NIKHEF/2007-016

SUPERGRAVITY SOLUTIONS WITH CONSTANT SCALAR INVARIANTS

A. COLEY, A. FUSTER AND S. HERVIK

Abstract. We study a class of constant scalar invariant (CSI) spacetimes, which belong to the higher-dimensional Kundt class, that are solutions of supergravity. We review the known CSI supergravity solutions in this class and we explicitly present a number of new exact CSI supergravity solutions, some of which are Einstein.

[PACS: 04.20.Jb, 04.65.+e]

1. Introduction

A D-dimensional differentiable manifold of Lorentzian signature for which all polynomial scalar invariants constructed from the Riemann tensor and its covariant derivatives are constant is called a constant scalar invariant (CSI) spacetime. The set of spacetimes with vanishing scalar (curvature) invariants will be denoted by VSI. The set of all locally homogeneous spacetimes will be denoted by Hom. Clearly, both VSI and homogeneous spacetimes are CSI spacetimes; hence, VSI $\subset$ CSI and Hom $\subset$ CSI.

Recently it was shown that the higher-dimensional VSI spacetimes with fluxes and dilaton are solutions of type IIB supergravity, and their supersymmetry properties [1] were discussed (also see [2, 3]). In this paper we shall study a (sub)class of CSI spacetimes and determine whether they are solutions of supergravity (and discuss whether they can admit supersymmetries). It is well known that $AdS_d \times S^{(D-d)}$ (in short $AdS \times S$) is an exact solution of supergravity (and preserves the maximal number of supersymmetries). Of course, $AdS \times S$ is an example of a CSI spacetime [4]. There are a number of other CSI spacetimes known to be solutions of supergravity and admit supersymmetries; namely, there are generalizations of $AdS \times S$ (for example, see [5]), (generalizations of) the chiral null models [6], and AdS gyratons [7, 8].

We wish to find a class of CSI which are solutions of supergravity and preserve supersymmetries. Clearly, we seek as general a subclass as possible, but that will include the simple generalizations of the $AdS \times S$ and AdS gyratons. There are two possible approaches. In the *top-down* approach, we can consider a subclass of known CSI spacetimes and investigate whether they can be solutions of supergravity. For example, we could consider product manifolds of the form $M \times K$ (where, for example, M is an Einstein space with negative constant curvature and K is a (compact) Einstein-Sasaki spacetime). We could then use previous work to investigate whether such spacetimes are solutions of supergravity and preserve supersymmetries (cf. [9]). Alternatively, we could use a *bottom-up* approach in which

Date: July 6, 2007.

we build CSI spacetimes using known constructions [4]. Although we are likely to find less general CSI spacetimes of interest, the advantage of this approach is that we can generate examples which by construction will be solutions of supergravity that preserve supersymmetries. We shall discuss both approaches below.

The set of all reducible CSI spacetimes that can be built from VSI and Hom by (i) warped products (ii) fibered products, and (iii) tensor sums [4] are denoted by CSI_R. The set of spacetimes for which there exists a frame with a null vector ℓ such that all components of the Riemann tensor and its covariants derivatives in this frame have the property that (i) all positive boost weight components (with respect to ℓ) are zero and (ii) all zero boost weight components are constant are denoted by CSI_F. Finally, those CSI spacetimes that belong to the (higher-dimensional) Kundt class, the so-called Kundt CSI spacetimes, are denoted by CSI_K. We note that by construction CSI_R, and by definition CSI_F and CSI_K, are at *most* of Riemann type II (i.e., of type II, III, N or O [10]). In [4] it was conjectured that if a spacetime is CSI, then the spacetime is either locally homogeneous or belongs to the higher-dimensional Kundt CSI class (i.e., CSI_K), and if a spacetime is CSI, then it can be constructed from locally homogeneous spaces and VSI spacetimes. This construction can be done by means of fibering, warping and tensor sums (i.e., CSI_R). Thus, it is plausible that for CSI spacetimes that are not locally homogeneous, the Weyl type is II, III, N or O, and that all boost weight zero terms are constant (i.e., CSI_F).

1.1. **Higher-dimensional Kundt spacetimes.** The generalized D-dimensional Kundt CSI_K metric can be written [4]

$$\mathrm{d}s^2 = 2\mathrm{d}u\left[\mathrm{d}v + H(v,u,x^k)\mathrm{d}u + W_i(v,u,x^k)\mathrm{d}x^i\right] + g^{\perp}_{ij}(x^k)\mathrm{d}x^i\mathrm{d}x^j, \tag{1}$$

where the metric functions H and W_i are given by

$$W_i(v,u,x^k) = vW_i^{(1)}(u,x^k) + W_i^{(0)}(u,x^k), \tag{2}$$

$$H(v,u,x^k) = v^2\tilde{\sigma} + vH^{(1)}(u,x^k) + H^{(0)}(u,x^k), \tag{3}$$

$$\tilde{\sigma} = \frac{1}{8}\left(4\sigma + W^{(1)i}W_i^{(1)}\right), \tag{4}$$

(and are subject to further differential constraints) and the transverse metric (where $\mathrm{d}S^2_{hom} = g^{\perp}_{ij}\mathrm{d}x^i\mathrm{d}x^j$ is a locally homogeneous space) satisfies the Einstein equations (where $i,j = 2,...,D-2$).

VSI spacetimes, with metric $\mathrm{d}s^2_{VSI}$, are of the form (1) with flat transverse metric (i.e., $g^{\perp}_{ij} = \delta_{ij}$) and the constant σ in (4) is zero (and where the metric functions H and W_i satisfy additional conditions) [11].

For a CSI_K spacetime the zero boost weight components of the Riemann tensor, $R_{ijmn} = R^{\perp}_{ijmn}$, where $R^{\perp}$ denotes the Riemann tensor components of the transverse metric, are all constant [4]. In general, the Weyl and Ricci types of the CSI_K spacetime is II [10]. A CSI_K spacetime is of Ricci type III when $R_{01} = R^{\perp}_{ij} = 0$, and is of Ricci type N if, in addition, $R_{1i} = 0$ (Ricci type O is vacuum).

The higher-dimensional Kundt metric (1) possesses a null vector field $\ell \equiv \partial/\partial v$ which is geodesic, non-expanding, shear-free and non-twisting [12]. The aligned, repeated, null vector ℓ is a null Killing vector (KV) in a CSI_K spacetime if and only if $H_{,v} = 0$ and $W_{i,v} = 0$, whence the metric no longer has any v dependence, and ℓ is, in fact, a covariantly constant null vector (CCNV) [11]. In this case the resulting spacetime is a product manifold with a CCNV-VSI Lorentzian piece of

Ricci and Weyl type III and a locally homogeneous transverse Riemannian space of Ricci and Weyl type II (in general).

2. Analysis

2.1. **Top-down approach.** It is well known that $AdS_d \times S^{(D-d)}$ is an exact solution of supergravity (for certain values of (D,d) and for particular ratios of the radii of curvature of the two space forms; in particular, $d = 5, D = 10$, $AdS_5 \times S^5$). Suppose the more general D-dimensional product spacetime $M_d \times K^{(D-d)}$ (in brief $M \times K$) is considered, where M is an Einstein space and K is compact (e.g., a sphere, or a compact Einstein space). We can ask: What are the most general forms for M and K such that the resulting product spacetime is an exact solution of some supergravity theory (for a particular dimension, and any particular fluxes)? Since $M \times K$ is a Freund-Rubin background, then if M is any Lorentzian Einstein manifold and K is any Riemannian Einstein manifold (with the same ratio of the radii of curvature as in the $AdS \times S$ case), then $M \times K$ is a solution of some supergravity theory (not worrying about whether the solution preserves any supersymmetry at the moment). The fluxes are given purely in terms of the volume forms of the relevant factor(s). In general, the supergravity equations of motion force M to have negative scalar curvature and K to have positive scalar curvature (in order to be able to take K to be hyperbolic space exotic supergravity theories need to be considered).

$AdS \times S$ is an example of a spacetime manifold in which all curvature invariants (including differential invariants) are constant. Indeed, it is even a Kundt spacetime; i.e., it is a CSI_K spacetime. There are many examples of CSI spacetimes in the Freund-Rubin $M \times K$ supergravity set. K could be a homogeneous space or a space of constant curvature. There are certainly many homogeneous examples of Freund-Rubin backgrounds.

The question then is whether these CSI solutions preserve any supersymmetry. Suppose that $M \times K$ is a Freund-Rubin background. The condition for preservation of supersymmetry demands that M and K admit Killing spinors (real for K (Riemannian) and imaginary for M (Lorentzian)). For K, the existence of such spinors implies that K is an Einstein space, whereas for M it must be imposed as an additional assumption. The analysis therefore reduces to determining which Riemannian and Lorentzian local metrics admit Killing spinors. The Riemannian case is well understood (at least in low dimension – for Freund-Rubin one needs $d < 8$), but the Lorentzian case is still largely open. For example, the amount of supersymmetry preserved in supergravity solutions which are the product of an anti-de Sitter space with an Einstein space was studied in [13]. We note that there are many homogeneous (CSI) examples of Freund-Rubin backgrounds.

More general results are possible. For example, suppose that $M \times K$ is a Freund - Rubin manifold in which M and the compact K are both Einstein spaces (and the signs and magnitudes of the cosmological constants are appropriately arranged), then if M admits a conformal Killing vector (spacelike, and a negative cosmological constant) then $M \times K$ is an exact solution of supergravity [14]. In a more general sense, any CSI spacetime of the form $M \times K$ for which the Ricci tensor is of type N [4] can be a solution of supergravity. In addition, in general if such a CSI spacetime admits a Killing spinor, it would then give rise to a null (or timelike) Killing vector (e.g., it would be a CCNV spacetime). These spacetimes would then be of interest if there exist source fields that support the supergravity solution and are consistent with the supersymmetry.

2.2. **Bottom-up approach.** We want to construct as general a subclass of CSI spacetimes as possible which are generalizations of $AdS \times S$ or AdS gyratons, perhaps restricting attention to CCNV and Ricci type N spacetimes. We shall start with a seed solution and then attempt to build up an appropriate solution. In particular, we shall build subsets of CSI_K and CSI_F, by constructing CSI_R spacetimes using a VSI seed and locally homogeneous (Einstein) spaces. Generalizations of $AdS \times S$ or AdS gyratons can be constructed in this way.

We construct a class of CSI_R spacetimes from VSI and locally homogeneous spacetimes as follows [4]. We begin with a general d-dimension VSI spacetime, with metric $\mathrm{d}s^2_{\mathrm{VSI}}$ given by (1). We then warp this metric with warp factor ω^2. If the VSI metric is Ricci flat (i.e., a d-dimensional vacuum solution; this implies certain differential conditions on H and W_i), and $\omega = l/z$ (where l is constant: curvature radius of AdS), then $\omega^2\mathrm{d}s^2_{\mathrm{VSI}}$ is an Einstein space with $\lambda = -(d-1)/l^2$ and therefore a d-dimensional vacuum solution with $\Lambda = -(d-1)(d-2)/(2l^2)$ (but where H and W_i satisfy now different equations). The CSI metrics constructed in this way have the same curvature invariants as pure AdS. Their Ricci type is II (and not lower). On the other hand, since $\omega^2\mathrm{d}s^2_{\mathrm{VSI}}$ is conformal to $\mathrm{d}s^2_{\mathrm{VSI}}$, their Weyl type is the same (III at most). By construction, all of these metrics have the same (constant) curvature invariants as AdS. Indeed, the spacetimes constructed from a CCNV-VSI (where the metric functions have no v-dependence; for example, the AdS gyraton) have a null Killing vector, which makes them attractive from a supersymmetry point of view. Note, however, that these spacetimes are not necessarily CCNV themselves. It is unlikely that spacetimes constructed from a non-CCNV VSI will have any null or timelike Killing vector.

We then consider a $(D-d)$-dimensional locally homogeneous space with metric $\mathrm{d}s^2_{\mathrm{Hom}} = \tilde{g}_{ab}(x^c)\mathrm{d}x^a\mathrm{d}x^b$; this space could be an Einstein space such as, for example, $\mathbb{E}^{D-d}$, S^{D-d} or $\mathbb{H}^{D-d}$. We then take the product manifold with metric

$$\mathrm{d}s^2_{\mathrm{CSI}} = \omega^2\mathrm{d}s^2_{\mathrm{VSI}} + \mathrm{d}s^2_{\mathrm{Hom}}, \tag{5}$$

where H and W_i are now possibly fibred (e.g., $H(v,u,x^k)$, $W_i(v,u,x^k)$) (i and k run possibly over all tranverse coordinates). If we take $\mathrm{d}s^2_{\mathrm{Hom}}$ to be Euclidean space, the Ricci tensor is of type II (the Lorentzian conformal-VSI part is of Weyl type III). These are CSI_R spacetimes (belonging to the higher-dimensional Kundt CSI class, CSI_K), and have been constructed in such a way as to be solutions of supergravity. There will be solutions that preserve supersymmetry. In particular, there is a subclass of these CSI_R spacetimes which is also CCNV (i.e., the subclass with $w^2 \equiv 1$ which is constructed from a CCNV VSI).

3. Supergravity examples

Let us provide some explicit examples of CSI supergravity spacetimes. The examples illustrate a useful method of constructing such spacetimes and, at the same time, are interesting as possible solutions of higher-dimensional gravity theories and supergravity. All of our examples are of the form of metric (1) satisfying eqns. (2) and (3). The way these are constructed is as follows. *(i)* First we find a homogeneous spacetime, $(\mathcal{M}_{\mathrm{Hom}}, \tilde{g})$ of Kundt form. Since there is a wealth of such spacetimes we will concentrate on those that are Einstein; i.e., they satisfy $\widetilde{R}_{\mu\nu} = \lambda\widetilde{g}_{\mu\nu}$. *(ii)* We then generalise to include inhomogeneous spacetimes, $(\mathcal{M}, g)$ by including arbitrary functions $W_i^{(0)}(u, x^k)$, $H^{(1)}(u, x^k)$ and $H^{(0)}(u, x^k)$. By construction, the curvature invariants of $(\mathcal{M}, g)$ will be identical to those of $(\mathcal{M}_{\rm Hom}, \tilde{g})$. These spacetimes can, for example, be used as the Lorentzian piece in the Freund-Rubin construction.

3.1. $(\mathcal{M}_{\mathbf{Hom}}, \tilde{g})$ **is a regular Lorentzian Einstein solvmanifold.** By using standard Einstein solvmanifolds, and Wick rotating, we can get many examples of homogeneous Einstein Kundt metrics [15]. All of these spacetimes can be written as follows:

$$g^{\perp}_{ij}(x^k)\mathrm{d}x^i\mathrm{d}x^j = \mathrm{d}w^2 + \sum_i \exp(-2q_i w)(\boldsymbol{\omega}^i)^2, \tag{6}$$

where $\{\boldsymbol{\omega}^i\}$ is a left-invariant metric of some subgroup[1],

$$W_i^{(1)}(u, x^k)\mathrm{d}x^i = 2p\mathrm{d}w, \quad \tilde{\sigma} = 0, \tag{7}$$

and $p = \sum_i q_i^2/(\sum_i q_i)$. The boost-weight decomposition of S (the trace-free Ricci tensor) and C (the Weyl tensor) is as follows:

- General $W_i^{(0)}(u, x^k)$, $H^{(1)}(u, x^k)$ and $H^{(0)}(u, x^k)$:
$$S = (S)_{-1} + (S)_{-2}, \quad C = (C)_0 + (C)_{-1} + (C)_{-2}.$$
- $W_i^{(0)}(u, x^k) = 0$, $H^{(1)}(u, x^k) = 0$, general $H^{(0)}(u, x^k)$:
$$S = (S)_{-2}, \quad C = (C)_0 + (C)_{-2}.$$
- An Einstein case: $W_i^{(0)}(u, x^k) = 0$, $H^{(1)}(u, x^k) = 0$, and
$$\Box^{\perp} H^{(0)} + \left(H^{(0)}W_i^{(1)}\right)^{;i} = 0,$$
where $\Box^{\perp}$ is the Laplacian on the transverse space, and
$$S = 0, \quad C = (C)_0 + (C)_{-2}$$

There is a cornucopia of examples of these metrics and the simplest one corresponds to $(\mathcal{M}_{\rm Hom}, \tilde{g})$ being AdS space (for which $(C)_0 = 0$). The corresponding inhomogeneous Einstein metric with $H^{(0)} \neq 0$ is the Siklos spacetime [16].

There are a few special metrics in this class worth mentioning. A special Siklos metric is the Kaigorodov spacetime [17] which is both Einstein and homogeneous (see section 3.3). Another special homogeneous metric is the conformally flat metric:

$$\mathrm{d}s^2 = 2e^{-2qz}\mathrm{d}u\,(\mathrm{d}v + ae^{qz}\mathrm{d}u) + e^{-2qz}\mathrm{d}y^2 + \mathrm{d}z^2. \tag{8}$$

This metric has vanishing Weyl tensor, $C = 0$, while $S = (S)_{-2}$. Both this metric, and the Kaigorodov metric, are homogeneous Kundt metrics having identical curvature invariants to AdS.

---

[1] If the solvmanifold is of rank one, this subgroup would be the nilpotent group corresponding to the Einstein nilradical.

There are many 'non-trivial' examples of this type as well. As an illustration, the following Kundt metric is a 7-dimensional regular Lorentzian Einstein solvmanifold:

$$\begin{aligned} ds^2 &= 2du\,(dv + 3pv dr) + e^{-4pr}(dx - y dw)^2 + e^{-3pr}(dy - z dw)^2 \\ &\quad + e^{-2pr}dz^2 + e^{-pr}dw^2 + dr^2, \end{aligned} \tag{9}$$

where $p = 1/(2\sqrt{2})$. This metric has $\widetilde{R}_{\mu\nu} = -(3/2)\widetilde{g}_{\mu\nu}$ and can be generalised to the inhomogeneous case by the standard procedure.

3.2. **Some 5D examples.** Let us consider some non-trivial examples which can not be obtained by a Wick-rotation of an Einstein solvmanifold.

3.2.1. *Transverse space is the Heisenberg group.* The transverse space is the Heisenberg group with a left-invariant metric:

$$g^{\perp}_{ij}(x^k)dx^i dx^j = \left(dx + \frac{b}{2}(y dz - z dy)\right)^2 + dy^2 + dz^2,$$

and

$$W^{(1)}_i dx^i = \sqrt{2}b\left(dx + \frac{b}{2}(y dz - z dy)\right), \quad \tilde{\sigma} = \frac{b^2}{4}.$$

Here, $\widetilde{R}_{\mu\nu} = -(b^2/2)\widetilde{g}_{\mu\nu}$. The Weyl tensor decomposes as

$$C = (C)_0 + (C)_{-1} + (C)_{-2}.$$

For the trace-free Ricci tensor:

- General $W^{(0)}_i(u, x^k)$, $H^{(1)}(u, x^k)$ and $H^{(0)}(u, x^k)$: $S = (S)_{-1} + (S)_{-2}$.
- $W^{(0)}_i(u, x^k) = 0$, $H^{(1)}(u, x^k) = 0$, general $H^{(0)}(u, x^k)$: $S = (S)_{-2}$.
- An Einstein case: $W^{(0)}_i(u, x^k) = 0$, $H^{(1)}(u, x^k) = 0$, and

$$\Box^{\perp}H^{(0)} + \left(H^{(0)}W^{(1)}_i\right)^{;i} = 0,$$

where $\Box^{\perp}$ is the Laplacian on the transverse space. This equation ensures that $S = 0$. The general solution to this equation can be found using standard methods (for example, separation of variables).

3.2.2. *Transverse space is $SL(2,\mathbb{R})$.* The transverse space is $SL(2,\mathbb{R})$ with a left-invariant metric:

$$g^{\perp}_{ij}(x^k)dx^i dx^j = \left(dx - a\frac{dz}{y}\right)^2 + \frac{b^2}{y^2}(dy^2 + dz^2),$$

and

$$W^{(1)}_i dx^i = \frac{\sqrt{2(a^2+b^2)}}{b^2}\left(dx - a\frac{dz}{y}\right), \quad \tilde{\sigma} = \frac{a^2}{4b^4}.$$

Here, $\widetilde{R}_{\mu\nu} = -[(a^2 + 2b^2)/(2b^4)]\widetilde{g}_{\mu\nu}$. The Weyl tensor decomposes as

$$C = (C)_0 + (C)_{-1} + (C)_{-2}.$$

For the trace-free Ricci tensor:

- General $W^{(0)}_i(u, x^k)$, $H^{(1)}(u, x^k)$ and $H^{(0)}(u, x^k)$: $S = (S)_{-1} + (S)_{-2}$.
- $W^{(0)}_i(u, x^k) = 0$, $H^{(1)}(u, x^k) = 0$, general $H^{(0)}(u, x^k)$: $S = (S)_{-2}$.

- An Einstein case: $W_i^{(0)}(u,x^k)=0$, $H^{(1)}(u,x^k)=0$, and

$$\Box^{\perp}H^{(0)}+\left(H^{(0)}W_i^{(1)}\right)^{;i}=0,$$

where $\Box^{\perp}$ is the Laplacian on the transverse space. This equation ensures that $S=0$. The general solution to this equation can be found using standard methods (for example, separation of variables).

3.2.3. *Transverse space is the 3-sphere, S^3.* The transverse space is the 3-sphere, S^3, with the Berger metric:

$$g^{\perp}_{ij}(x^k)\mathrm{d}x^i\mathrm{d}x^j=a^2\left(\mathrm{d}x+\sin y\mathrm{d}z\right)^2+b^2(\mathrm{d}y^2+\cos^2 y\mathrm{d}z^2),$$

and

$$W_i^{(1)}\mathrm{d}x^i=\frac{a\sqrt{2(a^2-b^2)}}{b^2}\left(\mathrm{d}x+\sin y\mathrm{d}z\right),\quad \tilde{\sigma}=\frac{a^2}{4b^4}.$$

Here, $\widetilde{R}_{\mu\nu}=-[(a^2-2b^2)/(2b^4)]\widetilde{g}_{\mu\nu}$, and hence, can be positive, zero or negative. The Weyl tensor always decomposes as

$$C=(C)_0+(C)_{-1}+(C)_{-2}.$$

For the trace-free Ricci tensor:

- General $W_i^{(0)}(u,x^k)$, $H^{(1)}(u,x^k)$ and $H^{(0)}(u,x^k)$: $S=(S)_{-1}+(S)_{-2}$.
- $W_i^{(0)}(u,x^k)=0$, $H^{(1)}(u,x^k)=0$, general $H^{(0)}(u,x^k)$: $S=(S)_{-2}$.
- An Einstein case: $W_i^{(0)}(u,x^k)=0$, $H^{(1)}(u,x^k)=0$, and

$$\Box^{\perp}H^{(0)}+\left(H^{(0)}W_i^{(1)}\right)^{;i}=0,$$

where $\Box^{\perp}$ is the Laplacian on the transverse space. This equation ensures that $S=0$.

3.3. **Examples in the literature.** A number of special cases of the examples discussed in the previous two subsections are known, and the supersymmetry properties of many of them have been discussed. All of the examples given below are in the subclass of CSI_R spacetimes. Let us review these examples briefly.

We give in the first place an example of a CCNV CSI. In [18] the following five-dimensional metric was considered:

$$\mathrm{d}s^2=2\mathrm{d}u\left[\mathrm{d}v+K(u,x^k)\,\mathrm{d}u\right]+\mathrm{d}\xi^2+\sin^2\xi\mathrm{d}\theta^2+\sin^2\xi\sin^2\theta\mathrm{d}\phi^2 \tag{10}$$

The transverse space is S^3 with unit radius and the function K satisfies

$$\Box^{\perp}K=0 \tag{11}$$

where $\Box^{\perp}$ is the Laplacian on S^3. The covariantly constant null Killing vector is ∂_v. Note that metric (10) is already in the Kundt form (1), with $W_i^{(1)}=W_i^{(0)}=\tilde{\sigma}=H^{(1)}=0$. The metric (10), together with a constant dilaton and appropiate antisymmetric field, is an exact solution to bosonic string theory[2].

The next two examples are not CCNV, but are constructed from a CCNV VSI (see section 2.2). As such they have the null Killing vector ∂_v; however, this vector is no longer covariantly constant due to the introduction of a warp factor. Recall

[2] However, it is not a vacuum solution of five-dimensional gravity.

that if the VSI seed metric is Ricci flat they are Einstein spaces. The first example is the d-dimensional Siklos spacetime

$$ds^2 = \frac{l^2}{z^2}\left[2\mathrm{d}u\mathrm{d}v + 2H(u,x^k)\,\mathrm{d}u^2 + (\mathrm{d}x^i)^2\right], \tag{12}$$

where $i = 1,\dots,d-2$. The Siklos metric can be cast in the Kundt form (1) by making a coordinate transformation $\tilde{v} = vl^2/z^2$

$$\mathrm{d}s^2 = 2\mathrm{d}u\left(\mathrm{d}\tilde{v} + \frac{l^2}{z^2}H(u,x^k)\,\mathrm{d}u + \frac{2\tilde{v}}{z}\,\mathrm{d}z\right) + \frac{l^2}{z^2}(\mathrm{d}x^i)^2 \tag{13}$$

In this way $\tilde{\sigma} = H^{(1)} = W_i^{(0)} = 0$, $H^{(0)} = (l^2/z^2)H(u,x^k)$ and $W_z^{(1)} = 2/z$; the transverse space is $\mathbb{H}^3$. In the new coordinates the null Killing vector is $l^2/z^2\partial_{\tilde{v}}$. The Kaigorodov metric K_d is a Siklos spacetime with $H = z^{d-1}$ [17, 20]. Since it is homogeneous, it has at least d Killing vectors (but only ∂_v can be null). The Siklos spacetime is of Weyl type N.

All of the Siklos metrics preserve 1/4 of the supersymmetries, regardless the form of the function H in (12) [19]. This was previously shown for the Kaigorodov metric in [20].

The second example is the d-dimensional AdS gyraton, with metric [8]

$$\mathrm{d}s^2 = \frac{l^2}{z^2}\left[2\mathrm{d}u\mathrm{d}v + 2H(u,x^k)\mathrm{d}u^2 + 2W_i(u,x^k)\mathrm{d}u\mathrm{d}x^i + (\mathrm{d}x^i)^2\right], \tag{14}$$

where $i = 1,\dots,d-2$ and H and W_i are independent of v. In the Kundt form we have (13) but additionally $W_i^{(0)} = \frac{l^2}{z^2}W_i$; the null Killing vector is $l^2/z^2\partial_{\tilde{v}}$ as for the Siklos metric. This is a metric of the form given in section 3.1 where the homogeneous space is AdS_d. The Weyl type is III. The five-dimensional AdS gyraton has been considered in the context of gauged supergravity, and both gauged and ungauged supergravity coupled to an arbitrary number of vector supermultiplets [7]. Some of these solutions preserve 1/4 of the supersymmetry [7, 21].

We consider now metrics of the form (5). The most well-known examples in this class are the $AdS \times S$ spaces. Let us discuss $AdS_5 \times S^5$

$$\mathrm{d}s^2 = \frac{1}{z^2}\left[2\mathrm{d}u\mathrm{d}v + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\right] + \mathrm{d}\Omega_5^2 \tag{15}$$

where $\mathrm{d}\Omega_5^2$ is the standard round metric on the unit[3] 5-sphere. This is clearly of the form (5), with the simplest VSI (Minkowski) spacetime. It is a (maximally symmetric) Einstein space. In the Kundt form (1)

$$\mathrm{d}s^2 = 2\mathrm{d}u\left(\mathrm{d}\tilde{v} + \frac{2\tilde{v}}{z}\mathrm{d}z\right) + \frac{1}{z^2}\left[\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\right] + \mathrm{d}\Omega_5^2 \tag{16}$$

with $\tilde{\sigma} = H^{(1)} = H^{(0)} = W_i^{(0)} = 0$, $W_z^{(1)} = 2/z$; the transverse space is $\mathbb{H}^3 \times S^5$. It is of Weyl type O (provided their sectional curvatures have equal magnitude and opposite sign, otherwise they are Weyl type D).

[3] We can multiply (15) by l^2; then $r^2 = 1/l^2$ is the radius of S^5.

Spaces of the form $AdS \times S$, together with appropriate five- or four-form fields, are maximally supersymmetric solutions of IIB and eleven-dimensional supergravities [22, 23, 24].

$AdS_5 \times S^5$ can be generalized by considering other VSI seeds. The resulting metrics are of Weyl type III at most[4]. For example,

$$\mathrm{d}s^2 = \frac{1}{z^2}\left[2\mathrm{d}u\mathrm{d}v + 2H(u,x,y,z,x^a)\mathrm{d}u^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2\right] + \mathrm{d}\Omega_5^2 \tag{17}$$

where x^a are the coordinates on S^5. In the Kundt form we have now $H^{(0)} = H/z^2$. Such spacetimes are supersymmetric solutions of IIB supergravity (and there are analogous solutions in $D = 11$ supergravity) [25]. Supersymmetric solutions of this type in $D = 5$ gauged supergravity were given in [26], where $\mathrm{d}s^2_{\mathrm{Hom}}$ was taken to be flat (Weyl type N).

The idea of considering spaces of the form $AdS \times M$, with M an Einstein (-Sasaki) manifold other than S^n, goes back to [27]. Such spaces have Weyl type II. In [27] supersymmetric solutions of $D = 11$ supergravity of Weyl type II are presented where, for example, M is the squashed S^7. Examples where M is taken to flat and hyperbolic space can be found in [28] (in the context of higher-dimensional Einstein-Maxwell theory). In ten dimensions, solutions of the form $AdS_5 \times T^{1,1}$ have been extensively studied. Recently, an infinite class of five-dimensional Einstein-Sasaki spaces (called $Y^{p,q}$) has attracted much attention[5] [30].

The final example concerns a warped product of AdS_3 with an 8-dimensional compact (Einstein-Kahler) space M_8:

$$\mathrm{d}s^2 = \omega^2[\mathrm{d}s^2(AdS_3) + \mathrm{d}s^2(M_8)]. \tag{18}$$

These metrics with non-vanishing 4-form flux are supersymmetric solutions of D=11 supergravity [5]. Similar constructions can be found in [31].

3.4. **Constructing homogeneous Einstein Kundt metrics.** Let us briefly discuss the general method for constructing the homogeneous Kundt metrics illustrated in subsections (3.1) and (3.2) (the examples given in 5D are easily generalized to higher dimensions). Consider a Lie group G equipped with a left-invariant frame $\mathbf{m}^i$. A class of Lorentzian Kundt metrics can then be written:

$$\mathrm{d}s^2 = 2\mathrm{d}u\left(\mathrm{d}v + v^2\tilde{\sigma}\mathrm{d}u + v\beta_i\mathbf{m}^i\right) + \delta_{ij}\mathbf{m}^i\mathbf{m}^j, \tag{19}$$

where $\tilde{\sigma}$ and β_i are constants. This is automatically a homogeneous space with left-invariant frame

$$\boldsymbol{\omega}^0 = v\mathrm{d}u, \quad \boldsymbol{\omega}^1 = \frac{\mathrm{d}v}{v} + v\tilde{\sigma}\mathrm{d}u + \beta_i\mathbf{m}^i, \quad \boldsymbol{\omega}^{i+1} = \mathbf{m}^i. \tag{20}$$

A set of transitively acting Killing vectors are:

$$\boldsymbol{\xi}_0 = v\frac{\partial}{\partial v} - u\frac{\partial}{\partial u}, \quad \boldsymbol{\xi}_1 = \frac{\partial}{\partial u}, \quad \boldsymbol{\xi}_{i+1} = \boldsymbol{\xi}_i^G,$$

---

[4] These spacetimes are of type II if the sectional curvatures are not of equal magnitude and opposite sign.

[5] However, these are not homogeneous and hence, not CSI. There are many homogeneous Einstein spaces on $S^2 \times S^3$ [29]; however, $T^{1,1}$ is the only one that is also Sasaki.

where $\boldsymbol{\xi}_i^G$ is a set of transitively acting Killing vectors on G.

Let $\widetilde{R}_{ij}$ be the Ricci tensor of $\delta_{ij}\mathbf{m}^i\mathbf{m}^j$. Then

$$R_{01} = \frac{1}{2}(4\tilde{\sigma} + C^j_{\ ji}\beta^i - \beta_i\beta^i), \tag{21}$$

$$R_{ij} = \widetilde{R}_{ij} + C_{(ij)k}\beta^k - \frac{1}{2}\beta_i\beta_j, \tag{22}$$

where $\beta_{(i;j)} = C_{(ij)k}\beta^k$ and $C^i_{\ jk}$ are the structure constants of the Lie group G. We have not written down the Ricci components of boost-weights -1 and -2.

The examples essentially split into two different cases according to whether G is unimodular or not. The regular Lorentzian solvmanifolds are not unimodular and can be found using the Riemannian analysis. The unimodular case, $C^i_{ij} = 0$, corresponding to true 'authentic' Lorentzian solutions and have to be found on a case-by-case basis.

3.4.1. *All 5D homogeneous Einstein manifolds of this type.* It can be shown that all 5D manifolds of this type with ($\beta_i \neq 0$) are given in the text or are Lorentzian versions of standard Einstein solvmanifolds. The classification of 3D Lie algebras is well-known and are enumerated I-IX using the Bianchi classification. The Lie algebras of the Heisenberg group, $SU(2) \cong S^3$ and $SL(2,\mathbb{R})$ are II, IX and VIII, respectively. The above method for the other Lie algebras also gives an Einstein metric for the type III algebra; however, the metric is the same as the $SL(2,\mathbb{R})$ since this also admits a simply transitive type III action.

3.4.2. *Other examples.* There have a few other examples.

- G is the $(2m+1)$-dimensional Heisenberg group. The spacetime is of dimension $(3+2m)$ and metric is similar to the $m=1$ case in subsection 3.2.1.
- G is $S^3 \times S^3$: Given the left-invariant one-forms σ^i and $\hat{\sigma}^i$ on the two S^3 so that
$$\mathrm{d}\sigma^i = \frac{1}{2}\varepsilon^i_{\ jk}\sigma^j \wedge \sigma^k, \quad \mathrm{d}\hat{\sigma}^i = \frac{1}{2}\varepsilon^i_{\ jk}\hat{\sigma}^j \wedge \hat{\sigma}^k,$$
the metic can be written:

$$\begin{aligned} \mathrm{d}s^2 &= 2\mathrm{d}u\left[\mathrm{d}v + \tilde{\sigma}v^2\mathrm{d}u + \alpha(\sigma^1 + \hat{\sigma}^1)\right] \\ &\quad + A^2\left[(\sigma^1)^2 + 2\lambda\sigma^1\hat{\sigma}^1 + (\hat{\sigma}^1)^2\right] + B^2\left[(\sigma^2)^2 + (\sigma^3)^2 + (\hat{\sigma}^2)^2 + (\hat{\sigma}^3)^2\right], \end{aligned} \tag{23}$$

where $A^2/2 < B^2 \leq A^2$ and
$$\lambda = \frac{2(A^2 - B^2)}{A^2}, \quad \tilde{\sigma} = \frac{3A^2 - 2B^2}{2B^4}, \quad \alpha^2 = \frac{2(A^2 - B^2)(3A^2 - 2B^2)}{B^4}.$$
This is a positively curved Einstein space.

4. Conclusion

In this paper we have discussed a (sub)class of CSI spacetimes which are solutions of supergravity. We have utilized two different approaches. In the *top-down* approach we considered a subclass of known CSI product manifolds of the form $M \times K$ and investigated the conditions under which they will be solutions of supergravity. In a *bottom-up* approach we built CSI Kundt spacetimes using a Lorentzian VSI spacetime and a known homogeneous spacetime as seeds [4], which by construction will automatically be solutions of supergravity. We also discussed which of these CSI supergravity solutions will preserve supersymmetries.

We have explicitly constructed a number of new exact CSI supergravity solutions, some of which are generalizations of $AdS \times S$ spacetimes and AdS gyratons. Indeed, in some of the simple generalizations of $AdS \times S$ spacetimes all of the curvature invariants are identical to those of $AdS \times S$, which may be of importance when considering higher order corrections [32] (i.e., it is plausible that these generalizations are also exact string solutions). The newly constructed spacetimes include solutions that are based on (warped) products of regular Lorentzian Einstein solvmanifolds (including the simple Siklos metric) and transverse spaces which are $(D-d)$-spheres (as well as squashed spheres and Euclidean and hyperbolic spaces). Finally, we have reviewed the known CSI supergravity solutions, and we have shown that they belong to the higher-dimensional Kundt class.

Acknowledgements: This work was supported by NSERC (AC), AARMS (SH) and the programme FP52 of the Foundation for Research of Matter, FOM (AF).

References

[1] A. Coley, A. Fuster, S. Hervik and N. Pelavas, JHEP **0705**, 032 (2007).
[2] D. Amati and C. Klimčík, Phys. Lett. B **219**, 443 (1989); G.T. Horowitz and A.R. Steif, Phys. Rev. Lett. **64** 260 (1990); A.A. Coley, Phys. Rev. Lett. **89**, 281601 (2002).
[3] R. R. Metsaev and A. A. Tseytlin, Phys. Rev. D **65**, 126004 (2002); M. Blau et al., JHEP **0201**, 047 (2002); P. Meessen, Phys. Rev. D **65**, 087501 (2002); J. G. Russo and A.A. Tseytlin, JHEP **0209**, 035 (2002); J. Maldacena and L. Maoz, JHEP **0212**, 046 (2002).
[4] A. Coley, S. Hervik and N. Pelavas, Class. Quant. Grav. **23**, 3053 (2006).
[5] J. Gauntlett et al., Phys. Rev. D **74**, 106007 (2006).
[6] G. T. Horowitz and A. A. Tseytlin, Phys. Rev. D **51**, 2896 (1995).
[7] M. Caldarelli et al., Class. Quant. Grav. **24**, 1341 (2007).
[8] V. P. Frolov and A. Zelnikov, Phys. Rev. D **72**, 104005 (2005); V. P. Frolov and D. V. Fursaev, Phys. Rev. D **71**, 104034 (2005).
[9] R. Kallosh et al., Phys. Rev. D **58**, 125003 (1998).
[10] A. Coley, R. Milson, V. Pravda and A. Pravdova, Class. Quant. Grav. **21**, L35 (2004).
[11] A. Coley, A. Fuster, S. Hervik and N. Pelavas, Class. Quant. Grav. **23**, 7431 (2006).
[12] A. Coley, R. Milson, V. Pravda and A. Pravdova, Class. Quant. Grav. **21**, 5519 (2004).
[13] B. S. Acharya, J. M. Figueroa-O'Farrill, C. M. Hull and B. Spence, Adv. Theor. Math. Phys. **2** 1249 (1999).
[14] R. Guven, Class. Quant. Grav. **23**, 295 (2006).
[15] S. Hervik, J. Geom. Phys. **52**, 298 (2004) & Class. Quant. Grav. **21**, 4273 (2004).
[16] S.T.C. Siklos, Lobatchevski plane gravitational waves, in *Galaxies, axisymmetric systems and relativity* ed. M.A.H. MacCallum, Cambridge University Press, 1985.
[17] V. Kaigorodov, Dokl. Akad. Nauk. SSSR **146**, 793 (1962); Sov. Phys. Doklady **7**, 893 (1963).
[18] G.T. Horowitz and A.A. Tseytlin, Phys. Rev. D **50**, 5204 (1994).
[19] D. Brecher, A. Chamblin and H. S. Reall, Nucl. Phys. B **607**, 155 (2001).
[20] M. Cvetic, H. Lu and C. N. Pope, Nucl. Phys. B **545**, 309 (1999).
[21] J. B. Gutowski and W. Sabra, JHEP **10**, 039 (2005); J. P. Gauntlett and J. B. Gutowski, Phys. Rev. D **68**, 105009 (2003).
[22] P. G. O. Freund and M. A. Rubin, Phys. Lett. B **97**, 233 (1980).
[23] K. Pilch, P. van Nieuwenhuizen and P. K. Townsend, Nucl. Phys. B **242**, 377 (1984).
[24] J. H. Schwarz, Nucl. Phys. B **226**, 269 (1983).
[25] A. Kumar and H. K. Kunduri, Phys. Rev. D **70**, 104006 (2004).
[26] J. Kerimo, JHEP **0509**, 025 (2005).
[27] M.J. Duff, H. Lu, C.N. Pope and E. Sezgin, Phys. Lett. B **371**, 206 (1996).
[28] V. Cardoso, O. J. C. Dias and J. P. S. Lemos, Phys. Rev. D **70**, 024002 (2004).
[29] D. Alekseevsky, I. Dotti and C. Ferraris, Pacific J. Math. **175**, 1 (1996).
[30] J. P. Gauntlett, D. Martelli, J. Sparks and D. Waldram, Adv. Theor. Math. Phys. **8**, 711 (2004).
[31] J. P. Gauntlett, D. Martelli, J. Sparks and D. Waldram, Class. Quant. Grav. **21**, 4335 (2004) & **23**, 4693 (2006).
[32] P. Meessen, arXiv:0705.1966.

Department of Mathematics and Statistics, Dalhousie University, Halifax, Nova Scotia, Canada B3H 3J5 (AC and SH); National Institute for Nuclear and High-Energy Physics (NIKHEF), Kruislaan 409, 1098 SJ, Amsterdam, The Netherlands (AF)

E-mail address: aac,herviks@mathstat.dal.ca; fuster@nikhef.nl