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The three-loop QCD β-function and anomalous dimensions

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Abstract

The analytic calculation of the three-loop QCD β-function and anomalous dimensions within the minimal subtraction scheme in an arbitrary covariant guage is presented. The result for the β-function coincides with the previous calculation [1].

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The calculation of the one-loop β-function in Quantum Cromodynamics (QCD) has lead to the discovery of the asymptotic freedom in this model and to the establishing QCD as the theory of strong interactions [2]. The two-loop QCD β-function was calculated in [3]. The first calculation of the three-loop β-function in QCD was done in [1]. Since then the perturbative calculations of physical quantities in QCD reached the α_s^3 (next-next-to-leading) order of perturbative expansion in the strong coupling constant α_s for the total cross section of e^+e^- annihilation into hadrons [4] and for the deep inelastic lepton-hadron scattering [5]. The phenomenological application of these calculations involves the three-loop approximation of the β-function. The three-loop β-function is also necessary for the investigation of the scheme-dependence problem in QCD and infrared behavior of the coupling constant [6],[7]. That is why it is important to have an independent check of such a complicated calculation as the calculation of the three-loop QCD β-function [1].

In this paper we present original three-loop results for QCD anomalous dimensions in an arbitrary covariant gauge and confirm the three-loop result for the β-function of the work [1].

Throughout the calculations we use the dimensional regularization [8] and the minimal subtraction (MS) scheme [9]. The definition of the β-function is:

$$\beta(a) \quad = \quad \mu^2 \frac{\partial a}{\partial \mu^2} = - \sum_{i=0}^{\infty} \beta_i a^{i+2}, \tag{1}$$

here $a = g^2/16\pi^2 = \alpha_s/4\pi$ and g is the strong coupling constant in the standard QCD lagrangian. To calculate the β-function we need to calculate those renormalization constants of the lagrangian which determine the renormalization constant of the charge. It was convenient for us (for technical reasons) to choose the following three renormalization constants: Z_1 of the quark-quark-gluon vertex and correspondingly Z_2 of the inverted quark propagator and Z_3 of the inverted gluon propagator. Note that the authors of the work [1] used another combination: ghost-ghost-gluon vertex and correspondingly inverted ghost and gluon propagators. So our calculation provides a really independent check of the three-loop β-function.

Renormalization constants within MS-scheme do not depend on dimensional parameters (masses, momenta) [10] and have the following structure:

$$Z(a) = 1 + \sum_{n=1}^{\infty} \frac{z^{(n)}(a)}{\epsilon^n}, \tag{2}$$

where ϵ is the parameter of the dimensional regularization, the dimension of the space-time being $D = 4 - 2\epsilon$. The renormalization constants define corresponding anomalous dimensions:

$$\gamma(a) = \mu^2 \frac{d \log Z(a)}{d\mu^2} = -a \frac{\partial z^{(1)}(a)}{\partial a}. \tag{3}$$

It is convenient to express the β-function through the corresponding anomalous dimensions:

$$\beta(a) \quad = \quad a \; [\; \gamma_3(a) + 2\gamma_2(a) - 2\gamma_1(a) \;]. \tag{4}$$

The calculation of the renormalization constants within MS-scheme can be reduced to the calculation only of massless propagator diagrams by means of the method of infrared rearrangement [11]. In the case of the quark-quark-gluon vertex it means in particular that we can safely nullify the gluon momentum reducing the calculation of the Z_1 to the propagator diagrams.

One way to find the constants Z_i is to obtain them as the sums of counterterms of individual diagrams. This implies application of R-operation to each individual diagram which is an enormous amount of work in the three-loop case. This way was used in [1]. But now we have en efficient program Mincer [12] written for the symbolic manipulation system Form [13]. This program computes analytically three-loop massless propagator diagrams within dimensional regularization. So we are able to obtain renormalization constants at the three-loop level straightforwardly from the multiplicative renormalizability of Green functions:

$$\Gamma_{Renormalized}(a) = Z(a)\ \Gamma_{Bare}(a_B), \tag{5}$$

where a_B is the bare charge.

Thus we compute with the program Mincer the unrenormalized three-loop one-particle-irreducible quark-quark-gluon vertex and inverted quark and gluon propagators. Having the two-loop bare charge we determine the necessary three-loop constants Z_1, Z_2 and Z_3 from the requirement that the poles in ϵ cancel in the r.h.s. of eq.(5). The computations were done in the covariant gauge with an arbitrary gauge parameter ξ to check directly the gauge independence of the β-function. So the diagrams were run with the gluon propagator $(g_{\mu\nu} - \xi \frac{q_\mu q_\nu}{q^2})/q^2$. (Note that the computations in [1] were done in the Feynman gauge $\xi = 0$.)

The results of our three-loop computation of the anomalous dimensions in the MS-scheme are:

$$\begin{aligned}
\gamma_1(a) \;=\;& +a\ [\ C_F + C_A + \xi(-C_F - 1/4 C_A)\ ] \\
& +a^2\ [\ -\frac{3}{2}C_F^2 + \frac{17}{2}C_F C_A + \frac{67}{24}C_A^2 - C_F n_f - \frac{5}{12}C_A n_f \\
& \quad + \xi(-\frac{5}{2}C_F C_A - \frac{15}{16}C_A^2) + \xi^2(\frac{1}{4}C_F C_A + \frac{1}{8}C_A^2)\ ] \\
& +a^3\ [\ \frac{3}{2}C_F^3 + C_F^2 C_A(12\zeta_3 - \frac{143}{4}) + C_F C_A^2(-\frac{15}{2}\zeta_3 + \frac{10559}{144}) + C_A^3(\frac{3}{4}\zeta_3 + \frac{10703}{864}) \\
& \quad + \frac{3}{2}C_F^2 n_f + C_F C_A n_f(6\zeta_3 - \frac{853}{36}) + C_A^2 n_f(-\frac{9}{2}\zeta_3 - \frac{205}{216}) + \frac{5}{9}C_F n_f^2 - \frac{35}{108}C_A n_f^2 \\
& \quad + \xi(C_F C_A^2(-\frac{3}{2}\zeta_3 - \frac{371}{32}) + C_A^3(-\frac{9}{16}\zeta_3 - \frac{127}{32}) + \frac{17}{8}C_F C_A n_f + \frac{1}{2}C_A^2 n_f) \\
& \quad + \xi^2(C_F C_A^2(\frac{3}{8}\zeta_3 + \frac{69}{32}) + C_A^3(\frac{3}{32}\zeta_3 + \frac{27}{32})) + \xi^3(-\frac{5}{16}C_F C_A^2 - \frac{7}{64}C_A^3)\ ],
\end{aligned} \tag{6}$$

$$\begin{aligned}
\gamma_2(a) \;=\;& +a\ [\ C_F + \xi(-C_F)\ ] \\
& +a^2\ [\ -\frac{3}{2}C_F^2 + \frac{17}{2}C_F C_A - C_F n_f + \xi(-\frac{5}{2}C_F C_A) + \xi^2(\frac{1}{4}C_F C_A)\ ] \\
& +a^3\ [\ \frac{3}{2}C_F^3 + C_F^2 C_A(12\zeta_3 - \frac{143}{4}) + C_F C_A^2(-\frac{15}{2}\zeta_3 + \frac{10559}{144}) + \frac{3}{2}C_F^2 n_f \\
& \quad - \frac{1301}{72}C_F C_A n_f + \frac{5}{9}C_F n_f^2 + \xi(C_F C_A^2(-\frac{3}{2}\zeta_3 - \frac{371}{32}) + \frac{17}{8}C_F C_A n_f)
\end{aligned}$$

$$+\xi^2(C_F C_A^2(\frac{3}{8}\zeta_3+\frac{69}{32}))+\xi^3(-\frac{5}{16}C_F C_A^2)\ ], \quad (7)$$

$$\begin{aligned}
\gamma_3(a) &= +a\ [\ -\frac{5}{3}C_A+\frac{2}{3}n_f+\xi(-\frac{1}{2}C_A)\ ] \\
&+a^2\ [\ -\frac{23}{4}C_A^2+2C_F n_f+\frac{5}{2}C_A n_f+\xi(-\frac{15}{8}C_A^2)+\xi^2(\frac{1}{4}C_A^2)\ ] \\
&+a^3\ [C_A^3(\frac{3}{2}\zeta_3-\frac{4051}{144})-C_F^2 n_f+C_F C_A n_f(12\zeta_3+\frac{5}{36})+C_A^2 n_f(-9\zeta_3+\frac{875}{36}) \\
&-\frac{11}{9}C_F n_f^2-\frac{19}{9}C_A n_f^2 \\
&+\xi(C_A^3(-\frac{9}{8}\zeta_3-\frac{127}{16})+C_A^2 n_f)+\xi^2(C_A^3(\frac{3}{16}\zeta_3+\frac{27}{16}))+\xi^3(-\frac{7}{32}C_A^3)\ ].
\end{aligned} \quad (8)$$

Here $C_F=(n_c^2-1)/2n_c$ and $C_A=n_c$ are the Casimir operators of the defining and adjoint representations of the color group $SU(n_c)$, n_f is the number of active quark flavors, ζ_3 is the Riemann zeta-function ($\zeta_3=1.202056903\ldots$), ξ is the gauge parameter.

Substituting the obtained γ-functions to the eq.(4) we find the desired β-function:

$$\begin{aligned}
\beta(a) &= -a^2[+\frac{11}{3}C_A-\frac{2}{3}n_f] \\
&-a^3[\frac{34}{3}C_A^2-2C_F n_f-\frac{10}{3}C_A n_f] \\
&-a^4[\frac{2857}{54}C_A^3+C_F^2 n_f-\frac{205}{18}C_F C_A n_f-\frac{1415}{54}C_A^2 n_f+\frac{11}{9}C_F n_f^2+\frac{79}{54}C_A n_f^2]
\end{aligned} \quad (9)$$

This result coincides with the previous calculation [1].

The β-function is gauge independent as it should be within the MS-scheme and does not contain ζ_3. The cancellation of the ξ-terms is a strong check of the validity of the calculations.

We repeat the result once more for the particular case of QCD ($n_c=3$):

$$\beta(a)=-a^2(11-\frac{2}{3}n_f)-a^3(102-\frac{38}{3}n_f)-a^4(\frac{2857}{2}-\frac{5033}{18}n_f+\frac{325}{54}n_f^2). \quad (10)$$

It is interesting to note that the poles of individual three-loop diagrams of the quark-quark-gluon vertex and the the total renormalized three-loop quark-quark-gluon vertex (which we have also calculated but do not present here because of the cumbersome result) contain the quartic Casimir operator (see e.g. [14]). But it cancels in the anomalous dimension γ_1 and correspondingly in the β-function which are expressed through the quadratic Casimir operators C_F and C_A only. Since the quartic Casimir operator is present in the finite part of the three-loop quark-quark-gluon vertex it can enter the four-loop β-function. But although existing techniques [11], [12] are sufficient to compute the four-loop β-function analytically, in practice it is a very difficult task because of enormous amount of diagrams to be calculated.

References

[1] O.V. Tarasov, A.A. Vladimirov, A.Yu. Zharkov, Phys. Lett. B 93 (1980) 429.

[2] D.J. Gross, F. Wilczek, Phys.Rev.Lett. 30 (1973) 1343;
H.D. Politzer, Phys.Rev.Lett. 30 (1973) 1346.

[3] W.E. Caswell, Phys.Rev.Lett. 33 (1974) 244;
D.R.T. Jones, Nucl.Phys. B 75 (1974) 531;
E.S. Egorian, O.V. Tarasov, Theor.Mat.Fiz. 41 (1979) 26.

[4] S.G. Gorishny, A.L. Kataev, S.A. Larin, Phys.Lett. B 212 (1988) 238, ibid B 259 (1991) 144.

[5] S.A. Larin, F.V. Tkachov, J.A.M. Vermaseren, Phys.Rev.Lett. 66 (1991) 862;
S.A. Larin, J.A.M. Vermaseren, Phys.Lett. B 259 (1991) 345.

[6] J. Chyla, A. Kataev, S.A. Larin, Phys.Lett. B 267 (1991) 269.

[7] A.C. Mattingly, P.M. Stevenson, Phys.Rev.Lett. 69 (1992) 1320.

[8] G. 't Hooft and M. Veltman, Nucl.Phys. B 44 (1972) 189;

[9] G. 't Hooft, Nucl.Phys. B 61 (1973) 455.

[10] J.C. Collins, Nucl.Phys. B 80 (1974) 341.

[11] A.A. Vladimirov, Theor.Mat.Fiz. 43 (1980) 210;
K.G. Chetyrkin, A.L. Kataev, F.V. Tkachov, Nucl.phys. B 174 (1980) 345.

[12] S.A. Larin, F.V. Tkachov, J.A.M. Vermaseren, Preprint NIKHEF-H/91-18 (1991).

[13] J.A.M. Vermaseren, Symbolic Manipulation with Form, Computer Algebra Nederland, Kruislaan 413, 1098 SJ Amsterdam, 1991, ISBN 90-74116-01-9.

[14] P. Cvitanovic, Phys.Rev. D 14 (1976) 1536 (see p.1540).