# Q&D QFT

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# 1 QFT for a single field in 0+0 dimensions

## 1.1 Generalities

The objects of primary interest in this section are:

| object | notation | remarks |
|---|---|---|
| Quantum constant | $\hbar$ | not yet Planck's constant! |
| Stochastic field variable | $\varphi$ | values in $(-\infty, +\infty)$ |
| Action | $S(\varphi)$ | assume $S(\varphi) \to \infty$ if $\lvert\varphi\rvert \to \infty$ |
| | | the action defines the physics |
| Probability density | $N \exp\left(-\frac{1}{\hbar}S(\varphi)\right)$ | normalized to unity |
| Green's functions: | | |
| Unconnected | $G_n$ | moment of prob. dens.: $G_n = \langle\varphi^n\rangle$ |
| | | $G_0 = \langle 1\rangle = 1$ always! |
| Connected | $C_n$ | cumulant of prob. dens., *e.g.* |
| | | $C_1 = G_1$, $C_2 = G_2 - {G_1}^2$ |
| Source | $J$ | counting tool |
| Path integral | $Z(J)$ | generating function for $G_n$: |
| | | $Z(J) = \sum\limits_{n\geq 0} \frac{1}{n!} G_n\,(J/\hbar)^n$ |
| (no name!) | $W(J)$ | generating function for $C_n$: |
| | | $W(J) = \sum\limits_{n\geq 1} \frac{1}{n!} C_n\,(J/\hbar)^n = \log(Z(J))$ |
| 'field function' | $\phi(J)$ | generating function for $C_n$: |
| | | $\phi(J) = \hbar W'(J) = \sum\limits_{n\geq 0} \frac{1}{n!} C_{n+1}\,(J/\hbar)^n$ |
| Effective action | $\Gamma(\phi)$ | classical description of full result |

The path integral has an integral form:

$$Z(J) = N \int d\varphi \;\exp\left(-\frac{1}{\hbar}\left(S(\varphi) - J\varphi\right)\right)$$

**Excercise 1** *Prove this.*

The field $\varphi$ and the field function $\phi(J)$ are related by

$$\phi(J) = \langle\varphi\rangle_J \quad \text{expectation value in the presence of the source}$$

**Excercise 2** *Prove this, using the fact that* $\phi(J) = \hbar Z'(J)/Z(J)$.

The connected Green's functions are obtained from

$$C_n = \hbar^n \left[ \frac{\partial^n}{(\partial J)^n} \phi(J) \right]_{J=0}$$

**Excercise 3** *Prove this.*

## 1.2 Perturbation theory

The usual form of the action is

$$S(\varphi) = K(\varphi) + V(\varphi) \quad , \quad K(\varphi) = \frac{1}{2}\mu\varphi^2 \quad , \quad V(\varphi) = \mathcal{O}(\varphi^3)$$

The interaction part $V(\varphi)$ is usually a polynomial. Expand, assuming $V(\varphi)$ to be 'small':

$$\begin{aligned} G_n &= N \int d\varphi \ \varphi^n \ \exp\left(-\frac{1}{\hbar}\left(K(\varphi) + V(\varphi)\right)\right) \\ &= \sum_{p\geq 0} \frac{N}{p!} \left(-\frac{1}{\hbar}\right)^p \int d\varphi \ \exp\left(-\frac{\mu\varphi^2}{2\hbar}\right) \ \varphi^n V(\varphi)^p \end{aligned}$$

The result is a sum of moments of a Gaussian distribution. From $G_0 \equiv 1$ one determines the form of $N$. Then the other $G_n$ are known, and from these follow the $C_n$. Note: the perturbation series is not convergent, but only asymptotic! In practice no problem.

**Excercise 4** *Prove that*

$$\frac{1}{\sqrt{2\pi\sigma^2}} \int dx \ x^{2n} \ \exp\left(-\frac{x^2}{2\sigma^2}\right) = \frac{(2n)!}{2^n n!}\sigma^{2n} \quad .$$

## 1.3 The Schwinger-Dyson equation

By partial integration it follows immediately that

$$\left[\frac{\partial}{\partial\varphi} S(\varphi)\right]_{\varphi \to \hbar \frac{\partial}{\partial J}} Z(J) = J Z(J)$$

This can be translated in an equation for $\phi(J)$, very useful for obtaining the $C_n$ directly.

**Excercise 5** *Prove the above equation.*

## 1.4 Example: $\varphi^4$ theory

$$S(\varphi) = K(\varphi) + V(\varphi) \quad , \quad K(\varphi) = \frac{1}{2}\mu\varphi^2 \quad , \quad V(\varphi) = \frac{1}{4!}\lambda_4\varphi^4 \qquad (\mu, \lambda_4 > 0)$$

Perturbation expansion for the $G_n$: $G_{2m+1} = 0$ and

$$\begin{aligned} G_{2m} &= \sum_{p\geq 0} \int d\varphi \ \exp\left(-\frac{\mu\varphi^2}{2\hbar}\right)\left(-\frac{\lambda_4}{24\hbar}\right)^p \varphi^{2m+4p} \\ &= N \sum_{p\geq 0} \left(-\frac{\lambda_4\hbar}{24\mu^2}\right)^p \left(\frac{\hbar}{\mu}\right)^p \frac{(2m+4p)!}{2^{m+2p}\ p!\ (m+2p)!} \end{aligned}$$

Results: see `MAPLE` output. For large $m$ and/or $p$, the coefficients *grow* roughly as $(p+m)!$.

The SD equation for $Z(J)$ for this theory reads

$$\frac{1}{3!}\lambda_4\hbar^3 Z'''(J) + \mu\hbar Z'(J) - JZ(J) = 0 \ .$$

Using $\phi(J) = \hbar Z'(J)/Z(J)$ this becomes

$$\phi(J) = \frac{J}{\mu} - \frac{\lambda_4}{6\mu}\left(\phi(J)^3 + 3\hbar\phi(J)\phi'(J) + \hbar^2\phi''(J)\right)$$

This allows an iterative solution starting with $\phi(J) = 0$, see `MAPLE` output.

**Excercise 6** *Extend these results to the case where* $V(\varphi) = \frac{1}{3!}\lambda_3\varphi^3 + \frac{1}{4!}\lambda_4\varphi^4$ .

## 1.5 Feynman diagrams

The Green's functions $G$ and $C$ can be computed diagrammatically, using Feynman diagrams. These are built up using lines ('propagators') and vertices. Each diagram has an algebraic value, given by the Feynman rules:

- A factor $\hbar/\mu$ for each propagator;
- A factor $-\lambda_k/\hbar$ for each vertex with $k \geq 3$ legs (provided the action contains a term $\lambda_k\varphi^k/k!$);
- A factor $+J/\hbar$ for each source term (a 'one-vertex');

- A symmetry factor based on the topology of the diagram:
  - A factor $1/p!$ for each group of $p$ equivalent (interchangeable) vertices. One-vertices in the same connected (piece of a) diagram are always equivalent to each other.
  - A factor $1/q!$ for each group of $q$ equivalent (interchangeable) internal lines;
  - A factor 1/2 for each 'leaf', that is, a line starting and ending at the *same* vertex[1];
  - A factor $1/r!$ for each group of $r$ equivalent (interchangeable) connected sub-pieces of a disconnected diagram
  - External lines (going off 'to infinity') are *not* equivalent.

The empty diagram counts as 1. Getting the symmetry factors correctly is the hardest part: fortunately, in many cases such as QED, or for tree diagrams (diagrams without closed loops) the symmetry factor is 1.

**Excercise 7** *Draw some Feynman diagrams and determine their value.*

**Excercise 8** *Show that tree diagrams always have symmetry factor 1.*

To obtain $G_n$, follow the following recipe:

1. Write down *all* Feynman diagrams with precisely $n$ external legs and no source terms;
2. Discard all diagrams (or disconnected parts of diagrams) with *no* external legs;
3. Compute the value of each diagram;
4. Sum them.

**Excercise 9** *Compute $G_2$ and $G_4$ to order ${\lambda_4}^2$ for the $\varphi^4$ theory using Feynman diagrams, and compare the answers to the* `MAPLE` *output.*

[1] This hold only for *unoriented* lines. For instance, for fermion lines, that are oriented, flipping this 'leaf' over does not lead to the same graph and hence such 'leaves' do not carry a factor 1/2.

To obtain the $C_n$, do the same but restrict to *connected* diagrams. To obtain $Z(J)$, put sources on all external legs of the diagrams for the $G_n$ (thus adding a factor $(J/\hbar)^n/n!$), and to obtain $W(J)$ do the same for the set of all $C_n$. From the symmetry factor involved in equivalent connected pieces of a disconnected diagram, it immediately follows that $Z(J) = \exp(W(J))$.

**Excercise 10** *Compute* $C_2$ *and* $C_4$ *to order* ${\lambda_4}^2$ *for the* $\varphi^4$ *theory using Feynman diagrams, and compare the answers to the* `MAPLE` *output.*

## 1.6 The loop expansion

For any diagram, define

$$\begin{aligned} P &= \text{number of connected pieces (P=1 for a connected diagram)} \\ E &= \text{number of external lines} \\ I &= \text{number of internal lines} \\ V_k &= \text{number of vertices with } k \text{ legs} \\ L &= \text{number of closed loops} \end{aligned}$$

There are two topological identities:

$$\sum_{k\geq 3} kV_k = 2I + E \quad , \quad \sum_{k\geq 3} V_k = I - L + P$$

Power of $\hbar$ for a given diagram:

$$\hbar^{I+E-\sum V_k} = \hbar^{E+L-P}$$

This does *not* depend on the internal details of the diagram. Truncating the perturbation series at a given power of $\hbar$ means truncating the series of diagrams at a given number of closed loops. This provides a unique perturbative expansion for theories with several coupling constants, *e.g.* $\lambda_{2+p}$ is formally of the same order as ${\lambda_3}^p$.

**Excercise 11** *Draw some Feynman diagrams and check the topological identities.*

## 1.7 Proof of the validity of Feynman diagrams

From the above recipes, it follows immediately that

- The set of all connected diagrams with arbitrary number of sources and precisely *one* external leg equals $\phi(J)$.
- The set of all connected diagrams with arbitrary number of sources and precisely *two* external legs equals $\phi'(J)$.
- The set of all connected diagrams with arbitrary number of sources and precisely *three* external legs equals $\phi''(J)$.
- ... and so on.

**Excercise 12** *Check this.*

For the $\varphi^4$ theory, consider $\phi(J)$, and classify what happens if one moves into the diagram over the single external leg. This leads to the *diagrammatic* SD equation for this theory:

By applying the Feynman rules (with the symmetry factors!) one sees that $\phi(J)$ obeys precisely the SD equation. The iterative procedure, starting with $\phi(J) = J/\mu$, therefore produces the correct answer.

**Excercise 13** *Show that in the above argument it is important that only* connected *diagrams are considered.*

**Excercise 14** *Determine the diagrammatic SD equation for $\varphi^3$ theory.*

## 1.8 Dyson summation

Consider a theory with action

$$S(\varphi) = \frac{1}{2}\mu\varphi^2 + V(\varphi) \quad , \quad V(\varphi) = \frac{1}{2}\lambda_2\varphi^2$$

The only connected Green's function is $C_2$. We may consider $V(\varphi)$ as an interaction, and get the diagrammatic SD equation which reads

$$\phi(J) = C_2 \frac{J}{\hbar} = \frac{J}{\mu} - \frac{\lambda_2}{\mu}\phi(J)$$

with solution

$$\phi(J) = \frac{J}{\mu + \lambda_2} \quad \Rightarrow \quad C_2 = \frac{\hbar}{\mu + \lambda_2}$$

Alternatively, we may immediately reinterpret the interacting theory with mass $\mu$ as a free theory with mass $\mu + \lambda_2$, with the same result. Hence, two-point interactions are just a shift in the mass term.

**Excercise 15** *Write down the explicit diagrams contained in the above SD equation. Arrive at the correct result for $C_2$ by explicit summation of the diagrams.*

## 1.9 The classical limit; instantons

For $\hbar \to 0$ the diagrammatic SD equation for the classical field $\phi_c = \lim_{\hbar \to 0} \phi(J)$ reads, in $\varphi^4$ theory:

Only tree diagrams contribute. In general, it is

$$S'(\phi_c) = J$$

The path integral is dominated by $\varphi$ values where the probability density has an minimum, *i.e.* by values $\varphi_c$ such that

$$\frac{\partial}{\partial \varphi_c} \exp\left(-\frac{1}{\hbar}\left(S(\varphi_c) - J\varphi_c\right)\right) = 0 \quad \Rightarrow \quad S'(\varphi_c) = J \ \ , \ \ \frac{\partial^2}{(\partial \varphi_c)^2} S(\varphi_c) > 0$$

This is the *classical field equation* (Maxwell, Klein-Gordon, Proca, Dirac, ...). It may have more than one solution. Let $\varphi_c^{(0)}$ be the dominant solution, and $\varphi_c^{(1)}$ be a subdominant one, that is, $S(\varphi_c^{(0)}) < S(\varphi_c^{(1)})$. The contribution to $Z(J)$ from the subdominant solution (an *instanton*) is suppressed by a relative factor

$$\exp\left(-\frac{1}{\hbar}\left(S(\varphi_c^{(1)}) - S(\varphi_c^{(0)})\right)\right)$$

This has *no* perturbative expansion in powers of $\hbar$: instanton effects are nonperturbative, and usually small.

**Excercise 16** *Consider the classical limit for a theory with action*

$$S(\varphi) = \frac{1}{24}\lambda_4\varphi^4 - \frac{1}{2}\mu\varphi^2$$

*Find the dominant and subdominant classical solutions for* $J > 0$ *and for* $J < 0$*, and show that in the classical limit* $\phi(J)$ *is a discontinuous function of* $J$*. Determine the nonperturbative suppression factor for the instanton.*

**Excercise 17** *All tree diagrams have symmetry factor 1. This allows us to count tree diagrams. Show that if we put* $\hbar = \mu = 1$ *and* $\lambda_j = -1$ *for all* $j \geq 3$*, the tree amplitude* $C_n$ *equals the number of tree graphs. Show that the Schwinger-Dyson equation for* $\varphi^3 + \varphi^4$ *theory then reads*

$$\phi = J + \frac{1}{2}\phi^2 + \frac{1}{6}\phi^3$$

*Show that* $\phi(J)$ *has a finite radius of convergence (Hint: write* $J$ *as a function of* $\phi$ *and look for* $dJ/d\phi = 0$*). Show that this means that the number of tree graphs contains a factor* $n!$ *for* $C_n$*.*

## 1.10 The effective action; 1PI diagrams

For a given action $S(\varphi)$ the effective action $\Gamma(\varphi)$ is that action whose *classical* solution equals the *full* solution for $S$, *i.e.*

$$\Gamma'(\varphi) = J \quad \text{for } \varphi = \phi(J)$$

It is considered to be a 'better approximation' to the real physics.

**Excercise 18** *Using excercise 2, prove that* $d\phi(J)/dJ > 0$*. From this, prove that* $d^2\Gamma(\varphi)/d\varphi > 0$*, hence the effective action is* concave *(bowl-shaped).*

**Excercise 19** *If we know* $\phi$ *as a function of* $J$*, we also know* $J$ *as a function of* $\phi$ *(at least in principle). Write* $J = Y(\phi)$*. Use this to prove that*

$$\Gamma(\phi) = \phi Y - \hbar \log(Z(Y)) \ .$$

A *one-particle irreducible* (1PI) diagram is a diagram that does *not* bedome disconnected upon cutting any internal line: it therefore consists of either a single vertex, or it contains 'only loops'. This leads to an alternative SD equation based on the 1PI character of the first encountered vertex:

which is precisely the form of the classical equation for an action$\Gamma(\varphi)$. This shows that the effective action is built up from the 1PI diagrams.

**Excercise 20** *Draw the one- and two-loop 1PI diagrams for the* $\varphi^3 + \varphi^4$ *theory with up to 3 external legas. Also, draw some diagrams that are* not *1PI.*

## 1.11 Renormalization

The action parameters are not given *a priori* but have to be extracted from measurements. Only after this has been done can one proceed to predict the outcome of new measurements. Consider $\varphi^4$ theory. By the 'amplitude' $A_n$ we denote connected Green's function $C_n$ with the leading power of $\hbar$ divided out: this will be justified later on. Any amplitude therefore starts at $\hbar^0$. Let the *measured values* of the connected 2,4,6-point amplitudes be $E_{2,4,6}$. The procedure of phenomenology is then:

1. Compute $A_2, A_4$ and $A_6$ to some order in perturbation theory.

2. Fit the values of the action parameters $\mu$ and $\lambda_4$ so that $A_2 = E_2$ and $A_4 = E_4$.

3. Use these values to compute $A_6$, and compare this prediction to the values $E_6$.

A next higher order in perturbation theory for $A_6$ is only useful if also the higher orders in $A_2$ and $A_4$ are accounted. Usually, therefore, one writes

$$\mu = \mu_{(0)} + \mu_{(1)}\hbar + \mu_{(2)}\hbar^2 + \cdots \quad , \quad \lambda_4 = \lambda_{(0)} + \lambda_{(1)}\hbar + \lambda_{(2)}\hbar^2 + \cdots$$

and uses $\mu_{(0)}$ and $\lambda_{(0)}$ to achieve $A_{2,4} = E_{2,4}$: the successive higher orders are then adjusted so as to leave these two results intact (this is not the only possibility, but it is the most convenient). Note that, in fact, $C_6$ is then expressed solely in terms of $E_2$ and $E_4$, and the action parameters $\mu$ ans $\lambda$ are just *bookkeeping devices*. For an example, see the `MAPLE` output. Note that if one only includes the higher-order terms in $A_6$ and not in $A_{2,4}$, the answer is quite different (and the corrections are much larger).

**Excercise 21** *Verify the following: if we would renormalize the* $C_n$ *rather than the* $A_n$*, the prediction for* $C_6$ *would contain no powers of* $\hbar$ *whatsoever, and perturbation theory would not survive renormalization: each next contribution from perturbation theory would be of the same order as the previous one.*

**Excercise 22** *Perform the renormalization procedure up to order* $\hbar^2$ *for* $\varphi^3$ *theory. Assume that amplitudes* $A_1$ *and* $A_2$ *are measured, and amplitude* $A_3$ *is the result to be predicted.*

## 1.12 Renormalizability: a primordial model

In higher dimensions, diagrams with loops a complicated and somtimes infinite. To model this, consider $\varphi^4$ theory. Introduce an additional ‘post-Feynman’ rule that says that loops with precisely one or two vertices on them pick up an additional (potentially infinite) contribution:

A dotted loop equals equals an undotted loop times a factor $c$.
This can be accomodated by a new SD equation:

where the shaded box-vertices $\Gamma_2$ and $\Gamma_4$ are given by

and contain *only* dotted loops (note: a factor $\hbar^2/\mu^2$ for the external legs, and a factor $-1/\hbar$ are taken out. This makes $\Gamma_{2,4}$ act precisely like vertices.). We have[2]

$$\Gamma_2 = \frac{\hbar\lambda_4 c_1}{2\mu^2}\left(1 - \frac{\hbar\lambda_4 c_2}{2\mu^2}\right)^{-1} - \frac{\hbar^2{\lambda_4}^2{c_2}^2}{6\mu^3} \quad , \quad \Gamma_4 = -\frac{\hbar\lambda_4^2 c_2}{2\mu}\left(1 - \frac{\hbar\lambda_4 c_2}{2\mu^2}\right)^{-1}$$

**Excercise 23** *Prove this.*

This new SD equation reads

$$\phi = \frac{J}{\mu} - \frac{\lambda_4 + 3\Gamma_4}{6\mu}\left(\phi^3 + 3\hbar\phi\phi' + \hbar^2\phi''\right) - \frac{\Gamma_2}{\mu}\phi$$

and is identical to the original one if we replace

$$\mu \to \mu^{(R)} \equiv \mu + \Gamma_2 \quad , \quad \lambda_4 \to \lambda_4^{(R)} \equiv \lambda_4 + 3\Gamma_4$$

The parameters $\mu^{(R)}$ and $\lambda_4^{(R)}$ are called the *renormalized* parameters: they are the only combinations that actually occur in any connected Green's function.

**Excercise 24** *Check this, paying special attention to the symmetry factors.*

Since $\mu$ and $\lambda_4$ (and $\mu^{(R)}$ and $\lambda_4^{(R)}$) are bookkeeping devices, this inclusion does not show up in the relation between $E_{2,4}$ and $E_6$ discussed above. The values of $c_{1,2}$ (and in particular the limit $c_{1,2} \to \infty$) are therefore irrelevant. Theories in which all loop divergences can be absorbed in a finite number of measurements are called *renormalizeable*. The Standard Model is an example.

**Excercise 25** *Extend the above discussion to the case of $\varphi^3 + \varphi^4$ theory. There are then shaded box-vertices with 1,2,3 and 4 lines. Give these vertices; write the modified SD equation; show that, if in the original action we also include a $\lambda_1\varphi^1$ vertex, the theory is renormalizeable; give the renormalized parameters.*

**Excercise 26** *Show that, with the above 'post-Feynman' rules, a $\varphi^6$ theory is* not *renormalizeable.*

**Excercise 27** *Consider an extension to a 'post-Feynman' rule where also loops with three vertices are dotted. Show that, for the $\phi^4$ theory discussed, this leads to a theory that is* not *renormalizeable.*

[2]The fact that $\Gamma_4$ is actually finite for $c_2 \to \infty$ is particular to this model, and is irrelevant to the conclusion.

# 2 QFT in Euclidean space

## 2.1 Many fields in 0+0 dimensions

Consider an infinite set of 0+0-dimensional fields, labelled $\varphi_n$, $-\infty < n < \infty$. Each field comes with its own source $J_n$. The action (including the sources) is chosen to be

$$S(\{\varphi\}) = \sum_n \left[\frac{1}{2}\mu{\varphi_n}^2 - \gamma\varphi_n\varphi_{n+1} + \frac{1}{4!}\lambda_4{\varphi_n}^4 - J_n\varphi_n\right]$$

Here $\mu$, $\gamma$ and $\lambda_4$ are all positive. For the generating field $\phi_n(\{J\})$ we have the diagrammatic SD equation

which reads

$$\phi_n = \frac{J_n}{\mu} + \gamma\left(\phi_{n-1} + \phi_{n+1}\right) - \frac{\lambda_4}{6\mu}\left({\phi_n}^3 + 3\,\hbar\phi_n\phi_n' + \hbar^2\phi_n''\right)$$

**Excercise 28** *Show the validity of this diagrammatic SD equation.*

**Excercise 29** *Show that, if $\gamma = 0$, the theory is* uninteresting *since then the $\varphi_n$ are all independent and hence uncorrelated.*

**Excercise 30** *Show that $\gamma > 0$ induces a* positive *correlation between $\varphi_n$ and $\varphi_{n\pm1}$.*

We now switch to a new propagator: rather than considering a propagator $\hbar/\mu$ connecting $\varphi_n$ to $\varphi_n$, we consider a propagator $\Pi_{n,m}$ connecting arbitrary $\varphi_n$ and $\varphi_m$ using *only* two-point vertices $\gamma$. With this new definition, the above SD equation is graphically written as and reads

$$\phi_n = \sum_m \left\{ \Pi_{n,m} \frac{J_m}{\hbar} - \Pi_{n,m} \frac{\lambda_4}{6\hbar} \left( {\phi_m}^3 + 3\hbar \phi_m \phi'_m + \hbar^2 \phi''_m \right) \right\}$$

**Excercise 31** *Show that this is correct.*

## 2.2 The propagator

Since the parameters $\mu$ and $\lambda_4$ do not depend on $n$, the propagator $\Pi_{n,m}$ can only depend on $|n - m|$. We may therefore denote it by $\Pi_{n-m} = \Pi_{m-n}$.

**Excercise 32** *Show this.*

The propagator obeys its own SD equation:

hence

$$\Pi_n = \frac{\hbar}{\mu} \delta_{0,n} + \gamma \left( \Pi_{n+1} + \Pi_{n-1} \right)$$

We solve this by Fourier transform, by introducing

$$R(z) \equiv \sum_n \Pi_n \, e^{-inz} \quad \Leftrightarrow \quad \Pi_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} dz \; e^{inz} \, R(z)$$

From the SD equation it follows that

$$R(z) = \frac{\hbar}{\mu} + 2\gamma \; \cos(z) \; R(z) = \frac{\hbar}{\mu - 2\gamma \cos(z)}$$

hence

$$\Pi_n = \frac{\hbar}{2\pi} \int_{-\pi}^{\pi} dz \; \frac{e^{inz}}{\mu - 2\gamma \cos(z)}$$

**Excercise 33** *Check the above derivation.*

**Excercise 34** *Compute an explicit form for* $\Pi_n$*, as follows. First, define* $\alpha = \exp(iz)$ *and* $\rho = \mu/2\gamma$*, and show that* $\Pi_n$ *is given by the contour integral*

$$\Pi_n = -\frac{\bar{h}}{2i\pi\gamma} \oint\limits_{|\alpha|=1} d\alpha \; \frac{\alpha^n}{\alpha^2 - 2\rho\alpha + 1}$$

*Show that the integrand has two single poles, at*

$$\alpha = \alpha_\pm \quad , \quad \alpha_\pm = \rho \pm \sqrt{\rho^2 - 1}$$

*Show that for* $\rho > 1$*,* $\alpha_-$ *lies inside the contour, and* $\alpha_+$ *outside it. Contract the contour integral around* $\alpha_-$ *to show that*

$$\Pi_n = \frac{\bar{h}}{\gamma(\alpha_+ - \alpha_-)} {\alpha_-}^n$$

*Check that this obeys the correct SD equation.*

## 2.3 The continuum limit

We now introduce a small number $\Delta$ with the dimension of length. We envisage the fields $\varphi_n$ not as sitting together in one single point, but rather spread out along a straight line, with a distance coordinate $x$. We define

$$x \equiv n\Delta \quad , \quad z \equiv k\Delta$$

so that we may write

$$\varphi_n \to \varphi(n\Delta) = \varphi(x) \quad , \quad \Pi_n \to \Pi(n\Delta) = \Pi(x) \quad , \quad \phi_n \to \phi(n\Delta) = \phi(x)$$

The integration element in the path integral becomes

$$\prod_n d\varphi_n \;\; \to \;\; \mathcal{D}\varphi$$

Assuming $\Delta$ to become infinitesimal, we then have

$$\Pi_n \to \Pi(x) = \frac{\Delta\bar{h}}{2\pi} \int\limits_{-\infty}^{\infty} dk \; \frac{e^{ixk}}{\mu - 2\gamma\cos(\Delta k)} \approx \frac{\bar{h}\Delta}{2\pi} \int\limits_{-\infty}^{\infty} dk \; \frac{e^{ixk}}{\left(\mu - 2\gamma\right) + \gamma\Delta^2 k^2}$$

We are essentially forced to take the following continuum limit as $\Delta \to 0$:

$$\gamma \to \frac{1}{\Delta} \quad , \quad \mu \to \frac{2}{\Delta} + m^2\Delta$$

and then the propagator becomes

$$\Pi(x) = \frac{\hbar}{2\pi} \int\limits_{-\infty}^{\infty} dk \; \frac{e^{ixk}}{k^2 + m^2}$$

**Excercise 35** *Show that $m$ must have the dimension of inverse length: argue that, in any reasonable relation to phenomenology, therefore $m$ ought to be the inverse Compton wavelength of a given particle.*

**Excercise 36** *By considering the integral over $k$ in the complex $k$-plane, show that $\Pi(x) = \exp(-m|x|)/(2m)$. Hint: close the contour in the appropriate way.*

The only way to arrive at a nontrivial, well-defined action is to take the additional continuum limits

$$\lambda_4 \to \Delta\lambda_4 \quad , \quad J_n \to \Delta J(x)$$

and, replacing $\Delta \sum\limits_n$ by $\int dx$, the action then becomes a *functional* of the field $\varphi(x)$:

$$S(\{\varphi\}) \to S[\varphi] = \int\limits_{-\infty}^{\infty} dx \; \left[\frac{1}{2}m^2\varphi(x)^2 + \frac{1}{2}\varphi'(x)^2 + \frac{1}{4!}\lambda_4\varphi(x)^4 - J(x)\varphi(x)\right]$$

where we have dropped negligible powers of $\Delta$.

**Excercise 37** *Check the above; convince yourself that this continuum limit is essentially the only one leading to well-defined and nontrivial physics. In the continuum limit for the action, be careful to write $\varphi_{n+1} = \varphi(x + \Delta)$ and use Taylor expansion.*

**Excercise 38** *Repeat the derivation of the propagator in case the original action does not contain the term $-\gamma\varphi_n\varphi_{n+1}$ but rather $-\gamma_1\varphi_n\varphi_{n+1} - \gamma_2\varphi_n\varphi_{n+2}$. Show that the continuum limit will generically be the same, except in case $\gamma_1 + 4\gamma_2 = 0$. Show that in that case the appropriate continuum limit is given by $\gamma_1 \sim 4/\Delta^3, \gamma_2 \sim -1/\Delta^3$, $\mu \sim 6/\Delta^3 + m^4\Delta$, upon which the propagator goes with $1/(k^4 + m^4)$ and the action contains not $\varphi'(x)^2$ but $\varphi''(x)^2$ (note:to obtain this last result, partial integration in the action may be necessary).*

## 2.4 The Euler-Lagrange equations

Consider the action (including the source) before the continuum limit. The classical equation for the field $\phi_n$ is given by

$$0 = \frac{\partial}{\partial \varphi_n} S(\{\varphi\}) = \mu\varphi_n - \gamma\left(\varphi_{n-1} + \varphi_{n+1}\right) + \frac{\lambda_4}{6}\varphi_n^3 - J_n\varphi_n$$

In the continuum limit this becomes, to leading order in $\Delta$:

$$\Delta\left(m^2\varphi(x) - \varphi''(x) + \frac{\lambda_4}{6}\varphi(x)^3 - J(x)\right) = 0$$

which we recognize as the Euler-Lagrange equation for the continuum version of this theory:

$$\frac{\delta S([\varphi])}{\delta\varphi(x)} - \frac{d}{dx}\frac{\delta S([\varphi])}{\delta\varphi'(x)} = 0$$

**Excercise 39** *Check that similar results hold for interactions other than $\varphi^4$, for instance for $\varphi^3 + \varphi^4$ theory. Also consider the case of an action with interaction term $\sigma{\varphi_n}^2(\varphi_{n+1} - \varphi_n)^2$ where $\sigma \to \kappa/\Delta$ in the continuum limit: in that case, the Euler-Lagrange equation should have the form*

$$m^2\varphi - \varphi'' - 2\kappa\varphi^2\varphi'' - 2\kappa\varphi{\varphi'}^2 = J$$

## 2.5 Feynman rules for the one-dimensional theory

It is more convenient to use a momentum representation for the physics than a position representation, since momentum is both conserved and more easily controlled in experiment. Actually, we shall use wave vectors k (having the dimension of inverse length) rather than momenta. We therefore use Fourier transforms of the field and the source:

$$\phi(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\; e^{ixk}\phi(k) \quad , \quad J(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\; e^{ixk}J(k)$$

The classical SD equation for the $\varphi^4$ theory thus becomes

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} dk\; e^{ixk}\phi(k) = \int_{-\infty}^{\infty} dy\; \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty} dk_1\; dk_2 e^{i(x-y)k_1 + ik_2 y}\frac{\hbar J(k_2)}{{k_1}^2 + m^2}$$
$$-\frac{\lambda_4}{6}\int_{-\infty}^{\infty} dy\; \frac{1}{(2\pi)^4}\int_{-\infty}^{\infty} dk_1\; dk_2\; dk_3\; dk_4\; e^{i(x-y)k_1 + iyk_2 + iyk_3 + iyk_4}\frac{\phi(k_2)\phi(k_3)\phi(k_4)}{{k_1}^2 + m^2}$$

from which we read off the Feynman rules:

- a factor

$$\hbar/(k^2 + m^2)$$

 for a line with wave number $k$;

- a factor

$$-\frac{(2\pi)\lambda}{\hbar}\delta(k_1 + k_2 + k_3 + k_4)$$

 for a vectex where lines with wave numbers $k_{1,2,3,4}$ meet (all counted incoming);

- a factor

$$+\frac{(2\pi)J(k_1)}{\hbar}\delta(k_1 + k_2)$$

 for a source term with wave number $k_1$ coupling to a line with wave number $k_2$;

- an integration with

$$\frac{1}{2\pi}\int\limits_{-\infty}^{\infty} dk$$

 for each line with wave number $k$;

- symmetry factors as before.

**Excercise 40** *Verify that these are indeed the correct Feynman rules for the above classical equation; verify also that they are appropriate for the full SD equation.*

## 2.6 Extension to more dimensions

Consider an infinite number of fields, labeled by more than one integer: $\varphi_{n_1,n_2,\ldots,n_D}$. Consider the following action:

$$S(\{\varphi\}) = \sum_n \Big[\frac{1}{2}\mu\varphi_n{}^2 + \frac{1}{4!}\lambda_4\varphi_n{}^4 - J_n\varphi_n \\ -\gamma\varphi_{n_1,n_2,\ldots,n_D}\varphi_{n_1+1,n_2,\ldots,n_D} - \gamma\varphi_{n_1,n_2,\ldots,n_D}\varphi_{n_1,n_2+1,\ldots,n_D} - \cdots \\ - \gamma\varphi_{n_1,n_2,\ldots,n_D}\varphi_{n_1,n_2,\ldots,n_D+1}\Big]$$

The appropriate picture is now that of fields spread out over a $D$-dimensional Cartesian grid, rather than a line. The propagator now has $D$ indices, and its Fourier transform is

$$R(z_1, z_2, \ldots, z_D) = \sum_{n_1, n_2, \ldots, n_D} e^{-i(n_1 z_1 + n_2 z_2 + \cdots n_D z_D)} \Pi_{n_1, n_2, \ldots, n_D}$$

By the same techniques as before we find

$$\Pi_{n_1, n_2, \ldots, n_D} = \frac{\hbar \Delta^D}{(2\pi)^D} \int\limits_{-\infty}^{\infty} d_1^k dk_2 \cdots dk_D \; \frac{e^{i(k_1 x_1 + k_2 x_2 + \cdots + k_D x_D)}}{\mu - 2\gamma \left( \cos(k_1 \Delta) + \cdots + \cos(k_D \Delta) \right)}$$

and we see that the continuum limit must read

$$\gamma \to \Delta^{D-2} \;\; , \;\; \mu \to 2D\Delta^{D-2} + m^2 \Delta^D \;\; , \;\; \lambda_4 \to \Delta^D \lambda_4 \;\; , \;\; J_n \to \Delta^D J(x)$$

The propagator then becomes

$$\Pi(\vec{x}) = \frac{\hbar}{(2\pi)^D} \int\limits_{-\infty}^{\infty} d^D k \; \frac{e^{i\vec{k}\cdot\vec{x}}}{\vec{k}^2 + m^2}$$

and the action is

$$S[\varphi] = \int\limits_{-\infty}^{\infty} d^D x \; \left[ \frac{1}{2} m^2 \varphi(\vec{x})^2 + \frac{1}{2} \left( \vec{\nabla} \varphi(\vec{x}) \right)^2 + \frac{1}{4!} \lambda_4 \varphi(\vec{x})^4 - J(\vec{x}) \varphi(\vec{x}) \right]$$

**Excercise 41** *Verify the above, in particular the $\Delta$-dependence of the continuum limit.*

## 2.7 The Feynman rules for Euclidean $D$-dimensional theory

From the above we establish the following Feynman rules:

- a factor

  $$\hbar / (\vec{k}^2 + m^2)$$

  for a line with wave vector $\vec{k}$;

- a factor
$$-\frac{(2\pi)^D\lambda}{\hbar}\delta^D(\vec{k}_1+\vec{k}_2+\vec{k}_3+\vec{k}_4)$$
for a vectex where lines with wave vectors $\vec{k}_{1,2,3,4}$ meet (all counted incoming);

- a factor
$$+\frac{(2\pi)^D J(\vec{k}_1)}{\hbar}\delta^D(\vec{k}_1+\vec{k}_2)$$
for a source term with wave vector $\vec{k}_1$ coupling to a line with wave vector $\vec{k}_2$;

- an integration with
$$\frac{1}{(2\pi)^D}\int_{-\infty}^{\infty} d^D k$$
for each line with wave vector $\vec{k}$;

- symmetry factors as before.

In practice one of course aims at $D = 4$.

**Excercise 42** *Check that these are the correct Feynman rules.*

## 2.8 Dimensional regularization

In sufficiently high dimension, loop integrals will become divergent. In order to manage and manipulate these divergencies, regularization is necessary. A very useful (although not the only) way to do this is by considering the number of dimensions as a continuously variable parameter. Many loop integrals can, by algebraic manipulations, be reduced to expressions involving the integral[3]

$$L_p(s) \equiv \frac{1}{(2\pi)^4}\int_{-\infty}^{\infty} d^4k\, \frac{1}{(\vec{k}^2+s)^p}$$

[3] This is a Euclidean rather than a Minkowskian integral. However, also in Minkoskian theories the loop inetrgals are first brought into Euclidean form before evaluation, by reversing the Wick rotation discussed later.

with some positive number $s$ and some power $p$. We now change from dimension $D = 4$ to a general dimension $D$: at the end of the calculations, $D \to 4$ is taken. In order to preserve the correct units of the loop integral, a scale parameter $\mu$ with the dimension of $|\vec{k}|$ must be taken, and then one can write

$$L_p(s) = \frac{\mu^{4-D}}{(2\pi)^D} \int_{-\infty}^{\infty} d^D k \, \frac{1}{(\vec{k}^2 + s)^p} = \frac{\mu^{4-D}}{(2\pi)^D} \int_0^{\infty} dt \, \frac{Q_D(t)}{(t+s)^p}$$

where

$$Q_D(t) \equiv \int_{-\infty}^{\infty} d^D k \, \delta(\vec{k}^2 - t) = \frac{\Gamma(1/2)^D}{\Gamma(D/2)} t^{-1+D/2}$$

**Excercise 43** *Prove this last result, using Euler's formula:*

$$\int_0^{\infty} \cdots \int_0^{\infty} dx_1 dx_2 \cdots dx_n \; {x_1}^{\alpha_1} {x_2}^{\alpha_2} \cdots {x_n}^{\alpha_n} \; \delta(x_1 + x_2 + \cdots + x_n - 1) = \frac{\Gamma(\alpha_1 + 1)\Gamma(\alpha_2 + 1) \cdots \Gamma(\alpha_n + 1)}{\Gamma(\alpha_1 + \alpha_2 + \cdots \alpha_n + n)}$$

The $t$ integral reads

$$\int_0^{\infty} dt \, \frac{t^{D/2-1}}{(t+s)^p} = \frac{\Gamma(p - D/2)\Gamma(D/2)}{\Gamma(p)} s^{-p+D/2}$$

**Excercise 44** *Prove this, again using Euler's formula.*

The final result for the loop integral reads

$$L_p(s) = \frac{\mu^{4-D}}{(4\pi)^{D/2}} \, \frac{\Gamma(p - D/2)}{\Gamma(p)} \, s^{-p+D/2}$$

For $D = 4$ and $p > 2$ this is finite and we may take $D = 4$. For $p = 2$ we take $D = 4 - 2\delta$, with $\delta$ infinitesimal, and arrive at

$$L_2(s) = \frac{1}{(4\pi)^2} \left( \frac{1}{\delta} + \log(4\pi) + \Gamma'(1) - \log\left(\frac{s}{\mu^2}\right) + \mathcal{O}(\delta) \right)$$

For $p = 1$ we find, using the same procedure,

$$L_1(s) = -\frac{s}{(4\pi)^2}\left(\frac{1}{\delta} + \log(4\pi) + 1 + \Gamma'(1) - \log\left(\frac{s}{\mu^2}\right) + \mathcal{O}(\delta)\right)$$

The fivergences now show up as poles for $\delta \to 0$. A renormalizeable theory can be formulated in such a way that the poles cancel in any physical quantity.

**Excercise 45** *Verify* $L_{1,2}(s)$, *using the fact that* $x^\delta \approx 1 + \delta\log(x) + \cdots$, *and also* $\Gamma(\delta) = \Gamma(1+\delta)/\delta$ *and* $\Gamma(\delta - 1) = -\Gamma(\delta)/(1-\delta)$.

**Excercise 46** *Show that the procedure of dimensional regularization requires us to define*

$$\int d^Dk\,\frac{1}{\vec{k}^2} = \int d^Dk = \int d^Dk\,\vec{k}^2 = \int d^Dk\,(\vec{k}^2)^2 = \int d^Dk\,(\vec{k}^2)^3 = \ldots = 0$$

# 3 QFT in Minkowski space

## 3.1 The Wick rotation in the path integral

From now onwe shall use $D = 4$. In Euclidean theoey, position is indcated by $\vec{x}_E = (x^1, x^2, x^3, x^4)$ where the 'physically relevant distance' between points separated by $\vec{x}$ is

$$(\vec{x}_E)^2 = (x^1)^2 + (x^2)^2 + (x^3)^2 + (x^4)^2$$

In reality, spacetime is (assumed to be) Minkowskian, with position indicated by $x^\mu = (x^0, x^1, x^2, x^3)$ where $x^0 = ct$, and the 'physically relevant distance' between points separated by $x^\mu$ is

$$-(x)^2 = -x^\mu x_\mu = (x^1)^2 + (x^2)^2 + (x^3)^2 - (x^0)^2 = \vec{x}^2 - (x^0)^2$$

The Euclidean path integral is given by

$$\begin{aligned} Z_E[J] &= N_E \int \mathcal{D}\varphi\ \exp\left(-\frac{1}{\hbar}S_E[\varphi]\right) \quad , \\ S_E[\varphi] &= \int d^4x_E\ \left[\frac{1}{2}(\vec{\nabla}\varphi)^2 + \frac{1}{2}m^2\varphi^2 + V(\varphi) - J\varphi\right] \end{aligned}$$

where $V(\varphi)$ is the interaction part of the action. It is postulated[4] that we may change the $x^4$ integral to run not from $-\infty$ to $+\infty$ but from $-i\infty$ to $+i\infty$. We then put $x^4 = ix^0$ so that $d^4x_E$ becomes $d^4x$ and the path integral becomes

$$\begin{aligned} Z[J] &= N\int \mathcal{D}\varphi \; \exp\left(\frac{i}{\hbar}S[\varphi]\right) \quad , \\ S[\varphi] &= \int d^4x_E \; \left[\frac{1}{2}\partial^\mu\varphi\partial_\mu\varphi - \frac{1}{2}\hat{m}^2\varphi^2 - V(\varphi) - J\varphi\right] \end{aligned}$$

where we have changed the sign convention in front of the source term.

**Excercise 47** *Check this, assuming* $\hat{m}^2 = m^2$.

Here we have adjusted the mass term by including a negative imaginary part:

$$m^2 \;\; \to \;\; \hat{m}^2 = m^2 - im\Gamma \quad , \quad m\Gamma > 0$$

otherwise the path integral would not converge (the physical interpretation of $\Gamma$ follows later).

**Excercise 48** *Check that* $m\Gamma > 0$ *ensures that the integrand goes to zero for large values of* $|\varphi(x)|$.

## 3.2 Wick rotation in the propagator

In the propagator we perform the same manipulations, with $\vec{k}_E$ denoting the Euclidean wave vector and $k_E^4 = ik^0$

$$\Pi_E(\vec{x}_E) = \frac{\hbar}{(2\pi)^4}\int\limits_{-\infty}^{\infty} dk_E^4 \int\limits_{-\infty}^{\infty} d^3\vec{k}\; \frac{\exp(i(\vec{x}\cdot\vec{k} + x_E^4 k_E^4))}{(k_E^4)^2 + \vec{k}^2 + m^2} \;\; \to$$

$$\begin{aligned} \Pi(x) &= \frac{\hbar}{(2\pi)^4}\int\limits_{+i\infty}^{-i\infty} dk_E^4 \int\limits_{-\infty}^{\infty} d^3\vec{k}\; \frac{\exp(i(\vec{x}\cdot\vec{k} + ix^0k_E^4))}{(k_E^4)^2 + \vec{k}^2 + \hat{m}^2} \\ &= \frac{i\hbar}{(2\pi)^4}\int\limits_{-\infty}^{\infty} d^4k \frac{\exp(-ik^\mu x_\mu)}{k^\mu k_\mu - \hat{m}^2} \end{aligned}$$

[4] This is called the Euclidean postulate.

**Excercise 49** *Show that if Re*$(\alpha) > 0$ *and Im*$(\alpha) < 0$*, the equation* $z^2 + \alpha = 0$ *has two solutions with complex argument lying between* $\pi/4$ *and* $\pi/2$ *and between* $5\pi/4$ *and* $3\pi/2$.

**Excercise 50** *Verify the result for* $\mathrm{Pi}(x)$*, by going over to a* $k_E^4$ *integration in the complex-*$k_E^4$ *plane. Show that the* $k_E^4$ *integral may be deformed to run not from* $-\infty$ *to* $+\infty$ *but from* $+i\infty$ *to* $-i\infty$ *because of the positions of the poles.*

## 3.3 The Feynman rules in Minkowski space

From the above we establish the following Feynman rules:

- a factor
$$i\hbar/(k^2 - m^2 + im\Gamma)$$
for a line with wave vector $k^\mu$;

- a factor
$$-i(2\pi)^4\frac{\lambda}{\hbar}\delta^4(k_1 + k_2 + k_3 + k_4)$$
for a vectex where lines with wave vectors $k^\mu_{1,2,3,4}$ meet (all counted incoming);

- a factor
$$-i(2\pi)^4\frac{J(k_1)}{\hbar}\delta^4(k_1 + k_2)$$
for a source term with wave vector $k_1^\mu$ coupling to a line with wave vector $k_2^\mu$;

- an integration with
$$\frac{1}{(2\pi)^4}\int\limits_{-\infty}^{\infty} d^4k$$
for each line with wave vector $k^\mu$;

- symmetry factors as before.

**Excercise 51** *Check that these are the correct Feynman rules.*

## 3.4 The decay width

The SD equation in the absence of interactions is

$$\phi(x) = \int_{-\infty}^{\infty} d^4y \, \frac{i\hbar}{(2\pi)^4} \int_{-\infty}^{\infty} d^4k \frac{\exp(-ik^\mu(x-y)_\mu)}{k^2 - m^2 + im\Gamma} \, \frac{-i}{\hbar} J(y)$$

where we have used the source in the position representation. We see that the field $\phi(x)$ arises as a response to the presence of the source $J(y)$ by Huygen's principle: this is the motivation for regarding the source as an 'external', experimenter-controlled ingredient: it 'produces' or 'absorbs' particles.

Let the source act instantaneously at time $t = 0$ over all of space, that is, $J(y) = \delta(y^0)$. We can the perform the integral over $y$, and see that only $\vec{k} = 0$ contributes. We are therefore producing particles at rest all over space. The result is

$$\phi(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk^0 \, \frac{\exp(-ik^0x^0)}{(k^0)^2 - \hat{m}^2}$$

**Excercise 52** *Check this.*

For $x^0 > 0$ the $k^0$ integral can be closed by a countour in the negative imaginary half-plane, and we find

$$\phi(x) = \frac{-i}{2\hat{m}} \, \exp(-ix^0\hat{m})$$

**Excercise 53** *Check this.*

Assuming tha $\Gamma$ is small compared to $m$, we have $\hat{m} = \sqrt{m^2 - im\Gamma} \approx m - i(\Gamma/2)$. In that case, the *particle density* at time $x^0 > 0$ is given by

$$|\phi(x)|^2 = |\phi(x^0 = 0)|^2 \, \exp(-x^0\Gamma)$$

The particle is *unstable*, with lifetime $\tau = (c\Gamma)^{-1}$. $\Gamma$ is called the *total decay width* of this particle. The case of stable particles is achieved by taking $\Gamma$ to be infinitesimal (but positive). The propagators in the momentum representation are therefore

$i\hbar/(k^2 - m^2 + i\epsilon)$ for a stable particle ($\epsilon \downarrow 0$)
$i\hbar/(k^2 - m^2 + im\Gamma)$ for an unstable particle with $\Gamma \ll m$

**Excercise 54** *Assuming that $m$ is the inverse Compton wavelength of a particle of mass $M$, show that $\Gamma \ll m$ implies $\tau \gg \hbar/(Mc^2)$. Argue that $\tau < \hbar/(Mc^2)$ is unlikely since this would involve superluminal signalling (probably).*

## 3.5 Classical kinematics from the propagator

Let us take an arbitrary energy $E > 0$ and an arbitrary momentum $\vec{p}$. Consider a source

$$J(x) \propto \exp\left(-\frac{(x^0)^2}{2c^2\tau^2} - \frac{\vec{x}^2}{2\sigma^2} - i\frac{Ex^0}{\hbar c} + i\frac{\vec{p}\cdot\vec{x}}{\hbar}\right)$$

This source is active in a region of size $\sigma^3$ around the origin, for a time of order $\tau$: for small $\sigma$ and $\tau$ it is localized in space. In momentum representation, we have

$$J(k) = \int_{-\infty}^{\infty} d^4x e^{ikx} J(x) \propto \exp\left(-\frac{\tau^2}{2\hbar^2}(E - k^0\hbar c)^2 - \frac{\sigma^2}{2\hbar^2}(\vec{p} - \vec{k}\hbar)^2\right)$$

So unless $\sigma$ and $\tau$ are too small (uncertainty relation!), the source is also localized in momentum.

**Excercise 55** *Verify the form of* $J(k)$*, and show the 'double localization'.*

The field for a stable particle, arising from this source, is

$$\begin{aligned}\phi(x^0, \vec{x}) &\propto \int_{-\infty}^{\infty} dk^0 \int_{-\infty}^{\infty} d^3k \, \frac{\exp(A)}{(k^0)^2 - \left(\vec{k}^2 + m^2 - i\epsilon\right)} \;, \\ A &= -\frac{\tau^2}{2\hbar^2}(E - k^0\hbar c)^2 - \frac{\sigma^2}{2\hbar^2}(\vec{p} - \vec{k}\hbar)^2 - ik^0x^0 + i\vec{k}\cdot\vec{x}\end{aligned}$$

For $x^0 > 0$ the $k^0$ integral is completely dominated by values around

$$k^0 = \omega(\vec{k}) \equiv \sqrt{\vec{k}^2 + m^2} > 0$$

**Excercise 56** *Show this by closing the contour in the complex-*$k^0$ *plane.*

The integral will only be appreciable if, for some $k^\mu$, the quantities $(E - \hbar c k^0)^2$ and $(\vec{p} - \hbar\vec{k})^2$ can vanish simultaneously. This requires $\vec{k} = \vec{p}/\hbar$ and $k^0 = E/(\hbar c)$, so that $E$ and $\vec{p}$ are related:

$$E = c\hbar\omega(\vec{p}/\hbar) = \sqrt{\vec{p}^2c^2 + m^2\hbar^2c^2}$$

This leads us to relate $m$ (units of inverse length) to the physical mass $M$ (units of mass) of the particle:

$$m = \frac{Mc}{\hbar}$$

Moreover, the integral will only be appreciable if the oscillatory part can become stationary, that is

$$\frac{\partial}{\partial k_j}\left(x^0\omega(\vec{k}) - \vec{x}\cdot\vec{k}\right) = x^0\frac{k_j}{\omega(\vec{k})} - x_j = 0 \quad , \quad j = 1,2,3$$

**Excercise 57** *Show this.*

The field is therefore only appreciable on trajectories in space with

$$\vec{x} = x^0\ \frac{\vec{k}}{\omega(\vec{k})} = t\ \frac{c\vec{p}}{E}$$

Conclusion: propagation over macroscopic distances is only possible for on-shell particles, and in the absence of interaction these move uniformly.

## 3.6 Antiparticles

The propagator:

$$\Pi(x) = \frac{i\hbar}{(2\pi)^4}\int_{-\infty}^{\infty} d^4k\frac{\exp(-i\vec{x}\cdot\vec{k})}{k^2 - \hat{m}^2}$$

is a sum over Fourier modes with momentum $\vec{p}$ and energy $E$. As seen, for $x^0 > 0$, only modes with $E > 0$ contribute; for $x^0 < 0$, only modes with $E < 0$ contribute.

**Excercise 58** *Check this statement*

The usual view of the world is that in which particles (a) carry positive energy, and (b) move forward in time. A momentum mode that corresponds to a particle carrying negative energy, moving backwards in time and carrying (say) a charge $Q$, is therefore interpreted as an *anti*particle with positive energy, moving forward in time but carrying the *opposite* charge $-Q$. Since the antiparticle interpretation follows from the same propagator that leads to a particle interpretation, we have the $CPT$ result that particle and antiparticle must have the identically same mass and width.

**Excercise 59** *Verify the above. Convince yourself that negative energies moving backwards in time do not imply a violation of causality.*

# 4 Phenomenology