

# Stability of Scalar Fields in Warped Extra Dimensions

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(Dated: June 16, 2010)

## Abstract

This work sets up a general theoretical framework to study stability of models with a warped extra dimension where  $N$  scalar fields couple minimally to gravity. Our analysis encompasses Randall-Sundrum models with branes and bulk scalars, and general domain-wall models. We derive the Schrödinger equation governing the spin-0 spectrum of perturbations of such a system. This result is specialized to potentials generated using fake supergravity, and we show that models without branes are free of tachyonic modes. Turning to the existence of zero modes, we prove a theorem relating the number of normalizable zero modes to the parities of the background scalar profiles. Constructions with definite parity and only odd scalar profiles are shown to be free of zero modes and are hence perturbatively stable. We give two explicit examples of domain-wall models with a soft wall, one which admits a zero mode and one which does not. The latter is an example of a model that stabilizes a compact extra dimension using only bulk scalars and does not require branes.

## I. INTRODUCTION

Extra dimensions are a plausible extension to the Standard Model (SM), and a vast array of specific realizations of models with extra dimensions exist. String theory provides a lot of the motivation to include extra dimensions, and there are many string-inspired phenomenological models, for example Refs. [1–5], as well as purely field-theoretic constructions, for example Refs. [6–8].

Much work has been devoted to the study of the type I Randall-Sundrum (RS) model [4], where the electroweak hierarchy can be naturally generated by the warping of spacetime in the extra dimension. Such warping is a generic feature of models with an extra dimension: the warp factor of the background metric is sensitive to *changes* in the energy density as one moves along the extra dimension, and so non-trivial background sources will generically induce a warped metric. In the simplest 5D set-up, the line element of a warped space is

$$ds^2 = e^{-2\sigma(y)} \eta_{\mu\nu} dx^\mu dx^\nu + dy^2, \quad (1)$$

where  $y$  is the extra dimension and  $\sigma(y)$  is the warp factor. Scalar fields with profile  $\phi_i(y)$  and fundamental branes with tension  $\lambda_\alpha$  located at  $y_\alpha$  will provide a source of energy density and act to drive the warp factor:

$$\sigma''(y) \propto \sum_i \phi_i'(y)^2 + \sum_\alpha \lambda_\alpha \delta(y - y_\alpha). \quad (2)$$

Here,  $i$  labels the scalars and  $\alpha$  labels the branes. Note that scalars can only provide ‘positive warping’, and only if they have a non-trivial profile in the extra dimension. Branes can provide both ‘positive warping’ and ‘negative warping’ since  $\lambda_\alpha$  can be of either sign.

In the original RS set-up, the extra dimension was compactified on a circle with  $y$  identified with  $-y$ . The two branes at the orbifold fixed points were required to have tensions of equal magnitude but opposite sign in order for the warp factor  $\sigma$  to join up consistently; the positive and negative warping needed to balance.<sup>1</sup> The type II RS model [5] had the extra dimension infinite in size, and required only a single brane of positive tension at the origin. The warp factor is driven to  $\pm\infty$  at large distances from the origin, localizing 4D gravity to

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<sup>1</sup> The model also required a bulk cosmological constant to keep  $\sigma'(y)$  constant between the two branes. This constant needed to be fine-tuned against the tension of the branes, a generic feature of warped extra dimensions with 4D Poincaré slices.

the brane. Since this model contains only a positive tension brane, one can replace it by a scalar field with a suitable  $y$ -profile. Such domain-wall models also localize 4D gravity [9] and form the basis of a large area of research, see for example Refs. [7, 8].

Recently there has been interest in a new type of RS-like warped spacetime: a compact spacetime where the negative tension brane is replaced with a physical singularity. These soft-wall models were originally designed to yield linear Regge trajectories in the context of the AdS/CFT correspondence [10], but have since been the basis of actual models beyond the Standard Model (BSM) [11–14]. Our ultimate aim is take the soft-wall model and go one step further by removing the final, positive-tension brane at the origin, and replacing it with a suitable scalar field profile. The model will then describe a compact extra dimension without the use of fundamental branes. This is interesting because it is a purely field theoretical construction and requires no appeal to string theory.

Such a step is not as straightforward as it may first seem, as it is crucial that one ensures the stability of the scalar field configuration. Part of this involves stabilizing the size of the compact extra dimension. For the RS model, one can utilize the Goldberger-Wise mechanism [15, 16] where a bulk scalar field stabilizes the set-up due to localized potentials on each brane. Here, the presence of branes and the ability to have a 4D potential for the 5D scalar localized to the brane is necessary for stabilization to work. Soft-wall models can also be stabilized in a similar way [14]; here, one needs a 4D potential localized to the brane at the origin. In order to build a soft-wall model without a brane, we shall need an alternative to the Goldberger-Wise mechanism. It is the aim of the present paper to show that, with the correct type of scalar background, one does not need any additional stabilization mechanism.

The techniques developed in this paper have wider application than just soft-wall models without a brane. The problem of stability of non-trivial scalar backgrounds is an important one, and in the case of a single scalar without gravity there exists a constructive way of finding the lowest energy, stable solution [17, 18]. For the case with gravity, things are not so simple, but some attempts have been made [19–21]. The problem is difficult because, as discussed above, non-trivial scalar profiles always induce warping of the metric, and, furthermore, modes of the scalars mix with spin-0 degrees of freedom in the metric. In the literature to date, all successful analyses have relied on the superpotential approach [22] also known as the fake supergravity approach [23]. Fake supergravity requires one to choose a

scalar potential such that it is invariant under supergravity transformations, even though the whole theory itself is not locally supersymmetric. In practice this is very simple to achieve: one just needs to generate the full potential from a more primitive superpotential. Using this fake supergravity approach, one can easily solve the background equations of motion, including Einstein's equations. It has been shown, using such a construction, that models with an arbitrary number of scalars are free of tachyonic spin-2 modes [22]. For a single scalar field, it has also been shown that this construction leads to models free of tachyonic modes in the spin-0 sector [24]. In this paper we extend this work to the case of an arbitrary number of background scalar fields. We find that the same result holds: there are no tachyonic modes in the spin-0 sector. Restricting ourselves to spaces that are symmetric under parity, we also show that set-ups where the scalar profiles are all odd do not contain spin-0 zero modes.

The paper is organized as follows. In Section II we discuss the gravity-free case as a warm-up and introduction to the superpotential approach. We show that a scalar field configuration has a mode spectrum which is always free of tachyons, and which always contains a zero mode of translation. In Section III we describe the model with gravity and the background solutions in the fake supergravity approach. The spectrum of spin-2 perturbations in this background are derived in Section IV. The usual result of a massless 4D graviton is reproduced. Section V contains the main results of this paper, and shows that, in the fake supergravity approach with an arbitrary number of background scalars, there are no tachyonic spin-0 perturbations. In Section VI we present a theorem regarding the existence of spin-0 zero modes, and analyze two specific models: one which supports such a zero mode, and one which does not. The latter is an example of a model with a stabilized compact extra dimension that does not require any branes. We conclude in Section VII.

## II. GRAVITY-FREE CASE

As discussed in the introduction, non-trivial scalar profiles always induce a warped metric, and so realistic models must include gravity. Nevertheless, the gravity-free case is interesting to study before moving to the case with gravity, and serves also to introduce our notation. For one scalar field the general analysis has been completed in Refs. [17, 18] and for two scalar fields, using the superpotential approach, in Ref. [25]. In this section we show that,

for  $N$  scalar fields without gravity, the superpotential approach yields solutions which are globally stable, for given boundary conditions, and up to zero-mode translations of the configuration.

The action is

$$\mathcal{S} = \int d^4x dy \left[ -\frac{1}{2} \sum_i \partial^M \Phi_i \partial_M \Phi_i - V(\{\Phi_i\}) \right], \quad (3)$$

where the sum is over  $1, \dots, N$ , and there are  $N$  scalar fields. The potential  $V$  is an arbitrary function of these  $N$  scalars. The equations of motion are

$$\partial^M \partial_M \Phi_i - V_i(\{\Phi_i\}) = 0. \quad (4)$$

Here we place a subscript  $i$  on  $V$  to denote partial differentiation with respect to the field  $\Phi_i$ . This notation is used heavily throughout the paper; in general, for a function  $X$  that depends on the scalar fields, we write

$$X_{ij\dots k}(\{\Phi_i\}) \equiv \frac{\partial}{\partial \Phi_i} \frac{\partial}{\partial \Phi_j} \dots \frac{\partial}{\partial \Phi_k} X(\{\Phi_i\}). \quad (5)$$

We are interested in finding static background solutions for the scalars, configurations that depend only on the extra dimension:  $\Phi_i = \phi_i(y)$ . In what follows, ‘configuration’ refers to the set of  $N$  scalar fields taking on the static solutions given by  $\phi_i(y)$ .

We can restrict ourselves to potentials generated by a superpotential  $W(\{\Phi_i\})$  such that

$$V(\{\Phi_i\}) = \sum_i \frac{1}{2} [W_i(\{\Phi_i\})]^2. \quad (6)$$

The subscript  $i$  denotes partial differentiation with respect to  $\Phi_i$ , as per Eq. (5). With this particular choice of potential, the Euler-Lagrange equations of motion, Eq. (4), are satisfied so long as the static configuration  $\phi_i(y)$  solve the first-order differential equation

$$\phi'_i(y) = W_i(\{\phi_i\}). \quad (7)$$

A prime denotes a  $y$  derivative, and we are evaluating  $W_i$  at  $\phi_i(y)$ . Note that to show this solves the equations of motion, and in many places throughout this paper, we use the relation

$$\frac{dW}{dy} = \sum_i \frac{\partial W}{\partial \Phi_i} \frac{d\Phi_i}{dy}. \quad (8)$$

It turns out that such background configurations are always globally stable, up to zero-mode translations. To prove this statement, we will look at three things: 1) the 4D energy density of the configuration; 2) perturbative modes; 3) the zero mode.

We compute the 4D energy density of the system by integrating the 5D stress energy over  $y$ . We shall assume that  $V$  takes the form given by Eq. (6) and that the scalar fields are functions of  $y$  only, given by  $\phi_i(y)$ . In particular we do not require the scalar fields to satisfy Eq. (7). We obtain

$$\begin{aligned}
E &= \int T_{00} dy \\
&= \int \left[ \frac{1}{2} \sum_i (\phi'_i)^2 + \frac{1}{2} \sum_i [W_i(\{\phi_i\})]^2 \right] dy \\
&= \int dW + \frac{1}{2} \sum_i \int [\phi'_i - W_i(\{\phi_i\})]^2 .
\end{aligned} \tag{9}$$

The first term in the last line here is just a surface term; it is difference of  $W$  evaluated at the boundaries of the extra dimension, and depends only on the values of  $\phi_i$  at these boundaries. Given a particular  $W$ , which completely determines  $V$ , and choices for the boundary values of  $\phi_i$ , the total energy density of the configuration is minimized precisely when Eq. (7) is satisfied. Hence, for given boundary conditions, the superpotential approach yields configurations which globally minimize the energy.

Since the configuration minimizes the energy, the solution must be perturbatively stable, and we can demonstrate this explicitly. This will be useful as a precursor to the case with gravity. Expanding in linear perturbations  $\varphi_i$  about the background

$$\Phi_i(x^\mu, y) = \phi_i(y) + \varphi_i(x^\mu, y) , \tag{10}$$

the equations of motion, Eq. (4), reduce to

$$-\varphi_i'' + (W_{ik}W_{kj} + W_{ijk}W_k)|_{\text{bg}} \varphi_j = \square \varphi_i . \tag{11}$$

Here, repeated indices are to be summed over, and  $X|_{\text{bg}}$  indicates that  $X$  is a function of  $\Phi_i$  and is to be evaluated using the background solutions  $\phi_i(y)$ ; that is,  $X(\{\phi_i\})$ . Performing a Fourier transform,  $\square \varphi_i \rightarrow E\varphi_i$ , we obtain a set of  $N$ , coupled, time-independent Schrödinger equations. We can write these Schrödinger equations as [25]

$$(\partial_y \delta_{ik} + W_{ik}|_{\text{bg}})(-\partial_y \delta_{kj} + W_{kj}|_{\text{bg}})\varphi_j = E\varphi_i , \tag{12}$$

and so, using the results of Appendix A, we see the system admits solutions only with  $E \geq 0$ . This is true so long as the perturbations vanish at the boundaries of the extra dimension which is a valid assumption.

Now let us consider the existence of the zero-mode perturbations with  $E = 0$ . For such a mode to exist, Eq. (12) implies that  $\varphi'_i = W_{ij}\varphi_j$ , which admits the solution  $\varphi_i = aW_i = a\phi'_i$ , with  $a$  a real, non-zero constant. This is the familiar result that the zero mode of translation is the first derivative of the background configuration. Note that when there are multiple background fields, the zero mode is the mode where all  $N$  perturbations  $\varphi_i$  are excited simultaneously with profiles proportional to  $\phi'_i$ .

As a quick example, consider  $W = -\sqrt{\lambda/2}(\phi^3/3 - v^2\phi)$ , which yields the familiar potential  $V = (\lambda/4)(\phi^2 - v^2)^2$ . Solutions are  $\phi = \pm v$  and the well-known kink:  $\phi = v \tanh \left[ v\sqrt{\lambda/2}(y - y_0) \right]$ . To get the anti-kink solution, one needs to start with  $-W$  instead. This demonstrates the fact that the superpotential encodes for both  $V$  and the static solution, and that different superpotentials can yield the same potential. Also noteworthy is the fact that, for this choice of  $V$ ,  $\phi = 0$  is a solution of the equation of motion but *cannot* be obtained from any superpotential. This is because the  $\phi = 0$  solution is unstable, and we have shown that the superpotential approach always yields globally stable solutions.

### III. WARPED BACKGROUND CONFIGURATION

We now come to the main topic of the paper and consider a general 5D theory with gravity coupled minimally to  $N$  scalar fields, including the possibility of fundamental brane terms. The corresponding action is given by

$$\mathcal{S} = \int d^4x dy \left[ \sqrt{-g} (M^3 R + \mathcal{L}_{\text{matter}}) - \sqrt{-g_4} \lambda \right]. \quad (13)$$

We are using a  $(- + + +)$  signature and  $M$  is the 5D Planck mass. The scalar-matter Lagrangian and brane terms are given respectively by

$$\mathcal{L}_{\text{matter}} = -\frac{1}{2} \sum_i g^{MN} \partial_M \Phi_i \partial_N \Phi_i - V(\{\Phi_i\}), \quad (14)$$

$$\lambda = \lambda(\{\Phi_i\}) = \sum_\alpha \lambda_\alpha(\{\Phi_i\}) \delta(y - y_\alpha). \quad (15)$$

The sum over  $i$  is from 1 to  $N$ , and  $V(\{\Phi_i\})$  is the potential that in general depends on all  $N$  scalars. The subscript  $\alpha$  indexes the branes,  $y_\alpha$  are their locations, and  $\lambda_\alpha(\{\Phi_i\})$  the scalar potentials localized to the branes, which includes the brane's tension. Our aim is to study the general stability conditions for non-trivial background configurations of this model. In

this section we discuss background solutions, and then in the following two sections analyze spin-2 and spin-0 perturbations.

For 4D Poincaré slices, the background metric ansatz is

$$ds^2 = e^{-2\sigma(y)} \eta_{\mu\nu} dx^\mu dx^\nu + dy^2, \quad (16)$$

where  $\mu, \nu$  index the 4D subspace. Einstein's equations are

$$G_{MN} = \frac{1}{2M^3} T_{MN}. \quad (17)$$

Assuming the scalar fields only depend on the extra dimension  $y$ , and denoting such a background configuration by  $\phi_i(y)$ , the Euler-Lagrange and Einstein's equations yield  $N + 2$  equations for  $N + 1$  functions:

$$\begin{aligned} 3\sigma'' &= \frac{1}{2M^3} \left( \sum_i \phi_i'^2 + \lambda(\{\phi_j\}) \right), \\ 12\sigma'^2 - 3\sigma'' &= -\frac{1}{2M^3} [2V(\{\phi_j\}) + \lambda(\{\phi_j\})], \\ \phi_i - 4\sigma'\phi_i' - V_i(\{\phi_j\}) - \lambda_i(\{\phi_j\}) &= 0. \end{aligned} \quad (18)$$

The first two equations are in fact the sum and difference of the  $(\mu\nu)$  and (55) components of Einstein's equations. The notation  $V(\{\phi_i\})$  means that the potential is to be evaluated with the background fields  $\phi_i$ . Also, we place a subscript  $i$  on  $V$  and  $\lambda$  to denote partial differentiation with respect to the field  $\Phi_i$ , as per Eq. (5). It is easy to show that one of these equations can be obtained using two others, therefore the equations are not all independent.

Solutions to the above equations can be obtained using the so-called fake supergravity approach. In this approach a superpotential  $W$  is introduced so that

$$V(\{\phi_i\}) = \sum_i \frac{1}{2} [W_i(\{\phi_i\})]^2 - \frac{1}{3M^3} [W(\{\phi_i\})]^2, \quad (19)$$

where  $W_i$  is obtained using Eq. (5). It was shown in Ref. [22] that the system of equations, Eqs. (18), can be written in terms of the superpotential  $W$  and solutions are given by

$$\begin{aligned} \sigma'(y) &= \frac{1}{6M^3} W(\{\phi_i\}), \\ \phi_i'(y) &= W_i(\{\phi_i\}). \end{aligned} \quad (20)$$

However, what we are interested in here are not the solutions of the system of equations. Instead, assuming solutions exist, we would like to see if the system is stable or not. For this purpose, we will study the spin-2 and spin-0 fluctuations of the metric and the scalar fields around the background solutions.

#### IV. SPIN-2 PERTURBATIONS

The general ansatz which takes into account both spin-0 and spin-2 perturbations is

$$\begin{aligned}
 ds^2 &= e^{-2\sigma(y)} [(1 - 2F(x^\mu, y)\eta_{\mu\nu} + h_{\mu\nu}(x^\mu, y)] dx^\mu dx^\nu + [1 + G(x^\mu, y)]^2 dy^2, \\
 \Phi_i(x^\mu, y) &= \phi_i(y) + \varphi_i(x^\mu, y).
 \end{aligned}
 \tag{21}$$

Here, we have chosen to work in the axial gauge,  $h_{\mu 5} = 0$ , with transverse traceless part  $\partial^\mu h_{\mu\nu} = \eta^{\mu\nu} h_{\mu\nu} = 0$ . The  $G_{ij}$  (spatial) components of Einstein's equations enforce  $G = 2F$  which we take from now on. The Einstein tensor has the following non-zero components

$$\begin{aligned}
 G_{\mu\nu} &= 3e^{-2\sigma}\eta_{\mu\nu} (2\sigma'^2 - \sigma'' - F'' + 6\sigma'F' - 12\sigma'^2F + 6\sigma''F) \\
 &\quad + e^{-2\sigma} \left( -\frac{1}{2}e^{2\sigma}\square h_{\mu\nu} - \frac{1}{2}h''_{\mu\nu} + 2\sigma'h'_{\mu\nu} + 3(2\sigma'^2 - \sigma'')h_{\mu\nu} \right), \\
 G_{\mu 5} &= 3\partial_\mu(F' - 2\sigma'F), \\
 G_{55} &= 6\sigma'^2 - 3e^{-2\sigma}\square F + 12\sigma'F'.
 \end{aligned}
 \tag{22}$$

The stress-energy tensor is

$$\begin{aligned}
 T_{\mu\nu} &= e^{-2\sigma}\eta_{\mu\nu} \left[ -\frac{1}{2}(1 - 6F)(\phi'_i)^2 - (V + \lambda)|_{\text{bg}} + (2V + 4\lambda)|_{\text{bg}}F - (V_i + \lambda_i)|_{\text{bg}}\varphi_i - \phi'_i\varphi'_i \right] \\
 &\quad + e^{-2\sigma}h_{\mu\nu} \left[ -\frac{1}{2}(\phi'_i)^2 - (V + \lambda)|_{\text{bg}} - (V_i + \lambda_i)|_{\text{bg}}\varphi_i \right], \\
 T_{\mu 5} &= \partial_\mu(\phi'_i\varphi_i), \\
 T_{55} &= \frac{1}{2}(\phi'^{\mu}_i)^2 - V|_{\text{bg}} - 4V|_{\text{bg}}F - V_i|_{\text{bg}}\varphi_i + \phi'_i\varphi'_i.
 \end{aligned}
 \tag{23}$$

Here, repeated  $i$  indices are to be summed over. Taking the 4-trace of the  $(\mu\nu)$  part of Einstein's equations shows that the spin-2 and spin-0 perturbations completely decouple from one another. See Ref. [16] for a discussion of the gauge degrees of freedom and the decoupling of spin-2 and spin-0 sectors. The spin-2 perturbations are the easiest to analyze, and we look at them first, returning to the spin-0 sector in the following section. The Euler-Lagrange equations for the  $N$  scalar fields do not contain  $h_{\mu\nu}$ ; these equations are given in the next section.

Using the background equations, Eq. (18), the equation for  $h_{\mu\nu}$  is

$$-e^{2\sigma}\square h_{\mu\nu} - h''_{\mu\nu} + 4\sigma'h'_{\mu\nu} = 0.
 \tag{24}$$

There is always a zero mode,  $h'_{\mu\nu} = 0$ , which is normalizable. It is the well-known 4D massless graviton [5]. In conformal coordinates  $z$  defined by  $dy = e^{-\sigma} dz$  with rescaled

$h_{\mu\nu} = e^{3\sigma/2}\tilde{h}_{\mu\nu}$  we have

$$-\tilde{h}''_{\mu\nu} + \left(\frac{9}{4}\sigma'^2 - \frac{3}{2}\sigma''\right) = \square\tilde{h}_{\mu\nu}. \quad (25)$$

This is a Schrödinger-like equation. It can be rewritten in a self-adjoint form as

$$(\partial_z + S_h)(-\partial_z + S_h)\tilde{h}_{\mu\nu} = \square\tilde{h}_{\mu\nu}, \quad (26)$$

with

$$S_h = -\frac{3}{2}\sigma'. \quad (27)$$

Appealing to Appendix A, we see that there are no tachyonic modes in the spin-2 sector.

## V. SPIN-0 PERTURBATIONS

In the previous section we have seen that the spin-2 sector decouples from the spin-0 sector, independent of the number of scalar fields involved. We now consider spin-0 perturbations around the background solutions  $\sigma(y)$  and  $\phi_i(y)$ . The metric ansatz is a restricted version of Eq. (21) with  $h_{\mu\nu} = 0$  [16]

$$\begin{aligned} ds^2 &= e^{-2\sigma(y)} [1 - 2F(x^\mu, y)] \eta_{\mu\nu} dx^\mu dx^\nu + [1 + 2F(x^\mu, y)]^2 dy^2, \\ \Phi_i(x^\mu, y) &= \phi_i(y) + \varphi_i(x^\mu, y). \end{aligned} \quad (28)$$

We work to linear order in the perturbations  $F$  and  $\varphi_i$ . The equations for these perturbations consist of two of Einstein's equations and the Euler-Lagrange equations:

$$6M^3(F' - 2\sigma'F) = \phi'_i\varphi_i, \quad (29)$$

$$6M^3(-e^{2\sigma}\square F - 2\sigma'F' + F'') = 2\phi'_i\varphi'_i + 2\lambda|_{\text{bg}}F + \lambda_i|_{\text{bg}}\varphi_i. \quad (30)$$

$$e^{2\sigma}\square\varphi_i + \varphi''_i - 4\sigma'\varphi'_i - 6F'\phi'_i - (4V_i + 2\lambda_i)|_{\text{bg}}F - (V_{ij} + \lambda_{ij})|_{\text{bg}}\varphi_j = 0. \quad (31)$$

Note that one of the redundant Einstein's equations has been omitted and it can be easily obtained by using Eq. (29) and Eq. (30).

We now go to conformal coordinates  $z$  to eliminate the  $e^{2\sigma}$  factor from the  $\square$ 's, and rescale the fields by a factor of  $e^{3\sigma/2}$  to help eliminate first derivatives. That is, we make the following change of variables and field redefinitions

$$\begin{aligned} dy &= e^{-\sigma(y)} dz, & \sigma(y) &= \sigma(z(y)), & \phi_i(y) &= \phi_i(z(y)), \\ F(y) &= e^{\frac{3}{2}\sigma(z(y))}\tilde{F}(z), & \varphi_i(y) &= e^{\frac{3}{2}\sigma(z(y))}\tilde{\varphi}_i(z). \end{aligned} \quad (32)$$

In terms of the conformal coordinate  $z$  and the new fields  $\tilde{F}$  and  $\tilde{\varphi}_i$ , Eqs. (29) through (31) become, respectively,

$$6M^3(-\frac{1}{2}\sigma'\tilde{F} + \tilde{F}') = \phi'_i\tilde{\varphi}_i, \quad (33)$$

$$6M^3\left[-\square\tilde{F} + \tilde{F}'' + 2\sigma'\tilde{F}' + \left(\frac{3}{4}\sigma'^2 + \frac{3}{2}\sigma''\right)\tilde{F}\right] = 2\phi'_i\tilde{\varphi}'_i + 2e^{-2\sigma}\lambda|_{\text{bg}}\tilde{F} \\ + \left(3\sigma'\phi'_i + e^{-2\sigma}\lambda_i|_{\text{bg}}\right)\tilde{\varphi}_i, \quad (34)$$

$$\square\tilde{\varphi}_i + \tilde{\varphi}''_i + \left(-\frac{9}{4}\sigma'^2 + \frac{3}{2}\sigma''\right)\tilde{\varphi}_i - 3\phi'_i\left(3\sigma'\tilde{F} + 2\tilde{F}'\right) \\ - e^{-2\sigma}\left(4V_i + 2\lambda_i\right)|_{\text{bg}}\tilde{F} - e^{-2\sigma}\left(V_{ij} + \lambda_{ij}\right)|_{\text{bg}}\tilde{\varphi}_j = 0. \quad (35)$$

We can make judicious use of Eq. (33), and its derivative, to eliminate all first derivatives of  $\tilde{F}$  and  $\tilde{\varphi}_i$  in the other two equations. We also make use of the background equation for  $\phi_i$ . This allows us to obtain

$$6M^3(\tilde{F}' - \frac{1}{2}\sigma'\tilde{F}) - \phi'_i\tilde{\varphi}_i = 0, \quad (36)$$

$$6M^3\left[\square\tilde{F} + \tilde{F}'' - \left(\frac{9}{4}\sigma'^2 + \frac{5}{2}\sigma''\right)\tilde{F}\right] + 2e^{-2\sigma}\lambda|_{\text{bg}}\tilde{F} + \left(-2\phi''_i + e^{-2\sigma}\lambda_i|_{\text{bg}}\right)\tilde{\varphi}_i = 0, \quad (37)$$

$$\square\tilde{\varphi}_i + \tilde{\varphi}''_i + \left(-\frac{9}{4}\sigma'^2 + \frac{3}{2}\sigma''\right)\tilde{\varphi}_i + \left(-4\phi''_i + 2e^{-2\sigma}\lambda_i|_{\text{bg}}\right)\tilde{F} \\ - \frac{1}{M^3}\phi'_i\phi'_j\tilde{\varphi}_j - e^{-2\sigma}\left(V_{ij} + \lambda_{ij}\right)|_{\text{bg}}\tilde{\varphi}_j = 0. \quad (38)$$

This is a set of  $N + 2$  equations for  $N + 1$  functions of  $z$ .

### A. Simplifying the perturbation equations

The main aim of this section is to manipulate and simplify the above  $N + 2$  equations in order to obtain a set of coupled Schrödinger-like equations, which can be solved to determine the spin-0 spectrum. Before simplifying the equations, we need to understand precisely their redundancy. Let  $\alpha$ ,  $\beta$  and  $\gamma_i$  be equal to the left-hand-sides of Eqs. (36) through (38) respectively. Then the Euler-Lagrange and Einstein's equations amount to

$$\alpha = \beta = \gamma_i = 0. \quad (39)$$

One can show that the following relation holds *in the bulk*:

$$\square\alpha + \alpha'' - \left(\frac{9}{4}\sigma'^2 + \frac{3}{2}\sigma''\right)\alpha - \beta' + \frac{1}{2}\sigma'\beta + \phi'_i\gamma_i = 0. \quad (40)$$

Thus, there is a certain amount of redundancy in the  $N + 2$  equations. The redundancy is quantified precisely by Eq. (40). In particular, if we solve  $\alpha = \beta = 0$  then we automatically have  $\phi'_i \gamma_i = 0$  and solving for  $N - 1$  of the  $\gamma_i$ 's is enough to solve for the system (that is, we have eliminated one of the scalars  $\tilde{\varphi}_i$ ). Counting the number of free integration constants (ICs), we have 2 for  $\alpha = 0$  and  $\beta = 0$  together, and  $2(N - 1)$  for  $N - 1$  of the  $\gamma_i$ 's. This is a total of  $2N$  free ICs.

We can make other choices for eliminating an equation. For example, if we choose to solve *all*  $\gamma_i = 0$  and  $\beta = 0$ , then we will have an equation left over for  $\alpha$ :

$$\square\alpha + \alpha'' - \left(\frac{9}{4}\sigma'^2 + \frac{3}{2}\sigma''\right)\alpha = 0. \quad (41)$$

Solving  $\beta = \gamma_i = 0$  only solves  $\alpha$  such that it is a solution to the above differential equation. Counting the total number of free ICs: 2 each from all  $N$  of the  $\gamma_i$ 's and 2 from  $\beta = 0$ . However, 2 of these ICs have to be used in order to pick the  $\alpha = 0$  solution of Eq. (41) and so we end up with a total of  $2N$  *free* ICs for the system, the same as in the alternative choice above.

We can also choose to solve  $\alpha = 0$  and  $\gamma_i = 0$  with  $2N + 1$  free ICs. Then Eq. (40) becomes  $-\beta' + \frac{1}{2}\sigma'\beta = 0$  and we need to have 1 constraining IC from demanding that we pick out the  $\beta = 0$  solution from this differential equation. Thus there are again only  $2N$  free ICs for the system.

These arguments hold for solutions of the system in the bulk. Without branes, the system is completely specified by  $2N$  integration constants. If there are branes in the set-up, the situation is the same, as the values of fields on the brane are continuous and hence the same as the values in the bulk just to the sides of the brane(s). Derivatives of the fields are discontinuous on the brane, and hence undefined. Instead, one has freedom to choose the field derivatives on one side of the brane or the other, and then uses the ‘jump’ conditions to determine the derivative on the opposite side. The jump conditions are found by integrating Eqs. (37) and (38) over each brane; see [16] for details.

We choose to solve the system  $\beta = \gamma_i = 0$ , as it leaves us with the most ‘symmetric’ system of equations. Solutions to this particular set of equations will include all solutions of the true system (by that we mean Eq. (39)), as well as some additional solutions which do not satisfy all the boundary conditions of Einstein’s equations. For us, this is all we shall require, as we are going to show that the extended set of solutions does not contain certain

modes (tachyonic and/or zero modes), and so the full system cannot therefore contain such modes.

If one makes the definitions

$$\tilde{\varphi}_i(z) = M^{3/2}\psi_i(z), \quad (42)$$

$$\tilde{F}(z) = \chi(z)/\sqrt{12}, \quad (43)$$

then the equations  $\beta = \gamma_i = 0$ , Eqs. (37), and (38), become

$$\square\chi + \chi'' - (\mathcal{V}_{00} + \mathcal{B}_{00})\chi - (\mathcal{V}_{0i} + \mathcal{B}_{0i})\psi_i = 0, \quad (44)$$

$$\square\psi_i + \psi_i'' - (\mathcal{V}_{ij} + \mathcal{B}_{ij})\psi_j - (\mathcal{V}_{0i} + \mathcal{B}_{0i})\chi = 0,$$

where

$$\begin{aligned} \mathcal{V}_{00} &= \frac{9}{4}\sigma'^2 + \frac{5}{2}\sigma'', \\ \mathcal{V}_{0i} &= \frac{2}{\sqrt{3M^3}}\phi_i'', \\ \mathcal{V}_{ij} &= \left(\frac{9}{4}\sigma'^2 - \frac{3}{2}\sigma''\right)\delta_{ij} + \frac{1}{M^3}\phi_i'\phi_j' + e^{-2\sigma}V_{ij}|_{\text{bg}}, \end{aligned} \quad (45)$$

and the brane terms are

$$\begin{aligned} \mathcal{B}_{00} &= \frac{1}{3M^3}e^{-2\sigma}\lambda|_{\text{bg}}, \\ \mathcal{B}_{0i} &= \frac{1}{\sqrt{3M^3}}e^{-2\sigma}\lambda_i|_{\text{bg}}, \\ \mathcal{B}_{ij} &= e^{-2\sigma}\lambda_{ij}|_{\text{bg}}. \end{aligned} \quad (46)$$

Note the symmetry of the cross-coupling in Eq. (44). We can use Eq. (44) to solve for the physical spin-0 spectrum, which includes the eigenvalues and corresponding extra-dimensional profiles.

A physical mode is defined as having the same  $x^\mu$  dependence in  $\chi$  and  $\psi_i$ , but possibly different  $z$  dependence. Separation of variables then proceeds by defining

$$\begin{aligned} \chi(x^\mu, z) &= f_0(z)\eta(x^\mu), \\ \psi_i(x^\mu, z) &= f_i(z)\eta(x^\mu), \end{aligned} \quad (47)$$

where  $f_m(z)$ ,  $m = 0, 1, \dots, N$  are the extra-dimensional profiles and  $\eta(x^\mu)$  the 4D mode. To find the spectrum of mass states of  $\eta$ , we perform a Fourier transform on the variable  $x^\mu$ :  $\square\eta \rightarrow E\eta$ , where  $E$  is the eigenvalue, proportional to the mass squared of the  $\eta$  mode.

Eq. (44) now becomes a coupled eigenvalue equation of the form

$$H \begin{pmatrix} f_0 \\ f_i \end{pmatrix} = \begin{pmatrix} -\partial_z^2 + \mathcal{V}_{00} + \mathcal{B}_{00} & \mathcal{V}_{0j} + \mathcal{B}_{0j} \\ \mathcal{V}_{0i} + \mathcal{B}_{0i} & -\partial_z^2\delta_{ij} + \mathcal{V}_{ij} + \mathcal{B}_{ij} \end{pmatrix} \begin{pmatrix} f_0 \\ f_j \end{pmatrix} = E \begin{pmatrix} f_0 \\ f_i \end{pmatrix}. \quad (48)$$

In the matrix multiplication there is a sum over repeated  $j$  indices from 1 to  $N$ , and we shall use such a notation in subsequent equations.

This equation is a system of  $N + 1$  coupled Schrödinger equations, with a symmetric coupling potential. The symmetry implies that the eigenvalues of the system will be real, so long as certain boundary conditions are satisfied. This equation is one of the main results of the paper. It allows one to determine the spectrum of physical spin-0 modes with  $N$  scalars coupled to gravity. It is valid for an arbitrary potential  $V$ , with or without branes at the edges of the extra dimension.

## B. The case without branes

We are now going to specialize to the case where the brane terms are absent;  $\lambda = 0$ . A particular example of such models is one where a domain wall replaces the fundamental brane [9]. In this case we know exactly how the perturbations behave at the boundaries, and can proceed to determine the spectrum. We shall show, using the fake supergravity approach, that there are no tachyonic modes for these models. We present conditions for the non-existence of tachyonic modes for models with fundamental branes in the following section.

In order to prove stability for a system with  $N$  scalar fields and the gravitational perturbation  $\chi$ , we need to show that the eigenvalues  $E$  are strictly positive for the given boundary conditions. Our analysis so far has been for a general potential  $V$  and general background configuration of scalar fields. If we specialize to configurations generated by the fake supergravity approach, we can prove that the eigenvalues  $E$  of this system are non-negative, and in some cases strictly positive.

The key for proving such positivity is the observation that one can write the perturbation potential as

$$\mathcal{V} = \begin{pmatrix} \mathcal{V}_{00} & \mathcal{V}_{0j} \\ \mathcal{V}_{0i} & \mathcal{V}_{ij} \end{pmatrix} = S^2 + S', \quad (49)$$

where

$$S = e^{-\sigma} \begin{pmatrix} \frac{1}{12M^3}W & \frac{1}{\sqrt{3M^3}}W_j \\ \frac{1}{\sqrt{3M^3}}W_i & -\frac{1}{4M^3}\delta_{ij}W + W_{ij}|_{\text{bg}} \end{pmatrix}. \quad (50)$$

Then,

$$H = (\partial_z + S^\dagger)(-\partial_z + S). \quad (51)$$

We have shown in Appendix A that the eigenvalues satisfy  $E \geq 0$  whenever

$$\Psi^\dagger (-\partial_z + S) \Psi|_{\text{boundary}} = 0, \quad (52)$$

where  $\Psi = (f_0, f_i)^T$ .

As is obvious from the above equation, boundary conditions play a crucial role on the positivity of the eigenvalues and therefore on the stability of the system. Here we discuss the boundary conditions with general  $N$  for the following cases:

**Full-interval space:** With a full interval, there need not be any parity conditions on the fields. The boundaries are at  $z_1$  and  $z_2$  which can be at infinity, or at finite values as in the case of a soft-wall model, where there is a physical singularity. We require that the perturbations are normalizable. As seen in Eq. (32) solutions in conformal coordinates  $z$  are related to the ones in  $y$  coordinates through the warp factor which diverges at large  $z$ . We see that in order to have normalizable solutions, the rescaled perturbations in conformal coordinates must vanish at the boundaries. Therefore,  $\Psi^\dagger (-\partial_z + S) \Psi|_{z=z_{1,2}} = 0$  is satisfied.

**Half-interval space:** The boundaries are at  $z = 0$  and  $z = z_1$  and we must choose parity for the scalars and the warp factor. The warp factor must be even in order to localize gravity. If a field is odd, it vanishes at  $z = 0$ , if it is even, then  $\partial_z \varphi_i$  vanishes at  $z = 0$ . It is also easy to verify that  $\Psi^\dagger S \Psi$  has odd parity, and therefore it also vanishes at  $z = 0$ . As such, the condition  $\Psi^\dagger (-\partial_z + S) \Psi|_{z=0} = 0$  is always satisfied. For the  $z = z_1$  boundary, the scalar perturbations vanish as for the full-interval case discussed above due to the normalizable condition on the perturbations.

Having shown that the boundary terms vanish, we can conclude that without branes, models with  $N$  scalars coupled to gravity do not have any tachyonic modes.

### C. The case with fundamental branes

For completeness we would also like to give sufficient conditions for the non-existence of tachyonic modes for models with  $N$  scalars coupled to gravity in the presence of fundamental branes with  $\lambda \neq 0$ . The profiles for the scalar fields for models with fundamental branes satisfy the effective Schrödinger equation given by Eq. (48) where the brane terms  $\mathcal{B}_{mn}$  are

given in Eq. (46). Repeating the analysis of Appendix A with brane terms we find that the effective Schrödinger equation can be written as

$$\int dz |\mathcal{S}\Psi|^2 + \int dz \partial_z (\Psi^\dagger \mathcal{S}\Psi) + \int dz (\Psi^\dagger \mathcal{B}\Psi) = E \int dz |\Psi|^2, \quad (53)$$

where  $\mathcal{S} = (-\partial_z + S)$  with  $S$  given by Eq. (50) and the brane terms are

$$\mathcal{B} = \sum_{\alpha} e^{-2\sigma} \begin{pmatrix} \frac{1}{3M^3} \lambda^{\alpha} & \frac{1}{\sqrt{3M^3}} \lambda_j^{\alpha} \\ \frac{1}{\sqrt{3M^3}} \lambda_i^{\alpha} & \lambda_{ij}^{\alpha} \end{pmatrix} \delta(z - z_{\alpha}), \quad (54)$$

where the sum over  $\alpha$  is over all fundamental branes in the model. For  $E \geq 0$  the sufficient condition is then modified to be

$$\Psi^\dagger (-\partial_z + S) \Psi|_{\text{boundary}} + \Psi^\dagger \mathcal{B}\Psi \geq 0. \quad (55)$$

Analyzing this requirement for different models with different brane potential terms  $\lambda^{\alpha}$  is outside the scope of this work.

## VI. ANALYZING ZERO-MODES

As we have seen in the previous section, the eigenvalues of the spin-0 sector are guaranteed to be non-negative when constructing configurations using the fake supergravity approach. We must still analyze the existence of zero mode solutions to show complete stability. In particular this will ensure that the size of the extra dimension is stabilized.

In this section, using the formalism we have developed, we will first show that for the  $N = 1$  case the eigensystem reduces to a single Schrödinger-like equation, and we determine the mass gap of the spin-0 spectrum. For configurations with  $N \geq 2$  scalar fields, analyzing the mass spectrum is a very difficult problem. Therefore we restrict our attention to models with specific superpotentials and analyze the existence of zero modes for these models. We shall explicitly construct a model with  $N = 2$  scalar fields that does not have a zero mode and therefore is stable. However, before our analysis, we would like to prove the following theorem, which gives a relation between the number of zero modes for a general theory with  $N$  scalar fields and the parities of the background solutions.

*Theorem: For a system of definite parity with  $N$  scalar fields that couple to gravity,*

the number of independent normalizable zero modes with  $E = 0$  is at most equal to the number of fields whose background solutions are even.

In order to prove this theorem, we first study the zero mode solutions. Looking at Eq. (A3) it is easy to see that a solution with  $E = 0$  is satisfied when  $\mathcal{S}\Psi = 0$ , that is

$$\begin{aligned} -f'_0 + S_{00}f_0 + S_{0i}f_i &= 0, \\ -f'_j + S_{0j}f_0 + S_{ij}f_i &= 0, \end{aligned} \tag{56}$$

where the matrix elements  $S_{mn}$  are given explicitly in Eq. (50). Now suppose there are normalizable zero mode solutions  $(f_0^{(0)}, f_i^{(0)})$  which satisfy the above equations. Working in  $y$  coordinates, we now add these zero mode solutions as perturbations to the background configuration to define the following new fields

$$\begin{aligned} \bar{\sigma}(\bar{y}) &= \sigma(y) + \epsilon \frac{e^{3/2\sigma}}{\sqrt{12}} f_0^{(0)}(y), \\ \bar{\phi}_i(\bar{y}) &= \phi_i(y) + \epsilon e^{3/2\sigma} M^{3/2} f_i^{(0)}(y), \end{aligned} \tag{57}$$

where we have defined a new coordinate  $\bar{y}$  in terms of the zero mode perturbations as

$$d\bar{y} = dy \left( 1 + 2\epsilon \frac{e^{3/2\sigma}}{\sqrt{12}} f_0^{(0)} \right). \tag{58}$$

$\epsilon$  in the above equations is a small parameter that parametrizes the small perturbations. Note that for models with fixed parity, the perturbations must have the same parity as the backgrounds. In the fake supergravity approach, the background fields are given by the solutions to

$$\begin{aligned} \frac{d}{dy}\sigma(y) &= \frac{1}{6M^3} W(\{\phi_i\}), \\ \frac{d}{dy}\phi_i(y) &= W_i(\{\phi_i\}). \end{aligned} \tag{59}$$

This is a system of linear differential equations for  $N + 1$  fields, which require  $N + 1$  initial conditions for a unique solution. The value of  $\sigma(0)$  can be chosen to be 0 since any other constant shift can be obtained by a redefinition of the coordinates  $x^\mu$ . This leaves us with  $N$  initial conditions to be specified for the  $N$  scalar fields  $\phi_i$ . The initial conditions for fields with odd parity are fixed to be  $\phi_i(0) = 0$  for  $\phi_i$  odd. Therefore the only free initial conditions to be fixed are those corresponding to fields with even parity. Once the remaining free initial conditions are fixed, the solutions to Eq. (59) are unique.

Solving for the derivatives of the zero mode solutions,  $f_0^{(0)'}$  and  $f_i^{(0)'}$ , in terms of the zero mode solutions themselves, by using Eq. (56), and using

$$\frac{d}{d\bar{y}} = \left( 1 - 2\epsilon \frac{e^{3/2\sigma}}{\sqrt{12}} f_0^{(0)} \right) \frac{d}{dy}, \quad (60)$$

it is easy to show that the new fields defined in Eq. (57) satisfy

$$\begin{aligned} \frac{d}{d\bar{y}} \bar{\sigma}(\bar{y}) &= \frac{1}{6M^3} W(\{\bar{\phi}_i\}), \\ \frac{d}{d\bar{y}} \bar{\phi}_i(\bar{y}) &= W_i(\{\bar{\phi}_i\}), \end{aligned} \quad (61)$$

up to first order in the perturbation parameter  $\epsilon$ . We see that the new fields defined in Eq. (57) are themselves background solutions with the *same* superpotential  $W$ . This shows us that the zero mode solutions for the system of  $N$  scalar fields coupled to gravity translates from one background solution to another taking us from one set of initial conditions in the  $y$  frame to another set of initial conditions in the  $\bar{y}$  frame. Note that this result is valid up to first order in  $\epsilon$  which means that the zero mode solutions have to be strictly much smaller than the background solutions. This restricts our reasoning to normalizable zero modes.

We have argued that, for models with fixed parity, the number of free initial conditions to be fixed is given by the number of background fields with even parity. This in particular means that if all the scalar background fields are chosen to have odd parity, there do not exist any free initial conditions, and the solution to the set of equations in Eqs. (59) is unique. This means that for such a system there can not exist any normalizable zero mode solutions. We have therefore proven that the number of zero mode solutions, which take us from one set of initial conditions to another, is given by at most the number of background fields with even parity.

### A. The $N = 1$ case

In this section, for completeness, we will analyze the system with one scalar coupled to gravity and show that it is stable with positive mass eigenvalues. Stability for models with one scalar field and no branes have been analyzed in Refs. [22, 24]. For models with a soft-wall and a fundamental brane stability has been proven in Ref. [14] and for RS models with two fundamental branes in Refs. [15, 16].

We start by analyzing the Einstein constraint equation, Eq. (36), and the second-order equation for  $\chi$  from Eq. (48). They are, respectively,

$$-f'_0 + S_{00}f_0 + S_{0i}f_i = 0, \quad (62)$$

$$-f''_0 + \mathcal{V}_{00}f_0 + \mathcal{V}_{0i}f_i = Ef_0. \quad (63)$$

As before, repeated  $i$ , and later  $j$ , indices are to be summed. Combining them in an obvious way yields

$$-f''_0 + \frac{S_{0i}\mathcal{V}_{0i}}{S_{0j}S_{0j}}f'_0 + \left(\mathcal{V}_{00} - \frac{S_{00}S_{0i}\mathcal{V}_{0i}}{S_{0j}S_{0j}}\right)f_0 + \left(\mathcal{V}_{0i} - \frac{S_{0i}S_{0k}\mathcal{V}_{0k}}{S_{0j}S_{0j}}\right)f_i = Ef_0. \quad (64)$$

We can eliminate the  $f'_0$  term, and obtain a Schrödinger-like equation, by defining

$$f_0 = \sqrt{S_{0i}S_{0i}}g. \quad (65)$$

This gives

$$-g'' + (A^2 - A' + S_{0i}S_{0i})g + \frac{1}{\sqrt{S_{0i}S_{0i}}}B_i f_i = Eg, \quad (66)$$

where

$$A = \frac{S_{0i}S_{ij}S_{0j}}{S_{0k}S_{0k}}, \quad (67)$$

$$B_i = \mathcal{V}_{0i} - \frac{S_{0i}S_{0k}\mathcal{V}_{0k}}{S_{0j}S_{0j}}. \quad (68)$$

Equation (66) can be used to prove the non-existence of a zero mode for a theory with one scalar field. For  $N = 1$  we have:

$$A = S_{11}, \quad (69)$$

$$B_i = 0. \quad (70)$$

Following the arguments in Appendix A, we see that for wave functions  $\Psi \neq 0$ , the eigenvalues of this system are non-negative since the boundary terms vanish as we have discussed previously. Equation (66) simplifies to

$$(\partial_z - A)(-\partial_z - A)g + S_{0i}S_{0i}g = Eg. \quad (71)$$

Now we will use the same trick as in the analysis of Appendix A and multiply Eq. (71) from the left by  $g^*$  and integrate over the extra dimension  $z$ . Up to a surface term that vanishes we have

$$\int dz \left| \left( -\frac{d}{dz} - A \right) g \right|^2 + \int dz |S_{01}g|^2 = E \int dz |g|^2. \quad (72)$$

Consider the existence of a zero mode, with  $E = 0$ . In order to satisfy Eq. (72) for an arbitrary superpotential  $W$ , and for the given boundary conditions by which the surface terms vanish, both of the terms on the left-hand side must vanish simultaneously. For an arbitrary  $W$  this means that the field  $g$ , and hence  $f_0$ , has to vanish. This shows that for nontrivial  $f_0$  there does not exist any zero mode.

We can push the analysis a little further and provide a lower bound on the mass gap to the first spin-0 state. Dropping one of the terms on the left-hand side of Eq. (72) gives the following inequality:

$$E \int dz |g|^2 \geq \int dz |S_{01}g|^2 \tag{73}$$

$$\geq \min[S_{01}^2] \int dz |g|^2, \tag{74}$$

where  $\min[S_{01}^2]$  is the minimum of  $S_{01}^2$  over  $z$ . Thus we obtain a bound on the mass of the first spin-0 state:

$$E \geq \min[S_{01}^2]. \tag{75}$$

Before ending this section we would like to note that, for a system with  $N = 1$  scalar fields coupled to gravity, the fact that there does not exist a zero mode solution is consistent with the theorem we have proven at the beginning of this section. Following Ref. [4], one would like to localize the graviton around the origin in the extra dimension and this in particular means that one needs to have a gravitational background solution  $\sigma$  that has even parity.  $\sigma$  is related to the superpotential through Eq. (20), so we must have a superpotential  $W$  that has odd parity. For a system with one scalar field, this can be achieved only with a scalar field that has odd parity background solutions. Therefore, using our theorem, we can see that for  $N = 1$  there can not be a zero mode solution.

## B. Explicit Examples

In this section we look at two specific domain wall models with  $N = 2$  scalar fields coupled to gravity. We have already shown that the spin-0 spectrum has strictly non-negative mass eigenmodes, and proven a theorem relating the existence of zero modes to the parities of the background scalar profiles. The two models to be presented will form explicit realizations of this theorem: one has an even scalar profile and a zero mode, the other all odd profiles

and no normalizable zero mode. Aside from this difference, both models are qualitatively the same. The field  $\Phi_1$  will play the role of a dilaton and has a background solution which diverges at finite  $y$ , generating a physical singularity and cutting off the extra dimension, effectively compactifying it. This is known as a soft wall [10, 11, 13, 14]. The second field  $\Phi_2$  takes the form of a kink, creating a domain-wall whose purpose is to replace the positive tension brane in usual soft-wall set-ups. The models we consider have definite parity, and the domain-wall sits at the origin, acting as an effective boundary of the extra dimension. A domain-wall soft-wall model is an appropriate name for this type of set-up. In this section we are concerned primarily in the stability of such models, and do not discuss any other phenomenology. Whether or not these models solve the hierarchy problem is an interesting question which we intend to address in future work.

### Example 1

The first model we consider has the following superpotential

$$W(\Phi_1, \Phi_2) = (a\Phi_2 - b\Phi_2^3) e^{\nu\Phi_1}, \quad (76)$$

where  $\Phi_1$  and  $\Phi_2$  are the dilaton and the kink fields respectively.  $a > 0$ ,  $b > 0$  and  $\nu$  are parameters in the model. We write the fields in terms of their background solutions and perturbations as

$$\begin{aligned} \Phi_1(x, y) &= \phi_1(y) + \varphi_1(x, y), \\ \Phi_2(x, y) &= \phi_2(y) + \varphi_2(x, y). \end{aligned} \quad (77)$$

The background fields satisfy

$$\begin{aligned} \frac{d}{dy}\phi_1 &= \nu (a\phi_2 - b\phi_2^3) e^{\nu\phi_1}, \\ \frac{d}{dy}\phi_2 &= (a - 3b\phi_2^2) e^{\nu\phi_1}, \\ \frac{d}{dy}\sigma &= \frac{1}{6M^3} (a\phi_2 - b\phi_2^3) e^{\nu\phi_1}. \end{aligned} \quad (78)$$

We choose the background solutions such that  $\phi_1$  has even and  $\phi_2$  has odd parity, that is

$$\frac{d}{dy}\phi_1|_{y=0} = 0, \quad \phi_2(0) = 0. \quad (79)$$

Using our theorem we know that this model will have at most one zero mode solution

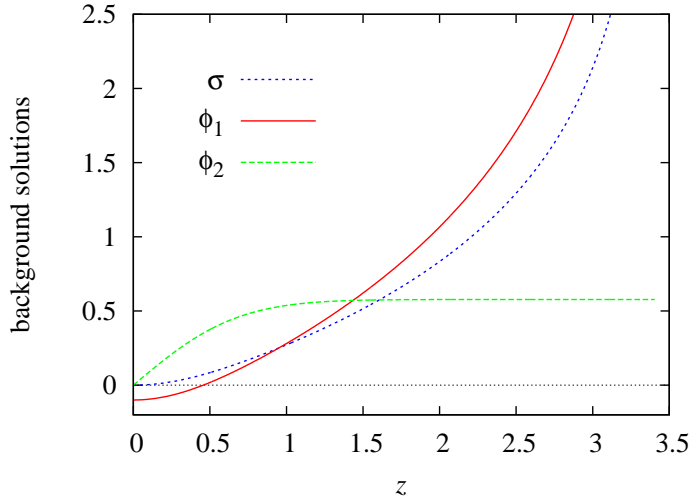


FIG. 1: Numerical solutions for the backgrounds  $\sigma$ ,  $\phi_1$  and  $\phi_2$  for example 1 with the parameter values  $a = 1$ ,  $b = 1$  and  $\nu = 1.4$ . The free initial condition for  $\phi_1(0)$  is chosen to be  $-0.1$ . The soft wall is at  $z = 3.41$  where  $\sigma$  and  $\phi_1$  diverge, ending the spacetime at a physical singularity. These solutions have definite parity and we have only plotted the  $z \geq 0$  half.

since we have one field whose background solution is even, which is the dilaton field. The nice thing about this simple superpotential choice is that the zero mode solution can be found analytically even though an analytic solution for the background fields is not possible. Fig. 1 shows the numerical solutions for the background fields for a particular parameter space point. Although we are not concerned here with the parameters of the model, an important thing to mention is that our choice of  $\nu = 1.4$  allows us to satisfy the equations of motion at the singularity; see Ref [14] for details.<sup>2</sup>

Going to the conformal coordinates using Eq. (32) and using the field redefinitions in Eq. (42), the zero mode satisfies

$$\begin{aligned}
 -f_0^{(0)'} + S_{00}f_0^{(0)} + S_{01}f_1^{(0)} + S_{02}f_2^{(0)} &= 0, \\
 -f_1^{(0)'} + S_{01}f_0^{(0)} + S_{11}f_1^{(0)} + S_{12}f_2^{(0)} &= 0, \\
 -f_2^{(0)'} + S_{02}f_0^{(0)} + S_{12}f_1^{(0)} + S_{22}f_2^{(0)} &= 0,
 \end{aligned}
 \tag{80}$$

where the matrix elements  $S_{mn}$  are given explicitly by Eq. (50) using the superpotential of Eq. (76). It is easy to verify that the normalizable zero mode solution to the above system

<sup>2</sup> Note that, in addition to  $\Phi_i$ , the parameters of our models are also dimensionful. For our plots we work with units where  $6M^3 = 1$ .

is given by

$$\begin{pmatrix} f_0^{(0)} \\ f_1^{(0)} \\ f_2^{(0)} \end{pmatrix} = \begin{pmatrix} -N \frac{\nu}{\sqrt{2}} e^{-3/2\sigma} \\ N e^{-3/2\sigma} \\ 0 \end{pmatrix}, \quad (81)$$

where  $N$  is a normalization constant. This zero mode physically corresponds to changes in the size of the extra dimension.

### Example 2

The second model we would like to analyze has the following superpotential

$$W(\Phi_1, \Phi_2) = \alpha \sinh(\nu \Phi_1) + (a \Phi_2 - b \Phi_2^3) \quad (82)$$

where  $\Phi_1$  and  $\Phi_2$  are again the dilaton and the kink fields. We write the fields in terms of their background and perturbations as in Eq. (77). As we have explained before the parity requirements on the gravitational background  $\sigma$  forces us to have an odd parity superpotential. To satisfy this, we choose the background kink solution to have odd parity, as in the previous example, as well as the background dilaton solution:

$$\phi_1(0) = 0, \quad \phi_2(0) = 0. \quad (83)$$

For this superpotential, the background solutions can be obtained analytically as

$$\begin{aligned} \phi_1(y) &= \frac{2}{\nu} \operatorname{arctanh} \left( \tan \frac{\alpha \nu^2 y}{2} \right), \\ \phi_2(y) &= \sqrt{\frac{a}{3b}} \tanh(\sqrt{3ab} y). \end{aligned} \quad (84)$$

The position of the soft-wall singularity in  $y$  coordinates can also be analytically determined to be

$$y_s = \frac{\pi}{2\alpha\nu^2}, \quad (85)$$

which is the location in the extra dimension where the dilaton field and warp factor diverge.

The zero mode solution is given as a solution to  $\mathcal{S}\Psi^{(0)} = 0$ . The system of equations are given by Eqs. (80). We solve this system numerically using the superpotential in Eq. (82). Parities of the perturbations must be the same as the parities for the background fields:

$$f_0^{(0)}(0) = 1, \quad f_1^{(0)}(0) = 0, \quad f_2^{(0)}(0) = 0, \quad (86)$$

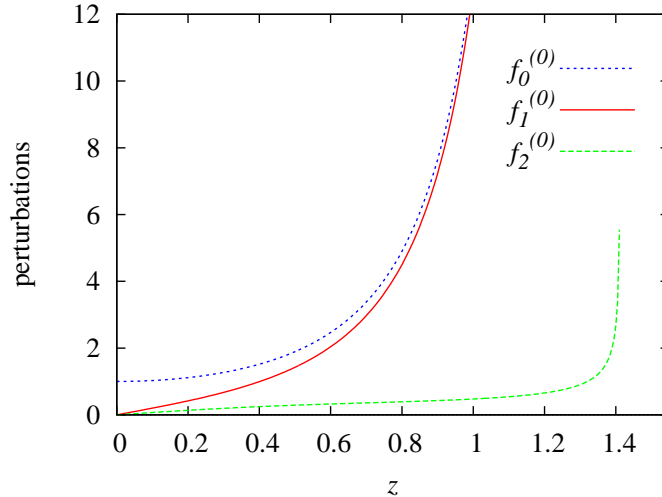


FIG. 2: Numerical solutions for the zero mode solutions  $f_0^{(0)}$ ,  $f_1^{(0)}$  and  $f_2^{(0)}$  for example 2 with the parameter values  $a = 0.5$ ,  $b = 0.3$ ,  $\alpha = 1$  and  $\nu = 1.4$ . The free initial condition for  $f_0(0)$  is chosen to be 1.0. The figure shows that the zero mode solutions diverge in the region close to the soft wall, making the solution not normalizable. These solutions have definite parity and we have only plotted the  $z \geq 0$  half.

where the scaling property of Eq. (80) allows us to choose  $f_0^{(0)}(0) = 1$  without loss of generality. If there exists a non-trivial zero mode in this model, then it must possess these initial conditions. Therefore, if these initial conditions lead to a non-normalizable solution, the only normalizable solution is the trivial one. We find numerically that the solution diverges at the location of the soft-wall, as shown in Fig. 2, and the solutions are indeed not normalizable. This lack of a zero mode is consistent with our theorem since, in this second example, all the scalars have odd parity. Furthermore, the existence of a non-normalizable solution to Eq. (80) is consistent with the proof of our theorem, which explicitly relied on the fact that perturbations must be small compared to the background solution.

In summary, the model presented here does not contain a zero mode and the physical size of the extra dimension is stabilized at the value given by Eq. (85).

## VII. CONCLUSIONS

Models with warped extra dimensions provide for interesting and rich extensions beyond the Standard Model. The classical backgrounds of such models generally contain bulk scalars

with non-trivial profiles. Stability of the background is an important theoretical issue, in particular, a model with a warped, compact extra dimension must have the size of the extra dimension stabilized. In this paper we studied perturbative stability of  $N$  scalars coupled to gravity by analyzing the spin-2 and spin-0 spectra. Although our results are quite general, we paid particular attention to the case of domain-wall models with a soft wall in an  $\text{AdS}_5$  background. These models are interesting to study because, as we have shown, they provide a purely field-theoretic mechanism for compactifying an extra dimension.

We have done part of our analysis using the fake supergravity approach, which has been widely exploited to engineer analytically tractable solutions to Einstein's equations. This approach has also been used to study stability of models where a single scalar field with non-trivial profile is coupled to gravity in a warped background [22–24]. In this paper we have extended these previous studies to the case where an arbitrary number of scalar fields couple minimally to gravity.

The first main result of this work is the derivation of Eq. (48), the coupled Schrödinger equation governing the spin-0 spectrum of  $N$  scalars coupled to gravity. This equation is valid for an arbitrary scalar potential, with or without additional brane terms. Following this, we specialized to potentials generated using fake supergravity and presented our second main result: a system with  $N$  scalar fields coupled to gravity and without branes has no tachyonic modes in the spin-0 sector. This general result is valid for all types of models, where the extra dimension may be infinite in size or finite with a soft-wall, and where there may or may not be definite parity. Extensions to models with branes that have brane potential terms were also briefly discussed. This result generalizes previous studies on stability of models with one scalar field coupled to gravity [22, 24]. Using our formalism we also studied the case with one scalar field and provide a lower bound on the mass of the first spin-0 mode, Eq. (75).

Our third main result is related to the existence of zero modes for models with  $N$  scalars coupled to gravity. Zero mode solutions in general destabilize the size of the extra dimension, and our aim was to determine criteria which guaranteed the absence of such zero modes. A general analytic study for models with an arbitrary number of scalar fields is rather complex, and we restricted ourselves to scenarios where the extra dimension has definite parity. We have proven a theorem that relates the number of zero modes to the parities of the scalars and used this result to show that zero modes are absent in models where all background

scalar profiles have odd parity. We demonstrated this by explicitly constructing two domain-wall models with a soft wall, one of which admitted a zero mode and the other not. The latter is an example of a model that stabilizes a compact extra dimension without using branes. Whether these models are realistic models that can solve the hierarchy problem is a question to be investigated in future work.

### Acknowledgments

We would like to thank J.W. van Holten and M. Postma for useful comments. This research was supported by the Netherlands Foundation for Fundamental Research of Matter (FOM) and the National Organization for Scientific Research (NWO).

### Appendix A: Analyzing the eigenvalues of the Hamiltonian

Take a system with an arbitrary number of scalars that satisfies Schrödinger's equation,

$$H\Psi = E\Psi. \quad (\text{A1})$$

Below we show that if one can write  $H$  as in supersymmetric quantum mechanics,

$$H = (\partial_z + S)(-\partial_z + S) + K^\dagger K, \quad (\text{A2})$$

with Hermitian  $S$ , the eigenvalues of  $H$  are non-negative for vanishing boundary terms. To see this we multiply Eq. (A1) from left with  $\Psi^\dagger$  and integrate over the extra dimension, which gives

$$\begin{aligned} E \int dz |\Psi|^2 &= \int dz \Psi^\dagger H \Psi \\ &= \int dz \Psi^\dagger (\vec{\partial}_z + S^\dagger)(-\vec{\partial}_z + S)\Psi + \int dz |K \Psi|^2 \\ &= \int dz \Psi^\dagger (-\overleftarrow{\partial}_z + S^\dagger)(-\vec{\partial}_z + S)\Psi + \int dz |K \Psi|^2 + \int dz \partial_z (\Psi^\dagger (-\partial_z + S)\Psi) \\ &= \int dz |\mathcal{S}\Psi|^2 + \int dz |K \Psi|^2 + \int dz \partial_z (\Psi^\dagger \mathcal{S}\Psi), \end{aligned} \quad (\text{A3})$$

where we defined

$$\mathcal{S} \equiv (-\partial_z + S). \quad (\text{A4})$$

The arrows on the partial derivatives indicate which way they act on. Notice that the last terms on the last two lines are the boundary terms. Since  $|\mathcal{S}\Psi|^2 \geq 0$  and  $|K\Psi|^2 \geq 0$ , for an arbitrary  $\Psi \neq 0$ , we can immediately see that  $E \geq 0$  if the boundary terms are zero, that is

$$\Psi^\dagger \mathcal{S}\Psi|_{\text{boundary}} = 0 \Rightarrow E \geq 0, \quad (\text{A5})$$

which is satisfied when either  $\Psi|_{\text{boundary}} = 0$  or  $\mathcal{S}\Psi|_{\text{boundary}} = 0$ . Note that these conclusions also hold when  $K = 0$ .

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