

Existence of a new instanton in constrained Yang-Mills-Higgs theory

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Abstract

Our goal is to discover possible new 4-dimensional euclidean solutions (instantons) in fundamental $SU(2)$ Yang-Mills-Higgs theory, with a constraint added to prevent collapse of the scale. We show that, most likely, there exists one particular new constrained instanton (I^*) with vanishing Pontryagin index. This is based on a topological argument that involves the construction of a non-contractible loop of 4-dimensional configurations with a certain upperbound on the action, which we establish numerically. We expect I^* to be the lowest action non-trivial solution in the vacuum sector of the theory. There also exists a related static, but unstable, solution, the new sphaleron S^* . Possible applications of I^* to the electroweak interactions include the asymptotics of perturbation theory and the high-energy behaviour of the total cross-section.

1 Introduction

We have conjectured [1] the existence of a new constrained instanton in the vacuum sector of euclidean $SU(2)$ Yang-Mills-Higgs theory. Our argument consists of two steps :

- (1) the construction of a non-contractible loop of 4-dimensional configurations of the Yang-Mills and Higgs fields, starting and ending at the classical vacuum;
- (2) the proof of a certain upperbound on the action over this non-contractible loop, which is crucial for having a genuine new solution, as will be explained later on.

For the last step, which is by far the most difficult of the two, we have to resort to numerical methods. The numerical results of our previous paper [1] were not entirely conclusive in establishing this upperbound on the actionprofile. Here, we present further numerical results that *are* conclusive, at least for certain values of the parameters in the theory.

Compared to our earlier work we have made the following improvements. First, the constraint on the scale is treated dynamically, whereas before the scale was fixed by hand. Second, the ansatz of the non-contractible loop is generalized, in order to give the fields more freedom to relax to a lower value of the action. Third, the efficiency and accuracy of the numerical methods for solving the variational equations from the ansatz were increased significantly. The combined effect of these three improvements allows us to establish the upperbound on the actionprofile, which turns out to be a rather delicate affair. Having established this upperbound does not rigorously prove the existence of a new instanton, but makes it very likely in our opinion. In that case, we also have, from our results for the non-contractible loop, an approximation of the exact solution I^* .

The present paper is primarily concerned with the existence of a new classical solution, and we refer the reader to [1] for the physics that motivates our search. The outline of the present article is as follows. In sect. 2 we discuss the constraint on the scale and what solutions precisely we are after. In sect. 3 we outline a general strategy for the search of new solutions, which is to find non-contractible loops in configuration space. One particular non-contractible loop is presented in sect. 4. First, we discuss, in subsect. 4.1, the basic ingredients of our construction; then, we give, in subsect. 4.2, the ansatz in full detail. The general behaviour of the action over this non-contractible loop is discussed in subsect. 4.3 . The actual numerical results for the actionprofile are presented in subsect. 5.2. These results show that, for all values of the Higgs mass (but not the gauge coupling), the

action profile holds, which is the main result of this paper. As a byproduct we obtain some numerical results for the well-known BPSTH instanton [2, 3]; these results are given in subsect. 5.1. Finally, we return, in sect. 6, to the possible existence of the I^* solution in $SU(2)$ Yang-Mills-Higgs theory and compare this with the situation in other theories. We also discuss, very briefly, potential applications of I^* to the physics of the electroweak interactions. There are two appendices. In the first of these, Appendix A, we give an outline of the calculations behind the numerical results of sect. 5. In the second, Appendix B, we describe a similar calculation for a static, 3-dimensional solution, the new sphaleron S^* , which is directly related to the new instanton I^* . The reader who is not interested in the details is advised to concentrate on sects. 2 and 3, possibly subsect. 4.1, and sect. 6.

2 Constrained Instantons

We consider a Yang-Mills-Higgs theory with $SU(2)$ gauge fields W and a single, complex Higgs doublet Φ . The euclidean action of this theory is

$$A_{\text{YMH}} = \int_{\mathbb{R}^4} d^4x \left[-\frac{1}{g^2} \frac{1}{2} \text{Tr} W_{\mu\nu}^2 + |D_\mu \Phi|^2 + \lambda \left(|\Phi|^2 - \frac{v^2}{2} \right)^2 \right], \quad (1)$$

where $W_{\mu\nu} \equiv \partial_\mu W_\nu - \partial_\nu W_\mu + [W_\mu, W_\nu]$, $D_\mu \equiv \partial_\mu + W_\mu$ and $W_\mu \equiv W_\mu^a \sigma^a / (2i)$, with σ^a the standard Pauli matrices. Here, and in the following, Greek indices run from 0 to 3 and Latin indices from 1 to 3. The masses of the three W vector bosons and the single Higgs scalar H are $M_W = gv/2$ and $M_H = \sqrt{2\lambda}v$. Finite action gauge field configurations can be classified by the Pontryagin index (or topological charge), which takes on integer values and is defined by

$$Q \equiv \frac{1}{8\pi^2} \int d^4x \left[-\frac{1}{2} \text{Tr} \left(\frac{1}{2} \epsilon_{\kappa\lambda\mu\nu} W_{\kappa\lambda} W_{\mu\nu} \right) \right]. \quad (2)$$

A simple scaling argument shows the absence of non-singular 4-dimensional classical solutions for the action (1): any configuration can lower its action by collapse to a point. For this reason one introduces a constraint on the scale and later integrates over the corresponding collective coordinate ρ . A convenient way to implement the constraint is to require [4]

$$\int d^4x O_d = 8 \pi^2 c \rho^{4-d}, \quad (3)$$

where $O_d = O_d(W, \Phi)$ is a field operator with canonical mass dimension $d > 4$. With a Lagrange multiplier $\tilde{\kappa}$ one then looks for the stationary points of

$$A_{\text{YMH}} + \tilde{\kappa} \left(\rho^{d-4} \int d^4x O_d - 8 \pi^2 c \right). \quad (4)$$

Concretely, we proceed as follows. First, we have to chose the constraint operator O_d . The actual choice is irrelevant in the end, as long as we integrate, in the path integral, over the collective coordinate ρ (with the appropriate Jacobian). For purely technical reasons, to be explained later on, we have chosen to work with the operator

$$O_8 \equiv \left(-\frac{1}{2} \text{Tr} \frac{1}{2} \epsilon_{\kappa\lambda\mu\nu} W_{\kappa\lambda} W_{\mu\nu} \right)^2. \quad (5)$$

Having made this choice, we solve the field equations that result from variations δW and $\delta \Phi$ in the constrained action

for a given value of the dimensionless coupling constant κ . Let us denote the fields of one particular solution by W^* and Φ^* , with a corresponding constrained action A^* . The scale of this solution is then determined by

$$\rho^* = \left[\frac{1}{8\pi^2 c} \int d^4x \text{O}_8(W^*) \right]^{-1/4} \quad (7)$$

and its Yang-Mills-Higgs action by

$$A_{\text{YMH}}^* = A_{\text{YMH}}(W^*, \Phi^*) . \quad (8)$$

These values ρ^* and A_{YMH}^* are functions of the coupling constant κ , as the fields W^* and Φ^* are. Eliminating κ one obtains A_{YMH}^* as a function of ρ^* .

This procedure is entirely straightforward. The problem is to *discover* the solutions of the field equations from the constrained action (6). Any such 4-dimensional euclidean solution will be called a “constrained instanton”.¹ The prototypical constrained instanton is the solution of Belavin et al. and ’t Hooft [2, 3]. We denote this basic instanton by I_{BPSTH} , or I for short. Its fields are given by (using differential forms)

$$\begin{aligned} W &= -f_I(x) dU_I U_I^{-1} \\ \Phi &= h_I(x) \frac{v}{\sqrt{2}} U_I \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ U_I &= \hat{x} \cdot \sigma , \end{aligned} \quad (9)$$

with $x^2 \equiv x_\mu x_\mu$, $\hat{x}_\mu \equiv x_\mu/x$ and $\sigma_\mu \equiv (1, i\sigma_m)$. The instanton has Pontryagin index $Q_I = 1$. By reflection there is also an anti-instanton \bar{I} , with equal action but opposite topological charge $Q_{\bar{I}} = -1$. Only in the limit $\rho, \lambda \rightarrow 0$, is it possible to find an analytical solution for the radial functions in the ansatz [2, 3]

$$\begin{aligned} f_I &= \frac{x^2}{x^2 + \rho^2} \\ h_I &= \sqrt{f_I} , \end{aligned} \quad (10)$$

with a corresponding action

$$A_{\text{YMH}} = \left(1 + \frac{1}{2} (\rho M_W)^2 \right) \frac{8\pi^2}{g^2} + \text{O}(\lambda) . \quad (11)$$

¹ In pure Yang-Mills theory “instantons” are sometimes meant to refer exclusively to solutions of the first-order self-duality equations, which then solve the second-order field equations by the

In sect. 5.1 we will give some numerical results for the action at finite values of ρ and λ .

In this article we look for constrained instantons that are not related to the BPST instanton ($|Q| = 1$) [2] or other self-dual solutions ($|Q| > 1$) [5] of pure Yang-Mills theory. In fact, our search is for solutions in the *vacuum sector* ($Q = 0$) of the constrained Yang-Mills-Higgs theory (6). No such solutions are known at present.

3 Strategy

Explicit construction of instantons in the vacuum sector of the constrained theory is not feasible at present. Instead, we present a topological argument for the possible existence of at least one such solution. This topological argument involves the construction of a suitable non-contractible loop (NCL) of 4-dimensional configurations of the fields W and Φ , starting and ending at the classical vacuum. In addition, we may hope to gain some insight into the structure of this solution, preliminary to an explicit construction of it.

The presence of non-contractible loops in configuration space implies, as Taubes [6] has shown in a somewhat different context, the existence of new solutions of the classical field equations. The intuitive idea is that by “shrinking” the NCL it gets “stuck” at a point in configuration space, which corresponds (or is close) to a stationary point of the action, i.e. a new solution of the field equations. Specifically, this is a mini-max procedure, where we take the maximum action on the NCL and try to minimize that action. It is essential for this topological argument that the euclidean action density (6), with positive coupling constants λ and κ , is a positive semi-definite functional of the fields, or, in other words, that it is bounded from below. In mathematics this general subject is called Morse theory, which aims to relate the critical points of a functional to the topology of the function space on which the functional is defined. The method of finding these stationary points by a mini-max principle on non-contractible loops goes under the name of Ljusternik-Snirelman theory. References to the mathematical literature can be found in [6].

For the case at hand there is, however, one obvious loophole in the argument. It could be, namely, that the mini-max procedure leads to an approximate solution consisting of the BPSTH instanton solution I and anti-instanton \bar{I} at infinite separation. In that case there would be no genuine new solution. Clearly, this possibility is ruled out if we are able to construct a particular NCL for which

$$\max A_{\text{NCL}}(\omega) < 2 A_I, \tag{12}$$

where ω parametrizes the position along the loop. This upperbound on the action-profile is a necessary condition for an existence proof of I^* and the main goal of this article is to establish it.

4 Non-contractible loop

In this section we present an ansatz for a non-contractible loop of configurations. We start with the basic ingredients that go into the construction, then give the details of the ansatz, and, finally, discuss the expected behaviour for the action over the loop. The actual profile of the action over the NCL has to be determined numerically, these results will be presented in sect. 5.

4.1 Basic ingredients

The first step in the construction of the NCL is to give the structure of the fields at infinity ($|x| \rightarrow \infty$). This will be based on a non-trivial mapping

$$\tilde{U} : S_3 \times S_1 \rightarrow SU(2) \sim S_3 , \quad (13)$$

where the first S_3 refers to the hypersphere at infinity, S_1 to the loop of configurations and $SU(2)$ to the gauge group. The mapping should belong to the non-trivial homotopy class in $\pi_4(S_3) = Z_2$, so that, later on, we have a loop of configurations that is indeed non-contractible. A specific choice for the $SU(2)$ matrix \tilde{U} is the following

$$\begin{aligned} \tilde{U} &= \exp[(\omega + \pi/2)i\sigma_3] (\hat{x} \cdot \sigma) \exp[-(\omega + \pi/2)i\sigma_3] (\hat{x} \cdot \sigma)^\dagger \\ \hat{x}_\mu &\equiv x_\mu / |x| \\ \sigma_\mu &\equiv (1, i\vec{\sigma}) , \end{aligned} \quad (14)$$

with a loopparameter $\omega \in [-\pi/2, +\pi/2]$. The fields at infinity (pure gauge, of course) are given by

$$\begin{aligned} W]_\infty &= -d\tilde{U} \tilde{U}^{-1} \\ \Phi]_\infty &= \frac{v}{\sqrt{2}} \tilde{U} \begin{pmatrix} 0 \\ 1 \end{pmatrix} . \end{aligned} \quad (15)$$

The second step is to extend these fields inwards. For this purpose we introduce two radial functions $\tilde{f}(x)$ and $\tilde{h}(x)$, which approach 1 at infinity and vanish at the origin, in order to ensure continuity. Also, we extend the range of the loop parameter ω to $[-\pi, +\pi]$, and make it into a real loop, starting and ending at the *same* point, i.e. the vacuum at $\omega = \pm\pi$. In this way we arrive at the following NCL of configurations

$$\begin{aligned}
\omega \in [-\pi, -\pi/2] \cup [+\pi/2, +\pi] : W &= 0 \\
\Phi &= (\tilde{h} \sin^2 \omega + \cos^2 \omega) \frac{v}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
\omega \in [-\pi/2, +\pi/2] : W &= -\tilde{f} d\tilde{U} \tilde{U}^{-1} \\
\Phi &= \tilde{h} \frac{v}{\sqrt{2}} \tilde{U} \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{16}
\end{aligned}$$

where the radial functions $\tilde{f}(x)$ and $\tilde{h}(x)$ have the boundary conditions

$$\begin{aligned}
\lim_{|x| \rightarrow \infty} \tilde{f}, \tilde{h} &= 1 \\
\tilde{f}(0) = \tilde{h}(0) &= 0. \tag{17}
\end{aligned}$$

The precise form of these radial functions $\tilde{f}(x)$ and $\tilde{h}(x)$ is arbitrary in principle, but the mini-max idea is to optimize them at the maximum point ($\omega = 0$) on the NCL. This is done by solving the variational equations for $\tilde{f}(x)$ and $\tilde{h}(x)$, that result from inserting the ansatz (16) with $\omega = 0$ in the constrained action (6) and making variations $\delta\tilde{f}$ and $\delta\tilde{h}$.

The NCL (16) is quite elegant, but not good enough for our purpose. The reason is that the actionprofile $A(\omega)$ has a maximum value $A(0)$ definitely greater than $2A_I$. Hence, the inequality (12) does not hold for the simplest possible NCL. This brings us to the third, and final, step in the construction of a suitable NCL. We start from the observation that there are really two ‘‘cores’’ in (14), each of which resembles the (anti)instanton (\bar{I}) I as given in (9). The idea now is to separate these cores, in order to profit from the attraction of the long-range fields present in the theory, the Higgs field in our case. This improved NCL, parametrized by $\omega \in [-3\pi/2, +3\pi/2]$, has the following structure

- Ia : $\omega \in [-3\pi/2, -\pi]$ build up of an initial Higgs configuration;
- IIa : $\omega \in [-\pi, -\pi/2]$ creation and separation of an $I\bar{I}$ pair;
- III : $\omega \in [-\pi/2, +\pi/2]$ relative isospin rotation of the $I\bar{I}$ pair;
- IIb : $\omega \in [+\pi/2, +\pi]$ collapse and annihilation of the $I\bar{I}$ pair;
- Ib : $\omega \in [+\pi, +3\pi/2]$ clean up of the remaining Higgs configuration.

The explicit construction of the NCL is somewhat involved and will given in the

4.2 Ansatz

The configurations of the NCL are given by

$$\begin{aligned}
\text{I} & : W = 0 \\
& \Phi = (h \cos^2 \omega + \sin^2 \omega) \frac{v}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\
\text{II, III} & : W = -f dU U^{-1} \\
& \Phi = h \frac{v}{\sqrt{2}} U \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{18}
\end{aligned}$$

with the following $SU(2)$ matrix

$$U = \exp[\Omega i \sigma_3] (\hat{y}_- \cdot \sigma) \exp[-\Omega i \sigma_3] (\hat{y}_+ \cdot \sigma)^\dagger \tag{19}$$

and notation

$$\begin{aligned}
\hat{y}_{\mu\pm} & \equiv y_{\mu\pm} / |y_{\pm}| & \sigma_\mu & \equiv (1, i \sigma_m) \\
y_{m\pm} & \equiv g_\pm x_m & y_{0\pm} & \equiv x_{0\pm} \equiv x_0 \pm D/2 \\
x_\pm^2 & \equiv x_{0\pm}^2 + r^2 & r^2 & \equiv x_m^2 \equiv \rho^2 + z^2 \\
z & \equiv x_3 & t & \equiv x_0.
\end{aligned}$$

Here $D = D(\omega)$ determines the core distance and $\Omega = \Omega(\omega)$ their relative isospin rotation

$$\begin{aligned}
\text{Ia} & : \omega \in [-3\pi/2, -\pi] & D = 0 & \Omega = 0 \\
\text{IIa} & : \omega \in [-\pi, -\pi/2] & D = d_{\max} \sin^2 \omega & \Omega = 0 \\
\text{III} & : \omega \in [-\pi/2, +\pi/2] & D = d_{\max} & \Omega = \omega + \pi/2 \\
\text{IIb} & : \omega \in [+ \pi/2, +\pi] & D = d_{\max} \sin^2 \omega & \Omega = \pi \\
\text{Ib} & : \omega \in [+ \pi, +3\pi/2] & D = 0 & \Omega = \pi.
\end{aligned} \tag{20}$$

The functions f , h and g_\pm in (18, 19) are taken to be axial functions $f(r, t)$, $h(r, t)$ and $g_\pm(r, t)$, with the following boundary conditions

$$\begin{aligned}
\lim_{|x| \rightarrow \infty} f, h, g_\pm & = 1 \\
f(0, \pm D/2) = h(0, \pm D/2) & = 0 \tag{21}
\end{aligned}$$

and reflection symmetry

$$f(r, t) = f(r, -t)$$

$$h(r, t) = h(r, -t)$$

The Pontryagin index (2) vanishes for all configurations of the NCL, precisely because of this reflection symmetry of the ansatz. This completes the basic construction of the NCL. We remark that the only difference compared to our previous paper [1] is the presence of the functions g_{\pm} in the matrix U (19), but this change will turn out to be essential.

It remains to specify the four axial functions $f(r, t)$, $h(r, t)$ and $g_{\pm}(r, t)$ that enter the ansatz. Just as for the simple NCL of the previous subsection, the mini-max procedure would be to insert the ansatz (18), for $\omega = 0$ and $D = d_{\max}$, into the action (6) and solve the variational equations for f, h and g_{\pm} . This turns out to be prohibitively difficult and, instead, we make an explicit choice for g_{\pm} . As will be explained in the next subsection, we want the cores to become independent for large values of their separation D . This can be achieved by the following choice, for example,

$$g_{\pm} = \frac{(x_{\mp}^2 D^2/4)^{\alpha} + (\beta \rho_{\text{I}})^{4\alpha}}{(x_{\mp}^2 D^2/4)^{\alpha} + (\beta \rho_{\text{I}})^{4\alpha} + (D/2)^{4\alpha}} \quad (23)$$

with x_{\pm}^2 defined below (19) and the parameter values $\alpha = 4.0$ and $\beta = 1.5$, obtained by trial and error. Here $\rho_{\text{I}} = \rho_{\text{I}}(\kappa)$ is the scale for the BPSTH instanton I, see sect. 5.1 below. Having fixed the functions g_{\pm} , we can solve numerically the variational equations for f and h . We will do this for both $\omega = 0$ and $\pi/2$, and for arbitrary values of the distance D . These solutions will be called $\bar{f}(\omega, D)$ and $\bar{h}(\omega, D)$, where the dependence on the spatial coordinates r and t is implicit. In order to keep the variational equations as small as possible, we have chosen the constraint operator (5), and not, for example,

$$\left(-\frac{1}{2} \text{Tr} W_{\mu\nu} W_{\mu\nu} \right)^2 ,$$

which would give significantly larger expressions. Furthermore, we set the constant c in the definition of the scale (7) to the value

$$c \equiv 288/21 , \quad (24)$$

in order to match the scale of the BPSTH instanton, which will be determined in sect. 5.1.

With the solutions $\bar{f}(\omega, D)$ and $\bar{h}(\omega, D)$ in hand, we can, at last, specify the functions f and h that enter the ansatz (18) for the NCL

$$\begin{aligned} \text{I, II} & : f = \bar{f}(\pi/2, D) \\ & h = \bar{h}(\pi/2, D) \end{aligned}$$

$$\begin{aligned}
\text{III} \quad : \quad f &= \bar{f}(0, d_{\max}) \cos^2 \omega + \bar{f}(\pi/2, d_{\max}) \sin^2 \omega \\
h &= \bar{h}(0, d_{\max}) \cos^2 \omega + \bar{h}(\pi/2, d_{\max}) \sin^2 \omega ,
\end{aligned} \tag{25}$$

with the distance $D = D(\omega)$ given by (20). Actually, we only need the $\omega = \pi/2$ solutions \bar{f} and \bar{h} down to some small value $D = d_{\min}$. We can then close the loop with a simple deformation of these functions and keep them non-singular, as explained in our previous paper [1] and in Appendix B at the end of this one. By choosing d_{\min} small enough, the action can be kept arbitrarily small.

To summarize, the field configurations of the NCL are given by (18 – 20) with the axial functions (23, 25), where \bar{f} and \bar{h} are the solutions of the variational equations for $\omega = 0$ or $\pi/2$ and $D \leq d_{\max}$.

4.3 Actionprofile

Our main interest lies in the actionprofile over the NCL. Here, we will give a general discussion of the possible behaviour, in order to prepare the way for the numerical results to be presented in sect. 5.

The profile of the constrained action over the NCL is essentially determined by the behaviour of the constrained action $A(\omega, D)$ for the solutions \bar{f} and \bar{h} at $\omega = 0$ or $\pi/2$. In fact, \bar{f} and \bar{h} solve the variational equations, so that the action attains its lowest value, within the ansatz, precisely for these functions. On physical grounds, we expect the following behaviour for A . For $D \rightarrow \infty$ the constrained action should approach twice that of the BPSTH instanton I, whereas for $D \rightarrow 0$ the effects of the Yang-Mills interactions should become important, which can be either repulsive ($\omega = 0$) or attractive ($\omega = \pi/2$). The Yang-Mills interactions for large values of D get suppressed exponentially, with a length scale set by M_W^{-1} . In this paper, we consider the case of vanishing Higgs mass $M_H = 0$ or $\lambda = 0$. This means that at very large distances D , where the Yang-Mills interactions drop out, only the effect from the Higgs field remains. Moreover, the Higgs interaction at large distances is attractive and basically independent of the relative isospin rotation Ω of the cores. So, the simplest behaviour we expect is the one sketched in fig. 1a. At this moment, we can also explain the need for the additional functions g_{\pm} in the ansatz (19). For $g_{\pm} = 1$, namely, there would be ‘‘tidal effects’’ from one core on the other, which decrease rather slowly with the distance D . If, instead, g_+ vanishes approximately near the $t = +D/2$, and g_- near $t = -D/2$, the cores become independent faster with increasing distance. This, then, is the reason behind our choice (23) for these

As mentioned before, the action profile over the NCL is essentially determined by the behaviour of $A(0, D)$ and $A(\pi/2, D)$. This follows from, in particular, (20) and (25) above. Concretely, the action profile, which is an even function of ω , is obtained as follows. We start in fig. 1a on the $\omega = \pi/2$ curve at $D = 0$ and move out to $D = d_{\max}$, then we go straight up to the $\omega = 0$ curve and back again, and finally we return in the same way to $D = 0$. We have verified that the action over part III of the NCL, with the functions (25), has indeed a single maximum at $\omega = 0$. From the curves of fig. 1a, then, we would conclude that the maximum action over the NCL can remain below $2A_{\text{I}}$, provided we have a large enough value of d_{\max} . In fact, the optimal choice (i.e. lowest possible maximum action) would be to move out to a distance d^* with a corresponding maximum action A^* , both of which are indicated in fig. 1a. However, another possible behaviour of $A(0, D)$ and $A(\pi/2, D)$ is sketched in fig. 1b. In that case, the maximum action would always stay above $2A_{\text{I}}$, regardless of the value of d_{\max} . A priori, there is no way to decide between the two types of behaviour shown in fig. 1 and we need an explicit calculation to settle the matter.

5 Numerical results

We present here some numerical results for the NCL constructed in the previous section. These results are for the case of vanishing quartic Higgs coupling constant $\lambda = 0$. We also give some results for the BPSTH instanton I. Distances will be expressed in units of M_W^{-1} and the action in units of $8\pi^2/g^2$, which is the action of the BPST instanton in pure Yang-Mills theory. A brief outline of the numerical methods is given in Appendix A.

5.1 BPSTH instanton

We have solved numerically the variational equations for the radial functions $f_I(x)$ and $h_I(x)$, that result from inserting the BPSTH ansatz (9) into the constrained action (6) and making variations δf_I and δh_I . As explained in sect. 2, these solutions \bar{f}_I and \bar{h}_I depend on the coupling constant κ of the constrained action A . We determine the scale ρ of the instanton from (7), with the constant $c = 144/21$, and the Yang-Mills-Higgs action A_{YMH} from (8), where W^* and Φ^* now refer to the ansatz (9) with the functions \bar{f}_I and \bar{h}_I . This particular choice for the numerical value of the constant c in (7) reproduces the scale ρ that enters the analytical solution (10) of pure Yang-Mills theory. Our numerical results are collected in Table 1, where, for future reference, we also give results for $\lambda/g^2 = 1/8$ ($M_H = M_W$) and $\lambda/g^2 = 100/8$ ($M_H = 10 M_W$). The $\lambda = 0$ results for A_{YMH} vs. ρ are compared, in fig. 2, with the expression (11), which is valid in the limit $\rho \rightarrow 0$. For values $\rho \gtrsim M_W^{-1}$ the radial functions of the numerical solution differ significantly from the analytical functions (10), resulting in a lower value of the action. Another consequence is that, for finite values of the scale ρ , the gauge fields are no longer self-dual.

The numerical results for the instanton I are relatively easy to obtain, as the variational equations for \bar{f}_I and \bar{h}_I are ordinary differential equations (ODEs). These results will serve as a check on those of the NCL, to which we turn now.

5.2 Non-contractible loop

We have solved numerically the variational equations for the axial functions $f(r, t)$ and $h(r, t)$, that result from inserting the ansatz of the NCL (18), for $\omega = 0$ or $\pi/2$, into the constrained action (6) and making variations δf and δh . These variational equations consist of two coupled, non-linear partial differential equations (PDEs),

solutions \bar{f} and \bar{h} , we also determine the Yang-Mills-Higgs action A_{YMH} and the scale ρ from the expressions (1) and (7), respectively, with the numerical value (24) for the constant c and the NCL configurations for W^* and Φ^* . As discussed in subsect. 4.3, we are primarily interested in the behaviour of the constrained action of the solution as a function of the core distance D . For $D \rightarrow \infty$ we reproduce the results of the previous subsection for $2A_{\text{I}}$ (Table 1). In fig. 3 we give the numerical results for the constrained action, normalized to its asymptotic value, for a relatively large value of the constraint coupling constant, namely $\kappa = 1$. Figure 4 shows the same results on an expanded scale. These numerical results seem to agree, for $\kappa = 1$, with the simple behaviour sketched in fig. 1a. In contrast, the results for $\kappa = 10^{-3}$, shown in fig. 5, display the alternative behaviour of fig. 1b. This different behaviour for small and large values of the scale was also seen in our previous results [1], but we cannot directly compare the scale $\bar{\rho}$ there with the scale ρ here.

We are reasonably confident that the results of figs. 3–5 are reliable, for the following reasons. First, the results are stable when the calculation is repeated on different grids (ranging from 25×50 to 100×200 points) and with different compactified coordinates for r and t , see Appendix A. The error of the data points in fig. 4 is estimated to be less than 0.1%, which is about a factor of 10 better than the numerical accuracy of our previous results [1]. Second, the variational equations for f and h are solved by relaxation, which implies that the exact solution can have a somewhat lower action. In other words, the “dip” of the $\omega = 0$ curve in fig. 4 can only get deeper. Third, and as is often the case with large numerical calculations, we believe our results because they behave in the way we expect them to do. In particular, the dip of fig. 4 becomes more shallow with λ/g^2 increasing. Also, we have an heuristic understanding of the origin, for small values of κ , of the “bump” on the $\omega = \pi/2$ curve in fig. 5. Fourth, a similar calculation for static fields gives comparable results, see Appendix B.

To summarize, we have constructed a NCL, for which the important upperbound (12) holds, provided the scale ρ is large enough. Specifically, we obtained for $\lambda/g^2 = 0$ and $\kappa = 1$ (see figs. 3-4)

$$\max A_{\text{NCL}}(\omega) = 1.994 A_{\text{I}} < 2 A_{\text{I}}. \quad (26)$$

The optimal maximum ($\omega = 0$) configuration of the NCL gives, moreover, an approximation of the conjectured constrained instanton I^* . This configuration leads to the following estimates for I^*

$$\begin{aligned}
d^* &\sim 10 M_W^{-1} \\
A_{\text{YMH}}^* &\sim 4.2 \, 8\pi^2/g^2.
\end{aligned}
\tag{27}$$

In fig. 6 we show the corresponding actiondensity $\tilde{a}_{\text{YMH}}(r, t)$ and Pontryagin-density $\tilde{q}(r, t)$, averaged over the polar angle θ , see Appendix A. This shows that our configuration is still a very loose molecule and we expect the exact solution I^* to be tighter and more cigar-like perhaps. But it is also clear, from fig. 4 especially, that the Yang-Mills cores are very hard and that d^* , which is in essence the distance between the points of vanishing Higgs field, cannot be much smaller than the width ($\sim 2 \rho^*$) of the configuration.

6 Discussion

We have constructed in this paper a non-contractible loop (NCL) with a maximum constrained action less than twice that of the BPSTH instanton, provided the length scale is fixed at a large enough value. This was established for the case of vanishing Higgs mass $M_H = 0$. We expect it to be possible to extend the result to all values $M_H < M_W$. Note that in the full electroweak theory there is also the photon field, which can provide the necessary attraction if the Higgs field becomes too short of range. As it stands, this upperbound on the actionprofile over the NCL is only a necessary condition in an eventual existence proof. Still, we are optimistic about the existence of I^* in $SU(2)$ Yang-Mills-Higgs theory. There exist, of course, analogous *static* solutions (sphalerons) in Yang-Mills-Higgs theory [6, 7, 8], but there are also encouraging results on *instantons* in pure Yang-Mills theory, which we will now discuss.

There has been a long-standing conjecture [9], based on the analogy with harmonic maps from S_2 to S_2 , that all solutions in euclidean $SU(2)$ Yang-Mills theory over S_4 are necessarily self-dual or anti-self-dual ($W_{\mu\nu} = \pm \frac{1}{2} \epsilon_{\mu\nu\kappa\lambda} W_{\kappa\lambda}$). In the vacuum sector, in particular, there would be no other solutions besides the classical vacuum itself. This conjecture has been proven false recently. In fact, Sibner et al. [10] showed the existence of infinitely many non-self-dual solutions in the vacuum sector, with an action

$$A_{\text{YM}} = m 16 \pi^2 / g^2 + \Delta A_{\text{YM}} , \quad (28)$$

for integers $m \geq 2$ and $\Delta A_{\text{YM}} < 0$, which, most likely, depends on m also. Later, several solutions were constructed explicitly by Sadun and Segert [11]. The existence proof of Sibner et al. [10] goes by a mini-max procedure over non-contractible loops, where an important ingredient is the inequality contained in (28), which is analogous to ours (12). The other main ingredient is an equivariant weak compactness theorem, where equivariant refers to an $U(1)$ symmetry of their ansatz. The integer m , which appears in (28), labels the embedding in $SU(2)$ of this $U(1)$ symmetry (rotation angle $\gamma \in [0, 2\pi]$), namely by the matrix $\exp[m\gamma\sigma_3/(2i)]$ for the gauge field transformations.

Equivariance is known to be a powerful tool in Morse theory. For this reason, it may be of importance to note that our ansatz (18, 19) also has an $U(1)$ symmetry, and precisely for the case $m = 1$ excluded in the existence proof [10] and the explicit construction [11] of the pure Yang-Mills solutions. The difference is that our theory

for the crucial upperbound (12) on the action. The constrained action (6) of this theory now has three terms extra, compared to the case of pure Yang-Mills theory. The constraint term, in particular, is designed to prevent collapse and, naively, we see no way how a regular solution I^* , related to the presence of non-contractible loops in configurations space, could fail to exist.

Just as the sphaleron S [7, 8] is associated with the BPSTH instanton I (loosely speaking S is a constant time slice of I), we expect a new sphaleron S^* [12] to be associated with I^* . In Appendix B we give some numerical results for a non-contractible *sphere* of 3-dimensional configurations, which support the existence of this new sphaleron S^* . After these results were obtained, we succeeded in constructing the solution S^* , on which we will report elsewhere. We expect the construction of I^* to proceed in the same way, only with greater technical complications. Henceforth, we take for granted the existence of the new constrained instanton I^* in $SU(2)$ Yang-Mills-Higgs theory.

An interesting question is to see what happens when massless fermions are introduced into the theory, or possibly fermions with Yukawa couplings to the Higgs. We then expect I^* (and S^*) to have fermion zero-modes. The reason is that, as explained in our previous paper [1], there is spectral flow of the Dirac eigenvalues along the NCL (18). This spectral flow was indirectly monitored in the numerical calculations of sect. 5 and our results support the claim that I^* has fermion zero-modes. Consequently, there should be a new ($B + L$ conserving) effective fermion vertex from I^* , with double the number of lines compared to the one from the BPSTH instanton [3]. We will now turn to different instanton solutions, which may be related to I^* in one way or another.

The pure $SU(2)$ Yang-Mills solutions [10, 11] discussed above can be expected to have counterparts in $SU(2)$ Yang-Mills-Higgs theory, i.e. there will be corresponding constrained instantons. In general, these solutions will have a rather large action, since for them we have already that $A_{\text{YM}} \gtrsim 32 \pi^2/g^2$, whereas $A_{\text{YM}} \sim 16 \pi^2/g^2$ for I^* . We expect I^* to be the lowest action non-trivial solution in the vacuum sector of the theory. More interesting, perhaps, is the inverse question, whether or not the constrained instanton I^* of $SU(2)$ Yang-Mills-Higgs theory has a corresponding solution in pure Yang-Mills theory. We conjecture there to be no such counterpart in pure Yang-Mills theory, because we see no obvious source of attraction. The fact that we found the upperbound on the action (12) to be violated for small values of the constraint coupling constant (see fig. 5) is suggestive, but is not really conclusive,

Different Yang-Mills-Higgs theories can also be considered. If W^* and Φ^* are the fields of the I^* solution in $SU(2)$ Yang-Mills-Higgs theory, then it is possible to embed them into a larger theory, provided that $SU(2)$ is a subgroup of the gauge group G and that the $SU(2)$ Higgs doublet (together with its vacuum expectation value) can be embedded in the larger Higgs representation. An example is $SU(3)$ Yang-Mills-Higgs theory with a complex triplet of Higgs. The embedding is then given by

$$W = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \times & \times \\ 0 & \times & \times \end{pmatrix} \quad \Phi = \begin{pmatrix} 0 \\ \times \\ \times \end{pmatrix} , \quad (29)$$

where the crosses indicate, symbolically, the $SU(2)$ solution. Note that these embeddings can lead to unexpected solutions of the field equations, since the larger theory may not even have non-contractible loops ($\pi_4(G) = 1$, as is the case for $G = SU(3)$, for example). The NCL was a tool to find the $SU(2)$ solution, but once we have found the solution, we can forget about the NCL and simply verify the fact that the ansatz solves the field equations, which is then carried over to the larger theory.

Finally, we comment on possible applications of the conjectured new instanton solution I^* . Of course, $SU(2)$ Yang-Mills-Higgs theory is at the heart of the Glashow-Weinberg-Salam model for the electroweak interactions. Evaluating euclidean path integrals for electroweak processes, it may be important to know that in addition to the classical vacuum there is a new stationary point, i.e. the constrained instanton I^* . It is well-known [13, 14] that such stationary points (and vacuum instability, in general) can play a role in the asymptotics of perturbation theory. Furthermore, this new stationary point could contribute directly to the euclidean path integrals of certain forward elastic scattering amplitudes, which control the total cross-sections for the corresponding processes. In our previous paper [1] we have shown that this contribution, evaluated classically ², suddenly becomes important as the parton center-of-mass energy \sqrt{s} increases. In fact, this threshold energy is determined by the structure of the I^* solution, and from our approximation of that solution we obtain

$$(\sqrt{s})_{\text{threshold}} \sim \frac{A_{\text{YMH}}^*}{d^*} = \left(\frac{A_{\text{YMH}}^*}{8\pi^2/g^2} \right) \left(\frac{2}{d^* M_W} \right) E_S , \quad (30)$$

² For a semiclassical calculation we also need to know the negative modes around the classical solution. In pure $SU(2)$ Yang-Mills theory it has been shown [15] that there must be at least two negative modes, instead of one; whether or not the same holds in Yang-Mills-Higgs theory is not known at the moment. Note that even a new, “non-perturbative” contribution to the real part

where we have used the definition

$$E_S \equiv \pi \frac{M_W}{\alpha_w} ,$$

which is close to the true value $3.04 M_W/\alpha_w$ for the sphaleron energy at $\lambda = 0$. With the numerical estimates given in (27) we find, not unexpectedly perhaps, that the threshold in the parton center-of-mass energy is of the order of $E_S \sim 10$ TeV. We intend to discuss the applications of I^* in a separate publication.

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Appendix A : Numerical methods

We describe in this appendix the algebraic and numerical calculations for the results reported in sect. 5. These calculations are straightforward, but cumbersome. Hence, we will give the main points only and leave many technical details out. In the first part of this appendix, we review the algebraic calculation of the action density for our ansatz. In the second part, we discuss the numerical solution of the variational equations from this action density.

A1 : Algebraic calculation

The main algebraic calculation consists of two steps. The first step is to insert the ansatz for the NCL into the constrained action. We use axial coordinates

$$\begin{aligned} x_0 &\equiv t , \\ x_3 &\equiv z \equiv r \cos \theta , \\ x_2 &\equiv \rho \cos \phi \equiv r \sin \theta \cos \phi , \\ x_1 &\equiv \rho \sin \phi \equiv r \sin \theta \sin \phi , \end{aligned} \tag{A.1}$$

and make all distances dimensionless with M_W^{-1} . The total constrained action (6) for the ansatz (18) takes the form

$$A = \frac{1}{g^2} \int_{-\infty}^{\infty} dt \int_0^{\infty} dr r^2 \int_0^{\pi} d\theta \sin \theta \int_0^{2\pi} d\phi a , \tag{A.2}$$

with a rotationally invariant action density

$$a = a(z, \rho, t) . \tag{A.3}$$

As a matter of fact, it is already clear from the ansatz (18, 19) that a rotation of (x_1, x_2) can be compensated by a global gauge transformation.

The second step is to perform two of the integrals in the action (A.2). The one over the azimuthal angle ϕ is trivial, of course. The integral over the polar angle θ can also be performed, since the functions f, h and g_{\pm} depend, by construction, on r and t only. In addition, we have the reflection symmetry $t \rightarrow -t$, so that the final expression of the constrained action for our ansatz takes the form

$$A = \frac{8\pi^2}{g^2} \int_0^{\infty} dt \int_0^{\infty} dr \frac{r^2}{\pi} \tilde{a} . \tag{A.4}$$

The averaged action density $\tilde{a} = \tilde{a}(r, t)$ has the following structure

$$\begin{aligned}
\tilde{a}_{\text{YM}} &= Z_{\text{YM}20} (\partial_t f)^2 + Z_{\text{YM}02} (\partial_r f)^2 + Z_{\text{YM}11} (\partial_t f \partial_r f) + Z_{\text{YM}00} (f(1-f))^2, \\
\tilde{a}_{\text{H}} &= 2 (\partial_t h)^2 + 2 (\partial_r h)^2 + Z_{\text{H}00} (h(1-f))^2 + 4 \frac{\lambda}{g^2} (h^2 - 1)^2, \\
\tilde{a}_{\text{C}} &= \kappa \left(Z_{\text{C}20} (\partial_t f)^2 + Z_{\text{C}02} (\partial_r f)^2 + Z_{\text{C}11} (\partial_t f \partial_r f) \right) (f(1-f))^2, \tag{A.5}
\end{aligned}$$

where ∂_r and ∂_t denote partial derivatives with respect to r and t . The eight coefficients Z are complicated rational functions of the variables r and t , together with the functions $g_{\pm}(r, t)$ and their various partial derivatives. These coefficients depend also on the loop parameter ω . With some further effort, (A.5) can be put in manifestly positive definite form. We have used the symbolic manipulation program FORM [16] for the algebraic calculation of the action density (A.5).

A2 : Numerical calculation

We now have to solve numerically the variational equations from the action (A.4). We proceed in three steps. The first step is to compactify the coordinates r and t to the variables x and y , respectively. Since we are interested in the long-range behaviour of the Higgs field, we choose for $y(t)$ a rather slow dependence on t , specifically

$$\begin{aligned}
y &= y_c + (1 - y_c) \frac{\bar{t} - \bar{t}_c}{1 + |\bar{t} - \bar{t}_c|} \\
y_c &\equiv \bar{t}_c / (1 + 2\bar{t}_c) \\
\bar{t} &\equiv t / t_{\text{scale}} \tag{A.6}
\end{aligned}$$

and similarly for $x(r)$. Furthermore, we let t_c correspond to the core position $t_c = D/2$ and set, typically, $r_c = \rho_{\text{I}}$ and $r_{\text{scale}} = t_{\text{scale}} = 2\rho_{\text{I}}$, where $\rho_{\text{I}} = \rho_{\text{I}}(\kappa)$ is the scale of the instanton I, see sect. 5.1. It is straightforward to make these changes of variables in the action (A.4), and we write the result as

$$A = \frac{8\pi^2}{g^2} \int_0^1 dy \int_0^1 dx \hat{a}, \tag{A.7}$$

with the Jacobians absorbed into $\hat{a} = \hat{a}(x, y)$. We look for the two functions $f(x, y)$ and $h(x, y)$ that minimize this action, with mixed Dirichlet and Neumann boundary conditions as indicated in fig. 7.

The second step is to discretize the integral (A.7). We use a rectangular grid for x and y

$$\begin{aligned} x &= i\Delta x & i &= 0, \dots, I \\ y &= j\Delta y & j &= 0, \dots, J \end{aligned} \quad (\text{A.8})$$

and write for the functions at the lattice points

$$f(i\Delta x, j\Delta y) \rightarrow f_{i,j} . \quad (\text{A.9})$$

Furthermore, we use central differences for the partial derivatives of f and h , for example

$$\partial_y f(i\Delta x, j\Delta y) \rightarrow \frac{f_{i,j+1} - f_{i,j-1}}{2\Delta y} . \quad (\text{A.10})$$

The resulting discretized action is, however, numerically unstable : minimization leads to functions $f_{i,j}$ and $h_{i,j}$ that take on alternating values of approximately 0 and 1. We have chosen to employ the following two countermeasures. First, we “double” the boundary conditions at infinity, namely

$$\begin{aligned} h_{i,J} &= h_{i,J-1} = 1 \\ h_{I,j} &= h_{I-1,j} = 1 \quad , \end{aligned} \quad (\text{A.11})$$

and similarly for f . Second, we “smear” the constraint term $\hat{a}_C(x, y)$ in the action-density. Specifically, we take for the first term in \hat{a}_C , cf. (A.5),

$$\kappa \hat{Z}_{C20} \left(\frac{f_{i,j+1} - f_{i,j-1}}{2\Delta y} \right)^2 \left(\frac{f_{i,j+1} + f_{i,j-1}}{2} \right)^2 \left(1 - \frac{f_{i,j+1} + f_{i,j-1}}{2} \right)^2 , \quad (\text{A.12})$$

and similarly for the other terms. In this way we end up with a discretized action-density $\bar{a}_{i,j}$ at the gridpoint (i, j) . For the total constrained action we have

$$A = \frac{8\pi^2}{g^2} \sum_{i=0}^I \sum_{j=0}^J \Delta x \Delta y w_{i,j} \bar{a}_{i,j} , \quad (\text{A.13})$$

with $w_{i,j}$ the weightfactors of, for example, the extended trapezoidal rule.

The third, and final, step is to solve numerically the variational equations for $f_{i,j}$ and $h_{i,j}$ coming from the discretized action (A.13). These equations are highly non-linear, especially for the case $\kappa = 1$ we are interested in, and we need a method that can handle this. We have successfully employed the method of non-linear over-relaxation (NLOR) [17], using grids of, typically, 25×50 points. Our FORTRAN program starts with some trial functions for $f_{i,j}$ and $h_{i,j}$, which are then relaxed, first with NLOR, but in the end also with some sweeps of *underrelaxation*, if necessary. In this way we obtain smooth configurations \bar{f} and \bar{h} , with a definite value for the constrained action. The exact solution may even have a somewhat lower action

Appendix B : Non-contractible sphere

We consider in this appendix static, 3-dimensional configurations of the gauge field W and the Higgs field Φ . The energy functional for these fields is

$$E_{\text{YMH}} = \int_{\mathbb{R}^3} d^3x \left[-\frac{1}{g^2} \frac{1}{2} \text{Tr} W_{mn}^2 + |D_m \Phi|^2 + \lambda \left(|\Phi|^2 - \frac{v^2}{2} \right)^2 \right], \quad (\text{B.1})$$

with the same notation as in (1). Henceforth, we set the quartic Higgs coupling constant $\lambda = 0$ and, for brevity, refer to E_{YMH} as E .

It is well-known, by now, that a non-contractible loop (NCL) of static configurations leads to the existence of a static, but unstable, classical solution, the sphaleron S [7, 8]. It is not difficult to construct also a non-contractible *sphere* (NCS) of static configurations [12]. A mini-max procedure over this NCS suggests the possible existence of a new sphaleron S^* , provided we exclude the case of two sphalerons S at infinite separation. This loophole is closed if we are able to construct a NCS, for which

$$\max E_{\text{NCS}}(\mu, \nu) < 2 E_S, \quad (\text{B.2})$$

where μ and ν parametrize the position on the sphere. Evidently, this discussion parallels the one for the new instanton I^* in the main part of this paper. Moreover, the ansatz of sect. 4 can easily be adapted to the present case. We will simply state the resulting ansatz for the NCS and give our numerical results, which establish the important inequality (B.2). These explicit results support the somewhat heuristic arguments given in our previous paper [12].

The NCS is parametrized by the square $\mu, \nu \in [-\pi, +\pi]$, with the boundary $|\mu| = \pi$ or $|\nu| = \pi$ corresponding to the classical vacuum. Writing $[\mu\nu] \equiv \max(|\mu|, |\nu|)$, the configurations of the NCS are

$$\begin{aligned} \pi/2 < [\mu\nu] \leq \pi & : W = 0 \\ & \Phi = (1 - (1 - h) \sin[\mu\nu]) \frac{v}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ \\ 0 \leq [\mu\nu] \leq \pi/2 & : W = -f dU U^{-1} \\ & \Phi = h \frac{v}{\sqrt{2}} U \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \end{aligned} \quad (\text{B.3})$$

with the following $SU(2)$ matrix U for $\mu, \nu \in [-\pi/2, +\pi/2]$

$$\begin{aligned}
U &= \exp[(\nu + \pi/2)i\sigma_3] \exp[(\mu + \pi/2)\hat{y}_- \cdot i\vec{\sigma}] \exp[-(\nu + \pi/2)i\sigma_3] \cdot \\
&\quad \exp[-(\mu + \pi/2)\hat{y}_+ \cdot i\vec{\sigma}]
\end{aligned} \tag{B.4}$$

and notation

$$\begin{aligned}
x_{m\pm} &\equiv (x_1, x_2, x_3 \pm D/2) \equiv (\rho \sin \phi, \rho \cos \phi, z \pm D/2) \\
y_{m\pm} &\equiv (g_{\pm} \rho \sin \phi, g_{\pm} \rho \cos \phi, z \pm D/2) .
\end{aligned} \tag{B.5}$$

The core distance $D = D(\nu)$ is given by

$$\begin{aligned}
\pi/2 < |\nu| \leq \pi & : D = 0 \\
0 \leq |\nu| \leq \pi/2 & : D = d_{\max} \cos^{\delta} \nu ,
\end{aligned} \tag{B.6}$$

with $\delta > 0$ a free parameter. The axial functions $f(\rho, z)$, $h(\rho, z)$ and $g_{\pm}(\rho, z)$ have the following boundary conditions and reflection symmetry

$$\begin{aligned}
\lim_{|x| \rightarrow \infty} f, h, g_{\pm} &= 1 \\
f(0, \pm D/2) &= h(0, \pm D/2) = 0 \\
f(\rho, z) &= f(\rho, -z) \\
h(\rho, z) &= h(\rho, -z) \\
g_{\pm}(\rho, z) &= g_{\mp}(\rho, -z) .
\end{aligned} \tag{B.7}$$

For $g_{\pm}(\rho, z)$ we take again the functions (23), with the same coefficients α and β , and where x_{\pm} is now defined by (B.5). It remains to specify the two axial functions f and h . We will give two alternative constructions, both of which lead to the desired inequality (B.2).

The first construction for the functions f and h in the ansatz (B.3) parallels the procedure followed for the new instanton in sect. 4, with the loop parameter ω there corresponding to the sphere parameter ν here. The procedure is to solve the variational equations for f and h at $\mu = 0$ and $\nu = 0$ or $\pi/2$, for arbitrary values of the core distance D . We denote these solutions by $\bar{f}(0, 0, D)$ and $\bar{f}(0, \pi/2, D)$, and similarly for \bar{h} , where the dependence on the spatial coordinates ρ and z is implicit. In fig. 8 we show the corresponding energy values. For $\nu = 0$ there is a clear minimum ³ at a core distance $d^* \sim 7 M_W^{-1}$. With these solutions \bar{f} and \bar{h} we

³ The dip for the instanton case (fig. 4) is more shallow, one reason being the fact that the

can specify the functions f and h in the ansatz (B.3)

$$\begin{aligned}
0 \leq |\nu| \leq \pi/2 & : f = \cos^2 \nu \bar{f}(0, 0, D) + \sin^2 \nu \bar{f}(0, \pi/2, D) \\
& h = \cos^2 \nu \bar{h}(0, 0, D) + \sin^2 \nu \bar{h}(0, \pi/2, D) \\
& D = d_{\max} \cos^\delta \nu \\
\pi/2 < |\nu| \leq \pi & : h = \bar{h}(0, \pi/2, 0) .
\end{aligned} \tag{B.8}$$

Choosing $d_{\max} = d^*$ and the parameter δ sufficiently small, we have verified that the energy stays everywhere below $2E_S$. In particular, the energy over slices of the NCS at constant values of ν have their maximum at $\mu = 0$ and drop to zero monotonically for $\mu \rightarrow \pm\pi$. The energy surface, however, is very steep for $\nu \sim \pm\pi/2$, and we prefer to show it for a somewhat different construction.

This second, alternative construction for the functions f and h in the ansatz (B.3) is as follows. The procedure is to solve the variational equations for f and h at $\mu = 0$ for the *whole* range of values $\nu \in [-\pi/2, +\pi/2]$, varying the core distance simultaneously $D = d_{\max} \cos^\delta \nu$. Actually, we do not need the complete range for ν , only the interval $[-\nu_{\min}, +\nu_{\min}]$, so that D runs from d_{\max} to some smaller value d_{\min} ($\equiv d_{\max} \cos^\delta \nu_{\min}$). We denote these solutions by $\bar{f}(0, \nu, D; \rho, z)$ and $\bar{h}(0, \nu, D; \rho, z)$, where now the dependence on the spatial coordinates ρ and z is explicit. For $\nu \in [-\pi/2, -\nu_{\min}] \cup [+ \nu_{\min}, +\pi/2]$ we use a simple deformation of the functions \bar{f} and \bar{h} at $\nu = \nu_{\min}$, in order to move the zeros of f and h appropriately. We denote the three segments of the NCS by

$$\begin{aligned}
\text{III} & : 0 \leq |\nu| \leq \nu_{\min} \\
\text{II} & : \nu_{\min} < |\nu| \leq \pi/2 \\
\text{I} & : \pi/2 \leq |\nu| \leq \pi
\end{aligned}$$

and take for the functions f and h in the ansatz (B.3)

$$\begin{aligned}
\text{III} & : f = \bar{f}(0, \nu, D; \rho, z) \\
& h = \bar{h}(0, \nu, D; \rho, z) \\
& D = d_{\max} \cos^\delta \nu \\
\text{II} & : f = \frac{x_+^2 \bar{f}(0, \nu_{\min}, d_{\min}; \rho, z + d_{\min}/2 - D/2) + x_-^2 \bar{f}(0, \nu_{\min}, d_{\min}; \rho, z - d_{\min}/2 + D/2)}{x_+^2 + x_-^2} \\
& h = \frac{x_+ \bar{h}(0, \nu_{\min}, d_{\min}; \rho, z + d_{\min}/2 - D/2) + x_- \bar{h}(0, \nu_{\min}, d_{\min}; \rho, z - d_{\min}/2 + D/2)}{x_+ + x_-}
\end{aligned}$$

$$D = d_{\max} \cos^\delta \nu$$

$$d_{\min} = D(\nu_{\min})$$

$$\text{I} : h = \frac{\bar{h}(0, \nu_{\min}, d_{\min}; \rho, z + d_{\min}/2) + \bar{h}(0, \nu_{\min}, d_{\min}; \rho, z - d_{\min}/2)}{2}. \quad (\text{B.9})$$

This construction leads to a smooth energy surface over the NCS, shown in fig. 9 for one particular set of parameters. The energy surface over the NCS is, by construction, invariant under $(\mu, \nu) \rightarrow (-\mu, -\nu)$. Again, the maximum energy over the NCS is reached at $\mu = \nu = 0$ and its value ($1.94 E_S$, for the parameters chosen in fig. 9) obeys the inequality (B.3).

To summarize, we have constructed in $SU(2)$ Yang-Mills-Higgs theory non-contractible spheres of static configurations, with energies everywhere below $2 E_S$. This suggests the existence of a new sphaleron S^* . Furthermore, we have, for the case of vanishing Higgs mass, an approximation of that solution from the optimal maximum configuration on the NCS, with a core distance and energy given by (see fig. 8)

$$\begin{aligned} d^* &\sim 7 M_W^{-1} \\ E^* &\sim 1.92 E_S, \end{aligned} \quad (\text{B.10})$$

where E_S takes the numerical value $3.04 M_W/\alpha_w$. In fig. 10 we show the energy density of this configuration. Finally, we note that the 3-dimensional configuration (B.3) at $\mu = \nu = 0$ is essentially equivalent to the slice $x_3 = 0$ of the instanton configuration (18, 19) at $\omega = 0$. It is in this sense that the new sphaleron S^* corresponds to a constant time slice of the new instanton I^* .

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	$M_H/M_W = 0$			$M_H/M_W = 1$			$M_H/M_W = 10$		
	ρ	A	A_{YMH}	ρ	A	A_{YMH}	ρ	A	A_{YMH}
$\kappa = 10^{-4}$	0.39	1.10	1.07	0.39	1.10	1.07	0.37	1.11	1.08
$\kappa = 10^{-3}$	0.58	1.19	1.13	0.57	1.20	1.14	0.54	1.24	1.16
$\kappa = 10^{-2}$	0.87	1.39	1.27	0.85	1.41	1.28	0.80	1.51	1.34
$\kappa = 10^{-1}$	1.30	1.79	1.55	1.26	1.86	1.58	1.19	2.07	1.73
$\kappa = 10^{+0}$	1.93	2.61	2.11	1.85	2.80	2.21	1.75	3.26	2.52
$\kappa = 10^{+1}$	2.85	4.30	3.26	2.69	4.88	3.56	2.58	5.77	4.21

Table 1: Numerical results for the BPSTH instanton (9) in the constrained Yang-Mills-Higgs theory (6). The classical theory has two dimensionless coupling constants κ and λ/g^2 , which control, respectively, the strength of the constraint term in the action and the mass ratio of the Higgs scalar H and vector bosons W . The scale ρ of the instanton is given in units of M_W^{-1} , the constrained action A and the standard Yang-Mills-Higgs action A_{YMH} in units of $8\pi^2/g^2$.

Figure captions

Fig. 1 : (a) Sketch of the constrained action A as a function of the core distance D , for the case of repulsive ($\omega = 0$) or attractive ($\omega = \pi/2$) Yang-Mills interactions. The Higgs mass vanishes and for large distances there is attraction between the cores. The horizontal dashed line corresponds to twice the action of the BPSTH instanton. Also indicated are the minimum value A^* on the $\omega = 0$ curve and the corresponding distance d^* .

(b) Alternative behaviour of A vs. D .

Fig. 2 : Yang-Mills-Higgs action A_{YMH} (in units of $8\pi^2/g^2$) of the numerical solution of the variational equations for the BPSTH ansatz (9), as a function of the scale ρ (in units of M_W^{-1}). The quartic Higgs coupling constant λ vanishes. The dashed curve represents the analytical result (11), valid for small values of ρ .

Fig. 3 : Constrained action A (normalized to its asymptotic value) of the numerical solution of the variational equations for the configurations of the non-contractible loop (18), as a function of the core distance D (in units of M_W^{-1}). Closed and open symbols correspond to, respectively, repulsive ($\omega = 0$) and attractive ($\omega = \pi/2$) Yang-Mills interactions. The coupling constants λ and κ in the constrained action (6) have the values 0 and 1, respectively.

Fig. 4 : Same as fig. 3, but with an expanded scale for the constrained action A .

Fig. 5 : Same as fig. 3, but for a smaller value of the constraint coupling constant κ .

Fig. 6 : Actiondensity $\tilde{a}_{\text{YMH}}(r, t)$ and Pontryagin-density $\tilde{q}(r, t)$ for the optimal maximum configuration of the non-contractible loop ($\omega = 0$, $D = 10 M_W^{-1}$, $\lambda = 0$, $\kappa = 1$), see fig. 4. Both densities have an arbitrary normalization of their maximum to 100. The coordinates r and t are in units of M_W^{-1} . The complete configuration is obtained by reflection $t \rightarrow -t$, under which the actiondensity is invariant, but the Pontryagin-density changes sign.

Fig. 7 : Boundary conditions for the functions $f(x, y)$ and $h(x, y)$, defined over the

and 1 at spatial infinity ($x = 1$ or $y = 1$). Neumann boundary conditions, for $x = 0$ or $y = 0$, are indicated by N.

Fig. 8 : Energy E (normalized to its asymptotic value) of the numerical solution of the variational equations for the configurations of the non-contractible sphere (B.3), as a function of the core distance D (in units of M_W^{-1}). Closed and open symbols correspond to different parameters of the ansatz, respectively ($\mu = 0, \nu = 0$) and ($\mu = 0, \nu = \pi/2$). The quartic Higgs coupling constant λ vanishes.

Fig. 9 : Energy E (with arbitrary normalization) over the non-contractible sphere (B.3, B.9). The energy surface over the whole sphere is obtained by reflection symmetry $(\mu, \nu) \rightarrow (-\mu, -\nu)$. The parameters for these results are $d_{\max} = 10 M_W^{-1}$, $\nu_{\min} = 7\pi/16$ and $\delta = 1$.

Fig. 10: Energydensity $e(\rho, z)$ (with arbitrary normalization) for the optimal maximum configuration of the non-contractible sphere ($\mu = \nu = 0, D = 7 M_W^{-1}, \lambda = 0$), see fig. 8. The coordinates ρ and z are in units of M_W^{-1} . The complete configuration is obtained by reflection $z \rightarrow -z$.