Zero modes on cosmic string loops

Marieke Postma [a] * and Betti Hartmann [b] †

a) *NIKHEF, Kruislaan 409, NL-1098 SJ Amsterdam, The Netherlands*

b) *Laboratoire de Mathématiques et Physique Théorique, Université de Tours, Parc Grandmont, 37200 Tours, France*

Abstract

We study the spectrum of fermionic modes on cosmic string loops. We find no fermionic zero modes nor massive bound states — this implies that vortons stabilized by fermionic currents do not exist. We have also studied kink-(anti)kink and vortex-(anti)vortex systems and find that all systems that have vanishing net topological charge do not support fermionic bound modes.

1 Introduction

Cosmic string loops have raised a lot of interest in the past decades. They are believed to form in the evolution of cosmic string networks. While standard loops of cosmic string eventually decay, it has been suggested that fermionic or bosonic currents "living" on the loop of string can stabilize it against decay. This was first put forward in [1] within a model of superconducting strings [2]. Such stable cosmic string loops are called *vortons* (see [3] and references therein). In the case of bosonic currents, a scalar field condensate builds up in the core of the string and leads to a conserved (Noether) current. Vortons have been constructed explicitly in a scalar field model with $U(1) \times U(1)$ symmetry [4].

Fermionic currents appear due to the coupling of the string forming Higgs and/or gauge field to fermions. The aim of this paper is the construction of the spectrum of both massless and massive fermionic bound states on cosmic string loops. For an infinitely long straight string the spectrum is well known: the number of zero modes is given by an index theorem and equals the winding number of the string [5, 6], and there are massive bound states as well [7, 8]. The equivalent question for the string loop is hardly studied. An incomplete analysis of this case was done in [9, 10]; they found that the number of zero modes is equal to the local winding number of the string. Although this seems to agree with the result as for the infinitely long straight string, we note that the straight string has a net winding whereas the loop does not. In [10] the existence of fermionic zero modes for a curved string was established analyzing small curvature

*E-mail:mpostma@nikhef.nl

†E-mail:betti.hartmann@lmpt.univ-tours.fr

corrections to the straight string solution. However, it is not clear whether this analysis can be extended to string loops. In this paper, we asymptotically solve the fermionic equations of motion in the presence of a string loop. In contrast to [9], we will show that no fermionic zero modes or massive bounds states exists on cosmic string loops. And thus no fermionic vortons exists.

In fact, we would not expect the existence of fermionic zero modes on string loops. The number of zero modes is an adiabatic invariant. One can adiabatically change the string loop to the trivial vacuum background, and the latter supports no bound states. Saying it in other words, the topological charge of the whole system is zero, and it follows from an index theorem [6] that there are no bound states. This statement can be verified explicitly in the much simpler case of a kink-antikink system. This system carries no net topological charge, it can be adiabatically deformed to the trivial vacuum. And indeed, solving the equations of motion, we find that the kink-antikink does not support fermionic bound states. The kink-antikink system is quite analogue to the set-up with two parallel straight strings with opposite winding number. For these simpler systems, we find that there are no fermionic zero modes for systems with vanishing net topological charge.

Our paper is organized as follows. In section 2 we discuss the spectrum of fermion bound states on kinks and antikinks. In section 3 we move on to cosmic strings, and discuss single straight strings, two parallel strings as well as string loops. We end with conclusions in section 4.

2 Kinks and Antikinks

We first examine the relatively simple system of kinks and antikinks. We expect that many of the results, which as we will see are topological in nature, carry over to the case of cosmic strings to be discussed in the next section.

2.1 Bosonic background

Consider a bosonic background of the form

$$\phi = \eta\Big[\tanh(a(z-z_1)) + B\Big(\tanh\left(a(z+z_1)\right)+1\Big)\Big]. \tag{1}$$

with $\eta > 0$. $B = 0$ corresponds to the well known kink solution of the ϕ^4-theory located at $z = z_1$; the antikink solution is obtained by replacing $\eta \to -\eta$. $B = -1$ describes a kink-antikink configuration. This is only a solution of the field equations in the limit $z_1 \to \infty$. Finally, $B = 1$ corresponds to a kink-kink solution. This set-up needs a bosonic potential with at least three minima, and thus does not occur in the ϕ^4-theory. In principle, the bosonic potential can be reconstructed from the Bogomolny equations via $\partial_z \phi = \sqrt{2V}$.

The "thin wall" approximation, i.e. the limit that the domain wall is infinitely thin, is equivalent to the limit $a \to \infty$. Eq. (1) then reduces to

$$\phi = \begin{cases} -\eta & z < -z_1, \\ \eta & -z_1 < z < z_1, \\ \eta(1+2B) & z_1 < z. \end{cases} \tag{2}$$

2.2 Zero mode solutions

Consider a fermion with a Yukawa interaction $\mathcal{L} \ni h\phi\bar{\psi}\psi$. The Dirac equation in the background of kinks and antikinks described by (1) reads

$$[i\gamma^\mu \partial_\mu - h\phi]\,\psi = 0. \tag{3}$$

To find the zero mode solutions we separate the longitudinal and transverse coordinates, and write

$$\psi = \alpha(t,x,y)\beta(z)\xi, \tag{4}$$

with ξ a constant spinor. Here $\alpha(t,x,y)$ solves the longitudinal Dirac equation and gives the dispersion relation. To find the zero mode solution we set $E=0$, $k_L = 0$ for the moment (so that it is trivially solved), and $\alpha = 1$; the case $E \neq 0$ is discussed in the next subsection. To solve the transverse part of the Dirac equations we introduce the eigenspinors

$$\gamma^z \xi_\pm = \pm i \xi_\pm. \tag{5}$$

The two projection eigenstates decouple in the Dirac equation, which becomes $\partial_z \beta_\pm = \mp h\phi\beta_\pm$. It has as solution

$$\beta_\pm(z) = N \exp(\mp B m_\psi z) \cosh\left[a(z-z_1)\right]^{\mp B m_\psi/a} \cosh\left[a(z+z_1)\right]^{\mp m_\psi/a}, \tag{6}$$

with N a constant normalization constant, and $m_\psi = h\eta$ the vacuum fermion mass.

In the chiral basis for γ matrices, see appendix A.2, the eigenspinors of the projection operator (5) are of the form

$$\xi_+^1 = \begin{pmatrix} 0 \\ i \\ 0 \\ 1 \end{pmatrix}, \quad \xi_+^2 = \begin{pmatrix} -i \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \xi_-^1 = \begin{pmatrix} 0 \\ -i \\ 0 \\ 1 \end{pmatrix}, \quad \xi_-^2 = \begin{pmatrix} i \\ 0 \\ 1 \\ 0 \end{pmatrix}. \tag{7}$$

There are four independent solutions $\psi^i_\pm = \beta_\pm \xi^i_\pm$ with $i = 1, 2$. They are normalizable if

$$\int_{-\infty}^{\infty} \mathrm{d}z\, \psi_m^\dagger \psi_m = \int_{-\infty}^{\infty} \mathrm{d}z\, |\beta_\pm|^2 \delta_{mn} < \infty \tag{8}$$

with ψ_m labeling the 4 states $\psi^i_\pm$.

Kink or antikink. To get the kink zero-mode solution we set $B = 0$ in (6) to obtain

$$\beta_\pm = N \cosh\left[a(z + z_1)\right]^{\mp m_\psi/a} . \tag{9}$$

The asymptotic behavior of the solutions is $\beta_\pm \propto \exp(\mp m_\psi |z|)$. The β_- solution is non-normalizable, while β_+ corresponds to a zero mode solution localized at the kink at $z = -z_1$. Two independent real solutions, or one complex solution, exist, constructed from $\psi^i_+ = \beta_+ \xi^i_+$ with $i = 1, 2$. To find the solution for an antikink, we replace $\eta \to -\eta$. Now β_- is the localized normalizable solution, while β_+ is non-normalizable. The profiles of the fermionic zero modes for kinks and antikinks respectively are shown in Fig. 1.

Kink-kink. The kink-kink configuration corresponds to $B = 1$. The asymptotic behavior of the solutions (6) then reads $\beta_- \propto \exp(m_\psi |z|)$ which is non-normalizable, and $\beta_+ \propto \exp(-3m_\psi |z|)$ which is normalizable. Both the bosonic background and the fermionic solutions are invariant under $z_1 \leftrightarrow -z_1$. This implies that we find only one complex zero mode solution, which is localized around $z = -z_1$. The profile of the zero mode solution is shown in Fig. 1.

Consider now the limit $z_1 \to 0$. The bosonic background then corresponds to a kink-configuration with ϕ interpolating between $-\eta$ and 3η. The zero mode solution becomes $\beta_\pm = \exp(\mp m_\psi z) \cosh[az]^{\mp 2m_\psi/a}$. As expected β_+ has the usual form of a normalizable zero mode on a kink (9). The factor 2 difference in the exponent of the cosh is due to the fact that the topological charge $Q \propto \phi(\infty) - \phi(-\infty)$ is twice as big as for a usual kink. The factor $\exp(\mp m_\psi x)$ comes from the fact that the $B = 1$ double kink solution is shifted horizontally by η compared to the $B = 0$ kink background, it is of the form $\phi = \eta + 2\eta \tanh(az)$.

Kink-antikink. The kink-antikink configuration corresponds to $B = -1$. Both zero mode solutions, although they are peaked at the kink or antikink, are non-normalizable since $\beta_\pm(\pm\infty) \propto \exp(m_\psi |z|)$. Thus no normalizable zero mode solution exists.

This result can be easily understood. The zero mode solution on a kink is a ξ_+-spinor, on an antikink a ξ_--spinor. So pasting the kink and antikink together, one can never match the zero mode solutions at the origin because of the orthogonal spinors. This problem is independent of the distance between the kinks, in particular it does not disappear in the limit $z_1 \to \infty$. And

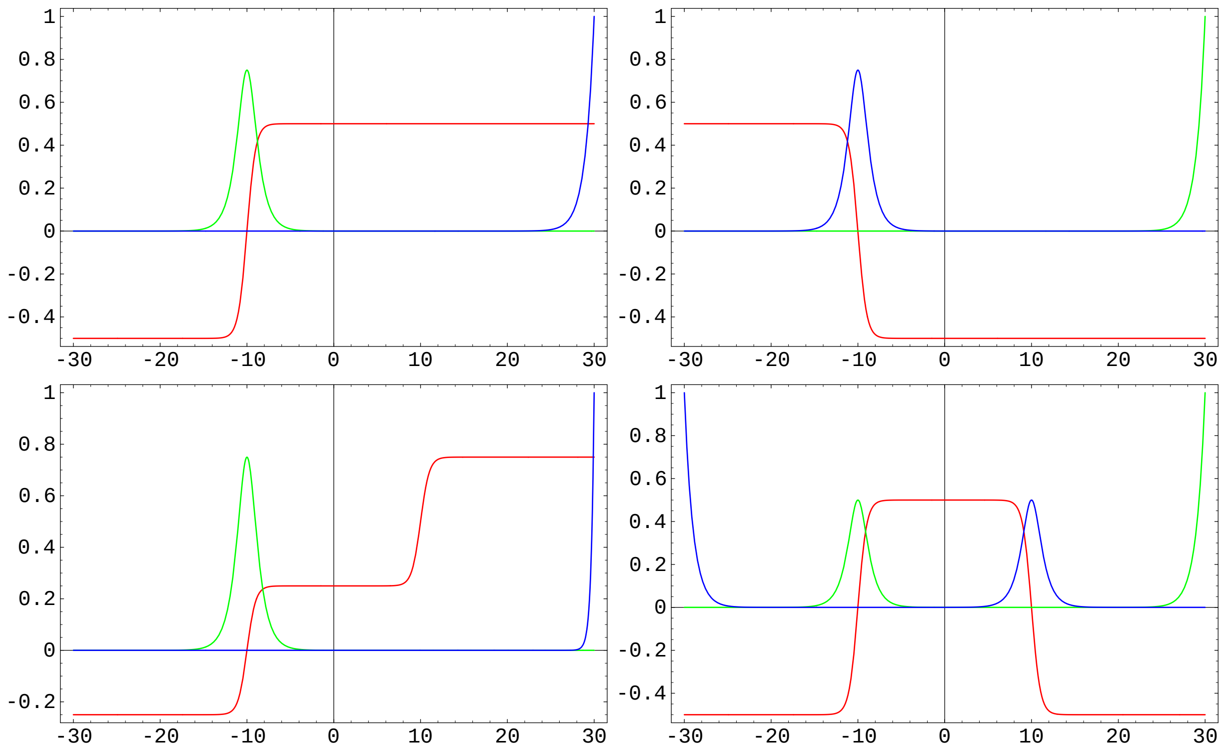

Figure 1: The profiles of the fermionic zero mode solutions β_+ (green) and β_- (blue) are shown together with the bosonic background ϕ (red) for the kink (top left), antikink (top right), kink-kink (bottom left) and kink-antikink (bottom right) configurations, respectively. We have chosen $z_1 = 10$, $\eta = h = a = 1$.

whereas the kink zero mode solution falls off exponentially at infinity $\beta_+ \propto \mathrm{e}^{-h\phi z}$ with $\phi = \eta$, this no longer holds for the kink-antikink system. Indeed, extending the wave function past the antikink, ϕ flips sign, and $\beta_+ \propto \mathrm{e}^{+h\eta z}$ blows up.

Maybe in a cosmological setting the zero mode solution does not have to be extended all the way to infinity as there is a natural cutoff. The cutoff can for example be due to the presence of yet another kink or anti-kink (cf. the argument for the existence of a network of global strings), or (in a higher dimensional setting) the boundary of spacetime. We can compare the height of the wave function localized at the kink at $z = -z_1$ with how it blows up at infinity if it is extended past the antikink at $z = z_1$. This gives

$$\frac{\beta_+(z = -z_1)}{\beta_+(z \to \infty)} = \frac{\mathrm{e}^{m_\psi z_1}}{\mathrm{e}^{m_\psi (2z - z_1)}}, \tag{10}$$

where we took the thin wall limit $a \to \infty$. It follows that $|\psi(-z_1)| = |\psi(z \to \infty)|$ for $z = 3z_1$. Thus the cutoff has to be smaller $z_{\rm cutoff} < 3z_1$ for the fermion to be localized on the kink at $z = -z_1$.

2.3 Massive bound states

We now proceed to consider non-zero energy solutions. In the rest frame $\vec{k} = 0$, and the Dirac equation reads

$$\left[i\gamma^0\partial_0 + i\gamma^z\partial_z - h\phi\right]\psi = 0. \tag{11}$$

Now $\xi^i_\pm$, which are eigenspinors of γ^z (see (5), (7)), are not eigenspinors of γ^0. The fact that the energy is non-zero mixes the spinors $\xi_\pm$.

We write the energy eigenstates as

$$\psi^k_E = \mathrm{e}^{-iEt}\left(\beta^k_+(z)\xi^k_- - i\beta^k_-(z)\xi^k_-\right) \tag{12}$$

with $k = 1, 2$. The factor $-i$ is put in for future convenience. ψ^1_E and ψ^2_E decouple in the Dirac equation. Their equations of motion are identical. In the following we will suppress the superscript, keeping in mind that the equations apply to both. The energy eigenstates satisfy $\hat{H}\psi_E = E\psi_E$ with

$$\hat{H} = i\partial_t = \gamma^0(-i\gamma^z\partial_z + h\phi). \tag{13}$$

This gives the following set of coupled equations

$$\begin{aligned} E\beta_+ &= (\partial_z - h\phi)\beta_- \\ -E\beta_- &= (\partial_z + h\phi)\beta_+ \end{aligned} \tag{14}$$

where we used $\gamma^0\xi_\pm = \pm i\xi_\mp$.

Let us analyze the asymptotic solutions. In the regions at spatial infinity, and for the case of multiple (anti)kink systems in between the (anti)kinks, the background field approaches a constant $\partial_z \phi = 0$. In this limit the equations of motion decouple. Eq. (14) can be written in the form

$$\partial_z^2 \beta_\pm = (m_f^2 - E^2)\beta_\pm \tag{15}$$

which has the solution

$$\tilde{\beta}_\pm = U_\pm \mathrm{e}^{\sqrt{m_f^2 - E^2} z} + D_\pm \mathrm{e}^{-\sqrt{m_f^2 - E^2} z} \tag{16}$$

with $m_f = h|\phi|$ the vacuum fermion mass, and $U_\pm, D_\pm$ integration constants. At spatial infinity one of the solutions is normalizable. Plugging back in (14) we find

$$\frac{U_-}{U_+} = \frac{-\sqrt{m_f^2 - E^2} - h\phi}{E}, \qquad \frac{D_-}{D_+} = \frac{\sqrt{m_f^2 - E^2} - h\phi}{E}. \tag{17}$$

In the core of a kink located at $z = 0$ the bosonic background is $\lim_{z\to 0} \phi(0) = \eta a z + \mathcal{O}(z^2)$. To find the asymptotic solutions we write $\beta_\pm = b_\pm z^{n_\pm}$ near the origin, with $n_\pm \geq 0$ to assure regularity at the core. The equations of motion (14) reduce to

$$\begin{aligned} E b_+ z^{n_+} &= n_- b_- z^{n_- - 1} - m_f a b_- z^{n_- + 1}, \\ -E b_- z^{n_-} &= n_+ b_+ z^{n_+ - 1} + m_f a b_+ z^{n_+ + 1}. \end{aligned} \tag{18}$$

The term proportional to m_f is higher order. There is no general solution that solves the equations. There are however two special solutions. The first is the zero mode solution (9) with $E = 0$, $b_- = 0, n_+ = 0$, which sets all lowest order terms to zero. The second special solution is a bound state solutions with $E \neq 0$ and $n_- = 0$, $n_+ = 1$. This choice allows the lowest order terms in the first equation in (18) to cancel, whereas in the second equation the higher order terms come in at lowest order as well.

Kink or antikink. The kink is known to have one bound state solution, which can be constructed explicitly. For simplicity consider a kink located at $z_1 = 0$. To find the bound state solution we substitute $\beta_+ = C\phi\beta_-$ in the Dirac equations (14). Provided we choose $C = -E/(a\eta)$ and $E^2 = a\eta(2h - a/\eta)$, the equations are degenerate and read

$$\partial_z \beta_- = \eta \tanh(az) \left(\frac{a}{\eta} - h\right) \beta_- \tag{19}$$

which has as solution $\beta_-(z) = N \cosh(az)^{1-h/a}$ with N a normalization constant. This bound state only exists for $h > a/(2\eta)$. There is no bound state solution in the thin wall limit $a/\eta \to \infty$. Unlike the zero mode solution, the number of bound states is not an adiabatic invariant, it is not determined by an index theorem.

The above result agrees with the asymptotic analysis. Indeed, consider a kink located at $z = -z_1$ in the thin wall approximation (see (2) with $B = 0$). In region I $z \in [-\infty : 0]$ the bosonic background is $\phi = -\eta$, and in region II $z \in [0 : \infty]$ it is $\phi = \eta$. The normalizable solution in region I is $\psi_E^I = \mathrm{e}^{\sqrt{m_f^2 - E^2} z}(U_+^I \xi_+ + U_-^I \xi_-)$, the solution in region II is $\psi_E^{II} = \mathrm{e}^{-\sqrt{m_f^2 - E^2} z}(D_+^{II} \xi_+ + D_-^{II} \xi_-)$. Matching the solutions at $z = 0$ requires

$$U_-^I / U_+^I = D_-^{II} / D_+^{II} \tag{20}$$

which is impossible for $E \neq 0$. Setting $E = 0$ in the equations of motion, we see that the first equation of (14) gives back the normalizable zero mode solution. Hence, in the thin wall approximation we find back the zero mode solution, but do not find additional massive bound states.

Kink-kink Consider now the kink-kink system. Following the same strategy as for the kink system we write $\beta_+ = C\phi\beta_-$. Now, however, we do not find a special solution to the equations of motion (14). The difference with the kink case is the appearance of cross terms $\tanh(a(z - z_1)) \tanh(a(z + z_1))$ which destroy the solution. One may argue that our ansatz for β_+ should be changed for the kink-kink system, but it is fixed by the requirement that at $z = -z_1$ the solution should approach the kink solution.

In the limit $z_1 \to 0$ the kink-kink system reduces to a single kink, which has a massive bound state. This bound state is very fragile. It dissapears in the thin wall limit. And from the kink-kink analysis it follows that it dissapears as well if the kink is deformed.

The kink-kink system has one zero mode and no bound states in the spectrum.

We have also tried to construct solutions to the equations (14) numerically, but could not find solutions.

Kink-antikink The story for the kink-antikink system is similar to the kink-kink system. We choose an Ansatz $\beta_+ = C\phi\beta_-$, which is dictated by the requirement that the solution approaches the kink solution at the kink at $z = -z_c$. No bound state solution is found, compared to the kink case it is the appearance of cross terms $\tanh(a(z - z_1)) \tanh(a(z + z_1))$ that destroy the solution.

The same conclusion also follows in the thin wall approximation (2) with $B = -1$. The solutions in the three distinct regions are given by (16). The boundary conditions constrain the solution in the asymptotic regions: $D_\pm = 0$ in the $z < -z_1$ region, and $U_\pm = 0$ in the $z > z_1$ region. Matching the solutions at both $z = -z_1$ and $z = z_1$, we find that the coefficients appearing in (16) in the middle region $-z_1 < z < z_1$ satisfy $D_+/D_- = U_+/U_- = 1$, which is impossible to satisfy, even for $E = 0$. Hence, there are no bound states at all.

This can be understood as follows. The decaying solution at minus infinity is $\propto \xi_+$, whereas the solution at plus infinity is $\propto \xi_-$, and these orthogonal spinors cannot be matched in the middle. This spinor structure of the decaying solution is determined by the value $\phi(\pm\infty)$, hence is directly related to topological arguments. The decaying solution at plus and minus infinity is the same spinor iff the topological charge is non-zero: $Q \propto \phi(\infty) - \phi(-\infty) \neq 0$. For the kink-antikink system, on the other hand $\phi(\infty) - \phi(-\infty) = 0$. The kink-antikink system can be adiabatically deformed to the trivial vacuum. As the vacuum carries no bound states, the kink anti-kink system has none either.

The Dirac equation describes a fermion with a spatially varying mass term, i.e. a fermion living in an effective potential $V_{\rm eff} = |h\phi|^2$. For the kink-antikink system the effective potential is a double well potential. Experience with the analogue quantum mechanical double well systems suggests there should be two bound states, split in energy with the energy difference going to zero in the limit that the two wells are taken infinitely far apart. It seems from our results above that this intuition fails for the fermion zero modes. Why?

The analogue with the quantum double well system breaks down. The double well potential in quantum mechanical problems can be adiabatically transformed by reducing the distance between the two wells, and for $z_1 = 0$ one ends up with a single well. The number of bound states is preserved. What happens is that the two lowest bound states of the single well become the two energy split bound states of the double wells. Now taking $z_1 \to 0$ in the kink-antikink system, they will annihilate, and one ends up with the trivial vacuum. The vacuum does not support bound states, so it is not surprising that there are no zero modes on the kink-antikink system. The number of zero modes is given by an index theorem, it is $(1/\eta)(\phi(\infty) - \phi(-\infty))$, which is zero for a kink-antikink configuration.

Note that our results disagree with those stated in [11, 12, 13]. In [12] the energy is calculated for $\Psi = (\psi_+ \pm \psi_-)_{B=0}$ states, which is split in energy and goes to zero in the $z_1 \to \infty$ limit, just as in a double well potential. However, these are zero mode solutions to a background with only one kink or antikink, they do not satisfy the Dirac equation for the kink-antikink system. Using instead $\Psi = (\psi_+ \pm \psi_-)_{B=-1}$, the wave function is non-renormalizable and the energy blows up.

3 Cosmic strings

Having built up intuition on the existence of fermionic modes on relatively simple domain wall system, let us see whether the same kind of arguments apply to systems with cosmic strings.

Consider two parallel strings, one with winding number (= topological charge) $+n$ and one with winding number $-n$. There is no net winding around the string, and thus — by analogy with domain walls — we do not expect fermionic zero modes in this system. The system can be deformed adiabatically to the trivial vacuum which does not support bound states. Looking at the spinor structure, zero modes on a string with $n > 0$ ($n < 0$), have positive (negative) chirality. Pasting a string and “antistring” together, the zero modes cannot be matched in the middle because of the orthogonal spinor structure — in close analogy with the domain wall. This problem is not ameliorated in the limit that the strings are taken far apart. The solution that is localized at the string does not resolve the angular dependency at the antistring, and vice versa. It might be that a bound state with non-zero energy can cure this problem since it mixes the two chiralities. But we have seen that for the kink-antikink system this was not the case.

The system of two parallel strings is closely related to the loop of cosmic string. In both cases there is no net winding number separating the system from the vacuum. Hence, we expect that the spectrum of fermionic bound states found on two anti-parallel strings (none!) is the same as for the string loop. This implies that there are no fermionic vortons, no loops of cosmic strings stabilized by fermionic currents.

In this section we study fermionic bound states on straight strings and loops of string. In the next subsection, we discuss the bosonic string background. In section 3.2.1 and 3.3 we derive the fermionic spectrum for straight strings and a string loop respectively. To count the number of bound states an asymptotic analysis suffices. A numerical analysis is needed to obtain the profile of the bound states solutions, which can be done in the full system including back reaction effects. This involves systems of nonlinear ordinary and partial differential equations for straight strings and string loops, respectively; it is left as a future project.

3.1 Bosonic string background

In this section we discuss the asymptotic behavior of local cosmic string, both for straight strings and for loops. We neglect gravity effects, as well as back reaction effects from fermions (which will not effect the number of bound states). Moreover, we restrict ourselves to strings satisfying the Bogomolny limit. Note that in this limit there is no force between parallel straight strings. Details on the cylindrical and toroidal coordinates used can be found in Appendix A.

The flat space-time metric is

$$\mathrm{d}s^2 = \mathrm{d}t^2 - h_1^2 \mathrm{d}x_1^2 - h_2^2 \mathrm{d}x_2^2 - h_3^2 \mathrm{d}x_1^2. \tag{21}$$

The Lagrangian describing the gauged string (Nielsen-Olesen string) reads

$$\mathcal{L} = (D_\mu \phi)(D^\mu \phi)^* - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{\lambda^2}{2} \left(|\phi|^2 - \eta^2 \right)^2 , \tag{22}$$

with $D_\mu\phi = (\partial_\mu + ieA_\mu)\phi$ and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The static cosmic string configuration aligned with the 3-axis, has a magnetic field confined to the string core and aligned with the 3-axis $B \equiv B_3 = F_{12}/(h_1 h_2)$. The Higgs field winds around the x_3 axis with winding number n (we take $n > 0$ without loss of generality). The string energy per unit length is

$$\begin{aligned}\mu &= \int \sqrt{-g_T}\mathrm{d}x_1\mathrm{d}x_2 \left\{(D_\mu\phi)(D^\mu\phi)^* - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{\lambda^2}{2}\left(|\phi|^2 - \eta^2\right)^2\right\} \\ &= \int \sqrt{-g_T}\mathrm{d}x_1\mathrm{d}x_2 \left\{\left|\frac{D_1\phi}{h_1}\right|^2 + \left|\frac{D_2\phi}{h_2}\right|^2 + \frac{1}{2}\left|\frac{F_{12}}{h_1h_2}\right|^2 + \frac{\lambda^2}{2}\left(|\phi|^2 - \eta^2\right)^2\right\} \end{aligned} \tag{23}$$

where the integration is over the transverse coordinates x_1, x_2 which span the plane perpendicular to the string. The metric on this transverse plane satisfies $\sqrt{-g_T} = h_1h_2$. Following the arguments of Bogomolny — using partial integration and the identity $[D_1, D_2]\phi = ieF_{12}\phi$ — we write the tension as

$$\begin{aligned}\mu = \int \mathrm{d}x_1\mathrm{d}x_2\, h_1h_2 \Bigg\{ &\left|\left(\frac{D_1}{h_1} + i\frac{D_2}{h_2}\right)\phi\right|^2 + \frac{1}{2}\left|\frac{F_{12}}{h_1h_2} - e(|\phi|^2 - \eta^2)\right|^2 \\ &+ \frac{\lambda^2 - e^2}{2}(|\phi|^2 - \eta^2)^2 - e\eta^2\frac{F_{12}}{h_1h_2}\Bigg\}. \end{aligned} \tag{24}$$

We have omitted the boundary term. The last term in the above expression is

$$Q = -e\eta^2 \int \mathrm{d}x_1\mathrm{d}x_2 F_{12} = -e\eta^2 \int B = 2\pi n\eta^2 \tag{25}$$

for a string with winding number n. Note that this is a topological number for a straight string, but not for a loop of string. For a loop, the string goes up and down through the transverse plane, and these contributions cancel. Hence, a string loop can decay, there is no topological barrier separating it from the vacuum. In the BPS limit $e = \lambda$, the third term cancels, and it follows that $\mu \geq Q$. The minimum energy configuration with $\mu = Q$ satisfies the BPS equations

$$\left(\frac{D_1}{h_1} + i\frac{D_2}{h_2}\right)\phi = 0 \tag{26}$$

$$\frac{F_{12}}{h_1h_2} - e(|\phi|^2 - \eta^2) = 0 \tag{27}$$

3.1.1 Straight string aligned with z-axis

We use cylindrical coordinates with $x_1 = r$, $x_2 = \varphi$ and $x_3 = z$. The results for this case are well known, however, we repeat them here for completeness. The scale factors are $h_r = h_z = 1$ and $h_\varphi = r$. Using the standard string Ansatz

$$\phi = \eta f(r)\mathrm{e}^{in\varphi} \tag{28}$$

$$A_\varphi = -\frac{n}{e}a(r) \tag{29}$$

the BPS equations become

$$
\begin{aligned}
(\partial_r + \frac{i}{r} D_\varphi)\phi &= 0 \qquad \Rightarrow \qquad \partial_r f - \frac{n}{r}(1-a)f = 0 \\
\frac{\partial_r A_\varphi}{r} - e(|\phi|^2 - \eta^2) &= 0 \qquad \Rightarrow \qquad n\frac{\partial_r a}{r} - e^2\eta^2(1-f^2) = 0
\end{aligned} \tag{30}
$$

with boundary conditions $f(0) = a(0) = 0$ and $f(\infty) = a(\infty) = 1$. Note that the energy of the configurations remains finite, because the gauge field cancels the kinetic energy of the Higgs field (due to the phase dependence) at infinity. The magnetic field is given by $B = \nabla \times A = \partial_r A_\varphi$. The solution is symmetric under rotations around the z-axis.

First, we consider the equations in the limit $r \to 0$. The term proportional to a in the first equation tends to zero, and it follows $f \propto r^{|n|}$. The second term in the second equation becomes r-independent and can cancel the first term for $a \propto r^2$, i.e.

$$
(\text{for } r \to 0) \quad \begin{cases} f \propto r^{|n|} \\ a \propto r^2 \end{cases} \tag{31}
$$

Far away from the string core the fields tend to their vacuum values $f \to 1$, $a \to 1$. To see how the fields approach these values we substitute $f = 1 - c_1 e^{-c_3 r}$ and $a = 1 - r c_2 \mathrm{e}^{-c_3 r}$ into the BPS equations. Upon taking the limit $r \to \infty$ we find $c_3 = \sqrt{2} e\eta = m_A$ the gauge boson mass (in the BPS limit $m_A = m_\phi$) and $c_2/c_1 = m_A/n$. We have:

$$
(\text{for } r \to \infty) \quad \begin{cases} f - 1 \propto \mathrm{e}^{-m_A r} \\ a - 1 \propto (m_A r)\mathrm{e}^{-m_A r} \end{cases} \tag{32}
$$

3.1.2 Cosmic string loop

Consider a loop of cosmic string lying in the x-y-plane. We introduce toroidal coordinates with $x_1 = v$, $x_2 = u$, $x_3 = \varphi$ such that the symmetry axis of the loop is the z-axis and the loop is aligned with the φ-direction. This set up is certainly less symmetric than that of the straight string since the fields now depend on the angle u winding around the string (in fact, to find the cosmic string loop, one has to solve partial differential equations in contrast to ordinary differential equations for the straight case). If the radius of the loop becomes very large, the u-dependence will practically disappear. The string Ansatz can be generalized to

$$
\phi = \eta f(u,v)\mathrm{e}^{inu} \tag{33}
$$

$$
A_u = -\frac{n}{e} a(u,v) \tag{34}
$$

and we used the "radial" gauge $A_v = 0$. The boundary conditions are

$$\begin{aligned}
\lim_{v\to\infty} f, a &= 0 \quad \text{string core,} \\
\lim_{(v,u)\to(0,0)} f, a &= 1 \quad \text{spatial infinity,} \\
\lim_{(v,u)\to(0,\pi)} f, a &= 1 \quad \text{origin.}
\end{aligned} \tag{35}$$

Note that we have assumed here that the loop radius is much larger than the string width $\kappa \gg m_A^{-1}$, so that at the origin (and along the full z-axis) the bosonic fields approach their vacuum values. This requirement can be relaxed, all that is needed is that the fields are regular at the origin. However, in that case it is doubtful how well a cosmic string loop pictures the situation.

The gauge field A_u is non-zero to cancel the kinetic term of the Higgs field ϕ at spatial infinity. The magnetic field is $B_\varphi = -\partial_v A_u(u,v)$. The BPS equations are

$$\partial_v f + n(1-a)f = 0, \qquad \partial_u f = 0 \tag{36}$$

$$n\partial_v a + h_u h_v e^2 \eta^2 (1 - f^2) = 0 \tag{37}$$

We get two equations for f from splitting the real and imaginary parts. It follows that the Higgs profile function is u-independent; the only u dependence of the Higgs is in the winding phase. The Higgs profile is independent of the scale factors h_i, and thus of the loop radius κ. The gauge profile function a is generically u-dependent since $h_u h_v$ depends on u. At infinity $h_u h_v$ is only a function of v and both f and a are functions of v only. Note that the u-independence of a far away from the string loop reflects the fact that there is no net winding around the string loop, no net topological charge.

Let us consider the asymptotic solutions. For details of the behavior of the toroidal coordinates in these limits see Appendix A. First consider the limit $v \to \infty$ in which the string core is approached and $f \to 0$, $a \to 0$. The BPS equations reduce to

$$\partial_v f + nf = 0, \qquad \partial_v a + \text{const.} \cdot \mathrm{e}^{-2v} = 0 \tag{38}$$

which has as solution

$$f \propto e^{-nv} \sim \left(\frac{\delta r}{\kappa}\right)^n, \qquad a \propto e^{-2v} \sim \left(\frac{\delta r}{\kappa}\right)^2 \tag{39}$$

with $\delta r = |r - \kappa|$ denoting the distance from the string core. The behavior is the same as in the core of a straight string (31). The fields approach their vacuum values far away from the string loop, corresponding to the limit $v \to 0$, $u \to 0$, and can be written in the form $f = 1 - \tilde{f}$, $a = 1 - \tilde{a}/n$. The BPS equations become

$$\partial_v \tilde{f} = \tilde{a}, \qquad \partial_v \tilde{a} = h_u h_v m_A^2 \tilde{f} \tag{40}$$

with $m_A = \sqrt{2}e\eta$. At spatial infinity, corresponding to $u = v \to 0$, the factor $h_u h_v$ tends to $(\kappa/v^2)^2$. The equations are then solved to get (up to a normalization constant)

$$
\begin{aligned}
f - 1 &\propto n\mathrm{e}^{-m_A\kappa/v} \sim n\mathrm{e}^{-m_A r} \\
a - 1 &\propto \frac{m_A\kappa}{v^2}\mathrm{e}^{-m_A a/v} \sim \frac{m_A r^2}{\kappa}\mathrm{e}^{-m_A r}
\end{aligned}
\tag{41}
$$

At the origin, corresponding to $u \to \pi, v \to 0$, the factor $h_u h_v$ tends to $(\kappa/2)^2$. The solutions are up to a renormalization constant

$$
\begin{aligned}
f - 1 &\propto \sinh\left(\frac{m_A\kappa v}{2}\right) \sim \sinh(m_A r) \sim (m_A r) \\
a - 1 &\propto \left(\frac{m_A\kappa}{2}\right)\left(\cosh\left(\frac{m_A\kappa v}{2}\right) - 1\right) \sim \left(\frac{m_A\kappa}{2}\right)(\cosh(m_A r) - 1) \sim \left(\frac{m_A\kappa}{2}\right)(m_A r)^2
\end{aligned}
\tag{42}
$$

The approach to the vacuum is exponentially fast at spatial infinity (just as it is in the case of a straight string), but slower — power law — at the origin. The solutions are valid for $v \ll 1$ which corresponds to $r \gg \kappa$ at spatial infinity, and $r \ll \kappa$ at the origin. The width of the string is $r \sim m_A^{-1}$.

3.2 Fermionic spectrum

Consider a Dirac spinor $\psi = \binom{\psi_L}{\psi_R}$, with ψ_L, ψ_R left and right handed two-component Weyl spinors. The fermion is charged under the $U(1)$ of the string. The Lagrangian reads

$$
\mathcal{L} = \psi_L^\dagger \bar{\sigma} \cdot D\psi_L + \psi_R^\dagger \sigma \cdot D\psi_R - \lambda\phi\psi_L^\dagger\psi_R - \lambda\phi^*\psi_R^\dagger\psi_L \tag{43}
$$

with $D_\mu\psi_{L,R} = (\partial_\mu + iq_{L,R}eA_\mu)\psi_{L,R}$. Gauge invariance requires $1 - q_L + q_R = 0$, where we have normalized the Higgs charge to unity. For a Dirac spinor the two Weyl fermions ψ_L, ψ_R are independent, and can have different charges. For a Majorana fermion $q_L = -q_R = 1/2$ and $\psi_R = i\sigma^2\psi_L^*$ are charged conjugate of each other, so that the Majorana spinor satisfies the reality condition $\psi_M = \psi_M^c$. The reality condition decreases the degrees of freedom. The equations of motion are

$$
\begin{aligned}
i\bar{\sigma} \cdot D\psi_L - \lambda\phi\psi_R &= 0 \\
i\sigma \cdot D\psi_R - (\lambda\phi)^*\psi_L &= 0
\end{aligned}
\tag{44}
$$

For a Majorana the two equations are not independent, but complex conjugate of each other.

3.2.1 Single straight string

The transverse and longitudinal part of the solution decouple, and we can write

$$
\psi_L = \alpha(z,t)\beta_L(r,\varphi)\xi_L, \qquad \psi_R = \alpha(z,t)\beta_R(r,\varphi)\xi_R. \tag{45}
$$

Here α and $\beta_{L,R}$ are scalar functions of the transverse and longitudinal coordinates respectively. The constant Weyl spinors $\xi_{L,R}$ are eigenvectors of the projection operator

$$\sigma^0\sigma^z\xi_L = \pm\xi_L, \qquad \sigma^0\sigma^z\xi_R = \mp\xi_R, \tag{46}$$

with the upper sign for a vortex ($n > 0$) and lower sign for an anti-vortex ($n < 0$). Explicitly $\xi_L = \binom{1}{0}$ and $\xi_R = \binom{0}{-i}$ for $n > 0$, and $\xi_L = \binom{0}{1}$ and $\xi_R = \binom{-i}{0}$ for $n < 0$, where the factor i in the right-handed spinors is for convenience. The Weyl spinors have the following properties: $i\sigma^0\sigma^1\sigma^2\sigma^3\xi_{L,R} = -\xi_{L,R}$ (chirality eigenstates), and $\sigma^r\xi_L = i\mathrm{e}^{\pm i\varphi}\xi_R$, $\sigma^r\xi_R = -i\mathrm{e}^{\mp i\varphi}\xi_L$, $\sigma^\varphi\xi_L = \mp e^{\pm i\varphi}\xi_R$ and $\sigma^\varphi\xi_R = \mp e^{\mp i\varphi}\xi_L$. Using this the transverse part of the Dirac equation can be written as

$$\begin{aligned}\left(\partial_r \pm i\frac{D_\varphi}{h_\varphi}\right)\beta_L &= \lambda\eta f(r)\mathrm{e}^{\pm i(|n|-1)\varphi}\beta_R \\ \left(\partial_r \mp i\frac{D_\varphi}{h_\varphi}\right)\beta_R &= \lambda\eta f(r)\mathrm{e}^{\mp i(|n|-1)\varphi}\beta_L\end{aligned} \tag{47}$$

with $f(r)$ the profile function of the Higgs field in (29), and as before the upper (lower) sign is for a vortex (anti-vortex). The phase-dependence can be resolved by choosing

$$\beta_{L,R}(r,\varphi) = b_{L,R}(r)\mathrm{e}^{il_{L,R}\varphi}, \qquad l_L = l_R \pm (|n|-1). \tag{48}$$

The equations reduce to

$$\begin{aligned}\left(\partial_r \mp \frac{(l_L - nq_L a)}{r}\right)b_L &= m_f f b_R \\ \left(\partial_r \pm \frac{(l_R - nq_R a)}{r}\right)b_R &= m_f f b_L\end{aligned} \tag{49}$$

where we used $h_\varphi = r$, and $m_f = \lambda\eta$ is the vacuum fermion mass.

We can solve the Dirac equation asymptotically. For simplicity we restrict the analysis to the vortex solution with $n > 0$. At $r \to \infty$ the functions $f \to 1$, $a \to 1$ and the $1/r$ term in the equations of motion is subdominant. The solution is

$$b_{L,R} = \mathrm{e}^{\pm m_f r} \tag{50}$$

There is one renormalizable solution. At the origin $f \sim r^n$ and $a \sim r^2$. We write $b_{L,R} \sim r^{c_{L,R}}$. Then the equations of motion reduce to

$$\begin{aligned}c_L r^{c_L-1} - l_L r^{c_L-1} &= m_f r^{c_R+n} \\ c_R r^{c_R-1} + l_R r^{c_R-1} &= m_f r^{c_L+n}\end{aligned} \tag{51}$$

A singular solutions exist for $c_L = l_L$, $c_R = -l_R$. Two regular solutions exist for $c_L = l_L$, $c_R = l_L + n + 1$ and $c_R = -l_R = n - 1 - l_L$, $c_L = l_L + 2n$. They connect to the one normalizable solution at infinity. Both need to be normalizable at the origin

$$\int \sqrt{-g_T} \mathrm{d}^2 x_T |\psi_{L,R}|^2 < \infty \qquad \Rightarrow \qquad \int \mathrm{d}r\, r |b_{L,R}|^2 < \infty \qquad \Rightarrow \qquad 2c_{L,R} + 1 > -1. \quad (52)$$

This is satisfied for

$$-1 < l_L < n. \quad (53)$$

Thus a string with winding number n has $|n|$ zero mode solutions with $l = 0, .., |n| - 1$. The zero mode spinor is proportional to ξ_+ for a vortex with $n > 0$, and proportional to ξ_- for an antivortex with $n < 0$.

For $|n| = 1$ the zero-mode solution does not have any angular dependence. The Majorana solution (with $\psi_R = \psi_L^c$) can be given analytically [5]:

$$\beta_\pm = \mathrm{e}^{\int_0^r \mathrm{d}r' \left(-\frac{a(r')}{2r'} + m_f f(r')\right)}. \quad (54)$$

The longitudinal part of the Dirac equation $(\sigma^0 \partial_0 - \sigma^z \partial_z)\psi_L = 0$ becomes

$$(\partial_0 \mp \partial_z)\, \alpha = 0 \quad \Longrightarrow \quad \alpha = \alpha(t \pm z). \quad (55)$$

The zero mode on a string (antistring) moves at the speed of light in the minus (plus) direction along the string and has a dispersion relation $E = |k|$.

3.2.2 Two parallel strings

Consider now two parallel strings, one string located at $x = x_1$ and one string or antistring at $x = -x_1$. If the distance between them is much larger than the string width, the bosonic background is well approximated by the two separated string solutions pasted together at $x = 0$.

String-string. Consider two parallel strings which both have winding number $n = +1$. The bosonic background is then of the form $\phi = f_1 \mathrm{e}^{i\varphi} + f_2 \mathrm{e}^{i\varphi}$ with $f_i \to 0$ at string i, and approaching the vacuum $f_i \to 1$ everywhere else.

The zero mode solutions localized at the string at $x = -x_1$ and the zero mode at the second string localized at $x = x_1$ have the same spinor structure. Moreover, near the string the angular dependence is resolved. Hence, they can be pasted together at $x = 0$. Indeed, we expect a zero-mode solution of the form $\psi_{R,L} = (\beta_1 \xi_+ + \beta_2 \xi_+)_{R,L}$, with β_i localized at string i. The equations of motion read:

$$\left(\partial_r + i\frac{D_\varphi}{h_\varphi}\right)(\beta_1 + \beta_2)_L = \lambda\eta(f_1 + f_2)(\beta_1 + \beta_2)_R \quad (56)$$

and a similar equation for the right handed spinors (interchanging $L \leftrightarrow R$ and a different sign in front of D_φ, see (47)). The two solutions decouple and are approximately given by the solutions for the single string; since $f_1\beta_2 \to 0$ at the location of the first string and $f_2\beta_1 \to 0$ at the location of the second, the corrections are small. The important thing to note is that the zero mode solution localized at one string can be extended without problems to the region where the other string is, as the angular dependence is still resolved there — this is in sharp contrast with the string antistring system to be discussed shortly.

This agrees with topological arguments. In the limit $x_1 \to 0$ the system reduces to a single string with winding number $n = 2$. Such a string has two zero mode solutions. As the strings are adiabatically separated, the number of zero modes remains constant. The two zero modes are the states localized at either one of the strings. This is different from the kink case. For strings, the number of fermionic zero modes equals the topological charge which is in $\mathbb{Z}$, while for kinks the topological charge is in $\mathbb{Z}_2$ (asymptotic vacua either different or the same) and the number of zero modes does not exceed unity.

String-Antistring. Consider now two parallel strings with opposite winding number, one string with $n = +1$ and the other with $n = -1$. The zero mode solution localized on the string and antistring have orthogonal spinor structure and decouple in the equations of motion. We can thus consider them separately. We write the zero-mode solution in the form $\psi_{L,R} = (\beta_1\xi_+ + \beta_2\xi_-)_{L,R}$, with β_i localized at string i. Let's concentrate on the β_1 solution; the β_2 solutions are similar. The equation of motion for β_1 is

$$\left(\partial_r + i\frac{D_\varphi}{h_\varphi}\right)\beta_{1L} = \lambda\eta(f_1 + f_2 \mathrm{e}^{-2i\varphi})\beta_{1R} \tag{57}$$

and a similar expression for $L \leftrightarrow R$. The β_1 solution does not solve the equations of motion near the antistring, where $n = -1$ and the angular dependence is unresolved. Although $\beta_1 \to 0$ near the antistring, it is not equal to zero. Thus, the zero mode solution disappears from the spectrum.

This agrees with the adiabatic argument. In the limit $x_1 \to 0$ one ends up with the vacuum $(n = 0)$. Since there is no topological charge, there are no zero modes. As the strings are slowly pulled apart the number of zero modes remains constant.

The adiabatic argument should also hold for bound state solutions with $E \neq 0$. The non-zero energy mixes ξ_+ and ξ_- spinors, which in principle might lead to a solution valid in the whole domain. We thus try the Ansatz

$$\psi_{L,R} = \mathrm{e}^{-iEt}\left(\beta_1\xi_+ + \beta_2\xi_-\right)_{L,R}. \tag{58}$$

The equations of motion are then of the form:

$$
\begin{aligned}
iE\beta_{1L} + (\partial_r - \frac{i}{2r}D_\varphi)\mathrm{e}^{-i\varphi}\beta_{2L} &= m_f(f_1\mathrm{e}^{i\varphi} + f_2\mathrm{e}^{-i\varphi})\beta_{2R} \\
iE\beta_{2L} + (\partial_r + \frac{i}{2r}D_\varphi)\mathrm{e}^{i\varphi}\beta_{1L} &= m_f(f_1\mathrm{e}^{i\varphi} + f_2\mathrm{e}^{-i\varphi})\beta_{1R}
\end{aligned}
\tag{59}
$$

and a similar equation for $L \leftrightarrow R$. To resolve the angular dependence we write $\beta_i = \mathrm{e}^{il_i\varphi}b_i$, with $i = \{1,2,3,4\} = \{1L, 2L, 1R, 2R\}$. Then the angular structure of the Dirac equation is of the form

$$
\begin{aligned}
\mathrm{e}^{il_1\varphi} + \mathrm{e}^{i(l_2-1)\varphi} &= \mathrm{e}^{i(l_4+1)\varphi} + \mathrm{e}^{i(l_4-1)\varphi} \\
\mathrm{e}^{il_2\varphi} + \mathrm{e}^{i(l_1+1)\varphi} &= \mathrm{e}^{i(l_3+1)\varphi} + \mathrm{e}^{i(l_3-1)\varphi}
\end{aligned}
\tag{60}
$$

Set $l_1 = l_4 + 1$ and $l_2 = l_4$. This resolves the angular dependence of the first equation above. Inserting this into the second equation, one finds that no l_3 exists that can solve the angular dependence. Likewise, for $l_1 = l_4 - 1$ and $l_2 = l_4 + 2$, the angular dependence for the first equation can be solved, but again no l_3 exists that solves the angular dependence of the second equation. There is no choice of l_i that resolves the angular dependence! And thus we conclude that the string antistring system does not support massive bound states either. [1]

3.3 Cosmic string loops

In this section we discuss fermionic bound states on a loop of cosmic string. The total winding number of the string loop is zero, the bosonic configuration can be continuously deformed towards the trivial vacuum. And thus we do not expect any bound states to exist, in direct analogy with the kink antikink and vortex anti-vortex system.

It is clear that the loop does not support zero mode solutions moving at the speed of light [14, 15]. Massless particles which go at the speed of light move along straight lines, and not along a curved path. (Indeed, along a curved path $\dot{k} \neq 0$ and a dispersion relation $E^2 = k^2$ is not possible). But a priori this does not exclude a zero mode solution in the limit $k \to 0$. In Ref. [9] the existence of zero mode solutions was claimed. However, their analysis is incomplete. They failed to check the behavior of the zero mode solutions at the center of the string, where the solutions is singular. We show this explicitly in the appendix B, where we give the asymptotic solutions of the Dirac equation.

The obstruction to bound states is the same as for the vortex anti-vortex system. For a loop lying in the (x, y)-plane, consider the solution in, say, the $x = 0$ plane. The angular dependence

[1] The single straight string is known to have massive bound states [7, 8]. These states do resolve the angular dependence in the Dirac equations.

of the equation of motion is exactly the same as for the vortex-antivortex. Since in the latter case the angular dependence could not be resolve, the same is true for the string loop. There are no bound state solution.

To see this all more explicitly, let's look at the Dirac equation. To describe bound states on loops of cosmic string we try the same strategy as before and split the solution in a longitudinal and transverse part

$$\psi_{L,R} = \alpha(t,\varphi)\beta_{L,R}(u,v)\xi_{L,R}, \tag{61}$$

with $\xi_{L,R}$ eigenstates of the projection operator

$$\sigma^0\sigma^\varphi\xi_L = \pm\xi_L \qquad \sigma^0\sigma^\varphi\xi_R = \mp\xi_R \tag{62}$$

Explicitly, the eigenspinors corresponding to the upper sign are

$$\xi_L = \begin{pmatrix} \mathrm{e}^{-i\varphi/2} \\ i\mathrm{e}^{i\varphi/2} \end{pmatrix}, \quad \xi_R = \begin{pmatrix} -i\mathrm{e}^{-i\varphi/2} \\ -\mathrm{e}^{i\varphi/2} \end{pmatrix}, \tag{63}$$

and the eigenspinors corresponding to the lower sign are

$$\xi_L = \begin{pmatrix} \mathrm{e}^{-i\varphi/2} \\ -i\mathrm{e}^{i\varphi/2} \end{pmatrix}, \quad \xi_R = \begin{pmatrix} i\mathrm{e}^{-i\varphi/2} \\ -\mathrm{e}^{i\varphi/2} \end{pmatrix}. \tag{64}$$

In contrast with the straight string, the eigenspinors are not constant but depend on the angle φ along the string loop. The φ dependence is needed to resolve the φ-dependence in the Dirac equation. For future use, we note that the spinors satisfy $\sigma^r\xi_{L,R} = -\xi_{L,R}$, $\partial_\varphi\xi_L = \pm\frac{1}{2}\xi_R$, and $\partial_\varphi\xi_R = \mp\frac{1}{2}\xi_L$.

The longitudinal part of the Dirac equation is $\left(\partial_t - \sigma^\varphi\frac{\partial_\varphi}{h_\varphi}\right)\alpha\xi_L = 0$ and a similar equation for ψ_R. Now since $\partial_\varphi\xi_\pm \neq 0$ an extra term appears and the equation reads

$$\left(\partial_t\alpha \mp \frac{\partial_\varphi}{r}\alpha\right)\xi_L + \frac{1}{2r}\alpha\xi_R = 0 \tag{65}$$

This cannot be solved as for the straight string. First of all, the equation depends through $h_\varphi = r(u,v)$ on the longitudinal coordinates, and the separation of variables is no longer valid. Secondly, there is an extra term since the eigenvectors are φ-dependent, but one could hope that this term can be "absorbed" in the longitudinal part of the Dirac equation.

Nevertheless, let us try to split the frequency part and see where we get. Try the Ansatz $\psi_{L,R} = \mathrm{e}^{-iEt}(\beta_1\xi_+ + \beta_1\xi_-)_{R,L}$, with $(\xi_\pm)_{L,R}$ the spinor eigenstates of (62). The transverse Dirac equation in cylindrical coordinates is

$$\begin{aligned} iE\beta_{1L} + (D_r - iD_z + \frac{1}{2r})\beta_{2L} + i\lambda\phi\beta_{2R} &= 0 \\ iE\beta_{2L} + (D_r + iD_z + \frac{1}{2r})\beta_{1L} - i\lambda\phi\beta_{1R} &= 0 \end{aligned} \tag{66}$$

and a similar equation for $L \leftrightarrow R$. Now transform to toroidal coordinates, see appendix A for details. Using $D_r + iD_z = -(2i/a)\sin^2\bar{\chi}(D_u + iD_v)$ and $1/(2r) = \sin\chi\sin\bar{\chi}/(a\sinh v)$ we get

$$
\begin{aligned}
i\frac{a}{2}E\beta_{1L} + \left(\sin^2\xi(D_u - iD_v) - \frac{i\sin\xi\sin\bar{\xi}}{2\sinh v}\right)\beta_{2L} \quad &+ \quad \frac{a\lambda\phi}{2}\beta_{2R} = 0 \\
i\frac{a}{2}E\beta_{2L} + \left(\sin^2\xi(D_u + iD_v) - \frac{i\sin\xi\sin\bar{\xi}}{2\sinh v}\right)\beta_{1L} \quad &+ \quad \frac{a\lambda\phi}{2}\beta_{1R} = 0
\end{aligned} \tag{67}
$$

If we follow the same strategy as for the straight string, we now want to remove the u dependence, i.e. the dependence on the coordinate winding around the string. This, however, seems impossible. Remember that χ, $\bar{\chi}$ depend on u, so $\sin\chi$ contains a $\mathrm{e}^{iu/2}$ and a $\mathrm{e}^{-iu/2}$ term. The phase structure of the equation above thus is

$$\beta_{1L} + (\mathrm{e}^{iu} + 1 + \mathrm{e}^{-iu})\beta_{2L} = \mathrm{e}^{inu}\beta_{2R} \tag{68}$$

and similar for $1 \leftrightarrow 2$. There is no way to remove the phase dependence. The situation is completely analogous to the vortex antivortex system, whether E is non-zero or not.

4 Conclusions

In this paper we have studied the fermionic spectrum on different topological defects including systems of soliton-antisoliton systems. We find that the number of zero modes is given by topological arguments, just as in the case of a single soliton. Any configuration that can be continuously deformed to the trivial vacuum does not support fermionic bound states. Although this sounds like a trivial conclusion, statements to the contrary can be found in the literature.

To built up intuition, we first studied the simpler kink-antikink systems. While a kink, antikink and kink-kink system all support one fermionic zero mode, a kink-antikink has none due to the different spinor structures of the fermionic mode associated with the kink and antikink. This is in complete agreement with topological arguments. Moreover, we find that no massive fermionic states on any system of kinks and anti-kinks exists.

Cosmic strings with winding number n are known to support n fermionic zero modes. Accordingly, we find that a system of two parallel BPS strings with $n = 1$ each supports two fermionic zero modes, while a string-antistring pair has no zero modes. In addition we find that a loop of cosmic string (which can be approximated by a string-antistring pair) has no fermionic zero modes. Note that this contrasts with the results in [9, 10].

Cosmic string loops stabilized by fermionic or bosonic currents, so-called *vortons*, are believed to be important for cosmology. Our results indicate that cosmic string loops with fermionic currents on them do not exist.

Of course, we have neglected the back reaction of the fermions on the bosonic fields and have only studied large loops. We believe, however, that taking these two points into account will not change our main conclusions.

Acknowledgments

We thank T. Vachaspati and S.C. Davis for helpful discussions. BH acknowledges S. Nicolis for bringing reference ([10]) to her attention. BH was supported by a CNRS grant, MP acknowledges support from FOM Netherlands.

A Cylindrical and toroidal coordinates

We use the flat space-time metric with signature $(+,-,-,-)$. Consider the three spatial dimensions with a metric of the form (21)

$$\mathrm{d}s^2 = -h_1^2 \mathrm{d}x_1^2 - h_2^2 \mathrm{d}x_2^2 - h_3^2 \mathrm{d}x_1^2. \tag{69}$$

In Cartesian coordinates $(x_1, x_2, x_3) = (x, y, z)$ with $h_i = 1$. Cylindrical coordinates are most convenient to describe a straight string aligned with the z-axis. Then $(x_1, x_2, x_3) = (r, \varphi, z)$ and $h_r = h_z = 1, h_\varphi = r$. The relation to Cartesian coordinates is $(x, y, z) = (r\cos\varphi, r\sin\varphi, z)$. The Pauli matrices in cylindrical coordinates are:

$$\sigma^r = \begin{pmatrix} 0 & \mathrm{e}^{-i\varphi} \\ \mathrm{e}^{i\varphi} & 0 \end{pmatrix}, \qquad \sigma^\varphi = \begin{pmatrix} 0 & -i\mathrm{e}^{-i\varphi} \\ i\mathrm{e}^{i\varphi} & 0 \end{pmatrix}, \qquad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{70}$$

Toroidal coordinates are most convenient to describe a loop of string lying in the x-y-plane. Then $(x_1, x_2, x_3) = (v, u, \varphi)$ with

$$\begin{aligned} h_v = h_u &= \frac{\kappa}{\cosh v - \cos u}, \\ h_\varphi &= \frac{\kappa \sinh v}{\cosh v - \cos u}. \end{aligned} \tag{71}$$

The relation with Cartesian coordinates is

$$(x, y, z) = (\frac{\kappa \sinh v \cos\varphi}{\cosh v - \cos u}, \frac{\kappa \sinh v \sin\varphi}{\cosh v - \cos u}, \frac{\kappa \sin u}{\cosh v - \cos u}). \tag{72}$$

At the core of the string the radius of the loop is κ, and its arc length $\kappa \mathrm{d}\varphi$ with $\varphi \in [0, 2\pi)$. The coordinate $u \in [0, 2\pi)$ winds around the string core, whereas v gives the radial distance away from the core (but note, for $v \to \infty$ the string core is approached, whereas $v \to 0$ corresponds to far away from the core). The Pauli matrices in toroidal coordinates read:

$$\sigma^v = \frac{1}{\cosh v - \cos u} \begin{pmatrix} \sin u \sinh v & \mathrm{e}^{-i\varphi}(\cos u \cosh v - 1) \\ \mathrm{e}^{i\varphi}(\cos u \cosh v - 1) & -\sin u \sinh v \end{pmatrix}, \tag{73}$$

$$\sigma^u = \frac{1}{\cosh v - \cos u} \begin{pmatrix} -\cos u \cosh v + 1 & \mathrm{e}^{-i\varphi} \sin u \sinh v \\ \mathrm{e}^{i\varphi} \sin u \sinh v & \cos u \cosh v - 1 \end{pmatrix}, \tag{74}$$

$$\sigma^\varphi = \begin{pmatrix} 0 & -i\mathrm{e}^{-i\varphi} \\ i\mathrm{e}^{i\varphi} & 0 \end{pmatrix}. \tag{75}$$

We also introduce the notation

$$\chi = (u + iv)/2. \tag{76}$$

The equations of motion may be simplified using $\sin\chi \sin\bar{\chi} = \frac{1}{2}(\cosh v - \cos u)$ and $\sin^2\bar{\chi} = \frac{1}{2}(1 - \cos u \cosh v - i \sin u \sinh v)$.

For 1-forms we use the notation

$$A = \sum A_i \mathrm{d}x^i = \frac{A_i}{h_i} e^i \tag{77}$$

with vierbein $e^i = h_i \mathrm{d}x^i$. For example, for a straight string in cylindrical coordinates

$$\begin{aligned} A &= A_\varphi \mathrm{d}\varphi = \frac{A_\varphi}{r} e^\varphi \\ &= A_\varphi \frac{\partial\varphi}{\partial x}\mathrm{d}x + A_\varphi \frac{\partial\varphi}{\partial y}\mathrm{d}y = -A_\varphi \frac{y}{r^2}\mathrm{d}x + A_\varphi \frac{x}{r^2}\mathrm{d}y = A_x \mathrm{d}x + A_y \mathrm{d}y. \end{aligned} \tag{78}$$

And similarly for higher forms: $F = F_{12}\mathrm{d}x^1\mathrm{d}x^2 = \frac{F_{12}}{h_1 h_2} e^1 e^2$.

A.1 Asymptotics

The core of the straight string aligned with the z-axis is approached in the limit $r = \sqrt{x^2 + y^2} \to 0$ (and φ ill defined), whereas asymptotically far corresponds to $r \to \infty$.

In toroidal coordinates the core of the string is approached in the limit $v \to \infty$ (and u ill defined). Then $(x, y, z) \to (\kappa \cos\varphi, \kappa \sin\varphi, \kappa \sin u \mathrm{e}^{-v})$ and $(h_v, h_u, h_\varphi) \to (\kappa \mathrm{e}^{-v}, \kappa \mathrm{e}^{-v}, \kappa)$. The core of the string loop is at $r = \sqrt{x^2 + y^2} = \kappa$. Write $r = \kappa + \delta r$, then we can express the limit $v \to \infty$ in terms of $\delta r \to 0$. More precisely

$$v = -\mathrm{arccosh}\left(\frac{r^2 + \kappa^2}{r^2 - \kappa^2}\right) \overset{\delta r \to 0}{\longrightarrow} \log\left(\frac{\kappa}{\delta r}\right), \tag{79}$$

and thus

$$\lim_{v\to\infty} e^{-v} \sim \lim_{\delta r \to 0} \frac{\delta r}{\kappa} \tag{80}$$

(r, κ have dimension of length, while u, v, φ are dimensionless.)

Spatial infinity corresponds to $u = v \to 0$. In this limit $\cosh v - \cos u \sim v^2$, $\sinh v \to v$ and thus

$$\begin{aligned} (x, y, z) &\overset{u=v\to 0}{\longrightarrow} \left(\frac{\kappa}{v}\cos\varphi, \frac{\kappa}{v}\sin\varphi, \frac{\kappa}{v}\right) \\ (h_v, h_u, h_\varphi) &\overset{u=v\to 0}{\longrightarrow} \left(\frac{\kappa}{v^2}, \frac{\kappa}{v^2}, \frac{\kappa}{v}\right) \end{aligned} \tag{81}$$

and

$$\lim_{u=v\to 0} v \propto \lim_{R\to\infty} \frac{\kappa}{R} \propto \lim_{r\to\infty} \frac{\kappa}{r} \tag{82}$$

with $R = \sqrt{x^2 + y^2 + z^2} = \sqrt{r^2 + z^2} \propto r$. In the limit $\kappa \ll R$, the distance from the string core approaches the distance from the origin R.

Finally, the limit $u = \pi, v \to 0$ corresponds to the origin, the center of the loop. Indeed, in this limit

$$
\begin{aligned}
(x, y, z) &\overset{u\to\pi,v\to 0}{\longrightarrow} (\frac{\kappa}{2}v\cos\varphi, \frac{\kappa}{2}v\sin\varphi, 0) \\
(h_v, h_u, h_\varphi) &\overset{u\to\pi,v\to 0}{\longrightarrow} (\frac{\kappa}{2}, \frac{\kappa}{2}, \frac{\kappa}{2}v)
\end{aligned}
\tag{83}
$$

and

$$
\lim_{u\to\pi,v\to 0} v \propto \lim_{r\to 0}\frac{r}{\kappa}. \tag{84}
$$

A.2 Notation and convention

Most of the solutions are given in terms of eigenspinors, and are independent of the basis of γ-matrices. When we give explicit solutions, we use the chiral basis:

$$
\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \tag{85}
$$

with $\sigma^\mu = (1, \sigma^i)$, $\bar{\sigma}^\mu = (1, -\sigma^i)$, and σ^i the Pauli matrices.

B Asymptotic solutions of [9]

In this appendix we show explicitly that there are no fermionic zero modes on a circular loop of cosmic string. This is in contrast with the results in [9]. Following [9] we remove half of degrees of freedom by considering Majorana fermions with $\psi_R = \psi_L^c = i\sigma\psi_L^*$. The equation to solve is then (67) with $E = 0$. Using the Ansatz

$$
\beta = X\mathrm{e}^{il(\chi+\bar{\chi})} + Y^*\mathrm{e}^{i(n-l)(\chi+\bar{\chi})} \tag{86}
$$

this gives the two coupled differential equations

$$
\begin{aligned}
\left(\sin^2\bar{\chi}\left(\partial_{\bar{\chi}} + i(l - \frac{n}{2}a)\right) + \frac{i\sin\chi\sin\bar{\chi}}{2\sinh v}\right) X &= \frac{\kappa m f}{2} Y \\
\left(\sin^2\chi\left(\partial_\chi - i(n - l - \frac{n}{2}a)\right) - \frac{i\sin\chi\sin\bar{\chi}}{2\sinh v}\right) Y &= \frac{\kappa m f}{2} X
\end{aligned}
\tag{87}
$$

with $m = \lambda\eta$ the vacuum fermion mass. These equations agree with those found in [9].

Infinity. Let us first consider the behavior at infinity and write

$$
\chi = \frac{1}{2}(u + iv) = \frac{1}{t}\mathrm{e}^{i\varphi}. \tag{88}
$$

Spatial infinity is approached in the limit $u = v \to 0$, or equivalently $t \to \infty, \varphi \to \pi/4$. Further, the gauge field and Higgs field approach their vacuum values and thus $a(v) \to 1$ and $f(v) \to 1$. Then using the approximation $\sin\chi \approx \chi$ and after redefining the fields

$$X = A\chi\sqrt{\frac{2i\chi\bar{\chi}}{\chi-\bar{\chi}}}, \qquad Y = B\bar{\chi}\sqrt{\frac{2i\chi\bar{\chi}}{\chi-\bar{\chi}}}, \tag{89}$$

(87) becomes

$$\begin{aligned} \chi\bar{\chi}\left(\partial_{\bar{\chi}} - ip\right) A &= \frac{\kappa m}{2} B \\ \chi\bar{\chi}\left(\partial_{\chi} - ip\right) B &= \frac{\kappa m}{2} A \end{aligned} \tag{90}$$

where we defined $p = n/2 - l$. Combining the two equations gives

$$\chi\bar{\chi}\left(\partial_{\chi} - ip\right)\chi\bar{\chi}\left(\partial_{\bar{\chi}} - ip\right) A = \left(\frac{\kappa m}{2}\right)^2 A. \tag{91}$$

Using (88) we write this in terms of t, φ to take the limit $r \to \infty$. The partial derivatives become

$$\partial_{\chi} = -\frac{t^2 \mathrm{e}^{-i\varphi}}{2}(\partial_t + \frac{i}{t}\partial_{\varphi}), \qquad \partial_{\bar{\chi}} = -\frac{t^2 \mathrm{e}^{i\varphi}}{2}(\partial_t - \frac{i}{t}\partial_{\varphi}) \tag{92}$$

and $\chi\bar{\chi} = t^{-2}$. Using these expressions in (91) and keeping only the terms with highest power in t gives:

$$\frac{1}{4}\partial_t^2 A - \frac{i}{4t}\partial_t\partial_{\varphi} A - \left(\frac{m\kappa}{2}\right)^2 A + \mathcal{O}(t^{-2}) = 0 \tag{93}$$

from which it follows that

$$\lim_{r\to\infty} A = C\mathrm{e}^{-m\kappa t} \propto \mathrm{e}^{\pm mr} \tag{94}$$

with C a normalization constant, and where we used $t \propto v^{-1} \propto \kappa/r$. There is a similar solution for $B = \mathrm{e}^{-i\pi/4}A$. Putting back all the factors we get for the normalizable solution:

$$\beta = \frac{C\mathrm{e}^{i\pi/4}\mathrm{e}^{-m\kappa t}}{t^{3/2}}\left(\mathrm{e}^{il\sqrt{2}/t} + \mathrm{e}^{i\pi/4}\mathrm{e}^{-i(n-l)\sqrt{2}/t}\right). \tag{95}$$

Loop core. The core of the loop is approached in the limit $v \to \infty$ or $\chi \to iv/2 \to i\infty$. In this limit $\sin\chi \to i\mathrm{e}^{v/2}$. Further $f, a \to 0$ at the core. Taking this limit in the Weyl equation (87), only the first term survives and gives

$$\begin{aligned} \left(\partial_{\bar{\chi}} + il\right) X = 0 \qquad &\Rightarrow \qquad X \sim f_X(\chi)\mathrm{e}^{-il\bar{\chi}} \\ \left(\partial_{\chi} - (n-l)\right) Y = 0 \qquad &\Rightarrow \qquad Y \sim f_Y(\bar{\chi})\mathrm{e}^{i(n-l)\chi} \end{aligned} \tag{96}$$

Then

$$\begin{aligned} \beta &= f_X(\chi)\mathrm{e}^{il\chi} + f_Y(\chi)\mathrm{e}^{i(n-l)\chi} \\ &= f_X(v)\mathrm{e}^{-\frac{lv}{2}} + f_Y(v)\mathrm{e}^{-\frac{(n-l)v}{2}} \quad \text{for } v \to \infty \end{aligned} \tag{97}$$

Since the functions $f_X(v), f_Y(v)$ are unconstrained, nothing much can be said about the number of solutions at the string core.

Origin. Finally consider the zero mode solution at the origin $(u,v) \to (\pi,0)$ or $(\chi,\bar{\chi}) \to (\pi/2,\pi/2)$, where $f \to 1$, $a \to 1$. Note that this case was not studied in [9]. The Weyl equation (87) becomes

$$\begin{aligned} \left((\partial_{\bar{\chi}} - ip) - \frac{1}{2(\chi-\bar{\chi})}\right) X &= \frac{\kappa m}{2} Y \\ \left((\partial_{\chi} - ip) + \frac{1}{2(\chi-\bar{\chi})}\right) Y &= \frac{\kappa m}{2} X \end{aligned} \tag{98}$$

where we have used the abbreviation $p = \frac{n}{2} - l$.

The problem is already apparent here: $\chi - \bar{\chi} \to 0$ and the second term on the left hand side of both equations blows up. Redefining the fields

$$X = A\sqrt{\frac{2i}{\chi-\bar{\chi}}}, \qquad Y = B\sqrt{\frac{2i}{\chi-\bar{\chi}}} \tag{99}$$

the equations simplify to

$$\begin{aligned} (\partial_{\bar{\chi}} - ip)A &= \frac{\kappa m}{2} B \\ (\partial_{\chi} - ip)B &= \frac{\kappa m}{2} A \end{aligned} \tag{100}$$

which can be combined into a single equation for A

$$(\partial_{\chi} - ip)(\partial_{\bar{\chi}} - ip)A = \left(\frac{\kappa m}{2}\right)^2 A \tag{101}$$

and likewise for B. Explicitly, we have:

$$\partial_{\chi}\partial_{\bar{\chi}}A - ip(\partial_{\chi} + \partial_{\bar{\chi}})A - \left(p^2 + (\kappa m/2)^2\right)A = 0. \tag{102}$$

We will try two type of Ansätze. First, we choose $A = A(\chi+\bar{\chi}) = A(u)$, which implies $A'' = 2ipA' + (p^2 + (\kappa m/2)^2)A$ and the prime denotes the derivative with respect to u. This has as solution

$$A = C\mathrm{e}^{(\pm\kappa m/2 + ip)u} \quad \overset{u\to\pi, v\to 0}{\longrightarrow} \quad \text{const.} \tag{103}$$

Further $B \propto A$ where the constant of proportionality is given by the equations.

The second type of Ansatz is $A = A(-i(\chi-\bar{\chi})) = A(v)$, which gives $A'' = (p^2 + (\kappa m/2)^2 A$ where the prime denotes derivative with respect to v. This has as solution

$$A = C\mathrm{e}^{\pm\sqrt{p^2+(\kappa m/2)^2}v} \quad \overset{u\to\pi, v\to 0}{\longrightarrow} \quad \text{const} \tag{104}$$

Both solutions do not tend to zero at the origin. On the contrary, plugging the above expressions back in, we find that zero mode solution

$$\beta = \sqrt{\frac{2}{v}}\left(A\mathrm{e}^{ilu} + B^*\mathrm{e}^{i(n-l)u}\right) \quad \overset{u\to\pi, v\to 0}{\longrightarrow} \propto 1/v \tag{105}$$

Although the solution blows up in the origin as $1/v$ the integral $\int \mathrm{d}v\mathrm{d}u\sqrt{g_2}\psi^\dagger\psi$ is finite. Nevertheless, this is not a valid solution. The spinors (63), (64) are only well defined at the origin if $\beta \to 0$. Note that a similar argument holds for the full z-axis with $v = 0$ and $u \neq 0$. The solution is thus singular on the symmetry-axis. Non-zero energy will affect the exponents in A, B, but not the dominant behavior of β in the limit $u \to \pi, v \to 0$. And thus non-zero energy cannot alleviate this problem.

References

[1] R. L. Davis, Phys. Rev. D **38** (1988) 3722.

[2] E. Witten, Nucl. Phys. B **249** (1985) 557.

[3] A. Vilenkin and E. P. S. Shellard, *Cosmic strings and other topological defects*, Cambridge University Press (1994); B. Carter, *Old and new processes of vorton formation*, Lect. Notes Phys. **541**, 71 (2000).

[4] Y. Lemperiere and E. P. S. Shellard, Phys. Rev. Lett. **91** (2003) 141601 [arXiv:hep-ph/0305156].

[5] R. Jackiw and P. Rossi, Nucl. Phys. B **190** (1981) 681.

[6] E. J. Weinberg, Phys. Rev. D **24**, 2669 (1981).

[7] S. C. Davis, W. B. Perkins and A. C. Davis, Phys. Rev. D **62** (2000) 043503 [arXiv:hep-ph/9912356].

[8] C. Ringeval, Phys. Rev. D **64** (2001) 123505 [arXiv:hep-ph/0106179].

[9] A. B. Gadde and U. A. Yajnik, arXiv:hep-th/0511235.

[10] L. Pismen and S. Rica, Phys. Rev. D **66** (2002) 045004.

[11] R. Jackiw and C. Rebbi, Phys. Rev. D **13**, 3398 (1976).

[12] B. Altschul, arXiv:hep-th/0111042.

[13] M. Y. Choi and C. Lee, Phys. Rev. D **67**, 107702 (2003).

[14] S. M. Barr and A. M. Matheson, Phys. Lett. B **198** (1987) 146.

[15] R. Jeannerot and M. Postma, JHEP **0412** (2004) 032 [arXiv:hep-ph/0411259].