

On the extended Poincaré Polynomial

Maximilian KREUZER*

Institut für Theoretische Physik, Technische Universität Wien,
Wiedner Hauptstraße 8-10/136, A-1040 Vienna, AUSTRIA

and

Christoph SCHWEIGERT#

NIKHEF-H/FOM, P.O. Box 41882
1009 DB Amsterdam, THE NETHERLANDS

ABSTRACT

We show that the numbers of generations and anti-generations of a (2,2) string compactification with diagonal internal theory can be expressed in terms of certain specifications of the elliptic genus of the untwisted internal theory which can be computed from the Poincaré polynomial. To establish this result we show that there are no cancellations of positive and negative contributions to the Euler characteristic within a fixed twisted sector. For our considerations we recast the orbifolding procedure into an algebraic language using simple currents. Turning the argument around, this allows us to define the ‘extended Poincaré polynomial’ $P(t, x)$, which encodes information on the orbits of the spinor current under fusion, for non-diagonal $N = 2$ superconformal field theories. As an application, we derive an explicit formula for $P(t, x)$ for general Landau-Ginzburg orbifolds.

*e-mail: kreuzer@tph16.tuwien.ac.at

#e-mail: t10@nikhef.nl

1 Introduction

The most accessible and most important quantities characterizing a string compactification arise in the description of the spectrum of massless excitations. A large part of this spectrum is topological: For a (2,2) vacuum the numbers n_{27} and $n_{\overline{27}}$ of generations and anti-generations can be computed in terms of the charge degeneracies of the Ramond ground states of the GSO-projected theory. In contrast, the numbers of gauge singlets and extra gauge bosons require additional information on the internal conformal field theory and may vary as we move in moduli space. The charge degeneracies of Ramond ground states of an $N = 2$ theory are conveniently encoded in their generating function, the Poincaré polynomial

$$P(t, \bar{t}) = (t\bar{t})^{c/6} \text{Tr}_{R_0} t^{J_0} \bar{t}^{\bar{J}_0}, \quad (1)$$

where the trace extends over the set R_0 of Ramond ground states in the theory. Note that, strictly speaking, P is not a polynomial in t and \bar{t} , since typically fractional powers occur in the normalization we have chosen. In practice one frequently only knows the Poincaré polynomial for the theory *before* GSO-projection; this polynomial is comparatively easy to compute even for rather complicated $N = 2$ theories like Kazama-Suzuki models [1, 2, 3]. To compute n_{27} and $n_{\overline{27}}$, it is necessary to know the Poincaré polynomial of the GSO-projected theory as well.

In [4] it was shown that the ‘Euler number’ $\chi = 2(n_{\overline{27}} - n_{27})$ of a string vacuum that results from the GSO-projection of a diagonal theory without additional twists can be obtained directly from the Poincaré polynomial $P(t) = P(t, 1)$ of the original internal conformal field theory. To compute n_{27} and $n_{\overline{27}}$ separately, the extended Poincaré polynomial was introduced [2] as an efficient bookkeeping device, which encodes information about the intersection of the orbits of the spinor current with the set of Ramond ground states.

It was observed [2, 3] that in all known cases the extended Poincaré polynomials of two superconformal field theories are identical whenever the Poincaré polynomials are the same. This suggests that the extended Poincaré polynomial may be fixed by the Poincaré polynomial alone, and that this holds independently of the underlying $N = 2$ theory. In particular, the numbers of generations and anti-generations might be computable directly from the Poincaré polynomial. In the present paper we show that for diagonal theories this is indeed the case. For a fixed twist of a diagonal theory, all Ramond ground states with the same right-moving $U(1)$ charge contribute with the same sign to the Euler number. As a consequence, we can generalize the Buturović formula and explicitly calculate n_{27} and $n_{\overline{27}}$ separately. This, in turn, tightly constrains the extended Poincaré polynomial.

In Section 2 we recall some basic facts about the implementation of the generalized GSO-projection using simple current modular invariants [5]. This construction is closely related to Gepner’s ‘shift vector’ method [6] and to Vafa’s orbifolding procedure [7]. In a brief digression we discuss the modding of ‘geometrical’ and ‘quantum’ symmetries from an algebraic point of view. The definition of the extended Poincaré polynomial and its use for the calculation of the non-singlet spectrum is a simple application of these ideas.

In Section 3 we analyse the index used by Buturović for the calculation of the Euler number. We show that the contributions coming from a fixed twisted sector never cancel. This result is used to derive formulae for n_{27} and $n_{\overline{27}}$.

The translation of the definition of the extended Poincaré polynomial into the orbifold

language allows to generalize it to arbitrary $N = 2$ superconformal field theories. It is, therefore, straightforward to derive an explicit formula for the extended Poincaré polynomial for orbifolds of Landau–Ginzburg models; this is done in Section 4. It is crucial that this works for conformal field theories with arbitrary central charge c . This has important practical consequences, since the extended Poincaré polynomial encodes all information that is relevant for the topological part of the spectrum of any vacuum that is built from a tensor product. Hence, if we reproduce the extended Poincaré polynomial of, say, a Kazama–Suzuki model with some Landau–Ginzburg orbifold, then we know that all possible spectra of tensor products containing this model must be identical to the spectrum of the tensor product that is obtained by replacing the Kazama–Suzuki model in this tensor product by the Landau–Ginzburg orbifold. As an example we consider the relation of an infinite series of non-Hermitian coset models to Landau–Ginzburg orbifolds.

2 Simple currents and $N = 2$ superconformal theories

To obtain a consistent string theory from an $N = 2$ superconformal theory, it is necessary to implement a number of projections, which can be described by means of integer spin simple currents. We will therefore first recall some basic facts about simple currents, and describe the ones that are present in any $N = 2$ theory and that constitute the ‘generic center’. For more details we refer the reader to the review [8].

A simple current J is a primary field whose fusion product with any primary field ϕ contains a single primary field $J\phi$ with multiplicity one [8]. The set of primary fields therefore decomposes into orbits with respect to the fusion product $(J)^n\phi$ whose maximal length N is called the order of the simple current; the set of all simple currents of a rational conformal field theory forms a finite abelian group, the *center*. For any primary field Φ_i we define the monodromy charge by $Q_J(\phi_i) \equiv h_i + h_J - h_{J_i}$, where h_i is the conformal weight of the primary field ϕ_i and h_J is the conformal weight of the simple current J . In operator products this charge is conserved modulo \mathbb{Z} . Thus the phase $\exp(2\pi i Q_J)$ is conserved under the operator product, which implies that the center acts as a discrete symmetry group on the conformal field theory. (Of course, not all symmetries have to be generated by simple currents and there are symmetries of a conformal field theory which are not described by its center.) The monodromy charges and conformal weights modulo \mathbb{Z} of the simple currents can be parametrized by a matrix R as

$$R_{ij} = r_{ij}/N_i \equiv Q_i(J_j) = Q_j(J_i), \quad h_{[\alpha]} \equiv \frac{1}{2} \sum_i r_{ii} \alpha^i - \frac{1}{2} \sum_{ij} \alpha^i R_{ij} \alpha^j \pmod{1}, \quad (2)$$

where $[\alpha] = \prod J_i^{\alpha^i}$, N_i is the order of the current J_i , and the diagonal elements R_{ii} are defined modulo 2.

2.1 Simple current modular invariants

Simple currents can be used to construct modular invariants, which are closely related to orbifolding with respect to the corresponding discrete symmetries. To construct a modular

invariant we need to choose a subgroup of the center for which all diagonal entries in the corresponding monodromy matrix r are even, so that the spin multiplied by the order is an integer (this condition is non-trivial only for simple currents with even order; it is related to the level matching condition in the orbifold framework). From the definition of the monodromy charge it follows that

$$h([\alpha]Ph) \equiv h(\phi) + h([\alpha]) - \alpha^i Q_i(\phi) \text{ mod } \mathbb{Z}, \quad (3)$$

while the fact that the monodromy charge is conserved modulo integers implies that

$$Q_i([\alpha]Ph) \equiv Q_i(\phi) + R_{ij}\alpha^j \text{ mod } \mathbb{Z}. \quad (4)$$

Furthermore, the S matrix elements for fields that are on the same orbits are related by phases,

$$S_{[\alpha]a, [\beta]b} = S_{a,b} e^{2\pi i(\alpha^k Q_k(b) + \beta^k Q_k(b) + \alpha^k R_{kl} \alpha^l)} \quad (5)$$

(the indices k and l are to be summed over the chosen subgroup of the center). It can now be checked that the matrix

$$M_{\phi, [\alpha]\phi} = \text{Mult}(\phi) \prod_i \delta_{\mathbf{z}} \left(Q_i(\phi) + X_{ij} \alpha^j \right) \quad (6)$$

commutes with the generators S and T of modular transformations, if X is properly quantized and $X + X^T \equiv R$ modulo \mathbb{Z} ; $\delta_{\mathbf{z}}(r)$ is 1 if $r \in \mathbb{Z}$ and 0 otherwise. $\text{Mult}(\phi)$ is the multiplicity of the primary field ϕ , i.e. the ratio of the size of the subgroup of the center that defines the modular invariant over the size of the orbit containing ϕ . It can be shown, using certain regularity assumptions, that (6) is the most general simple current modular invariant, i.e. a modular invariant that only relates primary fields on the same orbits of the center [9,10]. Note that the freedom in the choice of the anti-symmetric part of X corresponds to the freedom in the choice of phases of the projections in the twisted sectors of orbifolds, which are called discrete torsions.¹ The vector α of exponents can be interpreted as a twist, i.e. it tells us which sector of the orbifold a certain non-diagonal field comes from (this interpretation is consistent with the twist selection rules). It is interesting to remark that there are several infinite series of simple current modular invariants known [12] which are not of the form (6), and therefore do not possess a description in terms of orbifolds with discrete torsion; however, these modular invariants do not give rise to consistent conformal field theories.

2.2 Simple currents of $N = 2$ superconformal theories

Returning to superconformal field theory, the existence of a simple current with conformal weight $3/2$ and order 2, the supercurrent J_v , follows already from $N = 1$ supersymmetry; its monodromy charge is 0 for NS states and $1/2$ in the Ramond sector. Note that we are using the word supercurrent to denote a primary field J_v . However, it can happen that one can construct more than one supercharge out of descendants of this primary field.² For $N = 2$

¹ Because of the coincidence of the resulting string vacua, simple current modular invariants and orbifolds with discrete torsion were first conjectured to be equivalent in ref. [11].

² This is the case e.g. in $N = 2$ coset models [13] where both supercharges belong to the same primary field which has a representative with trivial quantum numbers except for the $\text{so}(d)$ part, where it is equal to the vector v .

superconformal models we have at least one more simple current, namely the Ramond ground state J_s of highest $U(1)$ charge, which in our normalization is $c/6$. This simple current, to which we will refer as the *spinor current*, implements the spectral flow: In terms of the bosonized $U(1)$ current of the $N = 2$ algebra $J(z) = \sqrt{\frac{c}{3}}\partial X(z)$ it is given by $J_s = \exp i\sqrt{\frac{3}{c}}X$. It follows from the operator product of the free boson X that the monodromy charge Q_s of the spinor current is related to the $U(1)$ charge Q by $Q_s \equiv -Q/2$ modulo 1. Thus, if $1/M$ is the charge quantum in the NS sector and $\hat{c} = c/3 = k/M$, then the order of the spinor current J_s is $2M$ if k is even and $4M$ if k is odd, because the charges in the Ramond sector are shifted by $c/6$. If k is odd, then $J_v = (J_s)^{2M}$, so that the order of the ‘generic’ center, which is present in any rational $N = 2$ theory, is $4M$ in both cases. J_s is a Ramond ground state, hence $h_s = c/24$ and $Q_s(J_s) = -c/12$. Putting the pieces together we find for the matrix of monodromies

$$R_{v,v} = 0, \quad R_{v,s} = 1/2, \quad R_{s,s} = n - c/12 \quad \text{with } n = \begin{cases} k/2 & k \text{ even} \\ 1 & k \text{ odd.} \end{cases} \quad (7)$$

In the case of minimal models k is the level and $M = k + 2$.

A consistent heterotic (or type II) vacuum is now obtained by the following procedure. Working in the bosonic framework (i.e. after the bosonic string map) we have to tensor the internal $c = 9$ $N = 2$ superconformal theories with a D_5 Kac–Moody algebra at level 1 (we omit a factor of E_8 at level 1, which is irrelevant for our considerations). For having a well-defined supersymmetry generator it is essential that we project out all mixed states, which do not have all factors in the NS sector or all in the Ramond sector, from the tensor product. Furthermore, the spectrum of the string should be space-time supersymmetric.

Both requirements can be implemented with simple currents: Note that the total spinor current $J_s^{tot} := s \otimes J_s$, i.e. the product of the spinor of D_5 with the spinor current J_s of the internal $c = 9$ theory, and the product $J_v^{tot} = v \otimes J_v$ of the vector v of D_5 with the supercurrent J_v both have integral spin since $h(J_s) = c/24 = 3/8$ and $h(s) = 5/8$. Moreover, all monodromies vanish. Hence, if we choose no torsion between J_s^{tot} and J_v^{tot} , i.e. $X = 0$ in eq. (6), then all fields with non-integral monodromy charges or, equivalently, $U(1)$ charges that are not even³ are projected out and both simple currents extend the chiral algebra of the theory. This is exactly what we want to achieve, since J_s^{tot} can be combined with right moving bosons to yield the gravitino vertex; on the gauge side of the heterotic string it extends the gauge group from D_5 to E_6 . Primary fields with mixed boundary conditions have half-integral monodromy charge with respect to J_v^{tot} ; if the internal theory is itself a tensor product of $N = 2$ theories, then the group generated by all bilinears in the vector currents has to extend the chiral algebra in order to align Ramond states and NS states (see [2, 14]).

2.3 The extended Poincaré polynomial

We are now in a position to define the extended Poincaré polynomial. Since the non-singlet E_6 representations all come from Ramond ground states of the GSO-projected theory, we encode, as for the ordinary Poincaré polynomial, only information on these states and their charges. Both the alignment of boundary conditions and the generalized GSO-projection

³ We have to project on *even* charges since we work after having applied the bosonic string map.

correspond to integral spin simple currents and we may, in a first step, disregard the projections corresponding to the product of δ functions in the expression (6) and consider the ‘unprojected orbifolds’. Eventually, to obtain the projected orbifold, we just have to omit the contributions with non-integral monodromy charges. The aim of the extended Poincaré polynomial is to encode all information about an $N = 2$ superconformal theory which is necessary to compute the massless spectrum of any tensor product with $c = 9$ that contains this model as one factor. To this end we also need to encode information on the twists: We first define the ‘full extended’ Poincaré polynomial

$$\mathcal{P}(t, \bar{t}, x, \sigma) = \sum_{l \geq 0} \sum_{k=0}^1 x^l \sigma^k P_{l,k}(t, \bar{t}), \quad (8)$$

where $P_{l,k}(t, \bar{t})$ is the Poincaré polynomial of the unprojected sector twisted by $J_s^{2l} J_v^k$. Hence, $P_{l,k}$ is obtained by looking for all pairs (α, α') of Ramond ground states with $\alpha' = J_s^{2l} J_v^k \alpha$; the charges of α and α' are encoded in the exponents of t and \bar{t} , respectively.

The information on the location of Ramond ground states on the simple current orbits of J_s and J_v is important if we consider tensor products of $N = 2$ factor theories. For simplicity we restrict ourselves to the case that the tensor product contains only two factors; what we are really interested in in this situation is the modular invariant obtained for the total spinor current $J_s^{(1)} J_s^{(2)}$ after the alignment of R and NS sectors. For the tensor product we thus obtain the ‘full extended’ Poincaré polynomial by the prescription

$$\mathcal{P}(t, \bar{t}, x, \sigma) = \sum_{l \geq 0} x^l \left(\sum_{k=0}^1 P_{l,k}^{(1)}(t, \bar{t}) P_{l,k}^{(2)}(t, \bar{t}) + \sigma \sum_{k=0}^1 P_{l,k}^{(1)}(t, \bar{t}) P_{l,1-k}^{(2)}(t, \bar{t}) \right). \quad (9)$$

Hence, (8) indeed encodes all the information from a factor theory that enters the computation of the generation numbers in arbitrary tensor products. In fact, this information is still redundant: Consider a pair of R ground states (α, α') whose contribution to $P_{l,k}$ is $t^{Q(\alpha) + \frac{c}{6}} \bar{t}^{Q(\alpha') + \frac{c}{6}}$. Then eq. (4) implies that

$$Q(\alpha') \equiv -2Q_s(\alpha') \equiv Q(\alpha) - 2(2lR_{ss} + kR_{sv}) \equiv Q(\alpha) + l\frac{c}{3} - k \pmod{2}. \quad (10)$$

Hence the exponent

$$k \equiv Q(\alpha) + l\frac{c}{3} - Q(\alpha') \pmod{2} \quad (11)$$

of σ is fixed in terms of the other exponents. So we can set σ to -1 ; the negative sign is a convenient choice because a twist by an odd number of supercurrents J_v implies a negative contribution to the index.

In the original definition of the extended Poincaré polynomial [2] Schellekens, in addition, put $\bar{t} = 1$. If we are only interested in applications to heterotic (2,2) string vacua built from diagonal theories, this is still a sufficient amount of information for the following reason. We can turn the above argument around and conclude that for given exponents of t , x and σ the charge $Q(\alpha')$ of α' is known modulo 2. For a symmetric (2,2) vacuum this is all we need to know. In [2] it was implicitly assumed that, for a twist with a given power of the spinor current, all contributions with a fixed charge $Q(\alpha)$ contribute with the same sign to the index. In the next section we will show that this is the case for all diagonal $N = 2$ superconformal field theories. Therefore, for this application, the extended Poincaré polynomial

$$P(t, x) := \mathcal{P}(t, 1, x, -1) \quad (12)$$

indeed encodes sufficient information for the calculation of the massless non-singlet modes for arbitrary tensor products (without additional twists).

It should be clear how the simple current construction of string vacua is related to the orbifold technique of ref. [7]. In that paper the theory is modded by the symmetry $j = \exp(2\pi i J_0)$, where J_0 is the zero mode of the left-moving $U(1)$ charge, in order to project to integral charges. The arguments concerning modular invariance are ‘modulo GSO-projection’, which means up to half-integral contributions to conformal weights in the Neveu-Schwarz sector and without specification of the action of the symmetry in the Ramond sector, i.e. up to a possible twist by the supercurrent. Our discussion shows that, in the specific situation which the extended Poincaré polynomial was invented for, the Poincaré polynomial $P(t, \bar{t})$ for the projected total internal conformal field theory and $P(t, x)$ encode equivalent information. In a more general situation, $P(t, \bar{t}, x) = \mathcal{P}(t, \bar{t}, x, -1)$ would be more appropriate, since the complete information about left- and right-charges allows to treat non-diagonal theories, whereas the information on the twist, encoded by the exponent of x , allows to compute the relevant information individually for each factor in a tensor product.

2.4 Orbifolds and chiral algebras

At this place we want to make a few general remarks on orbifolds and their description in an algebraic framework. In the cases of orbifolds we considered so far, the chiral algebra was always *extended* by some integer spin simple currents (in case of discrete torsion $X \neq X^T$ the left and right extension can be different [9]). There is, however, in the algebraic approach (compare e.g. [15]) a different use of the word ‘orbifold’ which denotes the case when one *restricts* the chiral algebra to some subalgebra of the original chiral algebra. This subalgebra has to contain the Virasoro algebra of the original theory; hence the conformal anomaly has the same value c in both conformal field theories. This is the case e.g. for the \mathbb{Z}_2 orbifold of a free boson X compactified on a circle: while the original algebra contains all polynomials in ∂X , the chiral algebra of the orbifold only contains even polynomials in ∂X , which are invariant under $\partial X \mapsto -\partial X$.

In the case of a general orbifold, however, the situation is much more involved: It may be that the new chiral algebra \mathcal{A}' neither contains the original chiral algebra \mathcal{A} nor is itself contained in \mathcal{A} ; however, in any case the intersection $\mathcal{A}' \cap \mathcal{A}$ is non-empty and contains the Virasoro algebra of \mathcal{A} . It may even be that there is no change in the chiral algebra at all and that the invariant is a pure automorphism invariant. This can be observed rather explicitly with the simple current modular invariants if the simple current has non-integer spin. An example are the D-type modular invariants of $\mathfrak{su}(2)$ at level $k = 2 \bmod 4$, which correspond to the automorphism of the fusion rule which maps the primary field with Dynkin label l onto itself if l is even and to $k - l$ if l is odd.

Note that any abelian orbifold has a symmetry group that is isomorphic to the twists defining the orbifold. It was observed for simple current invariants [8] that, in case of a cyclic center, the square of a modular invariant gives back the diagonal one if the modular invariant does not extend the algebra. In the orbifold language this corresponds to the modding by the ‘quantum symmetry’ (i.e. the symmetry implied by the twist selection rule), which also gives back the original theory. Since the latter does not refer to the chiral algebra at all it is natural

to ask if we can return to the original theory by such a modding also if the chiral algebra of a simple current modular invariant is extended. In that case we must not consider the maximal chiral algebra, since then different twists contribute to the same primary field and we cannot see the twist selection rule (this implies that, in such a situation, we must work with operator products instead of fusion rules, which are only well-defined with respect to the maximal chiral algebra). Here we are indeed in the situation discussed in [15]: The smaller algebra typically has more irreducible representations than the original one, and these must provide the twisted fields. By modding combinations of ‘classical’ and ‘quantum’ symmetries in an orbifold it is clear that we can have a mixed situation, where the chiral algebra is restricted and then re-extended by some fields from the twisted sectors.⁴ It would be interesting to extend this picture of going back and forth between the original CFT and the orbifold to non-abelian twists.

3 The massless spectrum

3.1 A non-cancellation theorem

We are now going to prove that for a fixed twist with respect to the spinor current J_s two Ramond ground states with the same charge can contribute to the index only with the same sign, in other words, that there are no cancellations within one fixed twisted sector. This result will allow us in the next subsection to compute $n_{\overline{27}}$ and n_{27} separately.

To prove this statement we fix a twist J_s^{2l} and assume that there are two Ramond ground states $|\alpha\rangle$ and $|\beta\rangle$ with the same $U(1)$ charge Q which both contribute to the extended Poincaré polynomial in that sector, but with different signs. We will show that this leads to a contradiction.

After possibly interchanging the role of $|\alpha\rangle$ and $|\beta\rangle$, we may assume that $|\alpha'\rangle = (J_s^{2l})_0|\alpha\rangle$ and $|\beta'\rangle = (J_s^{2l})_0|\beta\rangle$ both are non-vanishing. The index 0 indicates the zero modes of the operators, and hence $|\alpha'\rangle$ as well as $|\beta'\rangle$ are Ramond ground states as well. Since the superpartner of a Ramond ground state is not a Ramond ground state, but rather has conformal weight $h > \frac{c}{24}$ we conclude that $(J_s^{2l})_0|\alpha\rangle = 0$ has to vanish.

The main tool to derive a contradiction is an operator whose zero mode connects $|\alpha\rangle$ and $|\beta\rangle$. It can be constructed as follows: consider the primary fields \mathcal{O}_α and \mathcal{O}_β that generate $|\alpha\rangle$ and $|\beta\rangle$ from the vacuum. In any $N = 2$ superconformal field theory we can split off a $U(1)$ factor which is generated by the $U(1)$ current of the $N = 2$ algebra and write the theory as a tensor product of the $U(1)$ theory and some theory which only contains uncharged operators, with a non-product modular invariant. Therefore we can write $\mathcal{O}_\alpha = e^{iQ\sqrt{\frac{3}{c}}X} \hat{\mathcal{O}}_\alpha$ and $\mathcal{O}_\beta = e^{iQ\sqrt{\frac{3}{c}}X} \hat{\mathcal{O}}_\beta$ in terms of the bosonized $U(1)$ current and neutral operators $\hat{\mathcal{O}}$. These operators can a priori contain a polynomial in the derivative of X which is uncharged as well. However, the derivative of X is proportional to the $U(1)$ current in the $N = 2$ algebra, which is an element of the chiral algebra. Hence the presence of such operators in $\hat{\mathcal{O}}$ would contradict

⁴ Technically, the quantum symmetries can be implemented by introducing twists that act trivially on the original CFT, except for certain discrete torsions with the twists defining the orbifold [16].

the requirement that both $|\alpha\rangle$ and $|\beta\rangle$ are primary.

If we now use spectral flow to relate \mathcal{O}_β to a chiral operator \mathcal{O}_c and the conjugate operator \mathcal{O}_{α^c} to an anti-chiral operator \mathcal{O}_a , we observe that the zero mode of \mathcal{O}_a sends $|\alpha\rangle$ into the R ground state $|J_s^{-1}\rangle$ with lowest $U(1)$ charge and that the zero mode of \mathcal{O}_c sends that state into $|\beta\rangle$. Hence $|\beta\rangle = \psi_0|\alpha\rangle$ with $\psi_0 = (\mathcal{O}_c)_0(\mathcal{O}_a)_0$. Since ψ_0 does not change the X dependent part of \mathcal{O}_α it has to be uncharged and can therefore contain at most a polynomial in ∂X and its derivatives, which acts on $|\alpha\rangle$ as a c-number y . Since the only property of ψ_0 we are interested in is that it maps $|\alpha\rangle$ on $|\beta\rangle$ we can replace the polynomial in ∂X by y and assume that ψ_0 is chosen to be completely independent of X and its derivatives, and thus commutes with J_s^{2l} which is entirely build out of exponentials of X .

Putting the pieces together we find $|\beta'\rangle = (J_s^{2l})_0\psi_0|\alpha\rangle = \psi_0(J_s^{2l})_0|\alpha\rangle = 0$, which is a contradiction to our original assumption that α and β both contribute to the index with the same J_s^{2l} twist but with different signs.

3.2 Computation of the massless spectrum

In [4] it was shown how to compute the index of the GSO-projected theory directly from the Poincaré polynomial of the untwisted theory. To this end, the following quantities have been introduced:

$$\mathcal{P}_{r,s} := \text{tr}_{R_0^{(s)}} e^{2\pi i r(J_0 - \frac{c}{6})} e^{i\pi(Q_+ - Q_-)} = \text{tr}_{R^{(s)}} e^{i\pi(rc/3 + Q_+ - Q_-)} e^{2\pi i r J_0} q^{L_0 - c/24} \bar{q}^{\bar{L}_0 - c/24}. \quad (13)$$

Here $\text{tr}_{R_0^{(s)}}$ denotes the trace over all Ramond ground states in the s -th twisted sector and $\text{tr}_{R^{(s)}}$ the trace over the whole s -th twisted Ramond sector. Q_\pm denotes the left and the right moving $U(1)$ charge, respectively. The projection that is implemented by inserting the sum over r of $(\exp(2\pi i(J_0 - c/6)))^r$ along the second cohomology cycle of the torus takes into account the shift of the charges by $c/6$ under spectral flow to the Ramond sector.

The numbers $\mathcal{P}_{r,s}$ are index-like quantities and close relatives of the elliptic genus [17, 18]. To make the relation precise let us define for any twisted sector s the following trace:

$$Z_s(q, r; \bar{q}, \bar{r}) := \text{tr}_{R^{(s)}} (-1)^F q^{L_0 - c/24} \bar{q}^{\bar{L}_0 - c/24} e^{2\pi i(rJ_0 - \bar{r}\bar{J}_0)} \quad (14)$$

Our notation slightly differs from the usual definition by the factors of 2π and a relative minus sign for J and \bar{J} . The elliptic genus can be obtained from this expression by setting $\bar{r} = 0$, while r can take any value. Using index arguments one finds that this function does not depend on \bar{q} any more and one ends up with a character valued index which is a signed sum over the characters of Ramond ground states.

There is however, another set of index-like quantities, which can be obtained by restricting the values of r and \bar{r} in a different manner: Both are required to be integers. The usual index argument shows that for integral values of r and \bar{r} the contributions of two superpartners to Z_s cancel and that therefore Z_s does not depend on q and \bar{q} any more.

The numbers $\mathcal{P}_{r,s}$ can be obtained from (14) by setting \bar{r} to zero,

$$\mathcal{P}_{r,s} = Z_s(r, 0) e^{\pi i r c/3}; \quad (15)$$

as we have set $\bar{r} = 0$ this is equal to the elliptic genus, which at values of r we have chosen does not depend on q either and therefore simply is a number. Since these quantities are indices they are modular invariant and we can express those belonging to the twisted sectors using only expressions in the untwisted theory [4]: $\mathcal{P}_{r,s} = \mathcal{P}_{dr+bs,cr+as} = \mathcal{P}_{r \cap s, 0}$, where $r \cap s$ denotes the greatest common divisor of r and s .

In [4] it was argued that the contribution χ_s of the s -th twisted sector to the Euler number is

$$\chi_s = \frac{1}{M} \sum_{r=0}^{M-1} \mathcal{P}_{r \cap s}, \quad \text{where } \mathcal{P}_r := \mathcal{P}_{r,0} = P(t = e^{2\pi i r}, \bar{t} = 1). \quad (16)$$

Any of these numbers must be an integer; we will use this later to derive restrictions on the possible form of the extended Poincaré polynomial. The Euler number itself is given by

$$\chi = \sum_{s=0}^{M-1} \chi_s. \quad (17)$$

Since we know that within a fixed twisted sector the contributions to χ_s do not cancel, we can also derive an expression for the total number $\Sigma = |\mathcal{R}|$ of Ramond ground states in the GSO-projected theory in terms of the Poincaré polynomial of the untwisted theory:

$$\Sigma = \text{tr}_{R_0} \mathbf{1} = \sum_s |\chi_s| = \sum_s \left| \frac{1}{M} \sum_r \mathcal{P}_{r,s} \right|. \quad (18)$$

If we assume that space-time supersymmetry is not extended, we have the two relations

$$\chi = 2(n_{27} - n_{\bar{27}}) \quad \text{and} \quad \Sigma = 4 + 2(n_{27} + n_{\bar{27}}) \quad (19)$$

which determine both n_{27} and $n_{\bar{27}}$ separately. This assumption is no real obstacle in practice since space-time supersymmetry can only be extended if the Euler characteristic χ vanishes. More precisely, the Poincaré polynomial has to factorize into a $c = 3$ and $c = 6$ part, and the inverse charge quantum M must be the (co-prime) product of the inverse charge quanta of the factors.

3.3 Constraints on the indices \mathcal{P}_r

The indices \mathcal{P}_r are highly constrained by consistency requirements. First of all, they have all to be integers, since $\mathcal{P}_s = \mathcal{P}_{0,s}$ counts the number of Ramond ground states in the s -th twisted sector. In addition, the twist is mod M and hence $\mathcal{P}_{r,s}$ is periodic mod M in the second label. Hence we have

$$\mathcal{P}_r = \mathcal{P}_{r,0} = \mathcal{P}_{r,M} = \mathcal{P}_{r \cap M}, \quad (20)$$

and \mathcal{P}_r can depend only on $r \cap M$. As a consequence, also the contribution of the s -th twisted sector to the Euler number only depends on $s \cap M$.

In addition, there are divisibility constraints coming from formula (16) for the Euler number in the twisted sectors. If, e.g., p is a prime divisor of M , then $\chi_p = \frac{1}{p}(\mathcal{P}_p + (p-1)\mathcal{P}_1)$ must be an integer and hence $\mathcal{P}_p - \mathcal{P}_1$ must be a multiple of p .

We can also restrict the charges that occur in the s -th twisted sector: To begin with, assume that s and M are co-prime; then we have also that $(r \cap s) \cap M = 1$ for all r . We can use (16) to calculate

$$\chi_s = \frac{1}{M} \sum_{r=0}^{M-1} \mathcal{P}_{r \cap s} = \mathcal{P}_1 = \mathcal{P}_s = \mathcal{P}_{0,s}. \quad (21)$$

Note that the sum over r implements on the states in the Ramond sector a projection on charges for which $Q + \frac{c}{6} \in \mathbb{Z}$. Since there are no cancellations in χ_s , the calculation shows that this projection keeps all states in the s -th twisted sector and that therefore the charges of the Ramond ground states in this twisted sector obey $Q + \frac{c}{6} \in \mathbb{Z}$. It is straightforward to generalize this constraint to the case when s and M are not co-prime, $s \cap M = l$. A calculation analogous to (21) shows that in this case

$$\frac{l}{M} \sum_{r=0}^{M/l-1} \mathcal{P}_{lr,s} = \mathcal{P}_{0,s}, \quad (22)$$

which shows that for the Ramond ground states in these sectors the charges obey $Q + \frac{c}{6} \in \frac{1}{l}\mathbb{Z}$.

So far we have assumed that $\hat{c} := c/3$ is an integer. This condition is not really a restriction, because we can always fulfill it for a tensor product with a suitable number of minimal models: For $N = 2$ theories k has to be even if M is even [19]. Hence we get an integral total \hat{c} if we tensor with n minimal models at levels $k_i = M - 2$, where $n = k/2$ if k is even and $n = (k + M)/2$ if k is odd. The extended Poincaré polynomial for minimal models

$$P(x, t^M)^{(\text{mm})} = \sum_{s=1}^{M-1} t^{s-1} \frac{1 - (-)^s x^s}{1 - (-)^M x^M} \quad (23)$$

was derived in [2]. Note that there is a single contribution to the s^{th} twisted sector which has charge $(s - 1)/M$. Thus, for a central charge $c/3 = k/M \notin \mathbb{Z}$, the charges of Ramond ground states in the unprojected twisted sectors fulfill

$$Q + \frac{c}{6} \in -n \frac{s-1}{M} + \frac{\mathbb{Z}}{s \cap M}, \quad n \equiv k(M+1)/2 \pmod{M}. \quad (24)$$

The Poincaré polynomial of a minimal model evaluated at $e^{2\pi i r/M}$ is $-e^{-2\pi i r/M}$. Therefore $\mathcal{P}'_r = e^{-2\pi i nr/M} \mathcal{P}_r$ must be an integer, which, up to a sign, counts the numbers of Ramond ground states in the respective sector.

For the problem of constructing $P(t, x)$ from $P(t)$ this means that, due to the non-cancellation theorem, we have restricted the possible extended Poincaré polynomials for a given Poincaré polynomial to a finite set which is rather small. All remaining freedom is in the integral part of the charges (or up to multiples of $1/l$ if $l = s \cap M > 1$). In many cases the remaining freedom can be fixed by various consistency requirements. To illustrate this, we consider the E_7 invariant⁵ of the $N = 2$ minimal model at level 16, which has the Poincaré polynomial

$$P(t^9) = 1 + t^2 + t^3 + t^4 + t^5 + t^6 + t^8. \quad (25)$$

⁵ This is not a diagonal conformal field theory, but all contributions to the Poincaré polynomial are diagonal. Our proof of non-cancellation can be extended to this situation.

We find $\mathcal{P}'_1 = 1$ and $\mathcal{P}'_3 = -2$. Since the central charge c is smaller than 3, all charges are smaller than 1 and the extended Poincaré polynomial must be of the form

$$P(x, t^9) = P(t^9) + x + x^2 t^5 - x^3 (at + bt^4 + ct^7) + x^4 t^6 + x^5 t^2 - x^6 (ct + bt^4 + at^7) + x^7 t^3 + x^8 t^8. \quad (26)$$

$\mathcal{P}'_3 = -2$ implies that $a + b + c = 2$; note that charge conjugation relates the sectors with $s = 3$ and $s = 6$. The projection to ‘integral charges’ (after tensoring 4 minimal models at level 7) acts non-trivially only in the sectors 3 and 6, where it keeps $(\mathcal{P}'_1 + \mathcal{P}'_1 + \mathcal{P}'_3)/3 = 0$ states. This implies $a = 0$, since a is the coefficient of the only term which contributes to states with an integer charge in the tensor product. With a simple observation we can also fix the coefficients b and c : If we tensor our model with itself and in addition with 8 minimal models at level 7, then not only $(at)^2$, but also $2(bt^4)(ct^7)$ has the correct charge to survive the projection in the 3^{rd} sector. This time the projection keeps $((\mathcal{P}'_1)^2 + (\mathcal{P}'_1)^2 + (\mathcal{P}'_3)^2)/3 = 2$ states, which implies that $2bc = 2$. Since $b + c = 2$ we conclude $b = c = 1$.

If c is larger than 3 then we are (almost) always left with some ambiguity, because all information we obtained so far is insensitive to the integral part of all $U(1)$ charges. In that case we can, however, use the sum rule for $U(1)$ charges that was derived in [19] to constrain the integral parts of the charges (by applying this sum rule to tensor products we can derive further constraints, which was sufficient to fix all remaining freedom in a number of cases that we considered with $c > 3$). In [2, 3] consistency constraints for tensor products were also used to fix some ambiguity in the extended Poincaré polynomial that arose from field identification fixed point problems in Kazama–Suzuki coset models. Note, however, that the only information on the specific conformal field theory we used was the ordinary Poincaré polynomial.

4 Extended Poincaré polynomials for Landau–Ginzburg orbifolds

In this section we discuss how to compute the extended Poincaré polynomial in the orbifold framework and, as an application, derive an explicit formulae for the extended Poincaré polynomials of Landau–Ginzburg models and their orbifolds. Recall that the operator $j = \exp(2\pi i J_0)$ generates the symmetry group that leaves all states with integral charges invariant and that j^M is the identity in the NS sector. We denote by $P_l(t, \bar{t})$ the Poincaré polynomial in the unprojected sector twisted by j^l (up to superpartners, as discussed in section 2.3). In general $P_l(t, \bar{t})$ will be asymmetric in t and \bar{t} , even if the untwisted theory was diagonal. Up to a sign, the l -th twisted sector contributes to the extended Poincaré polynomial with $x^l P_l(t, 1)$. If the left and right charges Q_{\pm} of the states in the twisted sectors are known at least modulo two, we can use Equation (11) to compute this sign as $(-1)^s$, with $s = Q_+ - Q_- + r c/3$. This expression for s must be an integer even before the projection to invariant states. Hence we obtain the formula

$$P(x, t) = \sum_{r \geq 0} (e^{i\pi c/3} x)^r P_r(e^{i\pi} t, e^{-i\pi}) \quad (27)$$

for the extended Poincaré polynomial.

4.1 Untwisted Landau–Ginzburg models

For Landau–Ginzburg models, we are now in position to compute the extended Poincaré polynomial: The charges of the twisted ‘vacua’ in the Ramond sector are known [7, 20, 21] to be

$$Q_{\pm}|j^l\rangle_R = \left(\sum_{lq_i \in \mathbf{Z}} (q_i - \frac{1}{2}) \pm \sum_{lq_i \notin \mathbf{Z}} (\theta_i^{(l)} - \frac{1}{2}) \right) |j^l\rangle_R \quad (28)$$

where $q_i = n_i/M$ is the $U(1)$ charge of the i^{th} field and $\theta_i^{(l)} = lq_i - [lq_i]$ with $[x]$ denoting the greatest integer smaller than x . Thus the Poincaré polynomial in the l^{th} sector is *before* projection:

$$P_l(t, \bar{t}) = \prod_{lq_i \notin \mathbf{Z}} t^{\theta_i^{(l)} - q_i} \bar{t}^{1 - \theta_i^{(l)} - q_i} \prod_{lq_i \in \mathbf{Z}} \frac{1 - (t\bar{t})^{1 - q_i}}{1 - (t\bar{t})^{q_i}}, \quad (29)$$

the last factor comes from the chiral fields that are invariant under the twist and therefore contribute to Ramond ground states in the twisted sectors. Since $c/3 = k/M = \sum(1 - 2q_i)$ we obtain for the extended Poincaré polynomial

$$P(x, t) = \frac{1}{1 - (-)^k x^M} \sum_{r=0}^{M-1} (-)^{s_r} x^r \prod_{rq_i \notin \mathbf{Z}} t^{\theta_i^{(r)} - q_i} \prod_{rq_i \in \mathbf{Z}} \frac{t^{1 - q_i} - 1}{t^{q_i} - 1}, \quad (30)$$

where $s_r \equiv Q_+ - Q_- + rc/3 \equiv rN - N_{tw}(r)$, N is the total number of fields and $N_{tw}(r)$ is the number of fields that are not invariant under j^r .

Applying this formula to the $N = 2$ minimal models with the diagonal modular invariant we recover the expression (23); for the E_7 invariant we find agreement with (26). We also checked formula (30) for a large subclass of Grassmannian coset models, the CP_n models, for which a Landau–Ginzburg description is known [1]. These coset theories correspond to Hermitian symmetric spaces of the form

$$A(n, 1, k) = \frac{\text{su}(n+1)_k}{\text{su}(n) \oplus \text{u}(1)} \quad (31)$$

with central charge $c = \frac{3nk}{k+n+1}$. For our purpose the relevant data are the $U(1)$ charges $q_i = \frac{i}{m+n+1}$ of the chiral superfields with $1 \leq i \leq m$. Note that the minimal model at level k can be described as a CP_1 model at level k .

4.2 Landau–Ginzburg orbifolds

For LG orbifolds we can proceed analogously and use the more general formula

$$Q_{\pm}|h\rangle_R = \left(\pm \sum_{\theta_i^h \notin \mathbf{Z}} (\theta_i^h - \frac{1}{2}) + \sum_{\theta_i^h \in \mathbf{Z}} (q_i - \frac{1}{2}) \right) |h\rangle_R \quad (32)$$

for the charges of the vacua in the sectors twisted by group elements h . Here h acts on the fields X_i like $hX_i = \exp(2\pi i \theta_i^h) X_i$ with phases $0 \leq \theta_i^h < 1$. In order to compute the extended Poincaré polynomial for a LG orbifold with twist group \mathcal{G} we have to twist by all products

of group elements $h \in \mathcal{G}$ with powers of j . The projection, however, keeps all states that are invariant under the elements of the centralizer of h in \mathcal{G} , regardless of their charge (note that j commutes with all linear symmetries of the LG potential, so the centralizer is always independent of the j twist). The action of a group element g that commutes with h on the twisted ground states in the Ramond sector can be shown to be

$$g|h\rangle_R = (-)^{K_g(1+K_h)} \varepsilon(g, h) (\det g|_h) |h\rangle_R, \quad (33)$$

where the phases $\varepsilon(g, h)$ fulfill the usual constraints on discrete torsions [20]. For the concept of the extended Poincaré polynomial to make sense we have to make sure that the supercurrent survives the projection, i.e. all twists in \mathcal{G} must satisfy $\deg g = (-1)^{K_g}$ (see [20] for details). This fixes the group actions in the R sector, whose signs are parametrized by $(-1)^{K_g}$, and restricts the determinants of allowed twists to real values.

Using the above formula for the charges of the twisted vacua $|h\rangle_R$ and the fact that the chiral excitations are described by an effective LG theory consisting of the untwisted fields we find

$$P_h(t, \bar{t}) = \prod_{\theta_i^h > 0} t^{\theta_i^h - q_i} \bar{t}^{1 - \theta_i^h - q_i} \prod_{\theta_i^h = 0} \frac{1 - (t\bar{t})^{1 - q_i}}{1 - (t\bar{t})^{q_i}} \quad (34)$$

for the unprojected contribution of the h twisted sector to the Poincaré polynomial. In order to obtain the extended Poincaré polynomial we have to sum over all twists $h = \hat{h} j^r \in \mathcal{G} \times \mathbb{Z}_M$; the Ramond ground states in such a sector contribute to the coefficient of x^r . Then we need to project to states that are invariant under the group elements of \mathcal{G} .

For abelian groups this projection can be implemented directly in the above expression for P_h in a convenient and efficient way: Note that the denominators $1/(1 - (t\bar{t})^{q_i})$ describe the charge degeneracies of the free polynomial algebra and that the factors $1 - (t\bar{t})^{1 - q_i}$ subtract the contributions from the ideals that are generated by the gradients of the potential. It is essential that these gradients are independent, which guarantees the correct counting and hence that the complete expression is a polynomial. Therefore we can implement the group transformation in a diagonal basis $gX_i = \rho_i X_i$ by replacing the denominators by $1/(1 - \rho_i(t\bar{t})^{q_i})$ and the factors in the numerator by $1 - \rho_i^{-1}(t\bar{t})^{1 - q_i}$. Additional phases come from the transformation properties (33) of the twisted vacua. Eventually we can implement the projection by summing over $g \in \mathcal{G}$. It is, however, rather unpleasant to have polynomials with non-integral coefficient in the denominator. This can be avoided if we write $1/(1 - \rho_i(t\bar{t})^{q_i})$ as $(\sum_{n=0}^{|\mathcal{G}|-1} (\rho_i(t\bar{t})^{q_i})^n) / (1 - (t\bar{t})^{|\mathcal{G}|q_i})$, where we used that $\rho_i^{|\mathcal{G}|} = 1$. (Instead of summing over the group it is more efficient to work with formal variables describing the group action and to keep only those terms in the numerator that are invariant under all generators of the twist group \mathcal{G}).

As a simple example we consider the \mathbb{Z}_{2M} orbifold of the tensor product of two minimal models at levels $k_i = 2M - 2$. In order to have a real determinant, thus keeping spectral flow and supersymmetry, the two factors should transform with opposite phases. It is easy to see that this orbifold has the $U(1)$ charges quantized in units of $1/M$. Straightforward evaluation of the above formulas yields the extended Poincaré polynomial

$$P(x, t^M) = \frac{1}{1-x^M} \left(\frac{1-t^{2M-1}}{1-t} + (2M-1)t^{M-1} + \sum_{r=1}^{M-1} x^r \left((2r-1)t^{r-1} + (2M-2r-1)t^{M+r-1} \right) \right). \quad (35)$$

The choice of this example is motivated by the observation that, after tensoring with a third minimal model at level $M-2$, the Landau–Ginzburg orbifold

$$\underbrace{X^{2M} + Y^{2M}} / \mathbb{Z}_{2M} + Z^M \quad (36)$$

reproduces the Poincaré polynomial of the non-hermitian symmetric coset model

$$\frac{(C_2)_{2M-3} \oplus so(6)_1}{(C_1)_{2M-2} \oplus u(1)_{4M}}. \quad (37)$$

In fact, we find that the orbifold and the coset model also have the same extended Poincaré polynomial and hence yield the same spectra in tensor products. Nonetheless, the two theories cannot be isomorphic, because e.g. the numbers of simple currents are different: The coset has $16M$ simple currents while a single minimal model at level k already has $4(k+2)$ simple currents.

5 Conclusions

In this letter we have presented several new results concerning the computation of the topological part of the massless spectrum of a string compactification. For diagonal models we have proven a non-cancellation theorem which allows us to compute the number of generations and anti-generations using the Poincaré polynomial only.

Furthermore, we have generalized the definition of the extended Poincaré polynomial for non-diagonal theories and derived explicit formulae for it in the case of Landau–Ginzburg models and their orbifolds. Since the extended Poincaré polynomial is independent of the moduli, two theories with the same extended Poincaré polynomial need not be isomorphic. But at least we know that all the spectra of all tensor products which contain them as factors are the same (provided that no additional symmetries, which exist for only one of the two models with coinciding Poincaré polynomials, are modded).

If the left and right charges of all Ramond ground states in the twisted sectors are known, as is the case for Landau–Ginzburg orbifolds, we can extend the definition of the Poincaré polynomial to include also that information:

$$P(x, t, \bar{t}) = \sum_{r \geq 0} (e^{i\pi c/3} x)^r P_r(e^{i\pi} t, e^{-i\pi} \bar{t}), \quad (38)$$

where the information of the sign now becomes redundant. This could be useful for more general string vacua, as well as for other reasons, e.g. the comparison of $N=2$ theories that are given in a different formulation.

Acknowledgements:

We would like to thank J. Fuchs and A.N. Schellekens for helpful discussions. One of us (C.S.) would like to thank the Institut für Theoretische Physik der Technischen Universität Wien, where this work started, and both of us thank the theory division at CERN, where this work was finished, for hospitality. Partial support by the *Österreichische Nationalbank* under grant No. 5026 is gratefully acknowledged.

References

- [1] W. Lerche, C. Vafa, N.P. Warner, Nucl. Phys. **B324** (1989) 427
- [2] A.N. Schellekens, Nucl. Phys. **B366** (1991) 27
- [3] J. Fuchs and C. Schweigert, Nucl. Phys. **B411** (1994) 181
- [4] E. Buturović, Nucl. Phys. **B352** (1991) 163
- [5] A.N. Schellekens and S. Yankielowicz, Nucl. Phys. **B330** (1990) 103;
Tables Supplements CERN-TH.5440S/89 and CERN-TH.5440T/89 (unpublished)
- [6] D. Gepner, Nucl. Phys. **B296** (1988) 757;
N = 2 String Theory, in: Proceedings of the Trieste Spring School Strings 1989,
eds. M.Green et al. (World Scientific, Singapore, 1990)
- [7] C. Vafa, Mod. Phys. Lett. **A4** (1989) 1169; Superstring Vacua, HUTP-89/A057 preprint
- [8] A.N. Schellekens and S. Yankielowicz, Int. J. Mod. Phys. **A5** (1990) 2903
- [9] M. Kreuzer and A.N. Schellekens, Nucl. Phys. **B411** (1994) 97
- [10] B. Gato-Rivera and A.N. Schellekens, Nucl. Phys. **B353** (1991) 519;
Comm. Math. Phys. **145** (1992) 85
- [11] A. Font, L.E. Ibáñez, F. Quevedo and A. Sierra, Nucl. Phys. **B337** (1990) 119
- [12] J. Fuchs, A.N. Schellekens and C. Schweigert, Nucl. Phys. **B437** (1995) 667
- [13] Y. Kazama and H. Suzuki, Phys. Lett. **B216** (1989) 112, Nucl. Phys. **B321** (1989) 232
- [14] C. Schweigert, Theor. Math. Phys. **98** (1994) 326
- [15] R. Dijkgraaf, C. Vafa, E. Verlinde and H. Verlinde, Comm. Math. Phys. **123** (1989) 485
- [16] M. Kreuzer and H. Skarke, *Orbifolds with discrete torsion and mirror symmetry*,
ITP-UH-03/95 and TUW-95/01 preprint
- [17] A.N. Schellekens and N.P. Warner, Nucl. Phys. **B287** (1987) 317
- [18] E. Witten, Comm. Math. Phys. **109** (1987) 525
- [19] O. Aharony, S. Yankielowicz and A.N. Schellekens, Nucl. Phys. **B418** (1994) 157
- [20] K. Intriligator and C. Vafa, Nucl. Phys. **B339** (1990) 95
- [21] E. Witten, Int. J. Mod. Phys. **A9** (1994) 4783