

No isomorphism between the affine $\hat{sl}(2)$ algebra and the $N = 2$ superconformal algebras

Beatriz Gato-Rivera*

Instituto de Física Fundamental, CSIC

Serrano 123, Madrid 28006, Spain

NIKHEF-H, Kruislaan 409, NL-1098 SJ Amsterdam, The Netherlands

ABSTRACT

Since 1999 it became obvious that the would be ‘isomorphism’ between the affine $\hat{sl}(2)$ algebra and the $N=2$ superconformal algebras, proposed by some authors, simply does not work. However, this issue was never properly discussed in the literature and, as a result, some confusion still remains. In this article we finally settle down, clearly and unambiguously, the true facts: there is no isomorphism between the affine $\hat{sl}(2)$ algebra and the $N=2$ superconformal algebras.

September 2008

*Also known as B. Gato

1 Introduction

Subsingular vectors of the $N=2$ superconformal algebras were discovered, and examples given, in 1996 [1] [2]. Shortly afterwards Semikhatov and Tipunin claimed to have obtained a complete classification of the $N=2$ subsingular vectors in the paper ‘The Structure of Verma Modules over the $N=2$ Superconformal algebra’ [10]. Surprisingly, the only explicit examples of $N=2$ subsingular vectors known at that time did not fit into their classification. All the results presented in that paper, including the classification of subsingular vectors, were based on the assumption that there exists an isomorphism between the affine $\hat{sl}(2)$ algebras and the $N = 2$ superconformal algebras, proposed by the authors in earlier work [8][9] without proofs. Using this ‘isomorphism’ the authors: i) deduced that there were only two different types of submodules in $N=2$ Verma modules, and ii) claimed that they had constructed ‘non-conventional’ singular vectors with the property of generating the two types of submodules maximally, i.e. with no subsingular vectors left outside. The classification of the $N=2$ subsingular vectors then followed applying these two results. A couple of years later, in 1999, after some more papers by the same authors had appeared making use of the ‘isomorphism’, we proved, in a note sent to the archives [4], that both results were incorrect: there are four different types of submodules in $N=2$ Verma modules and the ‘non-conventional’ vectors do not generate the submodules maximally (we used one explicit example to see this). However, we did not emphasize enough the fact that our results provided a strong indication that the ‘would be isomorphism’ was incorrect, we just made a small comment about it. Although the authors did not make use of the ‘isomorphism’ again (as far as we know!), the lack of a clear discussion about this issue brought consequences: the ‘isomorphism’ remained, and still remains, as a belief by some colleagues who have not worked out the details, as we have had the chance to verify in several occasions, either in writing form or verbally. It is our intention now to clarify this issue, finally, and to settle down the true facts: there is no isomorphism between the affine $\hat{sl}(2)$ algebra and the $N=2$ superconformal algebras.

Our strategy will be to prove that the predictions of this ‘would be isomorphism’ regarding the Topological $N=2$ superconformal algebra, which is the $N=2$ algebra considered by Semikhatov and Tipunin, simply do not work. We do not find necessary to make a detailed analysis of the ‘isomorphism’ itself trying to understand why it does not work. The authors never provided a proof that their construction provided an isomorphism!¹ Otherwise we would have felt somehow compelled to find the mistakes. In what follows, in section 2 we will first introduce the Topological $N=2$ algebra, together with some important results regarding its representation theory. In section 3 we will describe the major claims made by Semikhatov and Tipunin in several papers (the first ones being [8] [9] [10]), regarding the different types and properties of the submodules of the Topological $N=2$ algebra, as deduced directly from the ‘isomorphism’ between this algebra and the affine $\hat{sl}(2)$ algebra. Then in section 4 we will prove that these claims are incorrect.

¹This was very much their style. For example, in ref. [10] they wrote eleven theorems, seven propositions, four lemmas and zero proofs (not even low level explicit examples).

2 The Topological N=2 superconformal algebra

The Topological N=2 superconformal algebra was deduced in 1990 as the symmetry algebra of two-dimensional topological conformal field theory (TCFT) [11]. It was the last N=2 superconformal algebra to be discovered and in fact can be obtained from the Neveu-Schwarz N=2 algebra by modifying the stress-energy tensor by adding the derivative of the U(1) current, a procedure known as *topological twist* [12][13]. It reads

$$\begin{aligned}
[\mathcal{L}_m, \mathcal{L}_n] &= (m-n)\mathcal{L}_{m+n}, & [\mathcal{H}_m, \mathcal{H}_n] &= \frac{c}{3}m\delta_{m+n,0}, \\
[\mathcal{L}_m, \mathcal{G}_n] &= (m-n)\mathcal{G}_{m+n}, & [\mathcal{H}_m, \mathcal{G}_n] &= \mathcal{G}_{m+n}, \\
[\mathcal{L}_m, \mathcal{Q}_n] &= -n\mathcal{Q}_{m+n}, & [\mathcal{H}_m, \mathcal{Q}_n] &= -\mathcal{Q}_{m+n}, & m, n \in \mathbf{Z}. & (2.1) \\
[\mathcal{L}_m, \mathcal{H}_n] &= -n\mathcal{H}_{m+n} + \frac{c}{6}(m^2+m)\delta_{m+n,0}, \\
\{\mathcal{G}_m, \mathcal{Q}_n\} &= 2\mathcal{L}_{m+n} - 2n\mathcal{H}_{m+n} + \frac{c}{3}(m^2+m)\delta_{m+n,0},
\end{aligned}$$

where \mathcal{L}_m and \mathcal{H}_m are the bosonic generators corresponding to the stress-energy tensor (Virasoro generators) and the U(1) current, respectively, and \mathcal{G}_m and \mathcal{Q}_m are the spin-2 and spin-1 fermionic generators, the latter being the modes of the BRST-current. The eigenvalues of the bosonic zero modes $(\mathcal{L}_0, \mathcal{H}_0)$ correspond to the conformal weight and the U(1) charge of the states. In a Verma module these eigenvalues split conveniently as $(\Delta + l, \mathbf{h} + q)$ for secondary states, where l and q are the *level* and the *relative charge* of the state and (Δ, \mathbf{h}) are the conformal weight and the charge of the primary state on which the secondary is built. The ‘topological’ anomaly c is the conformal anomaly corresponding to the Neveu-Schwarz N=2 algebra.

Due to the existence of the fermionic zero modes \mathcal{G}_0 and \mathcal{Q}_0 this algebra has two sectors: the \mathcal{G} -sector (states annihilated by \mathcal{G}_0) and the \mathcal{Q} -sector (BRST-invariant states annihilated by \mathcal{Q}_0), in analogy with the (+)-sector and the (-)-sector of the Ramond N=2 algebra, due to the fermionic zero modes G_0^+ and G_0^- . (As a matter of fact, these two N=2 algebras are exactly isomorphic, as was proven in [6].) However, the two sectors do not provide the complete description since there are also states which do not belong to any of the sectors [1][5][6]. That is, not all Verma modules and submodules decompose into the two sectors, but there are also indecomposable states, in particular indecomposable singular vectors. To see this one only needs to inspect the anticommutator of the fermionic zero modes $\{\mathcal{G}_0, \mathcal{Q}_0\} = 2\mathcal{L}_0$ acting on a given state $|\chi\rangle$. If the conformal weight of $|\chi\rangle$ is different from zero; i.e. $\mathcal{L}_0|\chi\rangle = (\Delta + l)|\chi\rangle \neq 0$, then $|\chi\rangle$ can be decomposed into a state $|\chi\rangle^{\mathcal{G}}$ annihilated by \mathcal{G}_0 , but not by \mathcal{Q}_0 , that we refer as \mathcal{G}_0 -closed and a state $|\chi\rangle^{\mathcal{Q}}$ annihilated by \mathcal{Q}_0 , but not by \mathcal{G}_0 , that we refer as \mathcal{Q}_0 -closed:

$$|\chi\rangle = \frac{1}{2\Delta}\mathcal{Q}_0\mathcal{G}_0|\chi\rangle + \frac{1}{2\Delta}\mathcal{G}_0\mathcal{Q}_0|\chi\rangle = |\chi\rangle^{\mathcal{Q}} + |\chi\rangle^{\mathcal{G}}. \quad (2.2)$$

If the conformal weight of $|\chi\rangle$ is zero, however, one only obtains $(\mathcal{G}_0\mathcal{Q}_0 + \mathcal{Q}_0\mathcal{G}_0)|\chi\rangle = 0$, which is satisfied in four different ways: i) The state is \mathcal{G}_0 -closed, $|\chi\rangle = |\chi\rangle^{\mathcal{G}}$, and $\mathcal{G}_0\mathcal{Q}_0|\chi\rangle^{\mathcal{G}} = 0$, ii) The state is \mathcal{Q}_0 -closed, $|\chi\rangle = |\chi\rangle^{\mathcal{Q}}$, and $\mathcal{Q}_0\mathcal{G}_0|\chi\rangle^{\mathcal{Q}} = 0$, iii) The state is chiral, $|\chi\rangle = |\chi\rangle^{\mathcal{G},\mathcal{Q}}$, annihilated by both \mathcal{G}_0 and \mathcal{Q}_0 , and iv) The state is indecomposable ‘no-label’, $|\chi\rangle = |\chi\rangle$, not annihilated by any of the fermionic zero modes.

In what follows we will use the standard definition of highest weight vectors and singular vectors for conformal algebras, i.e. they are the states with *lowest* conformal weight (*lowest* energy) in the Verma modules and in the null submodules, respectively, and therefore are annihilated by all the positive modes of the generators of the algebra (the lowering operators); i.e. $\mathcal{L}_{n \geq 1}|\chi\rangle = \mathcal{H}_{n \geq 1}|\chi\rangle = \mathcal{G}_{n \geq 1}|\chi\rangle = \mathcal{Q}_{n \geq 1}|\chi\rangle = 0$. Hence these annihilation conditions will be referred to as the conventional, standard highest weight (h.w.) conditions. Singular vectors that are not generated by acting with the algebra on other singular vectors are called *primitive*, otherwise they are called *secondary* singular vectors.

Subsingular vectors are also null but they do not satisfy the h.w. conditions, becoming singular, that is annihilated by all the positive generators, in the quotient of the Verma module by a submodule, however. As a consequence they are located *outside* that particular submodule (otherwise they would disappear after taking the quotient), although descending to it necessarily by the action of the lowering operators (so that they descend to ‘nothing’ once the submodule is set to zero). This implies that *the singular vectors cannot reach the subsingular vectors* going upwards by the action of the negative, rising operators, whereas *the subsingular vectors can reach the singular vectors* going downwards by the action of the positive, lowering operators.

Subsingular vectors for the N=2 algebras were discovered in 1996 in ref. [2] and the first examples for the case of the Topological N=2 algebra were published in January 1997 in ref. [1], together with the classification of all possible types of singular vectors taking into account the relative U(1) charge and the annihilation conditions with respect to the fermionic zero modes \mathcal{G}_0 and \mathcal{Q}_0 . This classification resulted in: 4 different types of singular vectors for chiral Verma modules built on chiral highest weight vectors $|0, \mathbf{h}\rangle^{G, Q}$, 10 different types of singular vectors for generic (standard) Verma modules built on \mathcal{G}_0 -closed h.w. vectors $|\Delta, \mathbf{h}\rangle^G$, another 10 types for generic Verma modules built on \mathcal{Q}_0 -closed h.w. vectors $|\Delta, \mathbf{h}\rangle^Q$, and 9 different types of singular vectors for ‘no-label’ Verma modules built on indecomposable h.w. vectors $|0, \mathbf{h}\rangle$. In generic Verma modules one can find \mathcal{G}_0 -closed, \mathcal{Q}_0 -closed, chiral and indecomposable singular vectors. In chiral and no-label Verma modules, however, only \mathcal{G}_0 -closed and \mathcal{Q}_0 -closed singular vectors can exist, with the exception of the chiral singular vectors at level zero in no-label Verma modules. For the case of the generic Verma modules built on \mathcal{G}_0 -closed h.w. vectors $|\Delta, \mathbf{h}\rangle^G$, which were the only generic Verma modules considered by Semikhatov and Tipunin (they ignored the ones built on \mathcal{Q}_0 -closed h.w. vectors as well as the no-label Verma modules built on indecomposable h.w. vectors), the possible types of singular vectors one can find are given by the following table [1] [5] :

	$q = -2$	$q = -1$	$q = 0$	$q = 1$
\mathcal{G}_0 -closed	–	$ \chi\rangle_l^{(-1)G}$	$ \chi\rangle_l^{(0)G}$	$ \chi\rangle_l^{(1)G}$
\mathcal{Q}_0 -closed	$ \chi\rangle_l^{(-2)Q}$	$ \chi\rangle_l^{(-1)Q}$	$ \chi\rangle_l^{(0)Q}$	–
chiral	–	$ \chi\rangle_l^{(-1)G, Q}$	$ \chi\rangle_l^{(0)G, Q}$	–
indecomposable	–	$ \chi\rangle_l^{(-1)}$	$ \chi\rangle_l^{(0)}$	–

In ref. [1] all singular vectors (i.e. 4 + 20 + 9) were written down explicitly at level 1. This classification of singular vectors was proven to be rigorous later in ref. [5], using the results for

the maximal dimensions of the corresponding spaces of singular vectors (1, 2 or 3 depending on the type of singular vector). Regarding subsingular vectors, in ref. [1] all the subsingular vectors in generic Verma modules that become singular in the chiral Verma modules were written down at levels 2 and 3. To understand this one has to take into account that chiral Verma modules are nothing but the quotient of generic Verma modules with zero conformal weight, $\Delta = 0$, by the submodules generated by the level-zero singular vectors (which are present in all generic Verma modules with $\Delta = 0$).

3 The Claims

Three months after the paper [1] was published in the archives (January 97), the paper [10] appeared also in the archives. As was mentioned before, in [10] as well as in earlier work the authors considered only the Topological N=2 algebra (among the four existing N=2 algebras). All the analysis and results presented by the authors were based on the claim that there exists an isomorphism between the affine $\hat{sl}(2)$ algebra and the N=2 superconformal algebra at hand, giving rise to the following two major assumptions that were described as proven facts:

i) In the N=2 Verma modules there are only two types of submodules. In particular, in the generic Verma modules built on \mathcal{G}_0 -closed h.w. vectors (called ‘massive’ Verma modules by the authors) one can find the two types of submodules, denoted as ‘massive’ (large) and ‘topological’ (small).

Let us notice already that, although ref. [1] appeared in the bibliography given by the authors, the classification of Verma modules (generic, no-label and chiral), with their possible existing types of singular vectors, was overlooked. In particular the authors ignored the indecomposable singular vectors in generic (‘massive’) Verma modules (see table (1.3)), which clearly generate a different type of submodule with no counterpart in the affine $\hat{sl}(2)$ algebra, as we will show. In other words, the very existence of the indecomposable singular vectors of the Topological N=2 algebra, written down explicitly at level 1 in ref. [1], lacking of a counterpart in the affine $\hat{sl}(2)$ algebra, is already sufficient to disprove any possible isomorphism between these two algebras.

ii) These two types of submodules are maximally generated (i.e. without letting any null states outside, like subsingular vectors) by some ‘non-conventional singular vectors’, constructed by the authors in refs. [8] [9], which satisfy ‘twisted’ h.w. conditions and coincide with the conventional singular vectors only in the case of ‘zero twist’. In more intuitive terms one can think of the ‘non-conventional’ singular vectors simply as certain null states which, unlike the conventional singular vectors, are not located at the bottom of the submodules, that is, they are not the null states with lowest conformal weight, except for the case of ‘zero twist’. (In our opinion, the authors use an unnecessary complicated notation: null states that generate bigger submodules than the h.w. singular vectors are nothing but subsingular vectors).

Based on these assumptions the authors presented a ‘complete’ classification of subsingular vectors for the Topological N=2 algebra (without giving explicit examples) where, surprisingly, the subsingular vectors given in ref. [1], which were the only explicit examples written down so far, did

not fit. If one takes into account that subsingular vectors do not exist for the affine $\hat{sl}(2)$ algebra, one might wonder whether the discovery of subsingular vectors for the N=2 superconformal algebras should have been already a strong indication against the existence of an isomorphism between these algebras and the affine $\hat{sl}(2)$ algebra. The strategy of the authors then was to claim that the isomorphism mapped the Verma modules and submodules of the two algebras between each other, in such a way that the singular vectors of the $\hat{sl}(2)$ algebra, which generate the submodules maximally in the absence of subsingular vectors, would correspond to the ‘non-conventional singular vectors’ of the Topological N=2 algebra, the N=2 subsingular vectors corresponding to some ordinary null states of $\hat{sl}(2)$ (or to singular vectors in the special case in which the N=2 subsingular vectors and the ‘non-conventional singular vectors’ coincide). An important observation is that the standard Verma modules of the $\hat{sl}(2)$ algebra were supposed to be isomorphic to the chiral Verma modules of the Topological N=2 algebra, built on chiral h.w. vectors $|0, \mathfrak{h}\rangle^{G,Q}$, that the authors called ‘topological Verma modules’. In order to construct the isomorphic counterparts of the generic N=2 Verma modules, the authors defined the ‘relaxed Verma modules’ built on non-standard h.w. vectors of $\hat{sl}(2)$.

4 The Facts

In what follows, in subsections 4.1 and 4.2, we will show that:

i) In generic (‘massive’) Verma modules one can find four different types of submodules with respect to their size and shape at the bottom/top.² Two of them fit, in principle, into the description of ‘massive’ and ‘topological’ submodules given by Semikhatov and Tipunin. The other two types do not fit into that description.

ii) The N=2 subsingular vectors written down in ref. [1] do not fit into the classification presented by the authors in ref. [10], providing in fact a proof that the ‘non-conventional singular vectors’ do not generate maximal submodules since one can find subsingular vectors outside which are pulled inside the submodule by the action of the positive lowering operators.

4.1 Different types of submodules

The determinant formulae for the Topological N=2 algebra were presented in [3] for the chiral (‘topological’) Verma modules, and in [6] for the generic (‘massive’) and ‘no-label’ Verma modules, together with a very detailed analysis of the singular vectors corresponding to the roots of the determinants. In addition it was proved – both theoretically and with explicit examples – that in generic Verma modules one can find four different types of submodules just by taking into account the size and the shape at the bottom of the submodules. Now we will review these results and argue that two of these types of submodules do not fit into the ‘massive’ and ‘topological’ submodules of Semikhatov and Tipunin, which according to the ‘isomorphism’ should be the only existing types

²We draw the Verma modules from the bottom upwards, Semikhatov and Tipunin draw them downwards.

of submodules of the Topological N=2 algebra. The argument goes as follows. The determinant formula for all the generic Verma modules – either with two h.w. vectors $|\Delta, \mathbf{h}\rangle^G$ and $|\Delta, \mathbf{h} - 1\rangle^Q$ ($\Delta \neq 0$) or with only one h.w. vector $|0, \mathbf{h}\rangle^G$ or $|0, \mathbf{h} - 1\rangle^Q$ – reads³

$$\det(\mathcal{M}_l^T) = \prod_{2 \leq r, s \leq 2l} (f_{r,s})^{2P(l-\frac{rs}{2})} \prod_{0 \leq k \leq l} (g_k^+)^{2P_k(l-k)} \prod_{0 \leq k \leq l} (g_k^-)^{2P_k(l-k)}, \quad (4.1)$$

where

$$f_{r,s}(\Delta, \mathbf{h}, t) = -2t\Delta + t\mathbf{h} - \mathbf{h}^2 - \frac{1}{4}t^2 + \frac{1}{4}(s - tr)^2, \quad r \in \mathbf{Z}^+, \quad s \in 2\mathbf{Z}^+ \quad (4.2)$$

and

$$g_k^\pm(\Delta, \mathbf{h}, t) = 2\Delta \mp 2k\mathbf{h} - tk(k \mp 1), \quad 0 \leq k \in \mathbf{Z}, \quad (4.3)$$

defining the parameter $t = (3 - \mathbf{c})/3$. For $\mathbf{c} \neq 3$ ($t \neq 0$) one can factorize $f_{r,s}$ as

$$f_{r,s}(\Delta, \mathbf{h}, t \neq 0) = -2t(\Delta - \Delta_{r,s}), \quad \Delta_{r,s} = -\frac{1}{2t}(\mathbf{h} - \mathbf{h}_{r,s}^{(0)})(\mathbf{h} - \hat{\mathbf{h}}_{r,s}), \quad (4.4)$$

with

$$\mathbf{h}_{r,s}^{(0)} = \frac{t}{2}(1+r) - \frac{s}{2}, \quad r \in \mathbf{Z}^+, \quad s \in 2\mathbf{Z}^+, \quad (4.5)$$

$$\hat{\mathbf{h}}_{r,s} = \frac{t}{2}(1-r) + \frac{s}{2}, \quad r \in \mathbf{Z}^+, \quad s \in 2\mathbf{Z}^+. \quad (4.6)$$

For all values of \mathbf{c} one can factorize g_k^+ and g_k^- as

$$g_k^\pm(\Delta, \mathbf{h}, t) = 2(\Delta - \Delta_k^\pm), \quad \Delta_k^\pm = \pm k(\mathbf{h} - \mathbf{h}_k^\pm), \quad (4.7)$$

with

$$\mathbf{h}_k^\pm = \frac{t}{2}(1 \mp k), \quad k \in \mathbf{Z}^+ \quad (4.8)$$

The partition functions are defined by

$$\sum_N P_k(N)x^N = \frac{1}{1+x^k} \sum_n P(n)x^n = \frac{1}{1+x^k} \prod_{0 < r \in \mathbf{Z}, 0 < m \in \mathbf{Z}} \frac{(1+x^r)^2}{(1-x^m)^2}. \quad (4.9)$$

The fact that $2P(0) = 2P_k(0) = 2$ indicates that the singular vectors come two by two at the same level, in the same Verma module. Generically one is in the \mathcal{G} -sector, annihilated by (at least) \mathcal{G}_0 , while the other is in the \mathcal{Q} -sector, annihilated by (at least) \mathcal{Q}_0 . Now comes an observation for the readers who are more acquainted with the Neveu-Schwarz N=2 algebra. The roots of the quadratic vanishing surface $f_{r,s}(\Delta, \mathbf{h}, t) = 0$ and of the vanishing planes $g_k^\pm(\Delta, \mathbf{h}, t) = 0$ are related to the

³The Verma modules built on \mathcal{G}_0 -closed h.w. vectors and the ones built on \mathcal{Q}_0 -closed h.w. vectors are not the same for zero conformal weight $\Delta = 0$ because in this case there is only one h.w. vector at the bottom of the Verma module together with one singular vector.

corresponding roots of the determinant formula for the Neveu-Schwarz N=2 algebra [14][15][16][17] via the topological twists. These transform the standard h.w. vectors of the Neveu-Schwarz N=2 algebra into \mathcal{G}_0 -closed h.w. vectors of the Topological N=2 algebra. As a consequence, under the topological twists, the Neveu-Schwarz singular vectors are transformed into the singular vectors of the \mathcal{G} -sector of the Topological algebra (see refs. [1][3] for a detailed account of the twisting and untwisting of primary states and singular vectors).

It is easy to check, by counting of states, that the partitions $2P(l - \frac{rs}{2})$, exponents of $f_{r,s}$ in the determinant formula, correspond to submodules of generic type, of the same size as the Verma module itself, so to speak, whereas the partitions $2P_k(l - k)$, exponents of g_k^\pm in the determinant formula, correspond to smaller submodules. Furthermore, as pointed out before, taking into account also the shape at the bottom one can distinguish four types of submodules. Two of these types correspond to the quadratic vanishing surfaces $f_{r,s}(\Delta, \mathbf{h}, t) = 0$, a third type corresponds to the vanishing planes $g_k^\pm(\Delta, \mathbf{h}, t) = 0$, and the fourth type corresponds to the ‘no-label’ submodules, built on indecomposable singular vectors, that one finds in certain intersections of $f_{r,s}(\Delta, \mathbf{h}, t) = 0$ and $g_k^\pm(\Delta, \mathbf{h}, t) = 0$, as we will explain.

The two types of submodules that correspond to the quadratic vanishing surfaces $f_{r,s}(\Delta, \mathbf{h}, t) = 0$ have therefore the same size, but they differ on the shape at the bottom, where they both have (in the case $\Delta \neq 0$) two *uncharged* singular vectors at level $l = \frac{rs}{2}$: $|\chi\rangle_l^{(0)G}$ in the \mathcal{G} -sector and $|\chi\rangle_l^{(0)Q}$ in the \mathcal{Q} -sector⁴. As shown in Figure I, on the left and in the center, in most cases the bottom of the submodule consists of two singular vectors connected by one or two horizontal arrows corresponding to the action of \mathcal{Q}_0 and/or \mathcal{G}_0 . There is only one arrow if one of the singular vectors is chiral, i.e. of type $|\chi\rangle_l^{(0)G,Q}$ instead, what happens generically for $\Delta = -l$. These submodules fit, in principle, into the description of ‘massive’ submodules given by Semikhatov and Tipunin. Namely, ‘massive’ submodules are supposed to correspond to the uncharged roots $f_{r,s}(\Delta, \mathbf{h}, t) = 0$, they have the same size as the generic (‘massive’) Verma module and they have two states at the bottom connected through \mathcal{Q}_0 and/or \mathcal{G}_0 , one of these states being the \mathcal{G}_0 -closed uncharged singular vector $|\chi\rangle_l^{(0)G}$ (they do not mention the possibility that this singular vector may be chiral for $\Delta = -l$, though).

It also happens, however, for $\Delta = -l, t = -\frac{s}{n}, n = 1, \dots, r$, that the two singular vectors at the bottom of the submodule are chiral both, and therefore disconnected from each other, as shown in Fig. I, on the right. Consequently these ‘chiral-chiral’ submodules, of the same size as the ‘massive’ submodules and corresponding also to the uncharged roots $f_{r,s}(\Delta, \mathbf{h}, t) = 0$, contain two disconnected pieces at the bottom and as a result do not fit into the description of ‘massive’ submodules. Nor do they fit into the description of two ‘topological’ (smaller) submodules together since these correspond to the charged roots $g_k^\pm(\Delta, \mathbf{h}, t) = 0$ with also two singular vectors at the bottom of the submodules connected by the action of \mathcal{Q}_0 and/or \mathcal{G}_0 : the charged singular vectors

⁴An important technical remark is that if one chooses as h.w. vector of the Verma module only the \mathcal{G}_0 -closed one $|\Delta, \mathbf{h}\rangle^G$, as Semikhatov and Tipunin do, regarding the \mathcal{Q}_0 -closed h.w. vector $|\Delta, \mathbf{h} - 1\rangle^Q$ simply as a descendant state, then the uncharged singular vectors $|\chi\rangle_l^{(0)Q}$ in the \mathcal{Q} -sector are necessarily described as negatively charged singular vectors $|\chi\rangle_l^{(-1)Q}$ built on the \mathcal{G}_0 -closed h.w. vector $|\Delta, \mathbf{h}\rangle^G$. For the case $\Delta = 0$ there is only one h.w. vector in the Verma module and therefore only one of the singular vectors can be described as ‘uncharged’ while the other must necessarily be described as charged with respect to the unique h.w. vector.

$|\chi\rangle_l^{(1)G}$ or $|\chi\rangle_l^{(-1)G}$ in the \mathcal{G} -sector plus their companions in the \mathcal{Q} -sector, as we will see.

Let us stress that the existence of ‘chiral-chiral’ submodules was obvious since January 1997 when the whole set of singular vectors of the Topological algebra at level 1 was written down in ref. [1]. For example, the chiral singular vectors $|\chi\rangle_1^{(q)G,Q}$ at level 1 built on \mathcal{G}_0 -closed h.w. vectors $|\Delta, \mathbf{h}\rangle^G$ (which are the only h.w. vectors considered by Semikhatov and Tipunin) were shown to be:

$$|\chi\rangle_1^{(0)G,Q} = (-2\mathcal{L}_{-1} + \mathcal{G}_{-1}\mathcal{Q}_0)|-1, -1\rangle^G, \quad (4.10)$$

$$|\chi\rangle_1^{(-1)G,Q} = (\mathcal{L}_{-1}\mathcal{Q}_0 + \mathcal{H}_{-1}\mathcal{Q}_0 + \mathcal{Q}_{-1})|-1, \frac{6-\mathbf{c}}{3}\rangle^G. \quad (4.11)$$

For $\mathbf{c} = 9$ ($t = -2$) these two chiral singular vectors are together in the same generic (‘massive’) Verma module built on the h.w. vector $|-1, -1\rangle^G$. Hence this example already proves the existence of ‘chiral-chiral’ submodules at level 1. (Observe what we indicated in footnote 3: the uncharged singular vectors $|\chi\rangle_l^{(0)Q}$ in the \mathcal{Q} -sector are necessarily described as negatively charged singular vectors $|\chi\rangle_l^{(-1)Q}$ when built on \mathcal{G}_0 -closed h.w. vectors $|\Delta, \mathbf{h}\rangle^G$. In this example, the uncharged singular vectors in the \mathcal{G} -sector and in the \mathcal{Q} -sector turn out to be chiral, i.e; annihilated by both \mathcal{G}_0 and \mathcal{Q}_0 .)

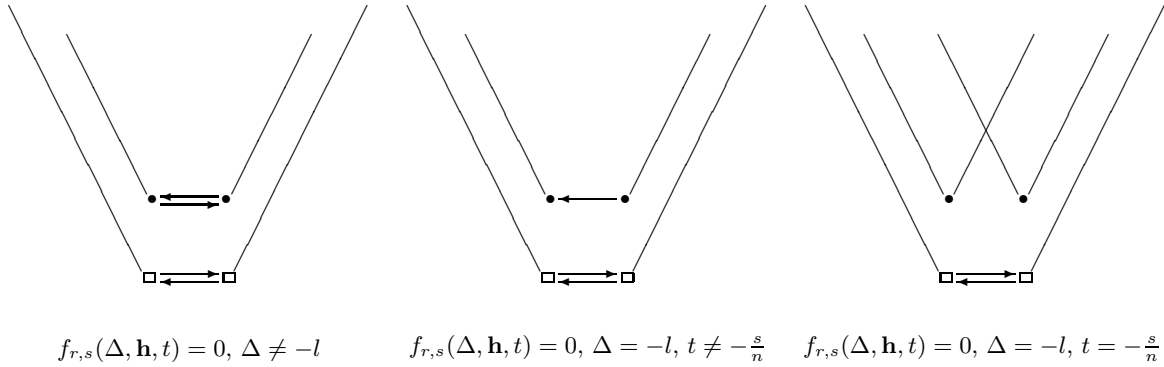


FIG. 1 . The singular vectors corresponding to the series $f_{r,s}(\Delta, \mathbf{h}, t) = 0$ belong to two different types of submodules of the same size but different shape at the bottom. In the first type, as shown in the figures on the left and in the center, the two singular vectors at the bottom of the submodules are connected by one or two arrows: the action of \mathcal{G}_0 and/or \mathcal{Q}_0 , depending on whether $\Delta \neq -l$ or $\Delta = -l, t \neq -\frac{s}{n}, n = 1, \dots, r$ (for which one of the singular vectors is chiral). In the second type, corresponding to $\Delta = -l, t = -\frac{s}{n}, n = 1, \dots, r$, the two singular vectors are chiral and therefore disconnected from each other, as shown in the figure on the right (the overlap of the two subsubmodules is another subsubmodule itself).

The third type of submodules, shown in Fig. II, on the left and in the center, corresponds to the charged roots of the vanishing planes $g_k^\pm(\Delta, \mathbf{h}, t) = 0$. As already pointed out, these submodules are smaller than the generic ones, with partition functions given by $P_k(l - k)$. In the most general case ($\Delta \neq 0$) the two singular vectors at the bottom of the submodule can be described as *charged*: positively charged $|\chi\rangle_l^{(1)G}$ in the \mathcal{G} -sector and $|\chi\rangle_l^{(1)Q}$ in the \mathcal{Q} -sector for $g_k^+(\Delta, \mathbf{h}, t) = 0$, and

negatively charged $|\chi\rangle_l^{(-1)G}$ in the \mathcal{G} -sector and $|\chi\rangle_l^{(-1)Q}$ in the \mathcal{Q} -sector for $g_k^-(\Delta, \mathbf{h}, t) = 0$. In each case one of the singular vectors becomes chiral for $\Delta = -l$ whereas the other singular vector does not. The bottom of these submodules is always connected therefore as there is no analog to the ‘chiral-chiral’ case of the uncharged singular vectors. When the bottom of the submodule is at level zero in the Verma module, then there is only one singular vector, which consequently is chiral. These submodules seem to fit well the description of ‘topological’ submodules given by Semikhatov and Tipunin. Namely, they correspond to the charged roots $g_k^\pm(\Delta, \mathbf{h}, t) = 0$, they are smaller than the generic ‘massive’ submodules and they have one or two states at the bottom. In the first case the unique state is charged and chiral (called ‘topological’) while in the second case the two states are connected through \mathcal{G}_0 and/or \mathcal{Q}_0 , one of these states being the \mathcal{G}_0 -closed charged singular vector $|\chi\rangle_l^{(1)G}$ (for $g_k^+(\Delta, \mathbf{h}, t) = 0$) or $|\chi\rangle_l^{(-1)G}$ (for $g_k^-(\Delta, \mathbf{h}, t) = 0$).

Finally, the fourth type of submodules, shown in Fig. II on the right, corresponds to the ‘no-label’ submodules built on indecomposable singular vectors. These are the widest submodules, with four singular vectors at the bottom. The indecomposable singular vectors are *primitive* singular vectors that only exist for discrete values of Δ, \mathbf{h}, t , in generic Verma modules in which there are intersections, at the same level l , of singular vectors corresponding to the series $f_{r,s}(\Delta, \mathbf{h}, t) = 0$ with singular vectors corresponding to one of the series $g_k^\pm(\Delta, \mathbf{h}, t) = 0$, with $\frac{rs}{2} = k = l$ and $\Delta = -l$. The values of \mathbf{c} for which indecomposable singular vectors exist are $\mathbf{c} = \frac{3r-6}{r}$, corresponding to $t = \frac{2}{r}$. These results were proved in ref. [6] although the existence of indecomposable singular vectors was established earlier, in January 1997 in ref. [1], as they were explicitly written down at level 1 (moreover, shortly afterwards indecomposable singular vectors were written down also at level 2 in ref. [7]).

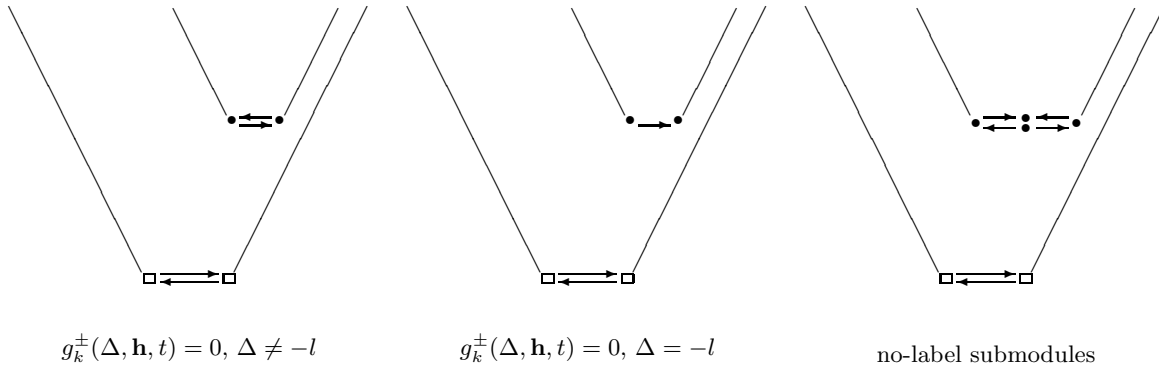


FIG. II . The singular vectors corresponding to the series $g_k^\pm(\Delta, \mathbf{h}, t) = 0$ belong to only one type of submodules, which are smaller than the generic ones. In the general case $k \neq 0$ there are two singular vectors at the bottom of the submodules, connected by \mathcal{G}_0 and/or \mathcal{Q}_0 , depending on whether $\Delta \neq -l$ or $\Delta = -l$. (However, for $k = 0$, that is level zero, the bottom of the corresponding submodule consists of only one singular vector which is chiral). On the right, the indecomposable singular vectors generate the ‘no-label’ submodules, which are the widest submodules with four singular vectors at the bottom.

The action of \mathcal{G}_0 and \mathcal{Q}_0 on an indecomposable singular vector $|\chi\rangle_l^{(g)}$ produce three secondary singular vectors (one \mathcal{G}_0 -closed, one \mathcal{Q}_0 -closed and one chiral) which cannot ‘come back’ to the

no-label singular vector by acting with \mathcal{G}_0 and \mathcal{Q}_0 :

$$\mathcal{Q}_0 |\chi\rangle_l^{(q)} \rightarrow |\chi\rangle_l^{(q-1)Q}, \quad \mathcal{G}_0 |\chi\rangle_l^{(q)} \rightarrow |\chi\rangle_l^{(q+1)G}, \quad \mathcal{G}_0 \mathcal{Q}_0 |\chi\rangle_l^{(q)} \rightarrow |\chi\rangle_l^{(q)G,Q}. \quad (4.12)$$

It happens that one of these secondary singular vectors corresponds to the series $f_{r,s}(\Delta, \mathbf{h}, t) = 0$, another one corresponds to the series $g_k^\pm(\Delta, \mathbf{h}, t) = 0$, and the remaining one corresponds to both series. Hence the bottom of the no-label submodules is connected, generated by the indecomposable singular vector and consists of four singular vectors: the primitive indecomposable singular vector and the three secondary singular vectors. Obviously, these submodules are wider than the ‘massive’ submodules (twice wider at the bottom, in fact) and do not fit into the description of ‘massive’ and ‘topological’ submodules. The no-label submodules cannot have a counterpart in the affine $\hat{sl}(2)$ algebra simply because there is no $\hat{sl}(2)$ counterpart of the indecomposable $N=2$ singular vectors.

In Fig. III one can see the case of an uncharged indecomposable singular vector $|\chi\rangle_l^{(0)}$ at level l , built on a \mathcal{G}_0 -closed h.w. vector $|-l, \mathbf{h}\rangle^G$, with the three secondary singular vectors that it generates by the action of \mathcal{G}_0 and \mathcal{Q}_0 .

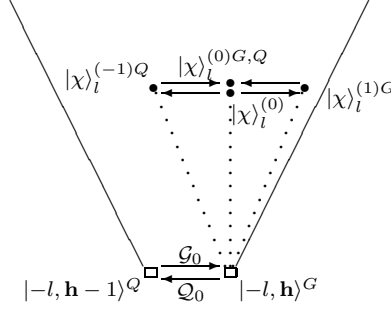


FIG. III . The uncharged indecomposable singular vector $|\chi\rangle_l^{(0)}$ at level l , built on the h.w. vector $|-l, \mathbf{h}\rangle^G$, is the primitive singular vector generating the three secondary singular vectors at level l : $|\chi\rangle_l^{(1)G} = \mathcal{G}_0 |\chi\rangle_l^{(0)}$, $|\chi\rangle_l^{(-1)Q} = \mathcal{Q}_0 |\chi\rangle_l^{(0)}$ and $|\chi\rangle_l^{(0)G,Q} = \mathcal{Q}_0 \mathcal{G}_0 |\chi\rangle_l^{(0)} = -\mathcal{G}_0 \mathcal{Q}_0 |\chi\rangle_l^{(0)}$. These cannot generate the indecomposable singular vector by acting with the algebra. However, they are the singular vectors detected by the determinant formula, corresponding to the series $f_{r,s}(\Delta, \mathbf{h}, t) = 0$ ($|\chi\rangle_l^{(-1)Q}$ and $|\chi\rangle_l^{(0)G,Q}$) and the series $g_k^+(\Delta, \mathbf{h}, t) = 0$ ($|\chi\rangle_l^{(1)G}$ and $|\chi\rangle_l^{(0)G,Q}$).

The corresponding uncharged indecomposable singular vector $|\chi\rangle_1^{(0)}$ at level 1, together with the three secondary singular vectors at level 1 read:

$$|\chi\rangle_{1,|-1,-1,t=2\rangle^G}^{(0)} = (\mathcal{L}_{-1} - \mathcal{H}_{-1})|-1, -1, t = 2\rangle^G, \quad (4.13)$$

$$|\chi\rangle_{1,|-1,-1,t=2\rangle^G}^{(1)G} = \mathcal{G}_0 |\chi\rangle_{1,|-1,-1,t=2\rangle^G}^{(0)} = 2\mathcal{G}_{-1}|-1, -1, t = 2\rangle^G, \quad (4.14)$$

$$|\chi\rangle_{1,|-1,-1,t=2\rangle^G}^{(-1)Q} = \mathcal{Q}_0 |\chi\rangle_{1,|-1,-1,t=2\rangle^G}^{(0)} = (\mathcal{L}_{-1} \mathcal{Q}_0 - \mathcal{H}_{-1} \mathcal{Q}_0 - \mathcal{Q}_{-1})|-1, -1, t = 2\rangle^G, \quad (4.15)$$

$$|\chi\rangle_{1,|-1,-1,t=2\rangle^G}^{(0)G,Q} = \mathcal{G}_0 \mathcal{Q}_0 |\chi\rangle_{1,|-1,-1,t=2\rangle^G}^{(0)} = 2(-2\mathcal{L}_{-1} + \mathcal{G}_{-1} \mathcal{Q}_0) |-1, -1, t = 2\rangle^G. \quad (4.16)$$

The indecomposable singular vector only exists for $t = 2$ ($\mathbf{c} = -3$) whereas the three secondary singular vectors are just the particular cases, for $t = 2$, of the one-parameter families of singular vectors of the corresponding types, which exist for all values of t and were written down in ref. [1].

We have shown that the two types of submodules of the Topological N=2 algebra proposed by Semikhatov and Tipunin in several papers – ‘massive’ and ‘topological’ submodules – as deduced from the ‘would be isomorphism’ between this algebra and the affine $\hat{sl}(2)$ algebra, fit into the ‘external’ description of the submodules of the first and third types that we have analyzed. That is, they fit into the description as regards size and shape at the bottom/top of the submodule. However, as we will see in next subsection using an explicit example, these submodules do not satisfy a crucial property derived from the ‘isomorphism’: they are not generated maximally by the ‘non-conventional singular vectors’ constructed by the authors. That is, one can find subsingular vectors outside the submodules generated by the ‘non-conventional singular vectors’.

For the readers not familiar with the concept of *subsingular* vector the following description can be quite clarifying: A given submodule may not be completely generated by the singular vectors at the bottom, that is, by the h.w. null vectors. These could generate only a subsubmodule of the whole (maximal) submodule, in which case one or more subsingular vectors generate the missing parts. Whereas the subsingular vectors can reach the singular vectors at the bottom by the action of the generators of the algebra, the contrary is not true: subsingular vectors cannot be reached by the action of the algebra on the singular vectors, therefore they are outside the submodules built on the singular vectors. As a result, when the submodules built on the singular vectors are set to zero the subsingular vectors surface as new singular vectors.

The submodules of the second and fourth types (‘chiral-chiral’ and ‘no-label’ submodules) should not exist were the ‘isomorphism’ correct. As a matter of fact, there is no $\hat{sl}(2)$ counterpart for the indecomposable singular vectors that generate the no-label submodules, and it is not even clear whether there is a $\hat{sl}(2)$ counterpart for the chiral *uncharged* singular vectors $|\chi\rangle_t^{(0)G,Q}$, as they have been systematically ignored by the authors.

Another important remark concerns the presentation of the singular vectors of the Topological N=2 algebra made by the authors, for convenience, in order to endorse the ‘isomorphism’. They claim that in the *conventional* approach the h.w. conditions imposed on the h.w. vectors and on *any* singular vector must include the annihilation by \mathcal{G}_0 (eq.(2.11) in ref. [10]). This statement is not only incorrect but also very misleading. First of all, in the conventional approach for the conformal and superconformal algebras, one defines the h.w. vectors and singular vectors (sometimes called simply null vectors) as the states with lowest conformal weight (lowest energy) in the Verma modules and submodules, respectively. As a result, in most Verma modules and submodules of the Ramond and the Topological N=2 algebras (they are isomorphic [6]) there are two sectors degenerated in energy, the + and – sectors for the Ramond algebra and the \mathcal{G} and \mathcal{Q} sectors for the Topological algebra, the corresponding states annihilated by the fermionic zero modes G_0^+ or G_0^- and \mathcal{G}_0 or \mathcal{Q}_0 , respectively [14][16][17][18][19][20][6]. That is, at the bottom of most Verma modules and

submodules of the Ramond and of the Topological N=2 algebras there are two h.w. vectors and two singular vectors, respectively, the fermionic zero modes interpolating between them. In addition, one can find indecomposable singular vectors not annihilated by any of the fermionic zero modes, that also must be called singular vectors following the conventional definition and generate the widest submodules, as we have just shown [1][7][5][6].

Second, and this is a crucial point, to break the symmetry between the \mathcal{G} and the \mathcal{Q} sectors, regarding the singular vectors of the \mathcal{Q} -sector simply as descendant states of ‘the singular vectors’ of the \mathcal{G} -sector, leads to a great deal of confusion in the case of zero conformal weight $\Delta + l = 0$. The reason is that for $\Delta + l = 0$ the \mathcal{Q}_0 -closed singular vectors $|\chi\rangle_{l=-\Delta}^{(q)Q}$ are in fact the primitive singular vectors generating the *secondary* singular vectors of the \mathcal{G} -sector, which are necessarily chiral of type $|\chi\rangle_{l=-\Delta}^{(q+1)G,Q}$ (see the details in ref. [6], Appendix A). In the conventions used by Semikhatov and Tipunin, however, the vectors $|\chi\rangle_{l=-\Delta}^{(q)Q}$ are not singular by definition, although they are necessarily null. As a result, since they are not descendant states of ‘the singular vector’ $|\chi\rangle_{l=-\Delta}^{(q+1)G,Q}$, but the other way around, the singular vectors of the \mathcal{Q} -sector $|\chi\rangle_{l=-\Delta}^{(q)Q}$ must be called *subsingular vectors* instead. For similar reasons, the indecomposable singular vectors must also be called subsingular vectors as they are null, not descendants of the singular vectors of the \mathcal{G} -sector, but the other way around, and they are not singular by definition.

4.2 The classification of subsingular vectors

Now we will see that the explicit examples of subsingular vectors of the Topological N=2 superconformal algebra given in ref. [1], which are singular in the chiral Verma modules, do not fit into the ‘complete classification’ of subsingular vectors presented in ref. [10]. As a bonus we will also deduce that the non-conventional singular vectors constructed in refs. [8] [9] do not generate maximal submodules, contrary to the claims of the authors who deduced this property directly from the ‘isomorphism’. This property, in addition, was used as a major tool for the classification of the subsingular vectors.

The authors classified the generic Verma modules built on \mathcal{G}_0 -closed h.w. vectors (‘massive’ Verma modules) according whether they have zero, one, two or more singular vectors from the uncharged and/or charged series associated to the roots of the determinant formula (in our notation $f_{r,s}(\Delta, \mathbf{h}, t) = 0$ and/or $g_k^\pm(\Delta, \mathbf{h}, t) = 0$, eqns. (2.2) and (2.3)). In every case they applied the assumption that there are only two types of submodules – ‘massive’ and ‘topological’ – and these are generated maximally by the non-conventional singular vectors constructed in refs. [8][9]. Namely, one ‘twisted topological’ non-conventional singular vector (where they mean twisted by the spectral flows⁵) is assumed to generate maximally one ‘topological’ submodule whereas one ‘twisted massive’ non-conventional singular vector is assumed to generate maximally one ‘massive’ submodule.⁶

⁵In our opinion, the authors used the spectral flows in a very dubious way. Apart from the ‘classical’ reference on the subject [21], the interested reader may also find useful the analysis done in refs. [22] and [23].

⁶As was mentioned before, if these non-conventional singular vectors could generate maximally, by the action of the algebra, the whole submodules whereas the conventional h.w. singular vectors failed to do the same, then the

These non-conventional singular vectors are null states that in general are not located at the bottom of the submodules unlike the conventional singular vectors. In fact, in the cases when they lie at the bottom then they coincide with the conventional singular vectors. An important remark is that the ‘twisted topological’ h.w. conditions satisfied by the ‘twisted topological’ non-conventional singular vectors reduce to the chirality h.w. conditions (i.e. annihilation by \mathcal{G}_0 , \mathcal{Q}_0 and by all the positive generators) in the case of the twist parameter equal to zero. As a result, the ‘zero twist topological’ non-conventional singular vectors coincide with the chiral *charged* conventional singular vectors at the bottom of the ‘topological’ submodules.

Using these assumptions and simple geometrical arguments the authors deduced in which cases the *conventional* singular vectors at the bottom of the submodules do not generate maximal submodules, and then using the non-conventional singular vectors they ‘identified’ the subsingular vectors, giving some general expressions in some cases. The subsingular vectors given by us in ref. [1] corresponded necessarily to the ones described by the authors in the case ‘codimension-2 charge-massive’, given⁷ in Proposition 3.9, for $n = 0$, since they are located in Verma modules with one charged chiral singular vector (at level zero, what gives $n = 0$) and one uncharged \mathcal{G}_0 -closed singular vector. In the notation of the authors, who draw the Verma modules from the top downwards, the charged singular vector is both a conventional ‘top-level’ singular vector and a non-conventional ‘twisted topological’ charged singular vector $|E(n)\rangle_{ch}$ with twist parameter $n = 0$ (i.e. the non-conventional singular vector is at the bottom of the submodule, in our notation, so that it coincides with the conventional singular vector). The uncharged \mathcal{G}_0 -closed singular vector is described as the conventional ‘top-level’ uncharged singular vector in the ‘massive’ submodule generated by the non-conventional ‘massive’ singular vector $|S(r, s)\rangle$, and is denoted as $|s\rangle$.

For this case, and in fact for all cases ‘described’ by Proposition 3.9, the authors deduced that a subsingular vector $|Sub\rangle$ must exist *inside* the maximal massive submodule generated by $|S(r, s)\rangle$ in the sense that $|Sub\rangle$ is located *outside* the non-maximal submodule generated by the conventional uncharged singular vector $|s\rangle$, becoming singular once $|s\rangle$ is set to zero. This implies that the subsingular vector $|Sub\rangle$ is ‘pushed down’ (‘up’ in the authors figures) by the action of the lowering operators inside the non-maximal submodule generated by $|s\rangle$, so that setting this submodule to zero is equivalent to push down the vector to nothing, i.e. the subsingular vector becomes singular. Observe that in this case the subsingular vector $|Sub\rangle$, once it reaches $|s\rangle$ by the action of the lowering operators, cannot go down (‘up’) anymore since $|s\rangle$ is the conventional singular vector at the bottom of the submodule annihilated by all the lowering operators. In other words, if the subsingular vector $|Sub\rangle$ becomes singular when $|s\rangle$ is set to zero, then acting with the lowering operators on $|Sub\rangle$ it cannot be pulled down beyond the level of $|s\rangle$, getting in fact ‘stuck’ in $|s\rangle$ (up to constants).

The subsingular vectors at level 3 given by us in ref. [1] do not follow the behaviour described by Proposition 3.9, however. Rather, they are pulled down beyond the uncharged conventional singular vector $|s\rangle$ that one finds at level 2 and, in fact, they can be pulled down until the very end, i.e. level zero, becoming singular only when the charged chiral singular vector $|E(0)\rangle_{ch}$ at level zero

non-conventional singular vectors would be nothing but subsingular vectors or some of their descendants, equivalently.

⁷The authors themselves claimed that the subsingular vectors given in ref. [1] were described by Proposition 3.9, case $n = 0$, although they did not explicitly mention this in the last revised, published version of ref. [10].

is set to zero. As a consequence, these subsingular vectors do not become singular when $|s\rangle$ is set to zero, what implies that they are not pulled inside the submodule generated by $|s\rangle$ by the action of the lowering operators (see Fig. IV), and therefore they are not located inside the maximal massive submodule supposed to be generated by the ‘massive’ singular vector $|S(r, s)\rangle$. But these subsingular vectors are neither located inside the submodule generated by $|E(0)\rangle_{ch}$ since they do not disappear when $|E(0)\rangle_{ch}$ is set to zero, becoming singular rather. In other words, as shown in Fig. IV, these subsingular vectors are pulled inside the submodule generated by $|E(0)\rangle_{ch}$ by acting with the lowering operators. This implies that the submodule generated by the non-conventional singular vector $|E(0)\rangle_{ch}$ is not maximal, in contradiction with the claims in ref. [10].

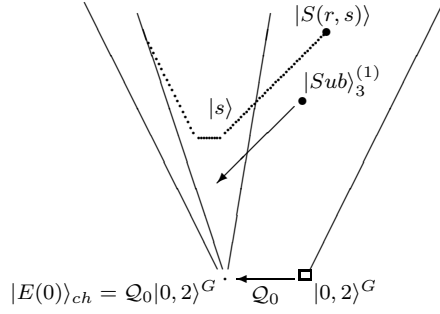


FIG. IV . When the charged level zero singular vector $|E(0)\rangle_{ch} = \mathcal{Q}_0|0, 2\rangle^G$ is set to zero, the generic (‘massive’) Verma module built on $|0, 2\rangle^G$ is divided by the submodule generated by this singular vector. As a result one obtains the chiral (‘topological’) Verma module built on the chiral h.w. vector $|0, 2\rangle^{G, Q}$. The subsingular vector $|Sub\rangle_3^{(1)}$ at level 3 is outside the submodule generated by $|E(0)\rangle_{ch}$, being pulled inside by the action of the lowering operators. Consequently, the submodule generated by the non-conventional ‘topological’ charged singular vector $|E(0)\rangle_{ch}$ (which being at the bottom of the submodule coincides with the conventional chiral singular vector $\mathcal{Q}_0|0, 2\rangle^G$) is not maximal since there is (at least) one subsingular vector left outside. This subsingular vector becomes singular, therefore, in the chiral Verma module $V(|0, 2\rangle^{G, Q})$ obtained after the quotient. Inside the submodule generated by $|E(0)\rangle_{ch}$ one finds the uncharged \mathcal{G}_0 -closed singular vector $|s\rangle$ (and its companion in the \mathcal{Q} -sector that is not indicated). The subsingular vector $|Sub\rangle_3^{(1)}$ is not pulled inside the submodule generated by $|s\rangle$ by the lowering operators and therefore it does not become singular once $|s\rangle$ is set to zero. Rather, it is pulled down to lower levels than $|s\rangle$. As a result $|Sub\rangle_3^{(1)}$ does not belong to the ‘massive’ submodule, supposedly to be maximally generated by the non-conventional ‘massive’ singular vector $|S(r, s)\rangle$, having $|s\rangle$ at the bottom.

One example given in ref. [1] is the subsingular vector $|Sub\rangle_3^{(1)}$ at level 3 with charge $q = 1$ built on the \mathcal{G}_0 -closed h.w. vector $|\Delta, \mathbf{h}\rangle^G$ with conformal weight $\Delta = 0$ and $U(1)$ charge $\mathbf{h} = 2$:

$$|Sub\rangle_3^{(1)} = \left\{ \frac{3-\mathbf{c}}{24} \mathcal{L}_{-1}^2 \mathcal{G}_{-1} - \frac{3}{4} \mathcal{L}_{-1} \mathcal{G}_{-2} - \frac{1}{4} \mathcal{L}_{-2} \mathcal{G}_{-1} + \frac{\mathbf{c}+9}{4(\mathbf{c}-3)} \mathcal{H}_{-2} \mathcal{G}_{-1} + \frac{27-\mathbf{c}}{4(3-\mathbf{c})} \mathcal{G}_{-3} + \frac{6}{\mathbf{c}-3} \mathcal{H}_{-1} \mathcal{G}_{-2} + \frac{3}{4} \mathcal{H}_{-1} \mathcal{L}_{-1} \mathcal{G}_{-1} + \frac{3}{3-\mathbf{c}} \mathcal{H}_{-1}^2 \mathcal{G}_{-1} \right\} |0, 2\rangle^G.$$

Acting with \mathcal{Q}_1 on this vector one does not hit the conventional uncharged singular vector $|s\rangle$ at level 2 but one reaches the state

$$\left\{ \frac{\mathbf{c}-12}{12} \mathcal{L}_{-1} \mathcal{G}_{-1} + \frac{3(11-\mathbf{c})}{4(3-\mathbf{c})} \mathcal{G}_{-2} + \frac{3(11-\mathbf{c})}{4(\mathbf{c}-3)} \mathcal{H}_{-1} \mathcal{G}_{-1} \right\} \mathcal{Q}_0 |0, 2\rangle^G, \quad (4.17)$$

which is a non-singular descendant of the level zero charged singular vector $|E(0)\rangle_{ch} = \mathcal{Q}_0 |0, 2\rangle^G$. That is, $|Sub\rangle_3^{(1)}$ is pulled inside the submodule generated by $|E(0)\rangle_{ch}$ by the action of \mathcal{Q}_1 . Acting further with \mathcal{L}_1 one reaches the state $\mathcal{G}_{-1}\mathcal{Q}_0 |0, 2\rangle^G$ at level 1 which, again, is not singular. Acting with \mathcal{Q}_1 on this state one reaches finally the level zero chiral charged singular vector: $\mathcal{Q}_1\mathcal{L}_1\mathcal{Q}_1 |Sub\rangle_3^{(1)} = \mathcal{Q}_0 |0, 2\rangle^G = |E(0)\rangle_{ch}$.

This example not only proves that Proposition 3.9 is incorrect, as $|Sub\rangle_3^{(1)}$ does not become singular when $|s\rangle$ is set to zero, and that the subsingular vectors presented in ref. [1] (the only examples known at that time!) do not fit into the ‘complete’ classification of subsingular vectors given in ref. [10]. As we have just discussed, this example also proves that the non-conventional topological singular vector $|E(0)\rangle_{ch} = \mathcal{Q}_0 |0, 2\rangle^G$ (which is located at the bottom of the submodule and therefore coincides with the conventional chiral singular vector) does not generate a maximal submodule since the subsingular vector $|Sub\rangle_3^{(1)}$ is outside this submodule, being pulled inside by the action of the lowering operators. This example disproves the claim that the non-conventional ‘massive’ and ‘topological’ singular vectors generate maximal submodules, i.e; with no space left outside for subsingular vectors. Indeed, we have shown that the subsingular vector $|Sub\rangle_3^{(1)}$ is neither generated by the ‘massive’ singular vector $|S(r, s)\rangle$ nor by the ‘topological’ singular vector $|E(0)\rangle_{ch}$, nor by both of them together.

Acknowledgements

We thank A. Belavin for discussions. This work has been partially supported by the Spanish Ministerio de Ciencia e Innovación, Research Project FPA2005-05046, and by the Project CONSOLIDER - INGENIO 2010, Programme CPAN (CSD2007 - 00042).

References

- [1] B. Gato-Rivera and J.I. Rosado, *Families of Singular and Subsingular Vectors of the Topological $N=2$ Superconformal Algebra*, Nucl. Phys. B514 [PM] (1998) 477, hep-th/9701041.
- [2] B. Gato-Rivera and J.I. Rosado, *Interpretation of the Determinant Formulae for the Chiral Representations of the $N=2$ Superconformal Algebra*, IMAFF-96/38, NIKHEF-96-007, hep-th/9602166 (1996).
- [3] B. Gato-Rivera and J.I. Rosado, *Chiral Determinant Formulae and Subsingular Vectors for the $N=2$ Superconformal Algebras*, Nucl. Phys. B503 (1997) 447, hep-th/9706041.
- [4] B. Gato-Rivera, *A Note concerning Subsingular Vectors and Embedding Diagrams of the $N=2$ Superconformal Algebras*, IMAFF-FM-99/09, hep-th/9910121 (1999).
- [5] M. Dörrzapf and B. Gato-Rivera, *Singular Dimensions of the $N=2$ Superconformal Algebras.I*, Comm. Math. Phys 206 (1999) 493, hep-th/9807234 (1998).
- [6] M. Dörrzapf and B. Gato-Rivera, *Determinant Formula for the Topological $N=2$ Superconformal Algebra*, Nucl. Phys. B558 [PM] (1999) 503, hep-th/9905063.
- [7] M. Dörrzapf and B. Gato-Rivera, *Transmutations between Singular and Subsingular Vectors of the $N=2$ Superconformal Algebras*, Nucl. Phys. B 557 [PM] (1999) 517, hep-th/9712085.
- [8] A.M. Semikhatov and I.Yu. Tipunin, *Singular Vectors of the Topological Conformal Algebra*, Int. J. Mod. Phys. A11 (1996) 4597, hep-th/9512079.
- [9] A.M. Semikhatov and I.Yu. Tipunin, *All Singular Vectors of the $N=2$ Superconformal Algebra via the Algebraic Continuation Approach*, hep-th/9604176 (last revised version in September 1998).

- [10] A.M. Semikhatov and I.Yu. Tipunin, *The Structure of Verma Modules over the $N=2$ Superconformal Algebra*, hep-th/9704111 V1 (1997). (A different version was published later in Comm. Math. Phys. 195 (1998) 129, where crucial comments had been removed, although the results were identical).
- [11] R. Dijkgraaf, E. Verlinde and H. Verlinde, Nucl. Phys. B352 (1991) 59.
- [12] T. Eguchi and S. K. Yang, Mod. Phys. Lett. A5 (1990) 1653.
- [13] E. Witten, Commun. Math. Phys. 118 (1988) 411; Nucl. Phys. B340 (1990) 281.
- [14] W. Boucher, D. Friedan and A. Kent, Phys. Lett. B172 (1986) 316.
- [15] S. Nam, Phys. Lett. B172 (1986) 323.
- [16] M. Kato and S. Matsuda, Phys. Lett. B184 (1987) 184.
- [17] P. Di Vecchia, J.L. Petersen and M. Yu, Phys. Lett. B172 (1986) 211;
P. Di Vecchia, J.L. Petersen, M. Yu and H.B. Zheng, Phys. Lett. B174 (1986) 280;
M. Yu and H.B. Zheng, Nucl. Phys. B288 (1987) 275.
- [18] E.B. Kiritsis, Phys. Rev. D36 (1987) 3048;
Int. J. Mod. Phys A3 (1988) 1871.
- [19] W. Lerche, C. Vafa and N. P. Warner, Nucl. Phys. B324 (1989) 427.
- [20] G. Mussardo, G. Sotkov and M. Stanishkov, Int. J. Mod. Phys. A4 (1989) 1135.
- [21] A. Schwimmer and N. Seiberg, Phys. Lett. B184 (1987) 191
- [22] B. Gato-Rivera and J.I. Rosado, *Spectral Flows and Twisted Topological Theories*, Phys. Lett. B369 (1996) 7, hep-th/9504056.
- [23] B. Gato-Rivera, *The Even and the Odd Spectral Flows on the $N=2$ Superconformal Algebras*, Nucl. Phys. B512 (1998) 431, hep-th/9707211 (1997).