NIKHEF-H/92-18

The renormalization of the axial anomaly in dimensional regularization

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Abstract

The prescription for the γ_5-matrix within dimensional regularization in multiloop calculations is elaborated. The three-loop anomalous dimension of the singlet axial current is calculated.

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Since the invention of dimensional regularization [1] and the minimal subtraction (MS) scheme [2] a lot of attention was paid to the problem of the γ_5-matrix within dimensional regularization. The following approaches to this problem were used in practical calculations: the prescriptions based on the original definition by 't Hooft and Veltman [1] [3] [4], keeping the four-dimensional anticommutation relation for γ_5 in D-dimensions [5] and dimensional reduction [6]. Discussions of the γ_5-prescriptions can be found e.g. in [7] [8].

The effective approach to perform multiloop calculations involving the non-singlet axial current in dimensional regularization was developed in [9] [10] where deep inelastic sum rules were calculated up to (and including) the three-loop level in QCD. The effectiveness of this approach is confirmed by its use in such an advanced calculation as the calculation of the deep inelastic structure function F_3 at the two-loop QCD level [11].

In the present paper this approach is elaborated for the cases of the pseudoscalar current and the singlet axial current. The three-loop anomalous dimension of the singlet axial current is calculated by imposing the requirement that the axial anomaly relation [12] [13] should preserve the one-loop character [14] in dimensional regularization.

Throughout the paper we use the MS-scheme [2] or its standard modification, $\overline{MS}$-scheme [15], to perform renormalizations. The dimension of space-time is defined in the standard way as $D = 4-2\epsilon$. All calculations are performed within massless perturbative QCD.

1. The non-singlet axial current. Let us first consider the case of the non-singlet axial current:

$$J^{5a}_\mu(x) = \overline{\psi}(x)\gamma_\mu\gamma_5 t^a\psi(x), \tag{1}$$

where ψ is a quark field and t^a is a generator of a flavor group.

In our opinion, the most practical definition of γ_5 for multiloop calculations in dimensional regularization (and the only one known to be self-consistent) is the original definition due to 't Hooft and Veltman [1]:

$$\gamma_5 = i\frac{1}{4!}\varepsilon_{\nu_1\nu_2\nu_3\nu_4}\gamma_{\nu_1}\gamma_{\nu_2}\gamma_{\nu_3}\gamma_{\nu_4}, \tag{2}$$

here the Levi-Civita ε-tensor is unavoidably a four-dimensional object and should be taken outside the R-operation where any object can be safely considered as a four-dimensional one; the indices $\nu_1 \dots \nu_4$ are D-dimensional inside the R-operation as all other indices within dimensional regularization. But γ_5 defined by eq.(2) does not anticommute anymore with the D-dimensional γ_μ. That is why in order to define the axial current correctly one should use (see below comments after eq.(10)) the symmetrical form of the axial current:

$$J^{5a}_\mu = \frac{1}{2}\overline{\psi}(\gamma_\mu\gamma_5 - \gamma_5\gamma_\mu)t^a\psi, \qquad \gamma_5 = i\frac{1}{4!}\varepsilon_{\nu_1\nu_2\nu_3\nu_4}\gamma_{\nu_1}\gamma_{\nu_2}\gamma_{\nu_3}\gamma_{\nu_4}. \tag{3}$$

In principle it is possible to perform the calculations using this definition of the axial current. But one can simplify the definition drastically. Let us commute γ_μ in the first term in (3) to the right. The D-dimensional metric tensors $g_{\mu\nu_i}$

arising during commutations can always be taken outside the R-operation where they can be safely contracted with the ε-tensor as four-dimensional objects. So we receive the following definition of the non-singlet axial current:

$$J_\mu^{5a} = i\frac{1}{3!}\varepsilon_{\mu\nu_1\nu_2\nu_3}\overline{\psi}\gamma_{\nu_1}\gamma_{\nu_2}\gamma_{\nu_3}t^a\psi. \tag{4}$$

This is exactly the prescription proposed in [3]. Thus we proved equivalence of the definitions (4) and (3) within dimensional regularization. To be sure that no holes are missed in this general proof we computed the axial vertex (see below eq.(8)) at the three-loop level, using both definitions. The results are identical. But the definition (4) is more compact and saves computational time enormously so we will use it in what follows.

Since the anticommutativity of γ_5 is violated by definition (2), the standard properties of the axial current and Ward identities which are valid e.g. in such a basic regularization as the Pauli-Villars regularization are also violated. In particular the renormalization constant Z_{MS}^{ns} of the non-singlet axial current in the MS-scheme is not equal to one any more. It was calculated in the three-loop approximation in QCD in [10]; we remind here the two-loop expression:

$$Z_{MS}^{ns} \quad = \quad 1 + a^2\frac{1}{\epsilon}(\frac{22}{3}C_F C_A - \frac{4}{3}C_F n_f), \tag{5}$$

where we use the notation $a = \frac{\alpha_s}{4\pi} = \frac{g^2}{16\pi^2}$ for the strong coupling constant, C_F and C_A are the Casimir operators of the defining and the adjoint representations of the color group and n_f is the number of quarks flavors. The relation between renormalized and bare operators is $(O)_R = Z\ (O)_B$.

To restore the renormalization invariance of the non-singlet axial current (i.e. to nullify its anomalous dimension) one should perform [16] an extra finite renormalization or in other words to introduce the extra finite renormalization constant $Z_5^{ns}(a)$. So the correct renormalized non-singlet axial current is:

$$(J_\mu^{5a})_R = Z_5^{ns}(a)\ Z_{MS}^{ns}(a)\ (J_\mu^{5a})_B = Z_5^{ns}\ Z_{MS}^{ns}\ i\frac{1}{3!}\varepsilon_{\mu\nu_1\nu_2\nu_3}\overline{\psi}_B\gamma_{\nu_1}\gamma_{\nu_2}\gamma_{\nu_3}t^a\psi_B, \tag{6}$$

where $\psi_B = Z_2^{\frac{1}{2}}\psi$ is a bare quark field. The full anomalous dimension can be now nullified:

$$\gamma_J^{ns}(a) = \mu^2\frac{d}{d\mu^2}\log(Z_5^{ns}\ Z_{MS}^{ns}) = \beta(a)\frac{\partial \log Z_5^{ns}}{\partial a} + \mu^2\frac{d}{d\mu^2}\log Z_{MS}^{ns} = 0. \tag{7}$$

Using this equation one can obtain Z_5^{ns} from the given Z_{MS}^{ns}. But since the renormalization group β-function starts with an a^2-term one can obtain $Z_5^{ns}(a)$ only in the approximation which is one order in a less than the given approximation of Z_{MS}^{ns}. That is why it is better to use the recipe of [9] and to find Z_5^{ns} from the requirement that the renormalized axial and vector vertices coincide:

$$Z_5^{ns} R_{MS} < \overline{\psi}\ J_\mu^{5a}(0)\ \psi >= R_{MS} < \overline{\psi}\ J_\mu^{a}(0)\ \psi > \gamma_5, \tag{8}$$

where $J_\mu^a(x) = \overline{\psi}(x)\gamma_\mu\gamma_5 t^a\psi(x)$ is the vector current. This relation means that anticommutativity of the γ_5-matrix is effectively restored, so the standard Ward identities are also restored. This prescription for Z_5^{ns} automatically ensures zero anomalous dimension (7) because the anomalous dimension of the vector vertex is naturally zero.

The first impression is that the calculation of the axial vertex is rather cumbersome within adopted prescription (4) for the axial current: taking the ε-tensor outside R-operation seems to leave inside R-operation three extra uncontracted indices ν_1, ν_2, ν_3. But it is always possible to contract first a quantity under consideration with an extra ε-tensor to produce a scalar object. In our case we first multiply the axial vertex with $\gamma_\mu \gamma_5 = i\frac{1}{3!}\varepsilon_{\mu\rho_1\rho_2\rho_3}\gamma_{\rho_1}\gamma_{\rho_2}\gamma_{\rho_3}$ and take the trace. Now outside the R_{MS}-operation the product of four-dimensional ε-tensors can be represented in a standard way as the determinant of the four-dimensional metric tensors:

$$\varepsilon_{\mu\nu_1\nu_2\nu_3}\varepsilon_{\mu\rho_1\rho_2\rho_3} = g_{\nu_1\rho_1}(g_{\nu_2\rho_2}g_{\nu_3\rho_3} - g_{\nu_2\rho_3}g_{\nu_3\rho_2}) + \ldots \tag{9}$$

Then outside the R_{MS}-operation these four-dimensional metric tensors can be safely considered as D-dimensional ones (which will add only inessential $O(\epsilon)$-terms to the renormalized axial vertex). These D-dimensional metric tensors $g_{\mu_i\rho_j}$ can be safely taken inside the R-operation. So one obtains finally a scalar expression inside the R-operation containing only D-dimensional objects which makes the practical calculations straightforward.

Calculating the axial vertex and the vector vertex in eq.(8) one finds Z_5^{ns}. The three-loop approximation for Z_5^{ns} was obtained in [10]; we recall here the two-loop expression:

$$Z_5^{ns} = 1 + a(-4C_F) + a^2(22C_F^2 - \frac{107}{9}C_F C_A + \frac{2}{9}C_F n_f). \tag{10}$$

The independence of Z_5^{ns} on the $\log(p^2/\mu^2)$ (where p is the momentum of quark legs in the vertices in eq.(8)) gives a strong check of the whole prescription. For example, trying to use the axial current in the naive form (1) with γ_5 defined in (2) one would obtain that $\log(p^2/\mu^2)$ does not cancel in Z_5^{ns} which excludes the possibility to use this naive form.

Note that both Z_{MS}^{ns} and Z_5^{ns} are gauge independent quantities (which was checked by calculations in an arbitrary covariant gauge). We would like to note also that the finite constant $Z_5^{ns}(a)$, like the usual ultraviolet renormalization constants, does not depend on the choice of the concrete modification of the MS-like schemes: whether it is calculated within the MS-scheme itself or $\overline{MS}$-scheme or G-scheme [17].

So to calculate any quantity involving the non-singlet axial current one can use the prescription (6) with Z_{MS}^{ns} and Z_5^{ns} given in (5) and (10). This prescription has all typical features, see [7], of the approach developed in [4], where D-dimensional indices split into 4-dimensional and $(D-4)$-dimensional indices. But in our approach all indices during the calculations are D-dimensional which avoids in practical calculations all complications connected with splitting indices.

In the case of several γ_5-matrices in one fermion line one can naively anticommute them (because of the validity of eq.(8)) as in the prescription with anticommuting γ_5 [5] and use the standard property $\gamma_5^2 = 1$. So e.g. the correlator of two axial non-singlet currents automatically coincides in the approach under consideration with the correlator of two vector non-singlet currents.

2. The pseudoscalar current. Let us apply now this prescription to the case of the pseudoscalar current:

$$P(x) = \overline{\psi}(x)\gamma_5\psi(x) = i\frac{1}{4!}\varepsilon_{\nu_1\nu_2\nu_3\nu_4}\overline{\psi}\gamma_{\nu_1}\gamma_{\nu_2}\gamma_{\nu_3}\gamma_{\nu_4}\psi \tag{11}$$

We will not distinguish singlet and non-singlet cases for the pseudoscalar current because the pseudoscalar current does not generate closed fermion loops in the massless case. As in the case of the non-singlet axial current, to obtain the correct renormalized pseudoscalar current we should introduce the finite constant Z_5^p in addition to the usual ultraviolet renormalization constant Z_{MS}^p in the MS-scheme:

$$(P)_R = Z_5^p(a)Z_{MS}^p(a)(P)_B = Z_5^p Z_{MS}^p i\frac{1}{4!}\varepsilon_{\nu_1\nu_2\nu_3\nu_4}\overline{\psi}_B\gamma_{\nu_1}\gamma_{\nu_2}\gamma_{\nu_3}\gamma_{\nu_4}\psi_B \tag{12}$$

We can calculate first in the standard way the renormalization constant Z_{MS}^p within the MS-scheme. In all calculations we use the program Mincer [18] written for the symbolic manipulation system Form [19]. This program computes analytically the three-loop massless diagrams of propagator type which is sufficient to calculate any renormalization constant within MS-scheme at the three-loop level.

The three-loop approximation for Z_{MS}^p in the MS-scheme is:

$$\begin{aligned}
Z_{MS}^p &= 1 + a(-C_F\frac{3}{\epsilon}) + a^2[C_FC_A(\frac{11}{2\epsilon^2}+\frac{79}{12\epsilon}) + C_Fn_f(-\frac{1}{\epsilon^2}-\frac{11}{6\epsilon}) + C_F^2(\frac{9}{2\epsilon^2}-\frac{3}{4\epsilon})] \\
&\quad + a^3[C_FC_An_f(\frac{44}{9\epsilon^3}+\frac{110}{27\epsilon^2}+\frac{8\zeta_3}{\epsilon}-\frac{58}{9\epsilon}) + C_FC_A^2(-\frac{121}{9\epsilon^3}-\frac{257}{54\epsilon^2}-\frac{599}{108\epsilon}) \\
&\quad + C_Fn_f^2(-\frac{4}{9\epsilon^3}-\frac{22}{27\epsilon^2}+\frac{17}{27\epsilon}) + C_F^2C_A(-\frac{33}{2\epsilon^3}-\frac{215}{12\epsilon^2}+\frac{3203}{36\epsilon}) \\
&\quad + C_F^2n_f(\frac{3}{\epsilon^3}+\frac{19}{6\epsilon^2}-\frac{8\zeta_3}{\epsilon}-\frac{107}{9\epsilon}) + C_F^3(-\frac{9}{2\epsilon^3}+\frac{9}{4\epsilon^2}-\frac{43}{2\epsilon})],
\end{aligned} \tag{13}$$

where ζ_3 is the Riemann zeta-function ($\zeta_3 = 1.202056903\ldots$). To restore the Ward identities we can, as in the case of the non-singlet axial current, define the finite renormalization constant Z_5^p from the requirement of coincidence of the pseudoscalar and scalar vertices:

$$Z_5^p R_{MS} < \overline{\psi}\ P(0)\ \psi >= R_{MS} < \overline{\psi}\ \ \overline{\psi}\psi(0)\ \ \psi > \gamma_5, \tag{14}$$

So the anticommutativity of the γ_5-matrix is effectively restored and it is anticommuted out of the pseudoscalar vertex. Calculating the three-loop pseudoscalar and scalar vertices we find the three-loop approximation for Z_5^p:

$$\begin{aligned}
Z_5^p &= 1 + a(-8C_F) + a^2(\frac{2}{9}C_FC_A + \frac{4}{9}C_Fn_f) + a^3[C_FC_An_f(\frac{64}{3}\zeta_3 + \frac{856}{81}) \\
&\quad + C_FC_A^2(-208\zeta_3 - \frac{958}{27}) + \frac{104}{81}C_Fn_f^2 + C_F^2C_A(608\zeta_3 - \frac{800}{27}) \\
&\quad + C_F^2n_f(-\frac{64}{3}\zeta_3 - \frac{580}{27}) + C_F^3(-384\zeta_3 + \frac{304}{3})]
\end{aligned} \tag{15}$$

Again the cancellation of $\log(p^2/\mu^2)$ in Z_5^p provides a good check of the calculations. Another check is the gauge independence of both Z_{MS}^p and Z_5^p (all calculations were done in an arbitrary covariant gauge). Note that Z_5^p for the pseudoscalar current differs from Z_5^{ns} for the non-singlet axial current.

Note also that the full renormalization constant $Z_5^p Z_{MS}^p$ of the pseudoscalar current does not coincide with the renormalization constant of the scalar current $Z_{\overline{\psi}\psi}$ but their anomalous dimensions do coincide:

$$\mu^2\frac{d}{d\mu^2}\log(Z_5^p\ Z_{MS}^p) = \mu^2\frac{d}{d\mu^2}\log Z_{\overline{\psi}\psi} = +a(3C_F) + a^2(\frac{3}{2}C_F^2 + \frac{97}{6}C_F C_A - \frac{5}{3}C_F n_f)$$
$$+a^3\ [\ \frac{129}{2}C_F^3 - \frac{129}{4}C_F^2 C_A + \frac{11413}{108}C_F C_A^2$$
$$+C_F^2 n_f(24\zeta_3 - 23) + C_F C_A n_f(-24\zeta_3 - \frac{278}{27}) - \frac{35}{27}C_F n_f^2]. \quad (16)$$

We would like to note that this our result agrees with the original calculation of the three-loop anomalous dimension of the quark mass in the MS-scheme in [20] and provides thus the independent check of that calculation.

Thus one can use in all calculations involving the pseudoscalar current the prescription (12) with Z_5^p and Z_{MS}^p given in (15) and (13) at the three-loop level. In the case of several pseudoscalar vertices in one fermion line one can (since the anticommutativity of the γ_5 is effectively restored by the prescription (14)) naively anticommute γ_5 and use the standard property $\gamma_5^2 = 1$. So e.g. the correlator of two pseudoscalar currents automatically coincides in the considered approach with the correlator of two scalar currents.

3. The singlet axial current. Let us consider now the case of the singlet axial current which we define in the analogy with the non-singlet current (4):

$$J_\mu^5 = \overline{\psi}\gamma_\mu\gamma_5\psi = i\frac{1}{3!}\varepsilon_{\mu\nu_1\nu_2\nu_3}\overline{\psi}\gamma_{\nu_1}\gamma_{\nu_2}\gamma_{\nu_3}\psi. \quad (17)$$

It is known that the singlet axial current is nontrivially renormalized because of the axial anomaly and the renormalization constant of the singlet axial current is nontrivial at the two-loop level [12],[21]. To receive the correct renormalized singlet axial current we need, as in the previous cases, to introduce the finite renormalization constant Z_5^s in addition to the standard ultraviolet renormalization constant Z_{MS}^s within the MS-scheme:

$$(J_\mu^5)_R = Z_5^s Z_{MS}^s (J_\mu^5)_B = Z_5^s Z_{MS}^s i\frac{1}{3!}\varepsilon_{\mu\nu_1\nu_2\nu_3}\overline{\psi}_B\gamma_{\nu_1}\gamma_{\nu_2}\gamma_{\nu_3}\psi_B. \quad (18)$$

We can calculate within the MS-scheme the renormalization constant Z_{MS}^s of the singlet current at the three-loop level:

$$Z_{MS}^s = 1 + a^2[C_F C_A(\frac{22}{3\epsilon}) + C_F n_f(\frac{5}{3\epsilon})] + a^3[C_F C_A n_f(-\frac{22}{27\epsilon^2} + \frac{149}{81\epsilon})$$
$$+C_F C_A^2(-\frac{484}{27\epsilon^2} + \frac{3578}{81\epsilon}) + C_F n_f^2(\frac{20}{27\epsilon^2} + \frac{26}{81\epsilon}) + C_F^2 C_A(-\frac{308}{9\epsilon}) + C_F^2 n_f(-\frac{22}{9\epsilon})]. \quad (19)$$

Now the problem is how to fix the finite renormalization constant Z_5^s. As in the previous cases one should restore within dimensional regularization the standard properties of the singlet current which exist in such a basic regularization procedure as the Pauli-Villars regularization. We can not anymore impose for this purpose the coincidence of the axial and vector vertices (8) because the singlet current generates closed fermion loops and we cannot anticommute γ_5 outside the singlet axial vertex.

To fix Z_5^s one can require within dimensional regularization the conservation of the one-loop character [14] of the operator relation of the axial anomaly which is valid in the Pauli-Villars regularization:

$$(\partial_\mu J_\mu^5)_R = a\frac{n_f}{2}(G\tilde{G})_R, \tag{20}$$

where $G\tilde{G} = \varepsilon_{\mu\nu\lambda\rho}G^a_{\mu\nu}G^a_{\lambda\rho}$ and $G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + gf^{abc}A^b_\mu A^c_\nu$ is the gluonic field strength tensor.

Let us consider in detail renormalizations of both sides of the anomaly relation. The divergence $\partial_\mu J_\mu^5$ is renormalized multiplicatively in the same way (18) as the current J_μ^5 itself:

$$(\partial_\mu J_\mu^5)_R = Z_5^s Z_{MS}^s(\partial_\mu J_\mu^5)_B. \tag{21}$$

But the operator $G\tilde{G}$ mixes under renormalization:

$$(G\tilde{G})_R = Z_{G\tilde{G}}(G\tilde{G})_B + Z_{GJ}(\partial_\mu J_\mu^5)_B. \tag{22}$$

It is known [22][23] that to explain why $\partial_\mu A_\mu$ does not mix under renormalization and $G\tilde{G}$ does mix it is convenient to represent $G\tilde{G}$ as the divergency of the axial gluon current:

$$\begin{aligned} G\tilde{G} &= \partial_\mu K_\mu, \\ K_\mu &= 4\varepsilon_{\mu\nu_1\nu_2\nu_3}(A^a_{\nu_1}\partial_{\nu_2}A^a_{\nu_3} + \frac{1}{3}gf^{abc}A^a_{\nu_1}A^b_{\nu_2}A^c_{\nu_3}). \end{aligned} \tag{23}$$

The current K_μ is not gauge-invariant. That is why the gauge invariant current J_μ^5 in some "good" gauges (e.g. axial gauge or background-field gauge) can not mix with the gauge-variant operator K_μ. Then the divergences $\partial_\mu J_\mu^5$ and $G\tilde{G}$ of the current J_μ^5 and K_μ are known to be renormalized as the current themselves. So $\partial_\mu A_\mu$ does not mix with $G\tilde{G}$ in these "good" gauges either. But since both divergences are gauge invariant this non-mixing is valid in any gauge.

To understand the restrictions on the renormalization constants $Z_{G\tilde{G}}$ and Z_{GJ} it is useful to take the renormalization group divergence of eq.(22):

$$\mu^2\frac{d}{d\mu^2}(G\tilde{G})_R = \gamma_{G\tilde{G}}(G\tilde{G})_R + \gamma_{GJ}(\partial_\mu J_\mu^5)_R, \tag{24}$$

where anomalous dimensions are defined for the case of operator mixing as follows:

$$(O_i)_R = Z_{ij}(O_j)_B,\ \gamma_{ij} = (\mu^2\frac{d}{d\mu^2}Z_{ik})(Z^{-1})_{kj} = -a\frac{\partial z_{ij}^{(1)}}{\partial a},\ Z_{ij} = 1 + \sum_{n=1}^{\infty}\frac{z_{ij}^{(n)}(a)}{\epsilon^n}. \tag{25}$$

From the renormalization invariance of the anomaly:

$$\mu^2\frac{d}{d\mu^2}(\partial_\mu J_\mu^5)_R = \mu^2\frac{d}{d\mu^2}a\frac{n_f}{2}(G\tilde{G})_R, \tag{26}$$

one receives now:

$$\gamma_J^s(\partial_\mu J^5_\mu)_R = a\frac{n_f}{2}[(\frac{\beta}{a} + \gamma_{G\tilde{G}})(G\tilde{G})_R + \gamma_{GJ}(\partial_\mu J^5_\mu)_R] = (\frac{\beta}{a} + \gamma_{G\tilde{G}} + a\frac{n_f}{2}\gamma_{GJ})(\partial_\mu J^5_\mu)_R. \quad (27)$$

From this equation one can assume the restrictions on the anomalous dimensions:

$$\gamma_{G\tilde{G}} = -\frac{\beta(a)}{a}, \quad \gamma_{GJ} = (a\frac{n_f}{2})^{-1}\gamma_J^s. \quad (28)$$

Strictly speaking eq.(27) itself permits a more general solution than the solution given in eq.(28). But the direct calculation at the two-loop level (see below eq.(31)) supports eq.(28).

To calculate Z_5^s one can calculate matrix elements of the l.h.s. and r.h.s. of the anomaly operator equation (20) between gluon states (so the famous anomalous triangle one-loop diagrams plus higher loops appear in the l.h.s.). To be more precise we calculate for the l.h.s of eq.(20) the following Green-function:

$$< A\ \partial_\mu J^5_\mu\ A >=$$

$$= R_{\overline{MS}}\ \varepsilon_{\lambda\rho\nu\sigma}\frac{p_\nu}{p^2}(\frac{\partial}{\partial q_\sigma})\int d^4x d^4y e^{ipx+iqy} < TA^a_\lambda(x)\partial_\mu J^5_\mu(y)A^a_\rho(0) >|^{amputated}_{q=0}, \quad (29)$$

where 'amputated' means that one-particle-irreducible diagrams with amputated external gluon legs are considered. As it was explained above, the essential point of the calculation is that the product of two ε-tensors can be substituted by the determinant of metric tensors which can be taken as D-dimensional ones inside R-operation. Some typical three-loop diagrams contributing to eq.(29) are shown in fig.1.

The result of the three-loop calculation in the $\overline{MS}$-scheme is:

$$\begin{aligned}< A\partial_\mu J^5_\mu A >= 24n_f a\{1 &+ a[C_F(4) + C_A(6 + 2\xi - \frac{1}{2}\xi^2)] + a^2[C_F C_A(\frac{323}{9} + 8\xi - 2\xi^2)\\ &+ C_F n_f(-\frac{349}{18} + 8\zeta_3) + C_F^2(-6) + C_A n_f(-\frac{343}{24} - 12\zeta_3 + \frac{22}{9}\xi - \frac{5}{9}\xi^2)\\ &+ C_A^2(\frac{4537}{48} + 4\zeta_3 + 3\zeta_3\xi - \frac{1}{4}\zeta_3\xi^2 + \frac{2467}{288}\xi - \frac{131}{72}\xi^2 - \frac{7}{8}\xi^3 + \frac{3}{16}\xi^4)]\}, \quad (30)\end{aligned}$$

where ξ is the gauge parameter in an arbitrary covariant gauge so the gluon propagator is $(g_{\mu\nu} - \xi\frac{q_\mu q_\nu}{q^2})/q^2$. We omitted in this result terms with $\log(p^2/\mu^2)$.

For the r.h.s of the anomaly equation (20) we calculate at the two-loop level the analogous matrix element, as for the l.h.s (29). Some typical two-loop diagrams contributing to the r.h.s can be obtained from the diagrams of fig.1 by shrinking the upper fermion loop into a point. Renormalization is done according to (22). The necessary approximations for the renormalization constants are:

$$\begin{aligned}Z_{G\tilde{G}} &= 1 + a[\frac{1}{\epsilon}(-\frac{11}{3}C_A + \frac{2}{3}n_f)]\\ &\quad + a^2[\frac{1}{\epsilon^2}(-\frac{44}{9}C_A n_f + \frac{121}{9}C_A^2 + \frac{4}{9}n_f^2) + \frac{1}{\epsilon}(C_F n_f + \frac{5}{3}C_A n_f - \frac{17}{3}C_A^2)],\\ Z_{GJ} &= a\frac{1}{\epsilon}12C_F. \quad (31)\end{aligned}$$

The validity of the restrictions of eq.(28) is confirmed in this approximation.
The two-loop result in the $\overline{MS}$-scheme is:

$$
\begin{aligned}
a\frac{n_f}{2} < A\ G\tilde{G}\ A > \quad = \quad & 24n_f a\{1 + a[C_A(6 + 2\xi - \frac{1}{2}\xi^2)] + a^2[+C_F n_f(-\frac{53}{3} + 8\zeta_3) \\
& + C_A n_f(-\frac{343}{24} - 12\zeta_3 + \frac{22}{9}\xi - \frac{5}{9}\xi^2) + C_A^2(\frac{4537}{48} + 4\zeta_3 + 3\zeta_3\xi \\
& - \frac{1}{4}\zeta_3\xi^2 + \frac{2467}{288}\xi - \frac{131}{72}\xi^2 - \frac{7}{8}\xi^3 + \frac{3}{16}\xi^4)]\}. \qquad (32)
\end{aligned}
$$

One can see that without the finite renormalization constant Z_5^s the l.h.s of the anomaly relation (30) and the r.h.s. (32) do not agree within the $\overline{MS}$-scheme. We can obtain the desired finite constant Z_5^s restoring the one-loop character of the anomaly relation, i.e. dividing (32) by (30):

$$
Z_5^s = 1 + a(-4C_F) + a^2(22C_F^2 - \frac{107}{9}C_A C_F + \frac{31}{18}C_F n_f). \qquad (33)
$$

We should stress that the difference between the singlet constant Z_5^s and the non-singlet constant Z_5^{ns} is only in the $C_F n_f$-term. This difference is due to only the light-by-light-scattering type diagrams (the type shown first on fig.1). The independence of the obtained finite constant Z_5^s on $\log(p^2/\mu^2)$ gives strong confirmation that both sides of the anomaly relation are really matched.

Thus both sides of the axial anomaly relation receive the non-trivial higher order corrections if one considers their matrix elements. But these corrections are matched and the one-loop character of the operator equation (which is valid in the Pauli Villars regularization) can be preserved in dimensional regularization.

Now we can calculate the full anomalous dimension of the singlet current in $O(a^3)$ approximation:

$$
\begin{aligned}
\gamma_J^s(a) \quad &= \quad \mu^2 \frac{d\log(Z_5^s Z_{MS}^s)}{d\mu^2} \\
&= \quad +a^2(-6C_F n_f) + a^3(-\frac{142}{3}C_F C_A n_f + \frac{4}{3}C_F n_f^2 + 18C_F^2 n_f). \quad (34)
\end{aligned}
$$

The first term agrees with the calculation [21].

It is interesting to consider now the transition to the QED case in the result (30) for the matrix element of the divergency $\partial_\mu J_\mu^5$ between gluon states, multiplied finally by the finite constant Z_5^s. The transition from the QCD case to the QED case can be done by simple substitutions: $C_A = 0,\ C_F = 1,\ n_f/2 = n_f,\ \alpha_s = \alpha$. Making this substitutions we find that the only surviving contributions to the matrix element of $\partial_\mu J_\mu^5$ between two photons are the famous one-loop triangle diagram and the three-loop diagrams of the light-by-light-scattering type (the first type on fig.1). The fact that these light-by-light-scattering diagrams give a non-zero contribution to the matrix element of $\partial_\mu J_\mu^5$ between two photon states was discovered originally by direct calculation in [24]. These three-loop diagrams give the correction to the width of the neutral pion decay into two photons.

Thus in all calculations involving the singlet axial current one can apply the calculational power of dimensional regularization by using for the singlet axial current the prescription (18) with corresponding renormalization constants given in (19) and (33).

The generalization of the considered γ_5-prescription for the massive case is straightforward. Within the MS-scheme the ultraviolet renormalization constants do not depend on masses [25]. The same is valid for the finite renormalization constants Z_5, so the obtained finite constants can be applied also in the massive case.

I am grateful to the collaborators of the Theory group of NIKHEF-H for helpful discussions.

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