag{92}$$

As was found by Carter [11], the Kerr-Newman space has two independent second rank Killing tensors. The metric tensor $g_{\mu\nu}$, here defined by eq.(89), is a Stackel-Killing tensor for any geometry and the corresponding conserved quantity is the Hamiltonian H. Furthermore there exists another Stackel-Killing tensor $K_{\mu\nu}$, which corresponds to a conserved quantity Z. A supersymmetric extension of this result, applying to spinning particles, is based on the anti-symmetric Killing-Yano tensor $f_{\mu\nu}$ found by Penrose and Floyd [14], which satisfies eq.(71):

$$D_\lambda f_{\mu\nu} + D_\mu f_{\lambda\nu} = 0,$$

and the covariant square of which is exactly the Stackel-Killing tensor $K_{\mu\nu}$. The new supersymmetry in spinning Kerr-Newmann space is then generated by a supercharge of the form given in eq.(59), with the Killing-Yano tensor as the f-symbol of the double vector ${f_\mu}^a$

$${f_\mu}^a = f_{\mu\nu} e^{\nu a},$$

and a corresponding three-index tensor c_{abc} obtained as in eq.(85).

We now calculate the explicit expression for the new supercharge and use this to find the Killing vector I_μ and the Killing scalar G which correspond to the Stackel-Killing tensor $K_{\mu\nu}$ in spinning Kerr-Newman space and which define the corresponding conserved charge Z.

The Killing-Yano tensor is defined by [14]

$$\begin{aligned}\frac{1}{2} f_{\mu\nu}\, dx^\mu \wedge dx^\nu &= a\cos\theta\, dr \wedge \left[dt - a\sin^2\theta\, d\varphi \right] + \\ &+ r\sin\theta\, d\theta \wedge \left[-a dt + (r^2 + a^2) d\varphi \right].\end{aligned} \tag{93}$$

Using the expressions for the vielbein $e_\mu{}^a(x)$ corresponding to the metric given in eq.(89)

$$
\begin{aligned}
e_\mu{}^0\, dx^\mu &= -\frac{\sqrt{\Delta}}{\rho}\left[dt - a\sin^2\theta\, d\varphi\right], \\
e_\mu{}^1\, dx^\mu &= \frac{\rho}{\sqrt{\Delta}} dr, \\
e_\mu{}^2\, dx^\mu &= \rho d\theta, \\
e_\mu{}^3\, dx^\mu &= \frac{\sin\theta}{\rho}\left[-adt + (r^2 + a^2) d\varphi\right],
\end{aligned}
\tag{94}
$$

one finds the following components of $f_\mu{}^a(x)$

$$
\begin{aligned}
f_\mu{}^0\, dx^\mu &= \frac{\rho}{\sqrt{\Delta}}\, a\cos\theta dr, \\
f_\mu{}^1\, dx^\mu &= -\frac{\sqrt{\Delta}}{\rho}\, a\cos\theta\left[dt - a\sin^2\theta\, d\varphi\right], \\
f_\mu{}^2\, dx^\mu &= -\frac{r\sin\theta}{\rho}\left[-adt + (r^2 + a^2) d\varphi\right], \\
f_\mu{}^3\, dx^\mu &= \rho r d\theta
\end{aligned}
\tag{95}
$$

One can check that this $f_\mu{}^a(x)$ indeed satisfies eq.(57). Finally, to find a conserved quantity we need to calculate $c_{abc}(x)$ from eq.(85). Its components read

$$
\begin{aligned}
c_{012} &= \frac{2a\sin\theta}{\rho} \qquad & c_{013} &= 0 \\
c_{023} &= 0 \qquad & c_{123} &= -\frac{2\sqrt{\Delta}}{\rho}
\end{aligned}
\tag{96}
$$

Inserting the quantities given in eqs.(95),(96) into eq.(59) we obtain the new supersymmetry generator $\mathcal{Q}_f$ for spinning Kerr-Newman space. From this expression and using eqs.(62) we can construct the Killing tensor, vector and scalar which define the conserved charge $Z = i/2\, \{\mathcal{Q}_f, \mathcal{Q}_f\}$. The results are

$$\begin{aligned}
K_{\mu\nu}(x)\, dx^\mu\, dx^\nu &= -\frac{\rho^2 a^2 \cos^2\theta}{\Delta} dr^2 + \frac{\Delta a^2 \cos^2\theta}{\rho^2}\left[dt - a\sin^2\theta\, d\varphi\right]^2 + \\
&\quad + \frac{r^2\sin^2\theta}{\rho^2}\left[-adt + (r^2+a^2)d\varphi\right]^2 + \rho^2 r^2 d\theta^2, \qquad (97)\\
I_\mu(x)\, dx^\mu &= \frac{2i}{\rho^2}\left[r\sin\theta\,\psi^1 + \sqrt{\Delta}\cos\theta\,\psi^2\right]\left[a\sin\theta\,\psi^0 - \sqrt{\Delta}\,\psi^3\right]\times \\
&\quad \times \left[-adt + (r^2+a^2)d\varphi\right] + \\
&\quad - i\sqrt{\Delta}\cos\theta\,\psi^2\left[a\sin\theta\,\psi^0 - \sqrt{\Delta}\,\psi^3\right] d\varphi \\
&\quad + i\sqrt{\Delta}\left[r\sin\theta\,\psi^1 + \sqrt{\Delta}\cos\theta\,\psi^2\right]\psi^3 d\varphi + \\
&\quad + \frac{ia\sin\theta}{\sqrt{\Delta}}\left[r\,\psi^0\psi^3 + a\cos\theta\,\psi^1\psi 2\right] dr \\
&\quad + i\sqrt{\Delta}\left[a\cos\theta\,\psi^0\psi^3 - r\,\psi^1\psi^2\right] d\theta, \qquad (98)\\
G &= -\frac{2Qa\cos\theta}{\rho^2}\,\psi^0\psi^1\psi^2\psi^3. \qquad (99)
\end{aligned}$$

The physical interpretation of these equations may become more clear if we recall, that the anti-commuting spin variables are related to the standard anti-symmetric spin tensor S^{ab}, which appears in the definition of the generators of the local Lorentz transformations, by

$$S^{ab} = -i\,\psi^a\psi^b. \qquad (100)$$

Indeed, from the Dirac-Poisson brackets (25) it can be verified straightforwardly that they satisfy the SO(3,1) algebra. The full Lorentz transformations are then generated by $M^{ab} = L^{ab} + S^{ab}$, L^{ab} being the orbital part. Like the generators of the Lorentz algebra, the generators of other symmetries like Z now also receive spin-dependent contributions. The Killing tensor $K_{\mu\nu}$ given in (97) is the one which was found in [2]. For spinless particles in Kerr-Newman space it defines a constant of motion directly, whereas for spinning particles it now requires the non-trivial contributions from spin which involve the Killing vector and Killing scalar computed above.

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