

On next-to-eikonal corrections to threshold resummation for the Drell-Yan and DIS cross sections

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Abstract

We study corrections suppressed by one power of the soft gluon energy to the resummation of threshold logarithms for the Drell-Yan cross section and for Deep Inelastic structure functions. While no general factorization theorem is known for these next-to-eikonal (NE) corrections, it is conjectured that at least a subset will exponentiate, along with the logarithms arising at leading power. Here we develop some general tools to study NE logarithms, and we construct an ansatz for threshold resummation that includes various sources of NE corrections, implementing in this context the improved collinear evolution recently proposed by Dokshitzer, Marchesini and Salam (DMS). We compare our ansatz to existing exact results at two and three loops, finding evidence for the exponentiation of leading NE logarithms and confirming the predictivity of DMS evolution.

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1 Introduction

Sudakov resummations are established in perturbative QCD for all logarithmic contributions, to leading power in the total momentum fraction carried by soft gluons. To illustrate this fact, consider as an example threshold resummation for the Drell-Yan process, or for a similar electroweak annihilation cross section at the hard scale Q . In this case, large logarithms arise in the hard partonic cross section when the total available center-of-mass energy, \hat{s} , is only slightly larger than the mass Q^2 of the selected electroweak final state. Gluon radiation into the final state is then forced to be soft, as gluons carry (at most) a total energy $(1-z)\hat{s}$, with $z \equiv Q^2/\hat{s}$. As a consequence, perturbative contributions at order α_s^n are enhanced by large logarithms in the form of ‘plus’ distributions, up to $[\ln^{2n-1}(1-z)/(1-z)]_+$. Upon taking a Mellin transform, these distributions turn into powers of logarithms of the Mellin variable N , conjugate to z , up to $\ln^{2n} N$. All these contributions can be resummed [1, 2], and they display a nontrivial pattern of exponentiation: the logarithm of the cross section in Mellin space, in fact, is enhanced only by single logarithms, up to $\ln^{n+1} N$ at order α_s^n .

It has been understood since the early days of QCD [3] that at least some non-logarithmic contributions (terms independent of N , which are Mellin conjugate to virtual corrections proportional to $\delta(1-z)$) also exponentiate. In fact, Ref. [4] later proved, at least for electroweak annihilation and DIS, that all such contributions can be organized in exponential form. One may naturally wonder to what extent this pattern of exponentiation can be extended beyond leading power in the Mellin variable N , or in the soft gluon energy fraction $1-z$.

There are several problems in attempting to extend the resummation formalism beyond leading power in N , or $1-z$. Indeed, resummation can be understood to be a consequence of Sudakov factorization, as discussed in [5]. To leading power in N , it can be shown that the Mellin moments of the cross section factorize into distinct functions responsible for infrared and collinear enhancements, times a hard remainder which is free of logarithms. Exponentiation follows from evolution equations that are dictated by this factorization. To date, no proof of such a Sudakov factorization is available beyond leading power in N . Part of this issue is the fact that, in order to achieve exponentiation, the phase space specific to the observable at hand, in the threshold limit, must itself factorize; this is achieved at leading power by taking the Mellin transform, thanks to the fact that the observable (essentially $1-z$ for the inclusive Drell-Yan cross section) is linear in soft gluon energies to leading power in $1-z$. Again, this simple property is lost beyond leading power.

Notwithstanding these obstacles, there is intriguing, if scattered, evidence that some of the mechanisms that lead to the resummation of leading power logarithms are still operating at next-to-leading power. Theoretically, evidence in this direction is provided by the Low-Burnett-Kroll theorem [6, 7], which states that (in QED) cross sections involving soft photons can be expressed in terms of radiation-less amplitudes not only at leading power in soft gluon energies (which corresponds to

the bremsstrahlung spectrum and to the eikonal approximation), but also at next-to-leading power. For such cross sections radiation is simply related to classical fields, and one expects some form of soft photon exponentiation to hold. In QCD, direct application of Low's theorem is complicated by the presence of collinear divergences [8], but one may still expect it to be relevant for soft emissions.

At a more practical level, one may observe that resummed cross sections are expressed in terms of integrals of certain anomalous dimensions, with integration limits dictated by the phase space available for soft radiation, and with the running coupling evaluated at the typical transverse momentum of the first gluon emission. These kinematical quantities are evaluated in the threshold limit, and one may expect that correcting their values in order to make them accurate at next-to-leading power in the soft momentum should lead to a physically meaningful improvement of the resummation.

This kind of reasoning has led to attempts to include certain sub-eikonal effects in practical implementations of Sudakov resummations, mostly in view of gauging the theoretical uncertainty of the resummation [9]. Typically, this involves including subleading terms in the collinear evolution kernel into the resummation, which is particularly appealing for Drell-Yan and related cross sections, where the entire singularity structure is determined by initial state soft and collinear radiation. This was applied in the case of Higgs production in Refs. [9, 11, 10, 12], and for prompt photon production in Ref. [13].

More recently, following the evaluation of collinear evolution kernels at three loops [14], a bold suggestion has been put forward by Dokshitzer, Marchesini and Salam (DMS) [15], who proposed a modified evolution equation for parton distributions, based on the idea that the proper ordering variable in the collinear shower should be the lifetime of parton fluctuations rather than the gluon transverse momentum. This modified evolution has remarkable consequences: it explains a previously mysterious numerical coincidence observed by [14], and it connects eikonal and sub-eikonal terms in the splitting function in a nontrivial way, consistent with the idea that all evolution effects which are non-vanishing as $z \rightarrow 1$ should be determined at one loop, with an appropriate definition of the coupling. The DMS proposal has later been refined by Basso and Korchemsky [16], who traced the recursive relation which determines the collinear anomalous dimension to the conformal invariance of the classical theory, and its breaking by the β function. The relations connecting eikonal and next-to-eikonal terms for parton evolution are then generalized to higher twist operators as well.

In this note, we begin to develop a systematic approach for the inclusion of next-to-eikonal terms in the resummation, inspired by the results of [15] and by the earlier work of [9]. We begin, in Section 2, by briefly reviewing the DMS approach, and describing how we intend to implement it in the context of Sudakov resummation. There, we also introduce some simple tools and definitions to evaluate the integrals that appear in resummed exponents to the desired accuracy. Then, in Section 3, we propose an ansatz to include in the resummation all next-to-eikonal effects that can

be argued to be under theoretical control. We do this for the Drell-Yan cross section and for the Deep Inelastic structure function F_2 . It is clear from the outset that our ansatz controls only a subset of all next-to-eikonal terms in the cross section: indeed, it may well be that not all such terms can be organized in exponential form. We believe however that the terms we include are physically well motivated, so we expect our ansatz to reproduce with reasonable accuracy higher order perturbative results, based on the evaluation of the exponent at lower orders. We proceed to test this expectation by comparing the results of expanding our proposed resummed expressions with the known exact results at two loops for the Drell-Yan cross section [17], and at two and three loops for DIS [18, 19].

As we will outline in our discussion, in Section 4, the results of this comparison are consistent with the assumption that at least leading next-to-eikonal logarithms do exponentiate, for all color structures. Furthermore, the implementation of the DMS approach reproduces with considerable accuracy (though not exactly) certain classes of subleading next-to-eikonal logarithms which could not have been generated by the standard resummation. We believe that these results are encouraging regarding the possibility that next-to-eikonal logarithms could be understood and organized to all orders, an effort which will ultimately require a full analysis of soft gluon effects beyond the eikonal approximation.

2 Tools for next-to-eikonal resummation

The task of probing the extension of the resummation formalism beyond the eikonal approximation requires both conceptual and practical tools. In this section we describe briefly the main conceptual progress that we are going to employ, which is the idea, put forward by DMS, that all NE terms in collinear evolution trace their origin to one loop effects, phase space, and the choice of an appropriate, physically motivated coupling. We present the DMS equation, and we show how it can be solved in exponential form, just like ordinary collinear evolution, to NE accuracy. Next, making use of a technique developed in [20], which we generalize to NE level, we present some simple results for the generic integrals that may appear in NE resummed cross section to any perturbative order.

2.1 The DMS evolution equation and its solution

Consider first the familiar collinear evolution equation for the non-singlet quark density

$$\mu^2 \frac{\partial}{\partial \mu^2} q(x, \mu^2) = \int_x^1 \frac{dz}{z} q\left(\frac{x}{z}, \mu^2\right) P_{qq}(z, \alpha_s(\mu^2)) . \quad (1)$$

As is well known, this simple convolution can be turned into a product by taking a Mellin transform,

$$\mu^2 \frac{\partial}{\partial \mu^2} \tilde{q}(N, \mu^2) = \gamma_N(\alpha_s(\mu^2)) \tilde{q}(N, \mu^2), \quad (2)$$

which leads to an exponential solution for the Mellin moments of the quark distribution,

$$\tilde{q}(N, \mu^2) = \exp \left[\int_{\mu_0^2}^{\mu^2} \frac{d\mu'^2}{\mu'^2} \gamma_N(\alpha_s(\mu'^2)) \right] \tilde{q}(N, \mu_0^2). \quad (3)$$

Note that here we express the solution in terms of a generic initial condition at some reference scale, as appropriate for the evolution of physical, measured parton distributions. When one instead considers parton-in-parton distributions, defined in QCD in terms of matrix elements of bilocal operators, one can use dimensional regularization to express the solution as a pure exponential (with no prefactor), using the fact that the dimensionally regularized coupling vanishes with the scale [5, 21]. Within the framework of dimensional regularization and in a minimal subtraction scheme, the structure of the anomalous dimension $\gamma_N(\alpha_s)$ at large values of N (corresponding to the $z \rightarrow 1$ limit) is known [22] to be single-logarithmic. It is of the form

$$\gamma_N(\alpha_s) = -A(\alpha_s) \ln \bar{N} + B_\delta(\alpha_s) - C_\gamma(\alpha_s) \frac{\ln \bar{N}}{N} + D_\gamma(\alpha_s) \frac{1}{N} + \mathcal{O}\left(\frac{1}{N^2}\right) \quad (4)$$

where the function $A(\alpha_s)$ is one half of the cusp anomalous dimension $\gamma_K(\alpha_s)$, and $\bar{N} = N e^{\gamma_E}$. The DMS proposal is that the functions $C_\gamma(\alpha_s)$ and $D_\gamma(\alpha_s)$ are not genuinely independent, but they can be derived from the knowledge of $A(\alpha_s)$. In turn, $A(\alpha_s)$ can be interpreted as a definition of the coupling in a suitable scheme, which has been variously described as ‘physical’, or ‘bremsstrahlung’, or ‘Monte Carlo’ scheme [23]. In order to implement this idea, DMS propose to replace Eq. (1) with

$$\mu^2 \frac{\partial}{\partial \mu^2} \psi(x, \mu^2) = \int_x^1 \frac{dz}{z} \psi\left(\frac{x}{z}, z^\sigma \mu^2\right) \mathcal{P}\left(z, \alpha_s\left(\frac{\mu^2}{z}\right)\right). \quad (5)$$

Here we have denoted by $\psi(x, \mu^2)$ a distribution which can be understood either as a fragmentation function or as a parton distribution; the parameter $\sigma = \pm 1$ serves to distinguish the two cases: $\sigma = +1$ for the space-like evolution of parton distributions, while $\sigma = -1$ for the time-like evolution of fragmentation functions. DMS argue (and verify at two loops) that with Eq. (5) the evolution kernel is the same for both kinematics. Furthermore, at least up to second order in α_s , the kernel \mathcal{P} has no contributions at order $(1-z)^0$, so that it can be written as

$$\mathcal{P}(z, \alpha_s) = \frac{A(\alpha_s)}{(1-z)_+} + B_\delta(\alpha_s) \delta(1-z) + \mathcal{O}((1-z)). \quad (6)$$

If one now chooses the cusp anomalous dimension (divided by the Casimir invariant of the appropriate representation, in this case C_F) as the definition of the coupling, setting $A(\alpha_s(\mu^2)) = C_F \alpha_{\text{PH}}(\mu^2)$, one may conclude that all contributions to the evolution kernel that do not vanish as $z \rightarrow 1$ appear at the first non-trivial order in this scheme.

In the physical scheme, writing $\mathcal{P}(z, \alpha_{\text{PH}}) = \mathcal{P}_1(z) \alpha_{\text{PH}}/\pi + \mathcal{O}(\alpha_{\text{PH}}^2)$, it is easy to construct an exponential solution, analogous to Eq. (3) but valid to NE order, for the distribution D . Indeed one may write

$$\mu^2 \frac{\partial}{\partial \mu^2} \psi(N, \mu^2) = \int_0^1 dz z^{N-1} \mathcal{P}_1(z) \alpha_{\text{PH}} \left(\frac{\mu^2}{z} \right) \psi(N, z^\sigma \mu^2). \quad (7)$$

The scale of the coupling can be shifted by using the β function, as

$$\begin{aligned} \mu^2 \frac{\partial}{\partial \mu^2} \psi(N, \mu^2) &= \int_0^1 dz z^{N-1} \mathcal{P}_1(z) \left[\frac{\alpha_{\text{PH}}}{\pi} \right. \\ &\quad \left. + (1-z) \left(\beta(\alpha_{\text{PH}}) - \sigma \frac{\alpha_{\text{PH}}}{\pi} \mu^2 \frac{\partial}{\partial \mu^2} \right) \right] \psi(N, \mu^2). \end{aligned} \quad (8)$$

One can now perform a Mellin transform, and introduce the anomalous dimensions

$$\widehat{\gamma}_1(N) = \int_0^1 dz z^{N-1} \mathcal{P}_1(z), \quad \widehat{\gamma}_1'(N) = \int_0^1 dz z^{N-1} (1-z) \mathcal{P}_1(z), \quad (9)$$

which clearly obey $\widehat{\gamma}_1'(N) = \widehat{\gamma}_1(N) - \widehat{\gamma}_1(N+1)$. One finds then

$$\psi(N, \mu^2) = \exp \left[\int_{\mu_0^2}^{\mu^2} \frac{d\mu^2}{\mu^2} \frac{\widehat{\gamma}_1(N) (\alpha_{\text{PH}}(\mu^2)/\pi) + \widehat{\gamma}_1'(N) \beta(\alpha_{\text{PH}}(\mu^2))}{1 + \sigma \widehat{\gamma}_1'(N) (\alpha_{\text{PH}}(\mu^2)/\pi)} \right] \psi(N, \mu_0^2), \quad (10)$$

which is valid up to corrections vanishing as $z \rightarrow 1$.

2.2 Moment integrals to $\mathcal{O}(1/N)$

Let us now turn to the practical issue of evaluating the generic integrals appearing in the exponents of threshold resummations, to our required accuracy, *i.e.* including all correction of order $1/N$. To this accuracy threshold-resummed partonic cross sections can be written as

$$\ln [\hat{\sigma}(N)] - H = \int_0^1 dz \frac{z^{N-1} - 1}{1-z} f_1[\ln(1-z)] + \int_0^1 dz z^{N-1} f_2[\ln(1-z)], \quad (11)$$

where H represents N -independent terms. Expanding the functions f_i in powers of their argument, as

$$f_i[\ln(1-z)] = \sum_{p=0}^{\infty} f_i^{(p)} \ln^p(1-z), \quad (12)$$

we can write

$$\ln [\hat{\sigma}(N)] - H = \sum_{p=0}^{\infty} \left[f_1^{(p)} \mathcal{D}_p(N) + f_2^{(p)} \mathcal{J}_p(N) \right], \quad (13)$$

in terms of the basic integrals

$$\mathcal{D}_p(N) = \int_0^1 dz \frac{z^{N-1} - 1}{1-z} \ln^p(1-z), \quad \mathcal{J}_p(N) = \int_0^1 dz z^{N-1} \ln^p(1-z). \quad (14)$$

In order to evaluate the integrals in Eq. (14), we follow [20] and introduce two generating functions, defined by

$$G_{\mathcal{D}}(\lambda, N) \equiv \int_0^1 (z^{N-1} - 1)(1-z)^{\lambda-1} = \frac{\Gamma(N)\Gamma(\lambda)}{\Gamma(N+\lambda)} - \frac{1}{\lambda}, \quad (15)$$

and by

$$G_{\mathcal{J}}(\lambda, N) \equiv \int_0^1 z^{N-1} (1-z)^{\lambda} = \frac{\Gamma(N)\Gamma(\lambda+1)}{\Gamma(N+\lambda+1)} = \frac{1}{N+\lambda} [\lambda G_{\mathcal{D}}(\lambda, N) + 1], \quad (16)$$

From these definitions, one sees that the integrals in Eq. (14) are given by

$$\mathcal{D}_p(N) = \frac{\partial^p}{\partial \lambda^p} G_{\mathcal{D}}(\lambda, N) \Big|_{\lambda=0}, \quad \mathcal{J}_p(N) = \frac{\partial^p}{\partial \lambda^p} G_{\mathcal{J}}(\lambda, N) \Big|_{\lambda=0}. \quad (17)$$

In order to evaluate the integrals explicitly to $1/N$ accuracy, we only need the first correction to Stirling's formula for the \mathcal{D} -type integrals,

$$\Gamma(z) = e^{-z} z^{z-1/2} \sqrt{2\pi} \left(1 + \frac{1}{12z} \right) \left(1 + \mathcal{O}\left(\frac{1}{z^2}\right) \right), \quad (18)$$

leading to

$$G_{\mathcal{D}}(\lambda, N) = \frac{1}{\lambda} \left[\frac{\Gamma(1+\lambda)}{N^\lambda} \left(1 + \frac{\lambda(1-\lambda)}{2N} \right) - 1 \right], \quad (19)$$

while for the \mathcal{J} -type integrals it suffices to take

$$G_{\mathcal{J}}(\lambda, N) = \frac{\Gamma(1+\lambda)}{N^{1+\lambda}}. \quad (20)$$

We note in passing that, to $1/N$ accuracy, there is a simple relation between the \mathcal{J} and the \mathcal{D} integrals; in fact

$$\mathcal{J}_p(N) = -\frac{d}{dN} \mathcal{D}_p(N) + \mathcal{O}\left(\frac{1}{N^2}\right), \quad (21)$$

which follows from an identical relation between the generating functions,

$$G_{\mathcal{J}}(\lambda, N) = -\frac{d}{dN}G_{\mathcal{D}}(\lambda, N) + \mathcal{O}\left(\frac{1}{N^2}\right). \quad (22)$$

A useful way to evaluate both sets of integrals in the large N limit is to map them into simpler integrals, where the dependence on N has been moved from the integrand to the upper limit of integration. This technique is well known [2, 20], and we extend it here to $1/N$ accuracy. Let the generating function of cutoff integrals be

$$G_L(\lambda, N) \equiv \int_0^{1-1/N} dz (1-z)^{\lambda-1} = \frac{1-N^{-\lambda}}{\lambda}. \quad (23)$$

It is then easy to relate this function to the functions $G_{\mathcal{D}}$ and $G_{\mathcal{J}}$. Expanding Eq. (19) in powers of λ one finds

$$G_{\mathcal{D}}(\lambda, N) = -G_L(\lambda, N) + \sum_{k=1}^{\infty} \frac{\Gamma_k(N)}{k!} \lambda^{k-1} \frac{1}{N^\lambda}, \quad (24)$$

where

$$\Gamma_k(N) = \frac{d^k}{d\lambda^k} \left[\Gamma(1+\lambda) \left(1 + \frac{\lambda(1-\lambda)}{2N} \right) \right]_{\lambda=0}. \quad (25)$$

This can be rewritten as

$$G_{\mathcal{D}}(\lambda, N) = \sum_{k=0}^{\infty} \frac{\Gamma_k(N)}{k!} (-1)^{k-1} \frac{\partial^k}{\partial(\ln N)^k} G_L(\lambda, N). \quad (26)$$

Using Eq. (22) one then immediately finds

$$G_{\mathcal{J}}(\lambda, N) = \frac{1}{N} \sum_{k=0}^{\infty} \frac{\Gamma_k(N)}{k!} (-1)^k \frac{\partial^{k+1}}{\partial(\ln N)^{k+1}} G_L(\lambda, N). \quad (27)$$

Eqs. (26) and (27) can be used to evaluate directly the \mathcal{D} and \mathcal{J} integrals to the desired accuracy, and indeed we will make use of this explicit evaluation in Section 3. One finds

$$\begin{aligned} \mathcal{D}_p &= \frac{1}{p+1} \sum_{k=0}^{p+1} \Gamma_k(N) \binom{p+1}{k} (-\ln N)^{p+1-k} + \mathcal{O}\left(\frac{\ln^m N}{N^2}\right), \\ \mathcal{J}_p &= \frac{1}{N} \sum_{k=0}^p \Gamma^{(k)}(1) \binom{p}{k} (-\ln N)^{p-k} + \mathcal{O}\left(\frac{\ln^m N}{N^2}\right), \end{aligned} \quad (28)$$

where $\Gamma^{(k)}$ is the k 'th derivative of the Euler gamma function. On the other hand, one can use Eqs. (26) and (27) to directly relate the logarithm of the cross section to a cutoff integral of the same functions f_1 and f_2 appearing in Eq. (11). This is

useful when one needs to correctly account for running coupling effects to all orders, as done in [2, 20]. To the present accuracy one can write

$$\begin{aligned} \ln [\hat{\sigma}(N)] - H &= \sum_{k=0}^{\infty} \frac{\Gamma_k(N)}{k!} (-1)^{k-1} \frac{\partial^k}{\partial(\ln N)^k} \int_0^{1-1/N} dz \frac{f_1[\ln(1-z)]}{1-z} \\ &- \frac{1}{N} \sum_{k=0}^{\infty} \frac{\Gamma^{(k)}(1)}{k!} (-1)^{k-1} \frac{\partial^{k+1}}{\partial(\ln N)^{k+1}} \int_0^{1-1/N} dz \frac{f_2[\ln(1-z)]}{1-z}. \end{aligned} \quad (29)$$

We now move on to applying these tools to the concrete example of threshold resummation for the Drell-Yan and DIS cross sections.

3 An ansatz for next-to-eikonal logarithms

In order to include NE effects in threshold resummation formulas we propose to modify the exponents in three ways. First of all, following DMS, we include sub-leading corrections in the argument of the running coupling. Second, we change the boundary of phase space accordingly. Third, and most relevant, we interpret the leading-logarithm function $A(\alpha_s)$ as arising from collinear evolution, and thus replace it with a NE generalization dictated by the DMS equation. This is done in the following way. While Eq. (5) cannot be diagonalized by means of a simple Mellin transform, it is however possible, as pointed out by DMS, to map the kernel $\mathcal{P}(z, \alpha_s)$ in Eq. (6) back to the conventional evolution kernel, order by order in perturbation theory, if one explicitly performs the shifts in the arguments of Eq. (5) by the action of differential operators. Indeed, one may rewrite Eq. (5) as

$$\mu^2 \frac{\partial}{\partial \mu^2} \psi(x, \mu^2) = \int_x^1 \frac{dz}{z} e^{-\ln z \left(\beta(\alpha_s) \frac{\partial}{\partial \alpha_s} - \sigma \frac{\partial}{\partial \ln \mu^2} \right)} \psi\left(\frac{x}{z}, \mu^2\right) \mathcal{P}(z, \alpha_s(\mu^2)), \quad (30)$$

where one should note that dependence on the coupling is only through the kernel \mathcal{P} , while explicit scale dependence arises only in the distribution ψ . Expanding the exponential and the kernel \mathcal{P} in perturbation theory one is led to an equation which can be diagonalized order by order. When solved in this way, the DMS equation can be understood as a framework to generate classes of higher-order contributions to collinear anomalous dimensions using low-order information. In this spirit, we will write conventional resummation formulas, but we will generalize the collinear evolution function $A(\alpha_s)$ by including all terms that are generated by the DMS equation. As we will see, this will lead to slightly different implementations for space-like and time-like kinematics. Let us now consider our two examples in turn.

3.1 The Drell-Yan cross section

We first consider the Drell-Yan hard partonic cross section in the $\overline{\text{MS}}$ factorization scheme, denoted $\hat{\omega}(N)$. We propose to generalize the exponentiation of threshold

corrections in the following way.

$$\begin{aligned} \ln \left[\widehat{\omega}(N) \right] &= \mathcal{F}_{\text{DY}}(\alpha_s(Q^2)) + \int_0^1 dz z^{N-1} \left\{ \frac{1}{1-z} D \left[\alpha_s \left(\frac{(1-z)^2 Q^2}{z} \right) \right] \right. \\ &\quad \left. + 2 \int_{Q^2}^{(1-z)^2 Q^2/z} \frac{dq^2}{q^2} P_s \left[z, \alpha_s(q^2) \right] \right\}_+ , \end{aligned} \quad (31)$$

where for simplicity we have set the factorization scale $\mu_F^2 = Q^2$. Here and below we adopt the convention that the ‘plus’ prescription applies only to singular terms in the expansion of the relevant functions in powers of $1-z$. In other words, for a singular function $f(z)$ with Laurent expansion $f(z) = \sum_{n=-1}^{\infty} f_n (1-z)^n$, and for any smooth function $g(z)$, regular as $z \rightarrow 1$, we define

$$\int_0^1 dz g(z) \left[f(z) \right]_+ \equiv f_{-1} \int_0^1 dz \frac{g(z) - g(1)}{1-z} + \int_0^1 dz g(z) \left(f(z) - \frac{f_{-1}}{1-z} \right) . \quad (32)$$

In Eq. (31), $\mathcal{F}_{\text{DY}}(\alpha_s)$ is responsible for the exponentiation of N -independent terms, in accordance with [4]. It comprises purely virtual contributions given in terms the quark form factor, and real emission terms, which were denoted by $F_{\overline{\text{MS}}}(\alpha_s)$ in [4]. The single-logarithm function $D(\alpha_s)$ can also be related to form factor data, and to the virtual part of the collinear evolution kernel $B_\delta(\alpha_s)$, as was done in [24], according to

$$D(\alpha_s) = 4 B_\delta(\alpha_s) - 2 \widetilde{G}(\alpha_s) + \beta(\alpha_s) \frac{d}{d\alpha_s} F_{\overline{\text{MS}}}(\alpha_s) , \quad (33)$$

where \widetilde{G} is constructed from single pole contributions to the quark form factor, as described in [4]. Finally, the DMS-improved space-like collinear evolution kernel $P_s(z, \alpha_s)$ is given in perturbation theory by $P_s(z, \alpha_s) = \sum_{n=1}^{\infty} P_s^{(n)}(z) (\alpha_s/\pi)^n$, where

$$P_s^{(n)}(z) = \frac{z}{1-z} A^{(n)} + C_\gamma^{(n)} \ln(1-z) + \overline{D}_\gamma^{(n)} . \quad (34)$$

Here $A^{(n)}$ and $C_\gamma^{(n)}$ are the perturbative coefficients of the functions appearing in Eq. (4), while $\overline{D}_\gamma^{(n)}$ is related to the perturbative coefficients of $D_\gamma(\alpha_s)$ by the simple shift $\overline{D}_\gamma^{(n)} = D_\gamma^{(n)} + A^{(n)}$; this takes into account the explicit factor of z multiplying $A(\alpha_s)$ in Eq. (34), which in turn is responsible for the inclusion of NE terms in the ordinary evolution kernel. In our normalization, $A^{(1)} = C_F$, $C_\gamma^{(1)} = \overline{D}_\gamma^{(1)} = 0$, while at two loops

$$\begin{aligned} A^{(2)} &= \frac{1}{2} \left[\left(\frac{67}{18} - \zeta(2) \right) C_A C_F - \frac{5}{9} n_f C_F \right] , & C_\gamma^{(2)} &= C_F^2 , \\ \overline{D}^{(2)} &= \frac{3}{4} C_F^2 - \frac{11}{12} C_A C_F + \frac{1}{6} n_f C_F . \end{aligned} \quad (35)$$

Notice in particular that the DMS procedure has brought into the resummation exponent abelian-like terms proportional to C_F^2 at two loops. As we will see, these terms do indeed find a match in the finite order expansion of $\widehat{\omega}(N)$. The ansatz (31) can be written in form of Eq. (11), and evaluated using the methods of Section 2. In Section 4, we will compare the perturbative expansion of Eq. (31), with the coefficients given in Eq. (35), to the exact results of [17]. In both cases, one may write the expansion

$$\widehat{\omega}(N) = \sum_{i=0}^{\infty} \left(\frac{\alpha_s}{\pi}\right)^n \left[\sum_{m=0}^{2n} a_{nm} \ln^m \bar{N} + \sum_{m=0}^{2n-1} b_{nm} \frac{\ln^m \bar{N}}{N} \right] + \mathcal{O}\left(\frac{\ln^p N}{N^2}\right), \quad (36)$$

and then compare the expressions for the coefficients a_{nm} and b_{nm} arising from the resummation to the exact ones.

3.2 DIS structure functions

We consider next the resummation for the DIS structure function $\widehat{F}_2(N)$, in the $\overline{\text{MS}}$ factorization scheme. Phase space and kinematics in this case are somewhat more complicated, since one has to deal with the final state jet, which is approximately massless near threshold, as well as with initial state soft and collinear radiation. We propose to generalize the conventional resummation formula as

$$\begin{aligned} \ln \left[\widehat{F}_2(N) \right] &= \mathcal{F}_{\text{DIS}}(\alpha_s(Q^2)) + \int_0^1 dz z^{N-1} \left\{ \frac{1}{1-z} B \left[\alpha_s \left(\frac{(1-z)Q^2}{z} \right) \right] \right. \\ &+ \left. \int_{Q^2}^{(1-z)Q^2/z} \frac{dq^2}{q^2} P_s[z, \alpha_s(q^2)] + \int_{(1-z)^2 Q^2/z}^{(1-z)Q^2/z} \frac{dq^2}{q^2} \delta P[z, \alpha_s(q^2)] \right\}. \quad (37) \end{aligned}$$

Here, as above, $\mathcal{F}_{\text{DIS}}(\alpha_s)$ is responsible for the exponentiation of N -independent terms. The case of the DIS cross section in the $\overline{\text{MS}}$ factorization scheme was not explicitly treated in Ref. [4], but it is easy to work out the relevant contributions from the information collected there. Indeed, one can reconstruct the structure function $\widehat{F}_2(N)$ from the moment space ratio of the Drell-Yan cross section computed in the $\overline{\text{MS}}$ scheme to that computed in the DIS scheme, both given in [4], as $\widehat{F}_2^{(\overline{\text{MS}})}(N) = \sqrt{\widehat{\omega}^{(\overline{\text{MS}})}(N)/\widehat{\omega}^{(\text{DIS})}(N)}$. One then easily verifies that $\mathcal{F}_{\text{DIS}}(\alpha_s)$ comprises a virtual part, given by the finite terms in the modulus squared of the space-like quark form factor, plus a combination of real emission contributions, which can be written as $(F_{\overline{\text{MS}}}(\alpha_s) - F_{\text{DIS}}(\alpha_s))/2$ in the notation of [4]. The single-logarithm function $B(\alpha_s)$ can be associated with the evolution of the final state jet. It is interesting to note here that $B(\alpha_s)$ can also be expressed in terms of form factor data, plus virtual corrections to the collinear evolution kernel, plus a total derivative of lower order contributions, just like the function $D(\alpha_s)$ in Eq. (33). Indeed, one verifies that

existing results up to three loops are consistent with

$$B(\alpha_s) = B_\delta(\alpha_s) - \tilde{G}(\alpha_s) + \beta(\alpha_s) \frac{d}{d\alpha_s} F_B(\alpha_s) , \quad (38)$$

with easily computed perturbative coefficients for the function $F_B(\alpha_s)$. Eq. (38) is in keeping with the general results of Ref. [25], where it was shown, at the amplitude level, that all IR and collinear singularities in massless gauge theories can be constructed from combinations of eikonal functions with the virtual collinear function $B_\delta(\alpha_s)$, up to total derivatives with respect to the scale. Finally, we turn to the second line of Eq. (37). There, we have used the fact that the integration over the scale q^2 has a range that can be split into two intervals, which correspond to different physical sources of radiation. Scales between the factorization scale Q^2 and the soft scale $(1-z)^2 Q^2$ correspond to Drell-Yan-like initial state radiation, while scales between the soft scale and the jet scale, $(1-z)Q^2$, correspond to the evolution of the final state jet. Accordingly, in the first range we use the same space-like evolution kernel $P_s(z, \alpha_s)$ that was employed in Eq. (31), while in the second range we use the time-like fragmentation kernel $P_t(z, \alpha_s)$. One may then define $\delta P(z, \alpha_s) \equiv P_t(z, \alpha_s) - P_s(z, \alpha_s)$, and thus get to Eq. (37). The function $\delta P(z, \alpha_s)$ begins at two loops, where it is given by [26]

$$\delta P^{(2)}(z) = -\frac{1}{2} C_F^2 (4 \ln(1-z) + 3) + \mathcal{O}(1-z) . \quad (39)$$

Once again, using the methods of Section 2, we can expand both the resummed and the exact results for $\widehat{F}_2(N)$ in powers of logarithms of \bar{N} , and in inverse powers of N , as

$$\widehat{F}_2(N) = \sum_{i=0}^{\infty} \left(\frac{\alpha_s}{\pi} \right)^n \left[\sum_{m=0}^{2n} c_{nm} \ln^m \bar{N} + \sum_{m=0}^{2n-1} d_{nm} \frac{\ln^m \bar{N}}{N} \right] + \mathcal{O} \left(\frac{\ln^p N}{N^2} \right) . \quad (40)$$

We can then compare the resummed and exact values of the coefficients c_{nm} and d_{nm} , up to two and three loops, using the results of [18, 19].

4 Discussion

We begin by checking the behavior of our ansatz at the one loop level. This is not trivial, since we have not added new coefficients in the exponent at one loop, and the only sources of $1/N$ terms are the expansions of the \mathcal{D}_p integrals, and the simple modifications of phase space. Using the one loop results for the functions $A(\alpha_s)$ and $D(\alpha_s)$, we find that for the Drell-Yan cross section the one-loop exact result is recovered, including all corrections down to $\mathcal{O}(1/N)$. Specifically, expanding Eq. (31), we find $b_{11} = 2C_F$ and $b_{10} = 0$, which is exact. Note that b_{10} vanishes as a consequence of a cancellation between subleading terms in the expansion of the \mathcal{D}_p

integrals and the modified phase space boundary. For DIS, including the one-loop value of the function $B(\alpha_s)$, we find that $d_{11} = C_F/2$ is correctly reproduced, while the non-logarithmic term at $\mathcal{O}(1/N)$ is underestimated: Eq. (37) yields $d_{10} = C_F/8$, while the exact result is $d_{10} = 21/8 C_F$. We take this as evidence (to be reinforced below) that our treatment of phase space for the final state jet is sufficiently precise to reproduce single NE logarithms, but not enough to fix NE constants (of course at this level non-factorizing effects for the observable, leading to a failure of exponentiation, at least in the form of Eq. (37), may also be a source of the discrepancy).

	C_F^2		$C_A C_F$		$n_f C_F$	
b_{23}	4	4	0	0	0	0
b_{22}	$\frac{7}{2}$	4	$\frac{11}{6}$	$\frac{11}{6}$	$-\frac{1}{3}$	$-\frac{1}{3}$
b_{21}	$8\zeta_2 - \frac{43}{4}$	$8\zeta_2 - 11$	$-\zeta_2 + \frac{239}{36}$	$-\zeta_2 + \frac{133}{18}$	$-\frac{11}{9}$	$-\frac{11}{9}$
b_{20}	$-\frac{1}{2}\zeta_2 - \frac{3}{4}$	$4\zeta_2$	$-\frac{7}{4}\zeta_3 + \frac{275}{216}$	$\frac{7}{4}\zeta_3 + \frac{11}{3}\zeta_2 - \frac{101}{54}$	$-\frac{19}{27}$	$-\frac{2}{3}\zeta_2 + \frac{7}{27}$

Table 1: Comparison of exact and resummed 2-loop coefficients for the Drell-Yan cross section. For each color structure, the left column contains the exact results, the right column contains the prediction from resummation.

At the two-loop level, we proceed as follows. Since our aim is to verify our ability to reproduce NE terms, suppressed by a power of N , we include in the exponent all terms that are required to reproduce ordinary Sudakov logarithms, *i.e.* the two-loop values of the functions $A(\alpha_s)$ and $D(\alpha_s)$ for the Drell-Yan cross section, and of the function $B(\alpha_s)$ for DIS. We include the two-loop DMS-induced contributions $C_\gamma^{(2)}$, $\overline{D}_\gamma^{(2)}$ and $\delta P^{(2)}(z)$ as well, since they are responsible for effects that originate at two loops, and can only be reproduced by their inclusion. Our results are summarized in Tables 1 (for the Drell-Yan cross section) and in Table 2 (for the DIS structure function).

We observe the following.

- The leading non-vanishing NE logarithms ($\ln^3 \bar{N}/N$ for the ‘abelian’ terms proportional to C_F^2 , and $\ln^2 \bar{N}/N$ for non-abelian terms) are correctly reproduced by the exponentiation, both for DY and for DIS, and separately for each color structure.
- Next-to-leading NE logarithms ($\ln^2 \bar{N}/N$ for terms proportional to C_F^2 , and $\ln \bar{N}/N$ for non-abelian terms) are reproduced with remarkable accuracy for the Drell-Yan process (in fact exactly for the $n_f C_F$ color structure), and reasonably well for the DIS process.
- The remaining NE logarithms, *i.e.* single logarithmic terms proportional to C_F^2 , are well reproduced by exponentiation for the Drell-Yan process, but only

roughly approximated for DIS. Non-logarithmic NE corrections are not well approximated by the exponentiation.

- More specifically, we note that for the Drell-Yan process the only source of terms proportional to $C_F^2 \ln^2 \bar{N}/N$ is the DMS-induced coefficient $C_\gamma^{(2)}$; indeed, the fact that $b_{10} = 0$ ensures that no such term can arise from the square of the one-loop contribution. This contribution, yielding $b_{22} = 4$, is an excellent approximation to the exact result, $b_{22} = 7/2$. For DIS, as might be expected, the situation is somewhat more intricate; indeed d_{22} receives contributions from three sources: the square of the one-loop exponent, $C_\gamma^{(2)}$, and $\delta P^{(2)}(z)$; also here, however, the final result, $d_{22} = 55/16$, is a fair approximation of the exact answer, $d_{22} = 39/16$.

	C_F^2		$C_A C_F$		$n_f C_F$	
d_{23}	$\frac{1}{4}$	$\frac{1}{4}$	0	0	0	0
d_{22}	$\frac{39}{16}$	$\frac{55}{16}$	$\frac{11}{48}$	$\frac{11}{48}$	$-\frac{1}{24}$	$-\frac{1}{24}$
d_{21}	$\frac{7}{4}\zeta_2 - \frac{49}{32}$	$-\frac{1}{4}\zeta_2 - \frac{105}{32}$	$-\frac{5}{4}\zeta_2 + \frac{1333}{288}$	$-\frac{1}{4}\zeta_2 + \frac{565}{288}$	$-\frac{107}{144}$	$-\frac{47}{144}$
d_{20}	$\frac{15}{4}\zeta_3 - \frac{47}{16}\zeta_2$ $-\frac{431}{64}$	$-\frac{3}{4}\zeta_3 + \frac{53}{16}\zeta_2$ $-\frac{21}{64}$	$-\frac{11}{4}\zeta_3 + \frac{13}{48}\zeta_2$ $-\frac{17579}{1728}$	$\frac{5}{4}\zeta_3 + \frac{7}{16}\zeta_2$ $-\frac{953}{1728}$	$\frac{1}{24}\zeta_2 - \frac{1699}{864}$	$-\frac{1}{8}\zeta_2 + \frac{73}{864}$

Table 2: Comparison of exact and resummed 2-loop coefficients for the DIS structure function. For each color structure, the left column contains the exact results, the right column contains the prediction from resummation.

Clearly, since some of the DMS modifications enter the stage at two-loops, our results verify that these contributions improve the approximation, but do not really test exponentiation. We can put at least our DIS ansatz to a more stringent test by comparing to the complete three-loop calculation performed by Moch, Vermaseren and Vogt [19]. In this case, since our aim is to test exponentiation at NE level, we have included the three-loop value of the function $B(\alpha_s)$, contributing to single Sudakov logarithms, but we have not included three-loop DMS-induced contributions such as $C_\gamma^{(3)}$ and $\delta P^{(3)}(z)$. We can then expect reasonable agreement only for a limited set of NE logarithms. Since at three loops one finds six independent color structures, up to five powers of NE logarithms, and transcendentals up to ζ_5 , we do not include here the lengthy tables of coefficients, but we give the most relevant results.

The three-loop analysis confirms that leading non-vanishing NE logarithms (in this case $\ln^5 \bar{N}/N$ for the color structure C_F^3 , $\ln^4 \bar{N}/N$ for the color structures $C_A C_F^2$ and $n_f C_F^2$, and $\ln^3 \bar{N}/N$ for the color structures $C_A^2 C_F$, $n_f^2 C_F$ and $n_f C_A C_F$) are

exactly reproduced by our resummation ansatz. Next-to-leading NE logarithms are reasonably well reproduced: specifically, for all color structures and separately for each degree of transcendentality the approximate results from the resummation have the same sign and similar numerical values to the corresponding exact results. In particular, this applies to the coefficient d_{34} , whose exact value is $57/64$, while the approximate result is $109/64$. Since d_{34} arises in part from interference between the NE coefficient $C_\gamma^{(2)}$ and the leading one-loop Sudakov logarithms in the exponent, we take this as mild evidence in favor of the exponentiation of DMS-induced corrections.

To summarize, we have provided an ansatz to include in threshold resummation a set of next-to-eikonal corrections, allowing for subleading phase-space effects, and including the modified collinear evolution proposed by Dokshitzer, Marchesini and Salam. It is understood that these modifications of conventional threshold resummation do not exhaust all possible sources of NE threshold logarithms, and indeed it may be expected that some such corrections might break Sudakov factorization and fail to exponentiate. By comparing our ansatz to finite order perturbative results for the Drell-Yan and DIS cross sections, up to three loops, we have however provided evidence that at least the leading non-vanishing NE logarithms do indeed exponentiate according to our proposal. We have furthermore provided evidence that the DMS equation induces a definite improvement for resummation at NE level: for example, abelian-like next-to-leading NE terms that conventional resummation completely fails to generate are accurately approximated when DMS evolution is implemented. In general, it is clear that our ansatz gives better results for the Drell-Yan process, presumably thanks to its simple phase space and kinematics. The presence of the final state jet in DIS, and the related constraints on phase space, may require a more detailed factorization analysis in order to collect all sources of NE terms, and indeed may well induce a breakdown of simple Sudakov factorization at NE level. To aid this preliminary exploration of NE exponentiation, we have provided here some practical tools that will be useful in future extensions of this work, and we have taken the opportunity to note a connection, given in Eq. (38), between the jet function $B(\alpha_s)$ and the virtual collinear function $B_\delta(\alpha_s)$, as was previously done for the soft function $D(\alpha_s)$ in the Drell-Yan cross section [24]. We believe that this work provides further motivation both to include leading NE correction in phenomenological resummation studies, and to pursue the corresponding theoretical work. Indeed, a full understanding of NE threshold logarithms must await a thorough analysis of soft gluon radiation beyond the eikonal approximation in the non-abelian theory, and specifically an adequate implementation of Low's theorem, mapping its boundaries of applicability in the case of massless QCD.

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