

The Role of Conformal Symmetry in the Jackiw-Pi Model

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Abstract

The Jackiw-Pi model in $2 + 1$ dimensions is a non-relativistic conformal field theory of charged particles with point-like self-interaction. For specific values of the interaction strengths the classical theory possesses vortex and multi-vortex solutions, which are all degenerate in energy. We compute the full set of first-order perturbative quantum corrections. Only the coupling constant g^2 requires renormalization; the fields and electric charge e are not renormalized. It is shown that in general the conformal symmetries are broken by an anomalous contribution to the conservation law, proportional to the β -function. However, the β -function vanishes upon restricting the coupling constants to values $g^2 = \pm e^2$, which includes the case in which vortex solutions exist. Therefore the existence of vortices also guarantees the preservation of the conformal symmetries.

1 Introduction

In this article the role of the conformal symmetries in the Jackiw-Pi model is studied at quantum level in 1-loop approximation. The Jackiw-Pi model, first described in [9–11], can be seen as an extension of the non-linear Schrödinger model with $U(1)$ Chern-Simons gauge fields. A similar study for the non-linear Schrödinger model was performed in [3].

Like the non-linear Schrödinger model, a variation on the Jackiw-Pi model has been used to study the physics of gasses of bosonic particles [1]. The Jackiw-Pi model has also been considered in the context of Aharonov-Bohm scattering [2], which essentially is the scattering of charged particles in the plane by a magnetic flux tube. In reference [2] the renormalization of coupling constant g^2 is studied. In this article the full renormalization at 1-loop level is presented, and at this order of perturbation theory only g^2 is seen to renormalize: the matter and gauge fields, mass and electric charge e are found not to renormalize.

The Jackiw-Pi model is a variation on the Abelian Higgs-model, with the Maxwell kinetic term replaced by a Chern-Simons term and with the matter Lagrangian taken to be non-relativistic. In the Abelian Higgs-model vortices arise as topological defects. For a specific choice of the coupling constants e and g^2 , i.e. $g^2 = -e^2$, stationary self-dual vortices arise in the Jackiw-Pi model as well [4, 7, 9–11].

Like the non-linear Schrödinger model, the classical Jackiw-Pi model possesses, next to translation, rotation, Galilei and gauge invariance, also scale invariance and special conformal invariance. Due to the 1-loop renormalization effects the scale and special conformal symmetries of the Jackiw-Pi model are seen to be broken precisely as in the non-linear Schrödinger model [3]. However, this breaking does not occur when the renormalized coupling constants e and g^2 are fixed to the value $g^2 = \pm e^2$. In particular, in the case of the minus sign both the conformal symmetries and the vortex solutions are seen to survive quantization.

This paper is structured as follows. Sect. 2 contains a short introduction to the Jackiw-Pi model and its vortex solutions. Sect. 3 reviews the symmetries of the model. Sect. 4 describes the quantization, whilst the 1-loop corrections are calculated in sect. 5. In sect. 6 the scale dependence of the coupling constants is computed, and sect. 7 contains a summary and our conclusions.

2 Introduction to the Jackiw-Pi Model

The Jackiw-Pi model combines a Chern-Simons kinetic term for $U(1)$ gauge fields with the action of the non-linear Schrödinger model for a complex matter field Ψ in 2-dimensional Euclidean space [4, 7, 9–11]. The full action is given by

$$S = \int dt d^2\mathbf{x} \frac{\sigma}{2} \epsilon^{\alpha\beta\gamma} A_\alpha \partial_\beta A_\gamma + i\Psi^* D_t \Psi - \frac{1}{2} |\mathbf{D}\Psi|^2 - \frac{g^2}{2} (\Psi^* \Psi)^2, \quad (1)$$

where the mass m of the matter field has been scaled away by rescaling the time variable by $t/m \rightarrow t$ [3]. Furthermore, $D_\alpha = \partial_\alpha - ieA_\alpha$ is the gauge covariant derivative; the indices in the Chern-Simons term run from 0 to 2 ($\epsilon^{012} = 1$), 0 denoting the temporal component of the gauge field and 1 and 2 its spatial components. The constant σ equals $+1$ or -1 , controlling the relative sign of the gauge kinetic term with respect to the other terms.

The equations of motion of this model are given by

$$-i\partial_t \Psi = \frac{1}{2} \mathbf{D}^2 \Psi + eA_0 \Psi - g^2 (\Psi^* \Psi) \Psi, \quad (2)$$

$$\sigma(\nabla_1 A_2 - \nabla_2 A_1) + e\Psi^* \Psi = 0, \quad (3)$$

$$\sigma(\partial_t A_i - \nabla_i A_0) - \frac{ie}{2} \epsilon_{ij} (\Psi^* D_j \Psi - (D_j \Psi)^* \Psi) = 0. \quad (4)$$

The equation of motion (2) is a gauged Schrödinger equation for the matter field. When the usual definitions for the magnetic field B , electric field E_i , charge density ρ and charge current J_i in two spatial dimensions are used,

$$B \equiv \nabla_1 A_2 - \nabla_2 A_1, \quad E_i \equiv \partial_t A_i - \nabla_i A_0, \quad (5)$$

$$\rho \equiv \Psi^* \Psi, \quad J_i \equiv \frac{1}{2i} (\Psi^* D_i \Psi - (D_i \Psi)^* \Psi). \quad (6)$$

the other two equations of motion (3) and (4) can be rewritten as

$$B = -\sigma e \rho, \quad E_i = -\sigma e \epsilon_{ij} J_j. \quad (7)$$

These equations relate the magnetic field B to the matter or charge density ρ via the so-called Gauss-Chern-Simons equation, and the electric field to the gauged matter or charge current J_i . The magnetic field is proportional to the matter density, and the electric field is perpendicular to the current, both with the factor $-\sigma e$.

Next to the equations of motion that follow from the action there is also the Bianchi identity for the electromagnetic field. Normally in four dimensions this identity leads to the two source-free Maxwell equations, however in $2 + 1$ dimensions they reduce to a single equation:

$$\partial_t B = \epsilon^{ij} \nabla_i E_j, \quad (8)$$

which in light of the equations above is nothing but the current conservation law:

$$\partial_t \rho + \nabla \cdot \mathbf{J} = 0. \quad (9)$$

It is well-known that under appropriate conditions the field equations (2)-(4) admit stationary vortex-type solutions [4, 7, 9–11], such that $\partial_t \rho = 0$. These solutions have strictly zero energy, as we show below. Under these circumstances the conditions for the existence of solutions can be derived from a Bogomol'nyi-type of argument.

Indeed, defining $D_{\pm} = (D_1 \pm iD_2)$, the hamiltonian of the Jackiw-Pi Model can be written in the form

$$\begin{aligned} H &= \int d^2 \mathbf{x} \frac{1}{2} |\mathbf{D}\Psi|^2 + \frac{g^2}{2} (\Psi^* \Psi)^2 \\ &= \int d^2 \mathbf{x} \frac{1}{2} |D_{\pm} \Psi|^2 \pm \frac{eB}{2} \Psi^* \Psi + \frac{g^2}{2} (\Psi^* \Psi)^2. \end{aligned} \quad (10)$$

Using the equation of motion (3), the energy of any classical solution is then given by

$$E = \int d^2 \mathbf{x} \frac{1}{2} |D_{\pm} \Psi|^2 + \frac{1}{2} (g^2 \mp \sigma e^2) (\Psi^* \Psi)^2. \quad (11)$$

Therefore the absolute minimum $E = 0$ of the energy is reached with $\Psi \neq 0$ when

$$D_{\pm} \Psi = 0, \quad g^2 \mp \sigma e^2 = 0. \quad (12)$$

Under these restrictions, solving the equations (2)-(4) reduces to solving the Liouville equation

$$(\nabla_1^2 + \nabla_2^2) \log \rho = \pm \sigma 2e^2 \rho. \quad (13)$$

Solutions exist for a specific choice of sign, depending on σ : $\pm\sigma = -1$. As a consequence it follows that

$$g^2 = -e^2 < 0. \quad (14)$$

This relation is physically sensible: it states that repulsive electromagnetic interactions are balanced by attractive contact interactions.

Explicitly, the solutions are given by

$$\rho = \frac{4}{e^2} \frac{|\partial_z f|^2}{(1 + |f|^2)^2}, \quad (15)$$

where f is a function of $z = x + iy$ only, allowing isolated poles (a meromorphic function). It is known that physically interesting solutions are given by f being equal to a ratio of polynomials [7, 8]. These solutions show vortex behaviour characterized by circulating currents \mathbf{J} .

3 The Symmetries of the Jackiw-Pi Model

The Jackiw-Pi Model has a rich set of symmetries. Next to being gauge invariant, it is invariant under exactly the same group of space-time symmetries as the non-linear Schrödinger model, the Schrödinger group, [3, 6, 9–12]. This group of transformations comprises space- and time-translations, rotations, Galilei transformations, scale transformations or dilatations and special conformal transformations. The generators of Galilean boosts \mathbf{G} , conformal scaling D and special conformal transformations K are respectively given by

$$\mathbf{G} = t\mathbf{P} + \int d^2\mathbf{x} \mathbf{x} |\Psi|^2; \quad (16)$$

$$D = 2tH + \int d^2\mathbf{x} x_i \left[\frac{i}{2} \Psi^* \overleftrightarrow{\nabla}_i \Psi + \frac{\sigma}{2} A_2 \overleftrightarrow{\nabla}_i A_1 \right]; \quad (17)$$

$$K = t^2 H - tD - \frac{1}{2} \int d^2\mathbf{x} \mathbf{x}^2 |\Psi|^2; \quad (18)$$

Noether's theorem then implies the conservation of the conformal charges:

$$\frac{dK}{dt} = -t\frac{dD}{dt}; \quad \frac{dD}{dt} = 0. \quad (19)$$

As in the case of the non-linear Schrödinger model these conservation laws imply relations between the field-dependent integrals

$$\begin{aligned} \mathbf{I} &\equiv \int d^d x \mathbf{x} \Psi^* \Psi = \mathbf{G} - t\mathbf{P}, \\ I_1 &\equiv \frac{1}{2} \int d^d x \mathbf{x}^2 \Psi^* \Psi = t^2 H - tD - K, \\ I_2 &\equiv \frac{i}{2} \int d^d x \mathbf{x} \cdot \Psi^* \overleftrightarrow{\mathbf{D}} \Psi = D - 2tH. \end{aligned} \quad (20)$$

Applying the conservation laws (19) and the conservation of the hamiltonian, it follows that [3, 5]

$$\frac{d\mathbf{I}}{dt} = -\mathbf{P}, \quad \frac{dI_1}{dt} = -I_2, \quad \frac{dI_2}{dt} = -2H. \quad (21)$$

As a result, the conformal invariance of the Jackiw-Pi model implies that any *stationary* solutions of the theory – with ρ time-independent, and for which \mathbf{I} , I_1 and I_2 are necessarily constant in time themselves – necessarily have zero energy and momentum. This is the basis for the derivation of the classical solutions presented above.

4 Quantization

In this section the Jackiw-Pi model is quantized using Feynman's path integral formulation.

To be able to find a suitable propagator for the gauge fields, the gauge is fixed via the Faddeev-Popov method. Following [2] the Coulomb gauge $\nabla_i A_i = 0$ is imposed by adding to the action a term

$$\Delta S_{GaugeFix} = \frac{1}{\xi} \int dt d^d \mathbf{x} (\nabla_i A_i)^2. \quad (22)$$

In the presence of this term the inverse of the kinetic term of the gauge fields in momentum space, following from action (1) with the additional gauge fixing term, is given by

$$\frac{\sigma}{\mathbf{k}^4} \begin{pmatrix} \xi k_0^2 & ik_2 \mathbf{k}^2 + \sigma \xi k_1 k_0 & -ik_1 \mathbf{k}^2 + \sigma \xi k_2 k_0 \\ -ik_2 \mathbf{k}^2 + \sigma \xi k_1 k_0 & \sigma \xi k_1^2 & \sigma \xi k_1 k_2 \\ ik_1 \mathbf{k}^2 + \sigma \xi k_2 k_0 & \sigma \xi k_1 k_2 & \sigma \xi k_2^2 \end{pmatrix} \quad (23)$$

Specializing to the Landau gauge $\xi = 0$, the only nonvanishing component of the propagator of the gauge field describes propagation from A_0 to either A_1 or A_2 and vice versa.

In [2] it is explained that there are no real gauge particles in the Jackiw-Pi model since the gauge fields are completely constrained due to the equations of motion (2)-(4). However, they can still be treated dynamically in internal lines. Another sign of this fact is that the propagator of the gauge field becomes independent of k_0 in the Landau gauge, meaning that the propagator in coordinate space is instantaneous in time.

Fixing the gauge using the Faddeev-Popov method introduces additional Grassmannian fields to the action, the *ghost fields*. However in a $U(1)$ gauge theory these fields do not couple to the matter or gauge fields in Coulomb- or Lorentz-type gauges, and can consequently be ignored.

Writing down the Feynman rules in momentum space in agreement with the conventions used in [3], it is seen that each Ψ -propagator comes with

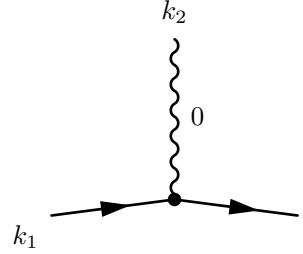
$$\begin{array}{c} \longrightarrow \\ k \end{array} = \frac{1}{(2\pi)^3} \frac{i}{k_0 - \mathbf{k}^2/2 + i\varepsilon}; \quad (24)$$

and each $A_0 A_i$ - or $A_i A_0$ -propagator with

$$\begin{array}{c} 0 \\ \text{~~~~~} \\ \text{~~~~~} \\ k \end{array} \begin{array}{c} i \\ \text{~~~~~} \\ \text{~~~~~} \\ k \end{array} = - \begin{array}{c} i \\ \text{~~~~~} \\ \text{~~~~~} \\ k \end{array} \begin{array}{c} 0 \\ \text{~~~~~} \\ \text{~~~~~} \\ k \end{array} = \frac{\sigma}{(2\pi)^3} \frac{\epsilon^{ij} k_j}{\mathbf{k}^2}. \quad (25)$$

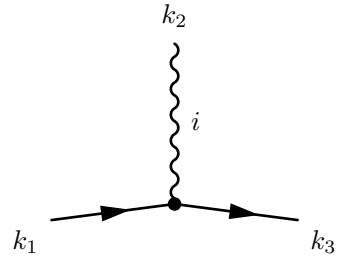
In the Jackiw-Pi model there are two 3-point vertices, one involving the A_0 field and one involving either A_1 or A_2 . The Feynman rule for the first is

given by



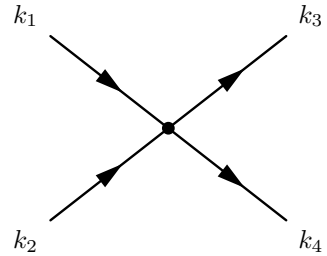
$$= -ie(2\pi)^{d+1}, \quad (26)$$

and for the latter by



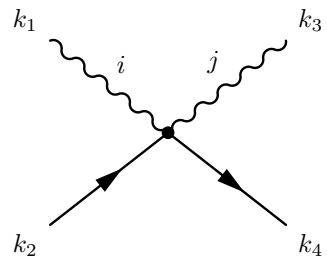
$$= -i\frac{e}{2}(\mathbf{k}_1 + \mathbf{k}_3)(2\pi)^{d+1}. \quad (27)$$

Also there are two 4-point vertices. Next to the one also present in the non-linear Schrödinger model,



$$= i\frac{g^2}{2}(2\pi)^{d+1}, \quad (28)$$

one involving twice the gauge field A_1 or A_2 :



$$= i\frac{e^2}{2}\delta^{ij}(2\pi)^{d+1}. \quad (29)$$

Finally, all vertices naturally come with energy and momentum conservation.

5 Renormalization Effects

Next the perturbative corrections to the tree-level propagators and coupling constants are calculated at the 1-loop level. The procedure is similar to that used in [3]: first the k_0 -integral is performed, and then dimensional regularization is used to perform the remaining \mathbf{k} -integral.

As it turns out, most Feynman diagrams vanish at 1-loop level. This can be traced back to the fact that the propagator of the gauge field does not depend on k_0 and is an odd function of \mathbf{k} , equation (25).

Corrections to the A -propagator

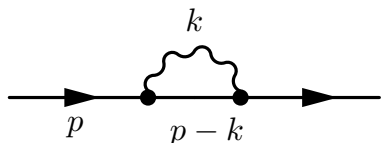
The only diagram that can be written down for the propagator of the gauge field at one loop level is given by


(30)

This *seagull diagram* is known to vanish, as was discussed in [3]. It is thus concluded that the gauge field is not renormalized.

Corrections to the Ψ -propagator

Knowing that the seagull diagram vanishes, the only possible correction to the Ψ propagator comes from the diagram


(31)

where the photon propagator either runs from A_0 to A_i or vice versa. Since these contributions are equal up to a minus sign, the sum of these two diagrams vanishes. However, on their own the diagrams are also seen to vanish. Indeed, each of the diagrams is proportional to

$$\int dk_0 d^2\mathbf{k} \frac{2p_i - k_i}{p_0 - k_0 - \frac{1}{2}(\mathbf{p} - \mathbf{k})^2 + i\varepsilon} \frac{\sigma\epsilon^{ij}k_j}{\mathbf{k}^2}. \quad (32)$$

After performing the k_0 integral, one is left with

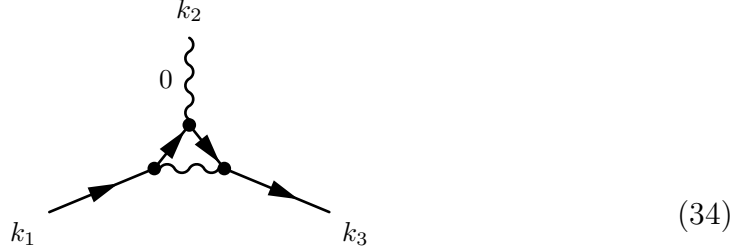
$$\pi i \int d^2 \mathbf{k} \frac{\sigma \epsilon^{ij} (2p_i - k_i) k_j}{\mathbf{k}^2} = 0, \quad (33)$$

which equals zero because $\epsilon^{ij} k_i k_j = 0$ and the oddness in \mathbf{k} of the remaining term.

It follows that the Ψ -field is not renormalized at this order in perturbation theory.

Corrections to the A_0 -3-point vertex

The only possible diagram is given by



where the k_i are the three-momenta ($k_{i,0}, \mathbf{k}_i$) running through the different external propagator lines.

Because the propagator of the gauge field doesn't depend upon k_0 , the integral over k_0 is of the form

$$\int dk_0 \frac{1}{k_0 + \dots - i\varepsilon} \frac{1}{k_0 + \dots - i\varepsilon} = 0. \quad (35)$$

Performing the integral via closing of the integration contour in the complex plane, there are two poles, both in the upper half plane. Closing the contour in the lower half plane, the integral is directly seen to vanish. This means the whole diagram vanishes and that the coupling constant e of the A_0 -3-point vertex, the electric charge, receives no corrections. For consistency this means that also the other 3-point vertex and the 4-point vertex with the gauge fields should receive no corrections, since they are proportional to e and e^2 respectively.

Corrections to the A_i -3-point vertex

Indeed, also the corrections to the 3-vertex involving A_i are seen to vanish. The possible diagrams are given by

The first vanishes due to the k_0 integral in the complex plane, the second and third because after the k_0 integral an integral odd in \mathbf{k} remains.

Corrections to the $A^2\Psi^2$ -4-point vertex

In this case the one loop level diagrams are given by

All three are seen to vanish because the photon propagator does not depend on k_0 .

Thus the complete set of relevant vertices consistently implies that the electric charge e is not renormalized at 1-loop level.

Corrections to the Ψ^4 -vertex

The only renormalization effects in the Jackiw-Pi model at 1-loop level appear when calculating the corrections to the coupling constant g^2 of the Ψ^4 -interaction.

At tree level the contributions to this process are given by

The expression corresponding to this sum of diagrams has been calculated in [2] working in the center of mass frame. Up to the terms that describe the external lines, it is in the present conventions given by

$$A^0(p_i, q_j) = i\frac{g^2}{2} - \frac{\sigma e^2}{2} \cot \theta, \quad (39)$$

where θ is here defined as the scattering angle between the colliding particles in the plane.

At one loop level the four contributing diagrams are given by

The upper right box diagram represents four different diagrams, depending on where the vertices involving A_0 are placed. The corresponding expressions have been examined in [2] and found to be finite. The total result is given by

$$A_{box}^{(1)} = \frac{i\Lambda^{-\epsilon}}{8\pi m} e^4 (\ln |2 \sin \theta| - i\pi). \quad (41)$$

Here we have extracted the dimension of the amplitude in $d = 2 - \epsilon$ dimensions using an arbitrary momentum scale factor Λ .

Also the two triangle diagrams have been studied in [2]. In that paper their divergence is controlled via a cut-off in momentum space. Here they are studied using dimensional regularization as was done in chapter [3].

The bottom left triangle diagram reads

$$A_{triangle,1}^{(1)} = (-ie)^2 \left(\frac{ie^2}{2} \right) \int \frac{dk_0 d^2 \mathbf{k}}{(2\pi)^3} B_{triangle,1} \quad (42)$$

$$B_{triangle,1} = \frac{\epsilon^{in} k_n \epsilon^i (q_2 - p_2 - k)_m}{\mathbf{k}^2 (\mathbf{q}_2 - \mathbf{p}_2 - \mathbf{k})^2} \times \frac{i}{p_{1,0} - k_0 - \frac{1}{2}(\mathbf{p}_1 - \mathbf{k})^2 + i\epsilon} \quad (43)$$

Performing the k_0 integral and writing the integrand into one denominator

using Feynman's trick, leads to

$$A_{triangle,1}^{(1)} = \frac{e^4 i}{4} \int \frac{d^2 \mathbf{k}}{(2\pi)^2} \int_0^1 dx \frac{\mathbf{k}^2 - x(1-x)(\mathbf{p}_2 - \mathbf{q}_2)^2}{[\mathbf{k}^2 + x(1-x)(\mathbf{p}_2 - \mathbf{q}_2)^2]^2}. \quad (44)$$

Using the technique of dimensional regularization, this integral becomes

$$A_{triangle,1}^{(1)} = \frac{e^4 i}{16\pi} \Lambda^{-\epsilon} \left\{ \frac{2}{\epsilon} - \gamma_E - \frac{1}{4\pi} + 2 - \ln \left(\frac{(\mathbf{p}_2 - \mathbf{q}_2)^2}{4\pi\Lambda^2} \right) \right\}, \quad (45)$$

where the momentum scale has been extracted. Similarly, the other triangle diagram leads to

$$A_{triangle,2}^{(1)} = \frac{e^4 i}{16\pi} \Lambda^{-\epsilon} \left\{ \frac{2}{\epsilon} - \gamma_E - \frac{1}{4\pi} + 2 - \ln \left(\frac{(\mathbf{p}_1 - \mathbf{q}_1)^2}{4\pi\Lambda^2} \right) \right\} \quad (46)$$

In the Center of Mass frame where $\mathbf{p}_1 = -\mathbf{p}_2 = \mathbf{p}_{CM}$ and $\mathbf{q}_1 = -\mathbf{q}_2 = \mathbf{q}_{CM}$ the results (45) and (46) are equal and given by

$$A_{triangle,CM}^{(1)} = \frac{e^4 i}{16\pi} \Lambda^{-\epsilon} \left\{ \frac{2}{\epsilon} - \gamma_E - \frac{1}{4\pi} + 2 - \ln \left(\frac{\mathbf{p}_{CM}^2 \sin^2 \theta / 2}{\pi\Lambda^2} \right) \right\}, \quad (47)$$

where (as before) θ is the scattering angle in the plane.

Using the result obtained in [3] for the remaining diagram, which in the Center of Mass frame and in the present conventions is given by

$$A_{NLS}^{(1)} = -\frac{g^4 i}{8\pi} \Lambda^{-\epsilon} \left\{ \frac{2}{\epsilon} - \gamma_E - i\pi - \ln \frac{\mathbf{p}_{CM}^2}{4\pi\Lambda^2} \right\}, \quad (48)$$

the total result at one loop level is given by

$$\begin{aligned} A^{(1)} &= \frac{ig^2}{2} - \frac{\sigma e^2}{2} \cot \theta + \frac{i\Lambda^{-\epsilon}}{8\pi} e^4 \ln |2 \sin \theta| \\ &\quad - \frac{i\Lambda^{-\epsilon}}{8\pi} [g^4 - e^4] \left(\frac{2}{\epsilon} - \gamma_E - i\pi \right) \\ &\quad + \frac{i\Lambda^{-\epsilon}}{8\pi} g^4 \ln \frac{\mathbf{p}_{CM}^2}{4\pi\Lambda^2} - \frac{i\Lambda^{-\epsilon}}{8\pi} e^4 \left[\frac{1}{4\pi} - 2 + \ln \frac{\mathbf{p}_{CM}^2 \sin^2 \theta / 2}{\pi\Lambda^2} \right]. \end{aligned} \quad (49)$$

This expression is the result of a subtle interplay between terms coming from the different diagrams. For example, the $2/\epsilon$ term comes from the triangle diagrams and the 1-loop diagram from the non-linear Schrödinger model, while the $i\pi$ term comes from this latter diagram and the box diagram, yet both $2/\epsilon$ and $i\pi$ come with the same coefficient.

The 1-loop result for A clearly diverges as $\epsilon \rightarrow 0$. This divergence can be repaired by introducing renormalized coupling constants:

$$g^2 \Lambda^{-\epsilon} = g_R^2 + \frac{1}{4\pi} \left(\frac{2}{\epsilon} - \gamma_E \right) (g_R^4 - e_R^4), \quad e^2 \Lambda^{-\epsilon} = e_R^2, \quad (50)$$

where g_R^2 and e_R^2 are dimensionless. In terms of this renormalized coupling constant g_R^2 , $A^{(1)}$ is finite and given by

$$\begin{aligned} \Lambda^{-\epsilon} A^{(1)} &= \frac{ig_R^2}{2} - \frac{\sigma e_R^2}{2} \cot \theta \\ &\quad - \frac{ie_R^4}{8\pi} \left(\frac{1}{4\pi} - 2 + \ln \left| \tan \frac{\theta}{2} \right| \right), \\ &\quad + \frac{i}{8\pi} (g_R^4 - e_R^4) \left[i\pi + \ln \frac{\mathbf{P}_{CM}^2}{4\pi\Lambda^2} \right] \end{aligned} \quad (51)$$

in the limit $\epsilon \rightarrow 0$.

6 The Running Couplings

The 1-loop renormalization (50) leads to a scale dependence of the coupling constants given in the limit $\epsilon \rightarrow 0$ by the β -functions

$$\beta_e(g_R^2, e_R^2) \equiv \Lambda \frac{\partial e_R^2}{\partial \Lambda} = 0, \quad \beta_g(g_R^2, e_R^2) \equiv \Lambda \frac{\partial g_R^2}{\partial \Lambda} = \frac{1}{2\pi} (g_R^4 - e_R^4). \quad (52)$$

Solving these equations leads to a constant electromagnetic coupling $\alpha = e_R^2$, and a running coupling constant $g_R^2(\Lambda)$, except for the special values

$$g_R^2(\Lambda) \equiv \pm \alpha, \quad (53)$$

one of which is precisely the condition necessary for the existence of time-independent self-dual solutions (14).

Whenever $g_R^2(\Lambda) \neq \pm\alpha$, the solution of equation (52) is given by

$$\left| \frac{g_R^2(\Lambda) - \alpha}{g_R^2(\Lambda) + \alpha} \right| = \left| \frac{g_*^2 - \alpha}{g_*^2 + \alpha} \right| \left(\frac{\Lambda^2}{\Lambda_*^2} \right)^{\alpha/2\pi}, \quad (54)$$

where $g_*^2 = g_R^2(\Lambda_*)$ at some reference scale Λ_* . This solution has three branches, one for which $g_R^2 > \alpha$, one for which $g_R^2 < -\alpha$ and one for which $|g_R^2| < \alpha$. The domains of these branches are determined by a scale Λ_s such that

$$\left(\frac{\Lambda_s^2}{\Lambda_*^2} \right)^{\alpha/2\pi} = \left| \frac{g_*^2 + \alpha}{g_*^2 - \alpha} \right|. \quad (55)$$

Defining g_s^2 by $g_s^2 \equiv g_R^2(\Lambda_s)$, it follows that

$$\left| \frac{g_s^2 + \alpha}{g_s^2 - \alpha} \right| = 1 \quad \Rightarrow \quad g_s^2 = 0, \pm\infty. \quad (56)$$

The various branches have been plotted in figures 1 and 2. Figure 1 shows both the branches for $g_R^2 > \alpha$ and for $g_R^2 < -\alpha$, having $g_s^2 = \pm\infty$. Figure 2 shows the branch for $|g_R^2| < \alpha$, having $g_s^2 = 0$.

It is clear that in the limits $\Lambda \rightarrow 0$, $\Lambda \rightarrow \infty$ the renormalized self coupling g_R^2 approaches the two special constant values

$$g_R^2(0) = \alpha, \quad g_R^2(\infty) = -\alpha. \quad (57)$$

In the limit $\alpha \rightarrow 0$, the branches with $g_R^2(\Lambda_s) = \pm\infty$, and hence $|g_R^2(\Lambda)| > \alpha$, reduce to the solution found for the β -function of the non-linear Schrödinger model [3]. This is as expected since the Jackiw-Pi model reduces to that model when taking $e_R = 0$ leading to $\alpha = 0$. Note, that for $|g_R^2(\Lambda_*)| > \alpha \geq 1$ the Jackiw-Pi model is never in the perturbative regime. However, when $\alpha < 1$ there is a perturbative domain $g_R^2(\Lambda) < 1$ for both small and very large Λ .

The new branch with $g_R^2(\Lambda_s) = 0$ and $|g_R^2(\Lambda)| < \alpha$ has been plotted in figure 2. From this solution it is clear that the model is in the perturbative regime for intermediate values of Λ when $\alpha \geq 1$, and for all values of Λ when $\alpha < 1$, at least in the 1-loop approximation.

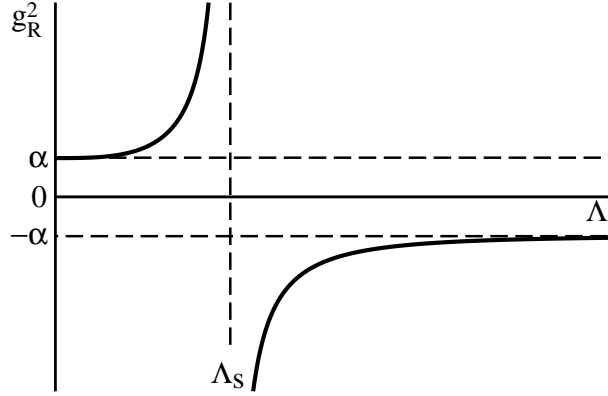


Figure 1: Plot of g_R^2 as a function of the momentum scale Λ for $|g_R^2(\Lambda_*)| > \alpha$.

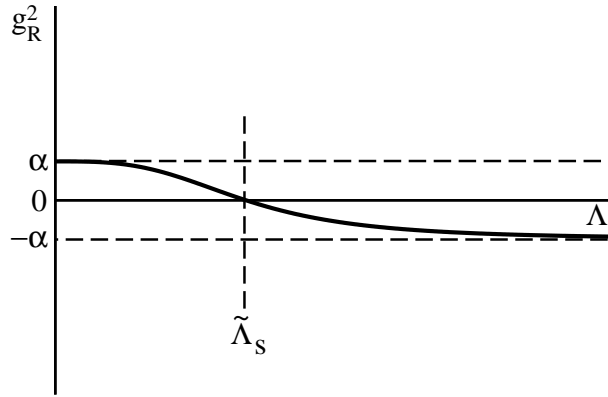


Figure 2: Plot of g_R^2 as a function of the momentum scale Λ for $|g_R^2(\Lambda_*)| < \alpha$.

7 Conclusion

It has been shown that to first order in perturbation theory only the coupling constant g^2 is renormalized: neither the fields, nor the electric charge are scale dependent. Moreover, if the coupling constants in the classical model are chosen such that $g^2 = \pm e^2$, this relation is preserved and $g^2 = g_R^2$ is constant as well.

In all other cases the renormalized coupling constant g_R^2 becomes a function of the scale Λ and the scale and special conformal symmetries are anomalous. Since only g^2 is renormalized, the argument given in [3] makes it clear

that the time-dependence of the corresponding charges D and K is given by

$$\begin{aligned}\frac{dK}{dt} &= -t \frac{dD}{dt}; \\ \frac{dD}{dt} &= \frac{1}{2} \beta(g_R^2) \int d^2x \Phi^{\dagger 2} \Phi^2 = \frac{1}{4\pi} (g_R^4 - e_R^4) \int d^2x \Phi^{\dagger 2} \Phi^2.\end{aligned}\tag{58}$$

From this expression it follows that the conformal symmetries indeed survive quantization in 1-loop approximation if and only if the condition $g^2 = g_R^2 = \pm e_R^2$ is satisfied.

It so happens that for the minus sign this condition is exactly the one the coupling constants need to satisfy such that the classical self-dual vortex solutions exist (14). In this special case not only the conformal symmetries survive quantization, but also the classical self-dual solutions. The condition $g^2 = -e^2$ is thus essential for the existence of self-dual vortex solutions in the classical theory and in the quantum theory as well.

Assuming $\alpha = e^2$ to be less than unity, and choosing g_R^2 less than α at some reference energy scale Λ_* , the Jackiw-Pi model is seen to have the unusual property that the model is in the perturbative regime for all values of Λ , as shown in figure 2. For small values of Λ g_R^2 is seen to be positive, signifying a repulsive interaction between the particles, and for large values of Λ g_R^2 is negative, signifying an attractive interaction between the particles.

In the case α less than unity, when g_R^2 is larger than α at the reference energy scale, the model has a perturbative regime either for small values of Λ or for very large values, as shown in figure 1. When α is larger than one, there only is a perturbative regime for intermediate values of Λ when g_R^2 at the reference scale is chosen less than α , as shown in figure 2.

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