

The Adapted Ordering Method for Lie Algebras and Superalgebras and their Generalizations

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ABSTRACT

In 1998 the Adapted Ordering Method was developed for the representation theory of the superconformal algebras in two dimensions. It allows: to determine maximal dimensions for a given type of space of singular vectors, to identify all singular vectors by only a few coefficients, to spot subsingular vectors and to set the basis for constructing embedding diagrams. In this article we present the Adapted Ordering Method for general Lie algebras and superalgebras, and their generalizations, provided they can be triangulated. We also review briefly the results obtained for the Virasoro algebra and for the $N = 2$ and Ramond $N = 1$ superconformal algebras.

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1 Introduction

In 1998 the Adapted Ordering Method was developed, by M. Dörrzapf and B. Gato-Rivera, for the study of the representation theory of the superconformal algebras in two dimensions¹. It allows: to determine maximal dimensions for a given type of space of singular vectors, to identify all singular vectors by only a few coefficients, to spot subsingular vectors and to set the basis for constructing embedding diagrams. This method was applied successfully to the $N = 2$ superconformal algebras^{1,2} (topological, Neveu-Schwarz, Ramond and twisted) and to the Ramond $N = 1$ superconformal algebra³, allowing to obtain rigorous proofs for several conjectured results, as well as many new results, especially for the case of the twisted $N = 2$ superconformal algebra and the case of the Ramond $N = 1$ superconformal algebra. The Adapted Ordering Method, however, can be applied to most Lie algebras and superalgebras, and their generalizations, provided they can be triangulated. The purpose of this article is precisely to provide the description of the Adapted Ordering Method for a general Lie algebra or superalgebra with a triangular decomposition into creation, annihilation and zero mode operators, like is the case for most of them⁴.

For the given algebra or superalgebra one defines freely generated modules over a highest weight vector, denoted as *Verma modules*. A Verma module is in general not irreducible, but it contains submodules which are freely generated over, at least, one highest weight vector different from the highest weight vector of the Verma module. These vectors are usually referred to as *singular vectors*. The irreducible highest weight representations are then obtained as the quotients of the Verma modules divided by all their submodules. Surprisingly, the complete set of singular vectors do not generate all the submodules in the case of Verma modules which contain *subsingular vectors*. The reason is that subsingular vectors are singular vectors of the quotient space, but not of the Verma module itself^{9–12}. In this case one has to divide further by the submodules generated by the subsingular vectors, repeating this division procedure successively, if necessary.

On the Verma modules one introduces a hermitian contravariant form. The vanishing of the corresponding determinant indicates the existence of at least one singular vector. The determinant may not detect the whole set of singular vectors, however, neither does it give the dimension of the space of singular vectors with some given weights. There could be in fact more than one linearly independent singular vectors with the same weights. Therefore, the dimensions of the spaces of singular vectors have to be found by an independent procedure, like the Adapted Ordering Method which puts upper limits on the dimensions of these singular spaces. For most weight spaces of a Verma module these upper limits on the dimensions are equal to zero and, as a consequence, we obtain a rigorous proof that there cannot exist any singular vectors for these weights. For some weights, however, one may find necessary conditions that allow spaces of singular vectors to exist, either only one-dimensional, as is the case for the Virasoro algebra, or even higher dimensional spaces, as it happens for the $N = 2$ and Ramond $N = 1$ superconformal algebras^{2,3,12–14}.

The idea for developing the Adapted Ordering Method originated, in rudimentary form, from a procedure due to A. Kent for the study of the representations of the Virasoro algebra⁵. For this purpose the author analytically continued the Virasoro Verma modules, yielding ‘generalised’ Verma modules, where he constructed generalised singular vectors expressions in terms of analytically continued Virasoro operators. This analytical continuation is not necessary, however, for the Adapted Ordering Method, nor is it necessary to construct singular vectors in order to apply it. The underlying idea is the concept of *adapted orderings* for all the possible terms of the ‘would be’ singular vectors. An adapted ordering is a criterion, satisfying certain requirements, to decide which of two given terms is the bigger one. To be more specific, a total ordering will be called

adapted to a subset of terms provided some conditions are met. The complement of that subset will be the *ordering kernel* and will play a crucial rôle since its size puts an upper limit on the dimension of the space of singular vectors.

In what follows, in section 2 we will describe the Adapted Ordering Method for a general Lie algebra or superalgebra which admits a triangulation and, as an example, we will apply this method to the Virasoro algebra. In section 3 we will review briefly the results obtained for the $N = 2$ and the Ramond $N = 1$ superconformal algebras, as an illustration of the possibilities of this method. Section 4 is devoted to conclusions.

2 The Adapted Ordering Method

Let \mathcal{A} denote a Lie algebra or superalgebra with a triangular decomposition into creation, annihilation and zero mode operators: $\mathcal{A} = \mathcal{A}^- \oplus \mathcal{A}^+ \oplus \mathcal{A}^0$. The Cartan subalgebra $\mathcal{H}_{\mathcal{A}}$ is contained in \mathcal{A}^0 but does not need to be identical to \mathcal{A}^0 . In general, an eigenvector with respect to the Cartan subalgebra with weights given by the set $\{l_i\}$, in particular a singular vector $\Psi_{\{l_i\}}$, can be expressed as a sum of products of creation operators acting on a highest weight vector with weights $\{\Delta_i\}$:

$$\Psi_{\{l_i\}} = \sum_{m_1, m_2, \dots \in \mathbb{N}_0} \sum_{a, b, c, \dots} k_{a_{-1}^{m_1}, a_{-2}^{m_2}, \dots, b_{-1}^{n_1}, b_{-2}^{n_2}, \dots} X_{\{l_i\}}^{a_{-1}^{m_1}, a_{-2}^{m_2}, \dots, b_{-1}^{n_1}, b_{-2}^{n_2}, \dots} |\{\Delta_i\}\rangle, \quad (1)$$

where $a_{-1}, a_{-2}, \dots, b_{-1}, b_{-2}, \dots$ are the creation operators of the algebra (some zero mode operators should also be included if they act as creation operators), $X_{\{l_i\}}^{a_{-1}^{m_1}, a_{-2}^{m_2}, \dots, b_{-1}^{n_1}, b_{-2}^{n_2}, \dots}$ are the products of the creation operators: $a_{-1}^{m_1} a_{-2}^{m_2} \dots b_{-1}^{n_1} b_{-2}^{n_2} \dots$, with total weights $\{l_i\}$, which will be denoted simply as *terms*, and $k_{a_{-1}^{m_1}, a_{-2}^{m_2}, \dots, b_{-1}^{n_1}, b_{-2}^{n_2}, \dots} \in \mathbb{C}$ are coefficients which depend on the given term. A non-trivial term Y then refers to a term with non-trivial coefficient k_Y .

Now let us define the set $\mathcal{C}_{\{l_i\}}$ as the set of all the terms with weights $\{l_i\}$:

$$\mathcal{C}_{\{l_i\}} = \{X_{\{l_i\}}^{a_{-1}^{m_1}, a_{-2}^{m_2}, \dots, b_{-1}^{n_1}, b_{-2}^{n_2}, \dots}, m_1, m_2, \dots, n_1, n_2, \dots \in \mathbb{N}_0\}, \quad (2)$$

and let \mathcal{O} denote a total ordering on $\mathcal{C}_{\{l_i\}}$, that is an ordering such that any two different terms in $\mathcal{C}_{\{l_i\}}$ are ordered with respect to each other. Thus $\Psi_{\{l_i\}}$ in Eq. (1) needs to contain an \mathcal{O} -smallest $X_0 \in \mathcal{C}_{\{l_i\}}$ with $k_{X_0} \neq 0$ and $k_Y = 0$ for all $Y \in \mathcal{C}_{\{l_i\}}$ with $Y <_{\mathcal{O}} X_0$ and $Y \neq X_0$. We define an *adapted ordering* on $\mathcal{C}_{\{l_i\}}$ as follows:

Definition 2.A A total ordering \mathcal{O} on $\mathcal{C}_{\{l_i\}}$ is called adapted to the subset $\mathcal{C}_{\{l_i\}}^A \subset \mathcal{C}_{\{l_i\}}$ in the Verma module $\mathcal{V}_{\{\Delta_i\}}$ if for any element $X_0 \in \mathcal{C}_{\{l_i\}}^A$ at least one annihilation operator Γ exists for which $\Gamma X_0 |\{\Delta_i\}\rangle$ contains a non-trivial term \tilde{X}

$$\Gamma X_0 |\{\Delta_i\}\rangle = (k_{\tilde{X}} \tilde{X} + \dots) |\{\Delta_i\}\rangle \quad (3)$$

which is absent, however, for all $\Gamma X |\{\Delta_i\}\rangle$, where X is any term $X \in \mathcal{C}_{\{l_i\}}$ such that $X_0 <_{\mathcal{O}} X$. The complement of $\mathcal{C}_{\{l_i\}}^A$, $\mathcal{C}_{\{l_i\}}^K = \mathcal{C}_{\{l_i\}} \setminus \mathcal{C}_{\{l_i\}}^A$ is the kernel with respect to the ordering \mathcal{O} in the Verma module $\mathcal{V}_{\{\Delta_i\}}$.

In this definition Γ should also include zero mode operators if they act as annihilation operators. A crucial point now is that the size of the kernel puts an upper limit on the dimension of the corresponding space of singular vectors, as stated in the following theorem:

Theorem 2.B *Let \mathcal{O} denote an adapted ordering on $\mathcal{C}_{\{l_i\}}^A$ at weights $\{l_i\}$ with kernel $\mathcal{C}_{\{l_i\}}^K$ for a given Verma module $\mathcal{V}_{\{\Delta_i\}}$. If the ordering kernel $\mathcal{C}_{\{l_i\}}^K$ has n elements, then there are at most n linearly independent singular vectors $\Psi_{\{l_i\}}$ in $\mathcal{V}_{\{\Delta_i\}}$ with weights $\{l_i\}$.*

Therefore, one needs to find suitable orderings in order to obtain the smallest possible kernels. Observe that the maximal possible dimension n does not imply that all the singular vectors of the corresponding type are n -dimensional. From this theorem one deduces that if $\mathcal{C}_{\{l_i\}}^K = \emptyset$ for a given Verma module, then there are no singular vectors with weights $\{l_i\}$ in it. That is:

Theorem 2.C *Let \mathcal{O} denote an adapted ordering on $\mathcal{C}_{\{l_i\}}^A$ at weights $\{l_i\}$ with trivial kernel $\mathcal{C}_{\{l_i\}}^K = \emptyset$ for a given Verma module $\mathcal{V}_{\{\Delta_i\}}$. A singular vector $\Psi_{\{l_i\}}$ in $\mathcal{V}_{\{\Delta_i\}}$ with weights $\{l_i\}$ is therefore trivial.*

In addition, the coefficients with respect to the terms of the ordering kernel $\mathcal{C}_{\{l_i\}}^K$ uniquely identify a singular vector with weights $\{l_i\}$. Since the size of the ordering kernels are in general small, as they put a limit on the dimensionality of the spaces of singular vectors, it turns out that just a few coefficients completely determine a singular vector no matter its size, what allows to find easily product expressions for descendant singular vectors. For example, in the case of the conformal and $N=1,2$ superconformal algebras the ordering kernels found for most weights have zero or one term, for some weights they have two terms and for some other weights they have three terms. This property is summarized in the following theorem:

Theorem 2.D *Let \mathcal{O} denote an adapted ordering on $\mathcal{C}_{\{l_i\}}^A$ at weights $\{l_i\}$ with kernel $\mathcal{C}_{\{l_i\}}^K$ for a given Verma module $\mathcal{V}_{\{\Delta_i\}}$. If two singular vectors $\Psi_{\{l_i\}}^1$ and $\Psi_{\{l_i\}}^2$ with the same weights $\{l_i\}$ have $k_X^1 = k_X^2$ for all $X \in \mathcal{C}_{\{l_i\}}^K$, then*

$$\Psi_{\{l_i\}}^1 \equiv \Psi_{\{l_i\}}^2. \quad (4)$$

Proof of Theorem 2.D: Let us consider $\Psi_{\{l_i\}} = \Psi_{\{l_i\}}^1 - \Psi_{\{l_i\}}^2$, which does not contain any terms of the ordering kernel $\mathcal{C}_{\{l_i\}}^K$, simply because $k_X^1 = k_X^2$ for all $X \in \mathcal{C}_{\{l_i\}}^K$. As $\mathcal{C}_{\{l_i\}}^K$ is a totally ordered set with respect to \mathcal{O} , the non-trivial terms of $\Psi_{\{l_i\}}$, provided $\Psi_{\{l_i\}}$ is non-trivial, need to have an \mathcal{O} -minimum $X_0 \in \mathcal{C}_{\{l_i\}}^A$. Thus the coefficient k_{X_0} of X_0 in $\Psi_{\{l_i\}}$ must be non-trivial. As \mathcal{O} is adapted to $\mathcal{C}_{\{l_i\}}^A$ one can find an annihilation operator Γ such that $\Gamma X_0 | \{\Delta_i\} \rangle$ contains a non-trivial term (for a suitably chosen basis depending on X_0) that cannot be created by Γ acting on any other term of $\Psi_{\{l_i\}}$ which is \mathcal{O} -larger than X_0 . But X_0 was chosen to be the \mathcal{O} -minimum of $\Psi_{\{l_i\}}$. Therefore, $\Gamma X_0 | \{\Delta_i\} \rangle$ contains a non-trivial term that cannot be created from any other term of $\Psi_{\{l_i\}}$. The coefficient of this term is obviously given by ck_{X_0} with c a non-trivial complex number. But $\Psi_{\{l_i\}}$ is also a singular vector and must be annihilated by any annihilation operator, in particular by Γ . It follows that $k_{X_0} = 0$, contrary to our original assumption. Thus, the set of non-trivial terms of $\Psi_{\{l_i\}}$ is empty and therefore $\Psi_{\{l_i\}} = 0$. This results in $\Psi_{\{l_i\}}^1 = \Psi_{\{l_i\}}^2$. \square

Theorem 2.D states, therefore, that if two singular vectors with the same weights, in the same Verma module, agree on the coefficients of the ordering kernel, then they are identical. If the ordering kernel is trivial we consequently find 0 as the only vector that can satisfy the highest weight conditions. This observation provides a proof for Theorem 2.C:

Proof of Theorem 2.C: The trivial vector 0 satisfies any annihilation conditions for any weights $\{l_i\}$. As the ordering kernel is trivial the components of the vectors 0 and $\Psi_{\{l_i\}}$ agree on the ordering kernel and using theorem 2.D we obtain $\Psi_{\{l_i\}} = 0$. \square

Using Theorem 2.D it is also easy to prove Theorem 2.B:

Proof of Theorem 2.B: Suppose there were more than n linearly independent singular vectors $\Psi_{\{l_i\}}$ in $\mathcal{V}_{\{\Delta_i\}}$ with weights $\{l_i\}$. We choose $n+1$ linearly independent singular vectors among them $\Psi_1, \dots, \Psi_{n+1}$. The ordering kernel $\mathcal{C}_{\{l_i\}}^K$ has the n elements X_1, \dots, X_n . Let k_{jk} denote the coefficient of the term X_j in the vector Ψ_k in a suitable basis decomposition. The coefficients k_{jk} thus form a n by $n+1$ matrix M . The homogeneous system of linear equations $M\lambda = 0$ thus has a non-trivial solution $\lambda^0 = (\lambda_1^0, \dots, \lambda_{n+1}^0)$ for the vector λ . We then form the linear combination $\Psi = \sum_{i=1}^{n+1} \lambda_i^0 \Psi_i$. Obviously, the coefficient of X_j for the vector Ψ is just given by the j -th component of the vector $M\lambda$ which is trivial for $j = 1, \dots, n$. Hence, the coefficients of Ψ are trivial on the ordering kernel. On the other hand, Ψ is a linear combination of singular vectors and therefore it is also a singular vector. Due to theorem 2.D one immediately finds that $\Psi \equiv 0$ and therefore $\sum_{i=1}^{n+1} \lambda_i \Psi_i = 0$. This, however, contradicts the assumption that $\Psi_1, \dots, \Psi_{n+1}$ are linearly independent. \square

Three observations come now in order. First, one may wonder whether there are any prescriptions in order to choose the most suitable orderings with the smallest kernels. The answer is that there are no general prescriptions since the possible orderings depend entirely on the given algebra or superalgebra. Whereas total orderings can always be constructed (for any two given terms, one simply defines which is the bigger one), to find the most suitable orderings is mainly a matter of trial and error. Second, every total ordering is adapted to, at least, a subset of terms. This includes the possibility that a given ordering is adapted only to the empty subset, in which case the ordering kernel is the whole set of terms: $\mathcal{C}_{\{l_i\}}^K = \mathcal{C}_{\{l_i\}}$. Obviously such an ordering does not give any information. Third, let us remark that the Adapted Ordering Method follows from the Definition 2.A of *adapted ordering* plus the Theorems 2.B, 2.C and 2.D, which are rigorously proven. There is nothing in the Definition 2.A, neither in the three theorems, that restricts the application of this method to infinite-dimensional algebras. For the same reason, it seems clear that the Adapted Ordering Method should be useful also for generalized Lie algebras and superalgebras such as affine Kac-Moody algebras, non-linear W-algebras, superconformal W-algebras, loop Lie algebras, Borchers algebras, F-Lie algebras for $F > 2$ ($F = 1$ are Lie algebras and $F = 2$ are Lie superalgebras), etc.

As a simple example of the Adapted Ordering Method we will see now the application of this method to the Virasoro algebra \mathbb{V} , which has been extensively studied in the literature⁶⁻⁸. This algebra is given by the commutation relations

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{C}{12}(m^3 - m)\delta_{m+n,0}, \quad [C, L_m] = 0, \quad m, n \in \mathbb{Z}, \quad (5)$$

where C commutes with all operators of \mathbb{V} and can hence be taken to be constant $c \in \mathbb{C}$. \mathbb{V} can be written in its *triangular decomposition*: $\mathbb{V} = \mathbb{V}^- \oplus \mathbb{V}^0 \oplus \mathbb{V}^+$, where $\mathbb{V}^+ = \text{span}\{L_m : m \in \mathbb{N}\}$ is the set of *annihilation operators*, $\mathbb{V}^- = \text{span}\{L_{-m} : m \in \mathbb{N}\}$ is the set of *creation operators*, and the *Cartan subalgebra* is given by $\mathbb{V}^0 = \text{span}\{L_0, C\}$. For elements of \mathbb{V} that are eigenvectors of L_0 with respect to the adjoint representation the L_0 -eigenvalue is called the *level* l . For the universal enveloping algebra $U(\mathbb{V})$, elements of the form $L_{-p_I} \dots L_{-p_1}$, $p_q \in \mathbb{Z}$ for $q = 1, \dots, I$, $I \in \mathbb{N}$, are at level $l = \sum_{q=1}^I p_q$. Note that annihilation operators $L_m \in \mathbb{V}^+$ have negative level $l = -m$.

A representation with L_0 -eigenvalues bounded from below contains a vector with L_0 -eigenvalue Δ which is annihilated by V^+ , a *highest weight (h.w.) vector* $|\Delta\rangle$:

$$V^+ |\Delta\rangle = 0, \quad L_0 |\Delta\rangle = \Delta |\Delta\rangle. \quad (6)$$

The *Verma module* \mathcal{V}_Δ built on $|\Delta\rangle$ is L_0 -graded in a natural way. The corresponding L_0 -eigenvalue is called the *conformal weight* and is written for convenience as $\Delta + l$, where l is the *level*. Any proper submodule of \mathcal{V}_Δ needs to contain a *singular vector* Ψ_l that is not proportional to the h.w. vector $|\Delta\rangle$ but still satisfies the *h.w. vector conditions* with conformal weight $\Delta + l$:

$$V^+ \Psi_l = 0, \quad L_0 \Psi_l = (\Delta + l) \Psi_l. \quad (7)$$

Now we will see the total ordering on \mathcal{C}_l defined by Kent⁵ for the Virasoro algebra. One has to take into account, however, that Kent used the following ordering to show that in his generalised Verma modules vectors at level 0 satisfying the h.w. conditions are actually proportional to the h.w. vector. Using the Adapted Ordering technology, though, one deduces that this ordering already implies that all Virasoro singular vectors are unique at their levels up to proportionality, simply because the ordering kernel for each $l \in \mathbf{N}$ has just one element: L_{-1}^l .

Definition 2.E *On the set \mathcal{C}_l of Virasoro operators we introduce the total ordering \mathcal{O}_V for $l \in \mathbf{N}$. For two elements $X_1, X_2 \in \mathcal{C}_l$, $X_1 \neq X_2$, with $X_i = L_{-m_{I_i}^1} \dots L_{-m_{I_i}^{n_i}} L_{-1}^{n_i}$, $n_i = l - m_{I_i}^1 \dots - m_{I_i}^{n_i}$, or $X_i = L_{-1}^l$, $i = 1, 2$ we define*

$$X_1 <_{\mathcal{O}_V} X_2 \quad \text{if} \quad n^1 > n^2. \quad (8)$$

If, however, $n^1 = n^2$ we compute the index $j_0 = \min\{j : m_j^1 - m_j^2 \neq 0, j = 1, \dots, \min(I_1, I_2)\}$. We then define

$$X_1 <_{\mathcal{O}_V} X_2 \quad \text{if} \quad m_{j_0}^1 < m_{j_0}^2. \quad (9)$$

For $X_1 = X_2$ we set $X_1 <_{\mathcal{O}_V} X_2$ and $X_2 <_{\mathcal{O}_V} X_1$.

The index j_0 describes the first mode, read from the right to the left, for which the generators in X_1 and X_2 (L_{-1} excluded) are different. For example, in \mathcal{C}_8 one has $L_{-2}L_{-2}L_{-2}L_{-1}^2 <_{\mathcal{O}_V} L_{-4}L_{-2}L_{-1}^2$ with index $j_0 = 2$. Observe that $L_{-1}^l \in \mathcal{C}_l$ is the global \mathcal{O}_V -minimum in \mathcal{C}_l . Now using the Adapted Ordering Method one finds the following theorem¹.

Theorem 2.F *The ordering \mathcal{O}_V is adapted to $\mathcal{C}_l^A = \mathcal{C}_l \setminus \{L_{-1}^l\}$ for each level $l \in \mathbf{N}$ and for all Verma modules \mathcal{V}_Δ . The ordering kernel is given by the single element set $\mathcal{C}_l^K = \{L_{-1}^l\}$.*

For example let us consider the set of terms at level 3, $\mathcal{C}_3 = \{L_{-1}^3, L_{-2}L_{-1}, L_{-3}\}$. One finds the total ordering $L_{-1}^3 <_{\mathcal{O}_V} L_{-2}L_{-1} <_{\mathcal{O}_V} L_{-3}$, which is adapted to $\mathcal{C}_3^A = \{L_{-2}L_{-1}, L_{-3}\}$ with the ordering kernel $\mathcal{C}_3^K = \{L_{-1}^3\}$. To see this one has to compute the action of the annihilation operators $\Gamma \in \{L_1, L_2, L_3\}$ on the three terms. In fact, the action of L_1 already reveals the structure of \mathcal{C}_3^A , as $L_1 L_{-2} L_{-1} |\Delta\rangle$ contains the term L_{-1}^2 that is absent in $L_1 L_{-3} |\Delta\rangle$. The action of the three annihilation operators on $L_{-1}^3 |\Delta\rangle$, however, produce terms that are also created by the action of these operators on $L_{-2}L_{-1} |\Delta\rangle$ and/or $L_{-3} |\Delta\rangle$.

Finally, from the previous theorem one now deduces the known result about the uniqueness of the Virasoro singular vectors⁵.

Theorem 2.G *If the Virasoro Verma module \mathcal{V}_Δ contains a singular vector Ψ_l at level l , $l \in \mathbf{N}$, then Ψ_l is unique up to proportionality.*

3 Results for the superconformal algebras

As an illustration of the possibilities of the Adapted Ordering Method, in this section we will review briefly the results obtained for the $N = 2$ and Ramond $N = 1$ superconformal algebras. This method has been applied to the topological, to the Neveu-Schwarz and to the Ramond $N = 2$ algebras in Ref. 1, to the twisted $N = 2$ algebra in Ref. 2 and to the Ramond $N = 1$ algebra in Ref. 3. As the representation theory of these superconformal algebras has different types of Verma modules, one has to introduce different adapted orderings for each type and the corresponding kernels also allow different degrees of freedom.

Let us start with the topological $N = 2$ superconformal algebra \mathbb{T} . It contains the Virasoro generators \mathcal{L}_m with trivial central extension, a Heisenberg algebra \mathcal{H}_m corresponding to the $U(1)$ current, and the fermionic generators \mathcal{G}_m and \mathcal{Q}_m corresponding to two anticommuting fields with conformal weights 2 and 1 respectively. \mathbb{T} satisfies the (anti-)commutation relations¹⁵

$$\begin{aligned}
[\mathcal{L}_m, \mathcal{L}_n] &= (m-n)\mathcal{L}_{m+n}, & [\mathcal{H}_m, \mathcal{H}_n] &= \frac{C}{3}m\delta_{m+n}, \\
[\mathcal{L}_m, \mathcal{G}_n] &= (m-n)\mathcal{G}_{m+n}, & [\mathcal{H}_m, \mathcal{G}_n] &= \mathcal{G}_{m+n}, \\
[\mathcal{L}_m, \mathcal{Q}_n] &= -n\mathcal{Q}_{m+n}, & [\mathcal{H}_m, \mathcal{Q}_n] &= -\mathcal{Q}_{m+n}, \\
[\mathcal{L}_m, \mathcal{H}_n] &= -n\mathcal{H}_{m+n} + \frac{C}{6}(m^2+m)\delta_{m+n}, & & \\
\{\mathcal{G}_m, \mathcal{Q}_n\} &= 2\mathcal{L}_{m+n} - 2n\mathcal{H}_{m+n} + \frac{C}{3}(m^2+m)\delta_{m+n}, & & \\
\{\mathcal{G}_m, \mathcal{G}_n\} &= \{\mathcal{Q}_m, \mathcal{Q}_n\} = 0, & m, n \in \mathbf{Z}. &
\end{aligned} \tag{10}$$

The set of *annihilation operators* \mathbb{T}^+ is spanned by the generators with positive index, the set of *creation operators* \mathbb{T}^- is spanned by the generators with negative index, and the *zero modes* are given by $\mathbb{T}^0 = \text{span}\{\mathcal{L}_0, \mathcal{H}_0, C, \mathcal{G}_0, \mathcal{Q}_0\}$. The Cartan subalgebra is generated by $\mathcal{H}_{\mathbb{T}} = \text{span}\{\mathcal{L}_0, \mathcal{H}_0, C\}$, where C can be taken to be constant $c \in \mathbf{C}$, and the fermionic generators $\{\mathcal{G}_0, \mathcal{Q}_0\}$ classify the different choices of Verma modules.

A h.w. vector $|\Delta, h\rangle^{\mathcal{N}}$ is an eigenvector of $\mathcal{H}_{\mathbb{T}}$ with \mathcal{L}_0 eigenvalue Δ , \mathcal{H}_0 eigenvalue h , and vanishing \mathbb{T}^+ action. Additional vanishing conditions \mathcal{N} are possible with respect to the operators \mathcal{G}_0 and \mathcal{Q}_0 , resulting as follows¹². One can distinguish four different types of h.w. vectors $|\Delta, h\rangle^{\mathcal{N}}$ labeled by a superscript $\mathcal{N} \in \{G, Q, GQ\}$, or no superscript at all: h.w. vectors $|\Delta, h\rangle^G$ annihilated by \mathcal{G}_0 but not by \mathcal{Q}_0 (\mathcal{G}_0 -closed), h.w. vectors $|\Delta, h\rangle^Q$ annihilated by \mathcal{Q}_0 but not by \mathcal{G}_0 (\mathcal{Q}_0 -closed), h.w. vectors $|0, h\rangle^{GQ}$ annihilated by both \mathcal{G}_0 and \mathcal{Q}_0 (*chiral*), with zero conformal weight necessarily, and finally undecomposable h.w. vectors $|0, h\rangle$ that are neither annihilated by \mathcal{G}_0 nor by \mathcal{Q}_0 (*no-label*), also with zero conformal weight. Hence we have four different types of Verma modules¹²: $\mathcal{V}_{\Delta, h}^G$, $\mathcal{V}_{\Delta, h}^Q$, $\mathcal{V}_{0, h}^{GQ}$ and $\mathcal{V}_{0, h}$, built on the four different types of h.w. vectors.

For elements X of \mathbb{T} which are eigenvectors of $\mathcal{H}_{\mathbb{T}}$ with respect to the adjoint representation one defines the *level* l as $[\mathcal{L}_0, X] = lX$ and the *charge* q as $[\mathcal{H}_0, X] = qX$. In particular, elements of the form $X = \mathcal{L}_{-l_L} \dots \mathcal{L}_{-l_1} \mathcal{H}_{-h_H} \dots \mathcal{H}_{-h_1} \mathcal{Q}_{-q_Q} \dots \mathcal{Q}_{-q_1} \mathcal{G}_{-g_G} \dots \mathcal{G}_{-g_1}$, and any reorderings of it, have level $l = \sum_{j=1}^L l_j + \sum_{j=1}^H h_j + \sum_{j=1}^Q q_j + \sum_{j=1}^G g_j$ and charge $q = G - Q$. The Verma modules are naturally $\mathbf{N}_0 \times \mathbf{Z}$ graded with respect to the $\mathcal{H}_{\mathbb{T}}$ eigenvalues relative to the eigenvalues (Δ, h) of the h.w. vector. For a vector $\Psi_{l, q}$ in $\mathcal{V}_{\Delta, h}^{\mathcal{N}}$ the \mathcal{L}_0 -eigenvalue is $\Delta + l$ and the \mathcal{H}_0 -eigenvalue is $h + q$ with the level $l \in \mathbf{N}_0$ and the relative charge $q \in \mathbf{Z}$.

The singular vectors are annihilated by \mathbb{T}^+ and may also satisfy additional vanishing conditions with respect to the operators \mathcal{G}_0 and \mathcal{Q}_0 . Therefore one also distinguishes singular vectors of the types¹² $\Psi_{l, q}^G$, $\Psi_{l, q}^Q$, $\Psi_{l, q}^{GQ}$ and $\Psi_{l, q}$. As there are 4 types of Verma modules and 4 types of singular vectors one might think of 16 different combinations of singular vectors in Verma modules. However,

no-label and chiral singular vectors do not exist neither in *chiral* Verma modules $\mathcal{V}_{0,h}^{GQ}$ nor in *no-label* Verma modules¹² $\mathcal{V}_{0,h}$ (with one exception: chiral singular vectors exist at level 0 in no-label Verma modules). Using the Adapted Ordering Method one has to introduce adapted orderings for the remaining 12 combinations, whose kernels give upper limits for the dimensions of the corresponding singular vector spaces. One finds that for most charges q singular vectors do not exist. For the case of the Verma modules $\mathcal{V}_{\Delta,h}^G$ built on \mathcal{G}_0 -closed h.w. vectors $|\Delta,h\rangle^G$, for $c \neq 3$, the maximal dimensions for the singular vector spaces $\Psi_{l,q}^G$, $\Psi_{l,q}^Q$, $\Psi_{l,q}^{GQ}$ and $\Psi_{l,q}$ are given as follows¹:

	$q = -2$	$q = -1$	$q = 0$	$q = 1$	$q = 2$
$\Psi_{l,q, \Delta,h\rangle^G}^G$	0	1	2	1	0
$\Psi_{l,q, \Delta,h\rangle^G}^Q$	1	2	1	0	0
$\Psi_{l,q, -l,h\rangle^G}^{GQ}$	0	1	1	0	0
$\Psi_{l,q, -l,h\rangle^G}$	0	1	1	0	0

TAB. a Maximal dimensions for singular vectors spaces in $\mathcal{V}_{\Delta,h}^G$.

Charges q that are not given have dimension 0 and hence do not allow any singular vectors. The maximal dimensions for the case of the Verma modules $\mathcal{V}_{\Delta,h}^Q$, for $c \neq 3$, are obtained simply by interchanging $G \leftrightarrow Q$ and $q \leftrightarrow -q$ in the previous table.

For the case of singular vectors in chiral Verma modules $\mathcal{V}_{0,h}^{GQ}$ and in no-label Verma modules $\mathcal{V}_{0,h}$, for $c \neq 3$, one obtains the following maximal dimensions¹:

	$q = -2$	$q = -1$	$q = 0$	$q = 1$	$q = 2$
$\Psi_{l,q, 0,h\rangle^{GQ}}^G$	0	0	1	1	0
$\Psi_{l,q, 0,h\rangle^{GQ}}^Q$	0	1	1	0	0
$\Psi_{l,q, 0,h\rangle}^G$	0	1	3	3	1
$\Psi_{l,q, 0,h\rangle}^Q$	1	3	3	1	0

TAB. b Maximal dimensions for singular vectors spaces in $\mathcal{V}_{0,h}^{GQ}$ and in $\mathcal{V}_{0,h}$.

Tables TAB. a and TAB. b prove the conjecture made in Ref. 12, using the algebraic mechanism denoted *the cascade effect*, about the possible existing types of topological singular vectors. In addition, low level examples were constructed¹² for all these types, what proves that all of them exist already at level 1. The four types of two-dimensional singular vector spaces of TAB. a also exist starting at level 2, and four examples at level 3 were constructed¹² as well. For the case of the three-dimensional singular vector spaces in no-label Verma modules in TAB. b, the corresponding types of singular vectors have been constructed at level 1 generating one-dimensional¹² as well as two-dimensional¹ spaces, but no further search has been done for the three-dimensional spaces.

Transferring the dimensions we have found in tables TAB. a and TAB. b to the Neveu-Schwarz $N = 2$ algebra¹⁷⁻²¹ is straightforward as this algebra is related to the topological $N = 2$ algebra through the topological twists T_W^\pm : $\mathcal{L}_m = L_m \pm 1/2H_m$, $\mathcal{H}_m = \pm H_m$, $\mathcal{G}_m = G_{m+1/2}^\pm$ and $\mathcal{Q}_m = G_{m-1/2}^\mp$, where $G_{m+1/2}^\pm$ are the half-integer moded fermionic generators. As a result, the standard Neveu-Schwarz h.w. vectors correspond to \mathcal{G}_0 -closed topological h.w. vectors, whereas the chiral (antichiral) Neveu-Schwarz h.w. vectors, annihilated by $G_{-1/2}^+$ ($G_{-1/2}^-$), correspond to chiral

topological h.w. vectors. This implies^{10,12} that the standard, chiral and antichiral Neveu-Schwarz singular vectors correspond to topological singular vectors of the types $\Psi_{l,q,|\Delta,h\rangle^G}^G$ and $\Psi_{l,q,|\Delta,h\rangle^{GQ}}^{GQ}$, whereas the ones built in chiral or antichiral Verma modules correspond to topological singular vectors of only the type $\Psi_{l,q,|\Delta,h\rangle^{GQ}}^G$. As a consequence, by untwisting the first row of table TAB. a one recovers the results^{13,14} that in Verma modules of the Neveu-Schwarz $N = 2$ algebra singular vectors only exist for charges $q = 0, \pm 1$ and two-dimensional spaces only exist for uncharged singular vectors. By untwisting the third row of table TAB. a one gets a proof for the conjecture¹² that chiral singular vectors in Neveu-Schwarz Verma modules only exist for $q = 0, 1$ whereas antichiral singular vectors only exist for $q = 0, -1$. The untwisting of the first row of table TAB. b, finally, proves the conjecture^{10,12} that in chiral Neveu-Schwarz Verma modules $\mathcal{V}_{h/2,h}^{NS,ch}$ singular vectors only exist for $q = 0, -1$, whereas in antichiral Verma modules $\mathcal{V}_{-h/2,h}^{NS,a}$ singular vectors only exist for $q = 0, 1$.

As to the representations of the Ramond $N = 2$ algebra¹⁸⁻²¹, they are exactly isomorphic to the representations of the topological $N = 2$ algebra. Namely, combining the topological twists T_W^\pm and the spectral flows one constructs a one-to-one mapping between the Ramond singular vectors and the topological singular vectors, at the same levels and with the same charges¹⁶. Therefore the results of tables TAB. a and TAB. b can be transferred to the Ramond singular vectors simply by exchanging the labels $G \rightarrow (+)$, $Q \rightarrow (-)$, where the helicity (\pm) denotes the vectors annihilated by the fermionic zero modes G_0^\pm , and by taking into account that the chiral and undecomposable *no-helicity* Ramond vectors^{12,11,16}, require conformal weight $\Delta + l = c/24$.

The twisted $N = 2$ superconformal algebra¹⁸⁻²¹ is not related to the other three $N = 2$ algebras. It has mixed modes, integer and half-integer, for the fermionic generators, and the eigenvectors have no charge, as the $U(1)$ current generators are half-moded, but they have *fermionic parity*. The Adapted Ordering Method was worked out for the twisted $N = 2$ algebra in Ref. 2. The maximal dimension for the singular vector spaces in standard Verma modules was found to be two and these two-dimensional singular spaces were shown to exist by explicit computation, starting at level $3/2$. In Verma modules built on G_0 -closed h.w. vectors, however, the singular vectors were found to be only one-dimensional. This method also allowed to propose a reliable conjecture for the coefficients of the relevant terms of all singular vectors, i.e. for the coefficients with respect to the ordering kernels, what made possible to identify all the cases of two-dimensional singular vector spaces for all levels, as well as to identify all G_0 -closed singular vectors. The resulting expressions, in turn, led to the discovery of subsingular vectors for this algebra, and several explicit examples were also computed. Finally, the multiplication rules for singular vectors operators were derived using the ordering kernel coefficients, what set the basis for the analysis of the twisted $N = 2$ embedding diagrams.

Finally let us consider the $N = 1$ superconformal algebras^{6,21-23}. The structure of the h.w. representations of the Neveu-Schwarz $N = 1$ algebra has been completely understood in Ref. 24. The corresponding Verma modules do not contain two-dimensional singular vector spaces neither subsingular vectors. In the case of the Ramond $N = 1$ algebra, however, the application of the Adapted Ordering Method in Ref. 3 has shown that its representations have a much richer structure than previously suggested in the literature. In particular, it was found that standard Verma modules may contain two-dimensional singular vector spaces and also subsingular vectors. Moreover, the two-dimensional ordering kernels allowed to derive multiplication rules for singular vector operators and led to expressions for two-dimensional singular spaces. Using these multiplication rules descendant singular vectors were studied and embedding diagrams were derived for the rational models. In addition, this allowed to conjecture the ordering kernel coefficients of all singular

vectors and therefore identify these vectors uniquely.

4 Conclusions and Final Remarks

We have presented the Adapted Ordering Method for general Lie algebras and superalgebras, and their generalizations, provided they can be triangulated, like is the case for most of them. This method is based on the concept of adapted orderings, leading to the definition of the ordering kernels, which play a crucial rôle since their sizes limit the dimensions of the corresponding spaces of singular vectors. As a result the adapted orderings must be chosen such that the ordering kernels are as small as possible. Weights for which the ordering kernels are trivial do not allow any singular vectors in the corresponding weight spaces. On the other hand, non-trivial ordering kernels give us the maximal dimension of a possible singular vector space and uniquely define all singular vectors through the coefficients with respect to them.

The Adapted Ordering Method follows from the Definition 2.A plus the Theorems 2.B, 2.C and 2.D, which are rigorously proven. There is nothing in the Definition 2.A, neither in the three theorems, that restricts the application of this method to infinite-dimensional algebras. For the same reason, it seems clear that the Adapted Ordering Method should be useful also for generalized Lie algebras and superalgebras such as affine Kac-Moody algebras, non-linear W-algebras, superconformal W-algebras, loop Lie algebras, Borcherds algebras, F-Lie algebras for $F > 2$ ($F = 1$ are Lie algebras and $F = 2$ are Lie superalgebras), etc.

One may wonder whether there are any prescriptions in order to construct the most suitable orderings with the smallest kernels. The answer to this question is that there are no general prescriptions or recipes as the orderings depend entirely on the given algebras. The way to proceed is a matter of trial and error. That is, one constructs one total ordering first, that can always be done since a total ordering is simply a definition establishing which of two given terms is the bigger one, then one computes the kernel and decides whether this kernel is small enough. In the case it is not, then one constructs a second ordering and repeats the procedure until one finds a suitable ordering. It may also happen, for a given algebra, that this procedure does not give any useful information because all the total orderings one can construct are adapted only to the empty subset, in which case the ordering kernel is the whole set of terms: $\mathcal{C}_{\{l_i\}}^K = \mathcal{C}_{\{l_i\}}$.

The Adapted Ordering Method has been applied so far to the $N = 2$ and Ramond $N = 1$ superconformal algebras, allowing to prove several conjectured results as well as to obtain many new results, as we have reviewed. For example, this method allowed to discover subsingular vectors and two-dimensional spaces of singular vectors for the twisted $N = 2$ and Ramond $N = 1$ algebras^{2,3}. (For the other three isomorphic $N = 2$ algebras two-dimensional singular spaces had been discovered¹²⁻¹⁴, as well as subsingular vectors⁹⁻¹², before the Adapted Ordering Method was applied to them). We are convinced therefore that this method should be of very much help for the study of the representation theory of many other algebras, in particular the $N > 2$ superconformal algebras, and some (at least) of the generalized Lie algebras listed above.

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