The CKM matrix and the Unitary Triangle(s)

Origin of the CKM matrix

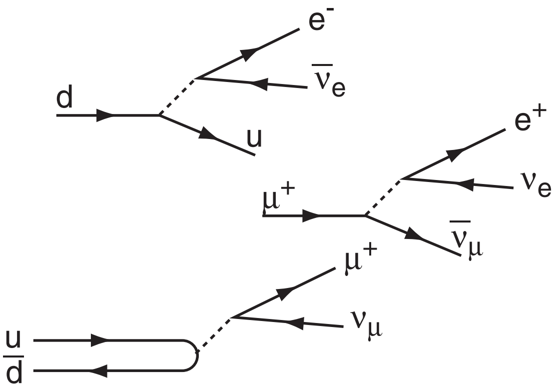
- Weak decays
- GIM
- Unification

Parametrization of CKM matrix

- Standard (?)
- Wolfenstein

The Unitary Triangles

Weak decays:



$n \rightarrow p + e^- + \bar{\nu}_e$

$\mu^+ \rightarrow e^+ + \bar{\nu}_\mu + \nu_e$

$\pi^+ \rightarrow \mu^+ + \nu_\mu$

All described by charged currents:

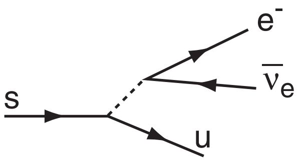
- exchange of $W^\pm$
- Fermi coupling constant $G_F/\sqrt{2} = g^2/(8 M_{W^2})$

Quark-lepton universality

Strangeness conserving charged currents: $\Delta S = 0$

Decay of strange particles:

$\Lambda^0 \rightarrow p + e^- + \bar{\nu}_e$



$\Delta S = 1$ decay: strength: factor 20 smaller:

$G_{F_K} / G_{F_\mu} = 0.22$; and also:

$G_{F_n} / G_{F_\mu} = 0.9740$

1963 Cabibbo: mixing down and strange quark:

$$\begin{pmatrix} u \\ d' \end{pmatrix} = \begin{pmatrix} u \\ d \cos\vartheta_C - s \sin\vartheta_C \end{pmatrix} \qquad \begin{pmatrix} u \\ s' \end{pmatrix} = \begin{pmatrix} u \\ d \sin\vartheta_C + s \cos\vartheta_C \end{pmatrix}$$

$\sin\vartheta_C = 0.22$

"rotated" charged weak currents

Problem: $K_0 - \bar{K}_0$:

$K_L - K_S$ mass difference much too large!

GIM

Glashown, Iliopoulos and Maiani - 1970

Weak Interactions with Lepton- Hadron Symmetry

Quarks: $\mathcal{P}$,$\mathcal{N}$ and λ Heavy vector boson

Problems:

- parity violation
- hypercharge conservation not conserved
- $\Delta s = -2$ amplitudes too large

Proposed new quark: $\mathcal{P}'$

	Q	Y	@
$\mathcal{P}'$	2/3	-2/3	1
$\mathcal{P}$	2/3	1/3	0
$\mathcal{N}$	-1/3	1/3	0
λ	-1/3	-2/3	0

Equivalence lepton $\Leftrightarrow$ quark:

$(\nu,\nu',e^-,\mu^-) \leftrightarrow (\mathcal{P}',\mathcal{P},\mathcal{N},\lambda)$

$$J_\mu^{\,L} = \bar{\ell}\, C_L\, \gamma_\mu\, (1+\gamma_5)\, \ell \qquad J_\mu^{\,H} = \bar{q}\, C_H\, \gamma_\mu\, (1+\gamma_5)\, q$$

$$C_L = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad C_H = \begin{pmatrix} 0 & 0 & \multicolumn{2}{c}{U} \\ 0 & 0 & & \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$U = \begin{pmatrix} -\sin\vartheta & \cos\vartheta \\ \cos\vartheta & \sin\vartheta \end{pmatrix}$$

Unitary matrices

- Divergencies resolved

  Leading contribution $K_L - K_S$ mass cancelled
- Conservation of hyper charge
- synthesis em and weak interactions

  Weinberg-Salam for lepton part in 1967/68

In GIM:

"Let us briefly consider a more daring speculation":
Introduction of triplet W^+, W^0 , W^-
and singlet W_s

combination of these gives γ and Z^0

GIM realizes: W^0 not necessary,
but:
- symmetric model
- aesthetically appealing

Unification electomagnetic and weak interactions

$W^{\pm}$ vertices:	V- A
photon and gluon:	V
Z^0:	both V + A and V - A

For interaction with Z^0, γ and G:

- FCNC: Flavour Changing Neutral Currents

- No flavour-changing neutral currents at tree level

Direct consequence unitary mixing matrix

The c-quark was discovered in 1974
at SLAC and Brookhaven

CP violation (1964) :

In two-generation system: one angle, no CP violation

Kobayashi and Maskawa (1973) proposed three-generation model with mixing of the d, s, b states.

Lepton:
Each lepton undergoes charge-changing transition to or from its own neutrino

Quark:
Mixing Q = 2/3 and Q = -1/3 type quarks

Weak eigenstates   Mass eigenstates

Standard model: $SU(3) \otimes SU(2) \otimes U(1)$

Yukawa matrices F and G for up and down-type quarks both systems not simultaneous diagonalized
- due to finite non-degenerate mass of the quarks

- u, c, t orthogonal, and
- d, s, b orthogonal!

u, c, t unmixed:
Choice purely historical!

$$\begin{pmatrix} d' \\ s' \\ b' \end{pmatrix} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \begin{pmatrix} d \\ s \\ b \end{pmatrix} = \hat{V}_{CKM} \begin{pmatrix} d \\ s \\ b \end{pmatrix}$$

For n generations:

n(n-1)/2	angles	3
(n-1)(n-2)/2	phases	1

$$\mathcal{L}_{CC} = \frac{g}{2\sqrt{2}} \{ W_\mu^\dagger J_W^\mu + h.c. \}$$

$$J_W^\mu = \sum \bar{u}_i \gamma^\mu (1-\gamma_5) V_{ij} d_j + \sum \bar{\nu}_\ell \gamma^\mu (1-\gamma_5) \ell$$

Flavour-changing transistions by charged currents: $\Delta Q = 1$

Local gauge invariance
Baryon number conservation

$\Rightarrow$ UNITARITY of V_{CKM}

$$\begin{pmatrix} d' \\ s' \\ b' \end{pmatrix} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \begin{pmatrix} d \\ s \\ b \end{pmatrix} = \hat{V}_{CKM} \begin{pmatrix} d \\ s \\ b \end{pmatrix}$$

Three angles ϑ_i and one complex phase δ

Original Kobayashi-Maskawa parametization: $s_i = \sin\vartheta_i$, $c_i = \cos\vartheta_i$

$$\begin{array}{ccc} c_1 & -s_1 c_3 & -s_1 s_3 \\ s_1 c_2 & c_1 c_2 c_3 - s_2 s_3 e^{i\delta} & c_1 c_2 s_3 + s_2 c_3 e^{i\delta} \\ s_1 s_2 & c_1 s_2 c_3 + c_2 s_3 e^{i\delta} & c_1 s_2 s_3 - c_2 c_3 e^{i\delta} \end{array}$$

And many more exist!

Results will not depend on choice

Proposal Particle Data Group:

Standard representation.

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13}e^{i\delta} \\ 0 & 1 & 0 \\ -s_{13}e^{i\delta} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} c_{12}c_{13} & s_{12}c_{23} & s_{13}e^{i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & s_{23}c_{13} \end{pmatrix}$$

Wolfenstein parametrization - 1983:

- Expansion in terms sin ϑ_C:
  $V_{us} = 0.22 = \lambda$ -> $V_{ud} = \ (1-\lambda^2) \quad 1-\lambda^2/2$
- From B-lifetime: $V_{cb} = 0.04 - 0.06 = A.\lambda^2$
- CP violation effects smaller third order: $A\lambda^3(\rho - i\eta)$
- Keep V_{ud}, V_{us}, V_{ts} and V_{tb} real;
  use unitarity to calculate λ^4

$$\begin{pmatrix} 1-\frac{\lambda^2}{2} & \lambda & A\lambda^3(\rho - i\eta + i\eta\frac{\lambda^2}{2}) \\ -\lambda & 1-\frac{\lambda^2}{2} - i\eta A\lambda^4 & A\lambda^2(1+i\eta\lambda^2) \\ A\lambda^3(1-\rho-i\eta) & -A\lambda^2 & 1 \end{pmatrix}$$

New experiments:

Much higher precision:

better parametrization

Several "Wolfenstein-parametrizations of higher order" exist.

Use the following parametrization:

Standard V_{CKM} from Particle Data Group with:

$s_{12} = \lambda$ $\qquad$ $s_{13} \sin \delta_{13} = A\lambda^3 \eta$

$s_{23} = A\lambda^2$ $\qquad$ $s_{13} \cos \delta_{13} = A\lambda^3 \rho$

$$\begin{pmatrix} 1-\frac{\lambda^2}{2}-\frac{\lambda^4}{8} & \lambda & A\lambda^3(\rho - i\eta) \\ -\lambda - A^2\lambda^5(\rho + i\eta - \frac{1}{2}) & 1-\frac{\lambda^2}{2}-(\frac{1}{8}+\frac{A}{2})\lambda^4 & A\lambda^2 \\ A\lambda^3[1-(\rho+i\eta)(1-\frac{\lambda^2}{2})] & -A\lambda^2 - A\lambda^4(\rho+i\eta-\frac{1}{2}) & 1-\frac{1}{2}A^2\lambda^4 \end{pmatrix} + \mathcal{O}(\lambda^6)$$

Unitarity requires:

$$|V_{ud}|^2 + |V_{us}|^2 + |V_{ub}|^2 = 1$$
$$|V_{cd}|^2 + |V_{cs}|^2 + |V_{cb}|^2 = 1$$
$$|V_{td}|^2 + |V_{ts}|^2 + |V_{tb}|^2 = 1$$
$$|V_{ud}|^2 + |V_{cd}|^2 + |V_{td}|^2 = 1$$
$$|V_{us}|^2 + |V_{cs}|^2 + |V_{ts}|^2 = 1$$
$$|V_{ub}|^2 + |V_{cb}|^2 + |V_{tb}|^2 = 1$$

$$V_{ud}^* V_{cd} + V_{us}^* V_{cs} + V_{ub}^* V_{cb} = 0$$
$$V_{ud}^* V_{td} + V_{us}^* V_{ts} + V_{ub}^* V_{tb} = 0$$
$$V_{cd}^* V_{td} + V_{cs}^* V_{ts} + V_{cb}^* V_{tb} = 0$$
$$V_{ud} V_{us}^* + V_{cd} V_{cs}^* + V_{td} V_{ts}^* = 0$$
$$V_{ud} V_{ub}^* + V_{cd} V_{cb}^* + V_{td} V_{tb}^* = 0 \quad \Leftarrow$$
$$V_{us} V_{ub}^* + V_{cs} V_{cb}^* + V_{ts} V_{tb}^* = 0$$

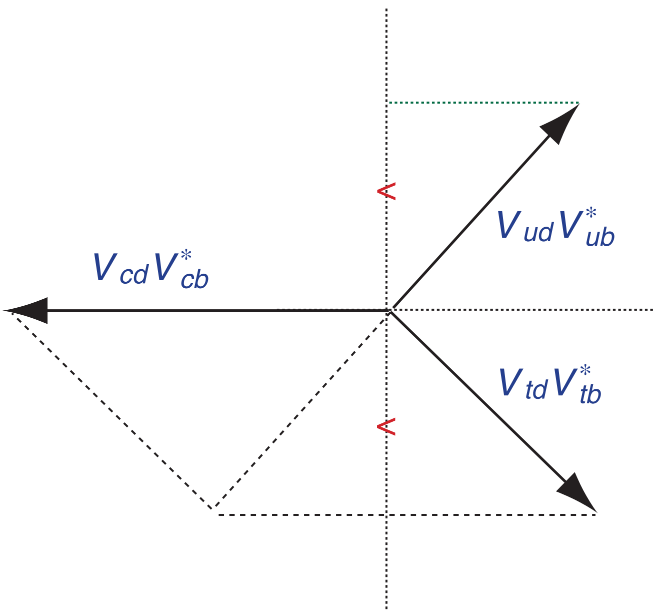
$$V_{ud}V^*_{ub} + V_{cd}V^*_{cb} + V_{td}V^*_{tb} = 0$$

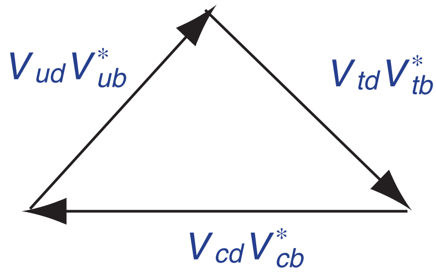
vectors in complex plane:

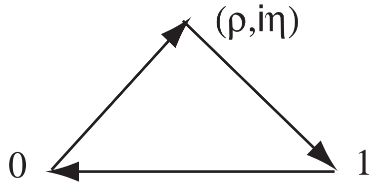
$V_{cd} = |V_{cd}|\, e^{i\vartheta}$ $\qquad$ $V_{cb}^* = |V_{cb}|\, e^{-i\varphi}$

Phase factor common to row or column can be eliminated!

The three vectors define a triangle:







Use Wolfenstein third order:

$V_{ud}\, V^*_{ub} = A\lambda^3(1 - \lambda^2/2)(\rho + i\eta)$

$V_{cd}\, V^*_{cb} = -A\lambda^3$

$V_{td}\, V^*_{tb} = A\lambda^3(1 - \rho - i\eta)$

Jarlskog's measure of CP-violation:

$V_{ud}V^*_{ub} + V_{cd}V^*_{cb} + V_{td}V^*_{tb} = 0 \qquad \text{x } V^*_{cd}V_{cb} / |V^*_{cd}V_{cb}|$

$$\Im m(V_{cb}V_{td}V^*_{cd}V^*_{tb}) = -\Im m(V_{cb}V_{ud}V^*_{ub}V^*_{tb})$$

$\Im m\,(A\lambda^2 \text{ x } A\lambda^3(1-\rho-i\eta) \text{ x } -\lambda \text{ x } 1) = A^2\lambda^6\eta = J_{CP} = 2 \text{ x Area UT}$

All six unitary orthogonality relations define triangles with the same area:

$$J_{CP} = \pm\Im m\,(V_{ik}V_{jl}V^*_{il}V^*_{jk}) \approx A^2\lambda^6\eta \qquad (i \neq j, l \neq k)$$

$$\Im m\,(V_{cb}V_{td}V^*_{cd}V^*_{tb}) = \Im m\,(V_{cb}(-V_{ts}V^*_{cs} - V_{tb}V^*_{cb})V^*_{tb})$$

$$= -\Im m\,(V_{cb}V_{ts}V^*_{cs}V^*_{cb}) - \Im m\,(V_{cb}V_{tb}V^*_{cb}V^*_{tb})$$

= 0 (pure real!)

$$V_{ud}^*V_{cd}+V_{us}^*V_{cs}+V_{ub}^*V_{cb}=0 \qquad \lambda,\ \lambda,\ \lambda^5$$

$$V_{ud}^*V_{td}+V_{us}^*V_{ts}+V_{ub}^*V_{tb}=0 \qquad \lambda^3,\ \lambda^3,\ \lambda^3$$

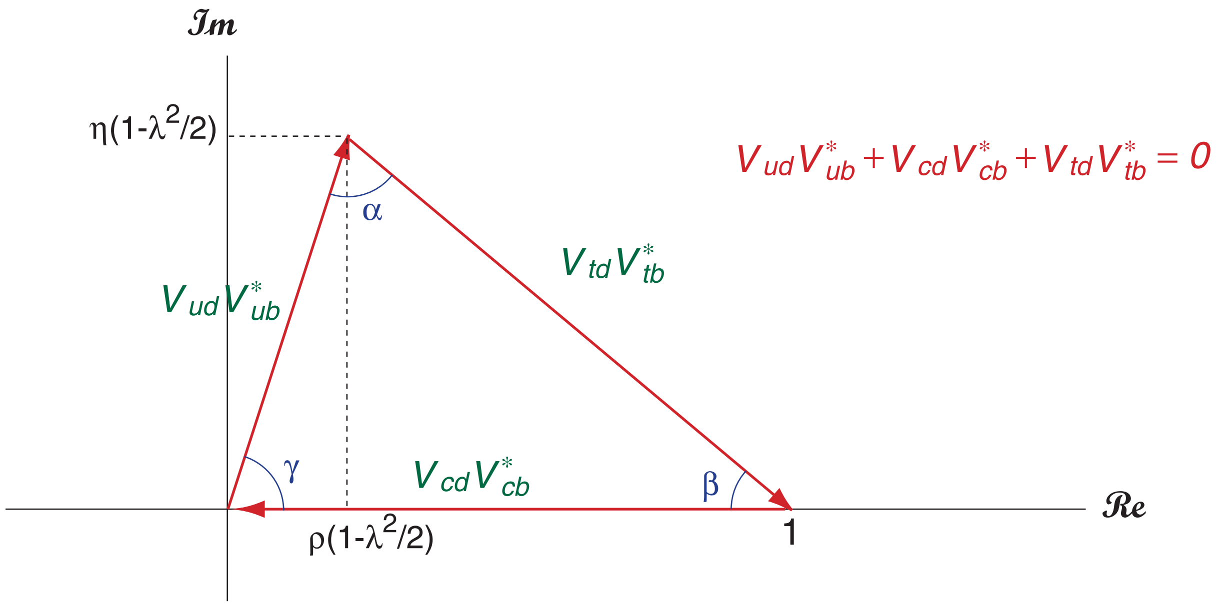
$$V_{cd}^*V_{td}+V_{cs}^*V_{ts}+V_{cb}^*V_{tb}=0 \qquad \lambda^4,\ \lambda^2,\ \lambda^2$$

$$V_{ud}V_{us}^*+V_{cd}V_{cs}^*+V_{td}V_{ts}^*=0 \qquad \lambda,\ \lambda,\ \lambda^5$$

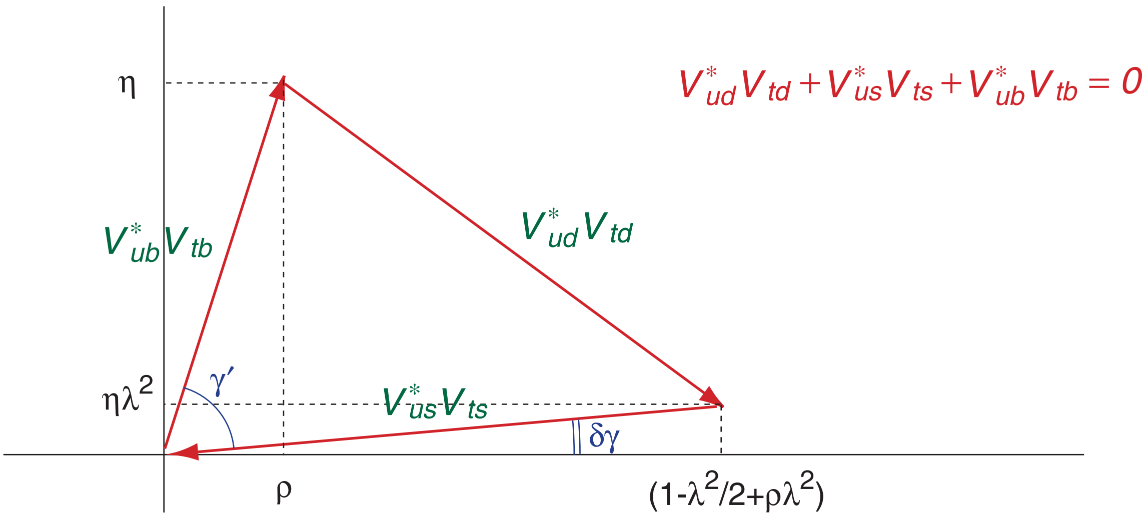
$$V_{ud}V_{ub}^*+V_{cd}V_{cb}^*+V_{td}V_{tb}^*=0 \qquad \lambda^3,\ \lambda^3,\ \lambda^3$$

$$V_{us}V_{ub}^*+V_{cs}V_{cb}^*+V_{ts}V_{tb}^*=0 \qquad \lambda^4,\ \lambda^2,\ \lambda^2$$

$$\begin{pmatrix} 1-\lambda^2/2 & \lambda & A\lambda^3(\rho-i\eta) \\ -\lambda & 1-\lambda^2/2 & A\lambda^2 \\ A\lambda^3(1-\rho-i\eta) & -A\lambda^2 & 1 \end{pmatrix}$$



$$\begin{array}{cccc} (\rho+i\eta) & -1 & (1-\rho-i\eta) & =0 \\ -\frac{1}{2}\lambda^2(\rho+i\eta)+O(\lambda^4) & +O(\lambda^4) & +\ \frac{1}{2}\lambda^2(\rho+i\eta)+O(\lambda^4) & \end{array}$$



$$\begin{array}{cccc} (1-\rho-i\eta) & -1 & +(\rho+i\eta) & =0 \\ \lambda^2(\rho+i\eta-\frac{1}{2})+O(\lambda^4) & +\lambda^2(\frac{1}{2}-\rho-i\eta)+O(\lambda^4) & +O(\lambda^4) & \end{array}$$

Physics with the Unitary Triangles:

Sides:

V_{ud}	β-decay	$(A,Z) \rightarrow (A,Z+1) + e^- + \bar{\nu}_e$	$\cos\vartheta_C$
V_{us}	K-decay	$K^+ \rightarrow \pi^0 + \ell^+ + \nu_\ell$	$\sin\vartheta_C$
		$K^0 \rightarrow \pi^- + \ell^+ + \nu_\ell$	
V_{cd}	ν-production of c's	$\nu_\ell + d \rightarrow \ell^- + c$	$\cos\vartheta_C$
V_{cs}		$D^\pm \rightarrow K^0 + \ell^\pm + \nu_\ell$	$\sin\vartheta_C$
V_{ub}	B-decay	$b \rightarrow u + \ell^- + \bar{\nu}_\ell$	
V_{cb}		$b \rightarrow c + \ell^- + \bar{\nu}_\ell$	
V_{td}	Δm in B^0-$\bar{B}^0$		

Is the triangle a triangle? Check on Standard Model/ New Physics!

LHCb: Measure asymmetries in B-decay:

$$V_{tb}^* V_{td} V_{cb}^* V_{cd} = |V_{tb}^* V_{td} V_{cd}^* V_{cb}|\, e^{-i\beta}$$

$B^0{}_d \rightarrow J/\Psi\, K_S$	$\sin 2\beta$
$B^0{}_d \rightarrow \pi^+\pi^-$	$\sin 2\alpha$
$B^0{}_s \rightarrow D_S{}^\pm K^\mp$	$\sin 2\gamma$

Other interesting angles: (In almost degenerate unitary triangle)

$\beta_S = \arg\,(-V_{cs}V_{cd}/V_{ts}V_{tb})$

$-\beta_K = \arg\,(V_{ud}V_{us}/V_{cd}V_{cs})$