Five Lectures on Field Theory for the Standard Model

presented at the CERN-CLAF School 2003,
San Miguel de Regla, Mexico

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version of September 19, 2005

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1 0+0 Field Theory and Feynman Diagrams

Among the ingredients of quantum field theory most directly relevant to particle phenomenology are the Feynman diagrams. These are most easily introduced in a context that is as simple as possible. To this end, it is useful to reduce the size of the universe to a single point, *i.e.* we start with quantum field theory in 0+0 dimensions.

1.1 Perturbation theory

The path integral describing quantum field theory for a zero-dimensional real (scalar) field φ is simply an integral involving a probability density. The measure is usually given by $\exp(-S(\varphi))$, where

$$S(\varphi) = \frac{\mu^2}{2}\varphi^2 + V(\varphi) \quad , \quad V(\varphi) = \sum_{n\geq 3} \frac{\lambda_n}{n!}\varphi^n \quad , \quad \lim_{\varphi\to\pm\infty} S(\varphi) = \infty \quad , \tag{1}$$

where φ denotes the single field value, μ is real and nonzero, and the real *coupling constants* λ_n may or may be nonzero. In practice, often $\lambda_n = 0$ for $n \geq 5$. In particle phenomenology one is usually interested in the *Green's functions*, that is, the expectation values of powers of φ:

$$G_n = \langle \varphi^n \rangle = \left(\int\limits_{-\infty}^{\infty} e^{-S(\varphi)}\ \varphi^n\ d\varphi \right) \left(\int\limits_{-\infty}^{\infty} e^{-S(\varphi)}\ d\varphi \right)^{-1} \quad . \tag{2}$$

The set of all Green's functions is combined in the *path integral*:

$$Z(J) = \sum_{n\geq 0} \frac{J^n}{n!} G_n = \int\limits_{-\infty}^{\infty} \exp(-S(\varphi) + J\varphi)\ d\varphi \quad \to \quad G_n = Z^{(n)}(0)/Z(0) \quad . \tag{3}$$

Even more significant are the *connected Green's functions*[1] C_n:

$$\sum_{n\geq 1} \frac{J^n}{n!} C_n \equiv W(J) = \log \frac{Z(J)}{Z(0)} \quad . \tag{4}$$

[1]The relation between G_n and C_n is that between moments and cumulants of a probability distribution: they encode the same information in different ways.

The path integral satisfies a *Schwinger-Dyson* (SD) equation:

$$S'\left(\frac{\partial}{\partial J}\right) Z(J) = \left(\mu\frac{\partial}{\partial J} + \sum_{n\geq 3}\frac{\lambda_n}{(n-1)!}\frac{\partial^{n-1}}{(\partial J)^{n-1}}\right) Z(J) = JZ(J) \ , \qquad (5)$$

where $Z(J)$ is that solution that is real for real J. The common approach to computing Green's functions is *perturbation theory*, in which the coupling constants in $V(\varphi)$ are treated as small[2]:

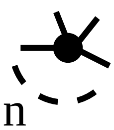
$$Z(J) \to \sum_{n\geq 0}\int_{-\infty}^{\infty} e^{-\mu^2\varphi^2/2}\frac{1}{n!}(J\varphi - V(\varphi))^n \, d\varphi \ , \qquad (6)$$

and the various powers of φ in the expansion integrate to

$$\int_{-\infty}^{\infty} e^{-\mu^2\varphi^2/2} \, \varphi^{2n} \, d\varphi = \frac{(2n)!}{n!2^n}\frac{1}{\mu^{2n}}\sqrt{\frac{2\pi}{\mu^2}} \quad , \quad \int_{-\infty}^{\infty} e^{-\mu^2\varphi^2/2} \, \varphi^{2n+1} \, d\varphi = 0 \ . \qquad (7)$$

1.2 Feynman diagrams and Dyson summation

A practical way of computing Green's functions is to use *Feynman diagrams*, That consist of vertices and lines ('propagators') connecting them. They correspond to algebraic quantities by way of *Feynman rules*, which in this case read

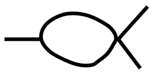
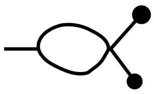
 $\to \dfrac{1}{\mu^2}$,  $\to J$, 

$\to -\lambda_n$. (8)

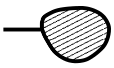
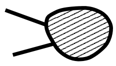
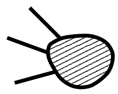
The propagators may or may not end in a vertex: propagator endings that do not are called *external lines*. Diagrams may be connected or disconnected: a disconnected diagram is considered as the product of its connected parts (and therefore the 'empty' diagram equals unity). Each Feynman diagram carries an additional *symmetry factor*, which is the product of a factor $1/p!$ for each set of p equivalent internal lines, and a factor $1/q!$ for each set of q equivalent vertices in the diagram: with 'equivalent' is meant that these

[2] In this expression, integral and summatino have been cavalierly interchanged. As a result, the perturbation series is *not* convergent, and has at most a formal meaning. However, it seems to work pretty well in practice...

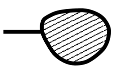
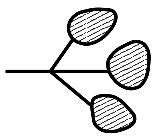
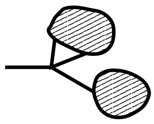
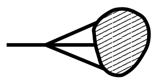
objects can be interchanged without affecting the topology of the diagram. This also implies a symmetry factor $1/r!$ for each set of r idential diagrams in a product. Two examples of diagrams, with their symmetry factors:

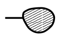
 $= \dfrac{\lambda_3\lambda_4}{(2!)\mu^{10}}$,  $= \dfrac{\lambda_3\lambda_4 J^2}{(2!)(2!)\mu^{10}}$. (9)

Let us now consider the (infinite) set $\phi(J)$ of all connected diagrams with precisely one external leg and any number of J vertices in addition to other allowed vertices. This is called the *amplitude-generating function* (AGF), and is denoted by a shaded blob. The set of all connected diagrams with more external legs are given by derivatives:

$\phi(J) =$  ,  $= \dfrac{\partial}{\partial J}\phi(J)$,  $= \dfrac{\partial^2}{(\partial J)^2}\phi(J)$ ... (10)

Let us specialize to a theory in which λ_4 is the only nonzero coupling. By inspection, it is clear that $\phi(J)$ satisfies a diagrammatic recursion relation:

 =  +  +  +  (11)

By starting with  = and iterating, we can thus build up the whole set of connected diagrams. Following the Feynman rules (including the symmetry factors for equivalent lines and blobs) this translates into the SD equation for the AGF:

$$\phi(J) = \frac{J}{\mu^2} - \frac{\lambda_4}{(3!)\mu^2}\phi(J)^3 - \frac{\lambda_4}{(2!)\mu^2}\phi(J)\frac{\partial}{\partial J}\phi(J) - \frac{\lambda_4}{(3!)\mu^2}\frac{\partial^2}{(\partial J)^2}\phi(J) \ . \quad (12)$$

If we now write

$$\phi(J) = \frac{\partial}{\partial J}W(J) = \frac{1}{Z(J)}\frac{\partial}{\partial J}Z(J) \ , \quad (13)$$

precisely the SD equation (5) for $Z(J)$ is recovered. Since

$$Z(J)_{\lambda_4=0} = \exp\left(\frac{J^2}{2\mu^2}\right) \quad \to \quad \phi(J)_{\lambda_4=0} = \frac{J}{\mu^2} \ , \quad (14)$$

we have thus proven that

1. C_n equals the set of all connected diagrams with precisely n external lines and no J vertices;

2. $W(J)$ equals the set of all connected Feynman diagrams with at least one J vertex and no external lines;

3. $Z(J)/Z(0)$ equals the set of all (connected and disconnected) Feynman diagrams without external lines and at least one J vertex.

For other theories the treatment is exactly the same, with the same results. The special rôle of the diagrams without any external lines or J vertices, the so-called *vacuum bubbles*, stems from the fact that they make up precisely $Z(0)$ and are therefore cancelled by the normalization of the path integral.

It may seem illogical that the 2-point vertex (μ^2) is treated differently from the interaction ones (λ_n). To show that the treatment is actually consistent, consider a theory in which, in addition to the μ^2 term we add another term $\lambda_2\varphi^2/2$. The SD equation then becomes

$$\phi(J) = \frac{J}{\mu^2} - \frac{\lambda_2}{\mu^2}\phi(J) - \frac{1}{\mu^2}(\text{interaction terms}) \ . \tag{15}$$

It is easy to see that this can be rewritten as

$$\phi(J) = \frac{J}{\mu^2+\lambda_2} - -\frac{1}{\mu^2+\lambda_2}(\text{interaction terms}) \ , \tag{16}$$

so that the addition of the quadratic interaction term is simply a modification of the μ^2 in the Feynman rules. This is *Dyson* summation: any two-point interaction may be subsumed in the definition of the propagator:

$$\begin{aligned} &\text{——} + \text{—}\bullet\text{—} + \text{—}\bullet\text{—}\bullet\text{—} + \cdots \\ &= \frac{1}{\mu^2} + \frac{1}{\mu^2}(-\lambda_2)\frac{1}{\mu^2} + + \frac{1}{\mu^2}(-\lambda_2)\frac{1}{\mu^2}(-\lambda_2)\frac{1}{\mu^2} + \cdots \\ &= \frac{1}{\mu^2}\frac{1}{1+\lambda_2/\mu^2} = \frac{1}{\mu^2+\lambda_2} \ . \end{aligned} \tag{17}$$

1.3 The loop expansion

In actually performing the (diagrammatic) perturbation expansion one has to decide how to truncate the perturbation series. When several different coupling constants are involved, this is to some extent ambiguous. It is therefore customary to order the diagrams topologically, according to the number of closed loops contained. Diagrams without closed loops are called *tree diagrams*. We can convieniently implement this by inserting, in the SD equation (12), a factor $\hbar$ for each closed loop involving the explicit vertex, so that (12) becomes

$$\phi(J) = \frac{J}{\mu^2} - \frac{\lambda_4}{(3!)\mu^2}\phi(J)^3 - \frac{\hbar\lambda_4}{(2!)\mu^2}\phi(J)\frac{\partial}{\partial J}\phi(J) - \frac{\hbar^2\lambda_4}{(3!)\mu^2}\frac{\partial^2}{(\partial J)^2}\phi(J) \quad . \tag{18}$$

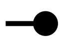
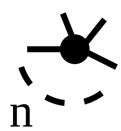
This so-called *loop expansion* requires a modification of what we have seen so far: we need to write

$$\phi(J) = \hbar\frac{\partial}{\partial J}\log Z(J) \quad , \quad Z(J) = \int\limits_{-\infty}^{\infty} \exp\left(-\frac{1}{\hbar}(S(\varphi) - J\varphi)\right)\, d\varphi \quad . \tag{19}$$

The SD equation then reads

$$S'\left(\hbar\frac{\partial}{\partial J}\right) Z(J) = JZ(J) \quad . \tag{20}$$

Accordingly, the Feynman rules now read

 $\rightarrow \dfrac{\hbar}{\mu^2}$,  $\rightarrow \dfrac{J}{\hbar}$, 

$\rightarrow -\dfrac{\lambda_n}{\hbar}$. (21)

A diagrams containing E external lines, I internal lines, V_n vertices of n-point type, and consisting of P connected parts, obeys the following two topological sum rules:

$$\sum_n V_n = I - L + P \quad , \quad \sum_n nV_n = 2I + E \quad . \tag{22}$$

The $\hbar$-dependence of a connected diagram ($P = 1$) is therefore given as $\hbar^{E-P+L}$ so that indeed each additional loop in a given C_n brings in another factor $\hbar$. The loop expansion also resolves the truncation ambiguity by making each λ_n equivalent to $\hbar^{-1+n/2}$.

1.4 Classical Limit, Instantons, and Effective Action

The limit $\hbar \to 0$ is of interest by itself. On the one hand, the diagrams involved in the SD equation (12) for $\phi_c(J) = \lim_{\hbar\to 0}\phi(J)$ are those without any closed loops, so that it reads

$$S'\left(\phi_c(J)\right) = J \ . \tag{23}$$

On the other hand, we may take $\hbar \to 0$ in the path integral. Then, the dominant contribution comes from that value φ_c where the integrand has a maximum:

$$S'(\varphi_c) - J = 0 \ . \tag{24}$$

It is seen that $\phi_c(J) = \varphi_c$ as expected. In more realistic theories, the *classical equation* (24) is for instnance the Klein-Gordon, Dirac or Maxwell equation.

The only theory with a single minimum point is the *free* theory, in which $\lambda_n = 0$, $n \geq 3$. Nonfree theories contain several solutions to the classical field equation, of which often a single one corresponds to the *global* minimum of the action. Let us denote this value by φ_0. Another, *local* minimum value φ_1 will give to the physics of the theory a contribution that is suppressed by a factor $\exp(-(S(\varphi_1) - S(\varphi_0))/\hbar)$: such contributions do not admit of a series expansion in powers of $\hbar$ and are therefore *nonperturbative*. Solutions to the classical field equation that give a local rather than the global minimum of the action are called *instantons*[3].

Since the classical field equation looks simpler than the SD equation, it is natural, for a given action $S(\varphi)$, to search for an *effective* action, denoted $\Gamma(\varphi)$, such that *its* classical field equation gives the actual, full, solution to the SD equation for $S(\varphi)$. That is, we require $\Gamma(\varphi)$ to satisfy $\Gamma'(\varphi) = J$. Partial intergation then leads to the definition of the effective action as a Legendre transform:

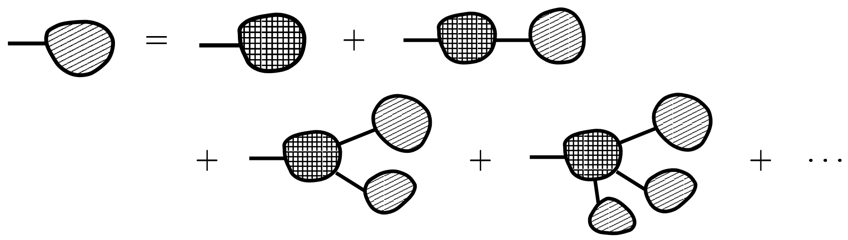
$$\Gamma(\phi) = J(\phi)\phi - \hbar W(J(\phi)) \ , \tag{25}$$

where J is now considered a function of ϕ rather than the other way around. Note that for this to make sense, $\phi(J)$ should be monotonic. From Eqs. (3) and (13) it is easy to show that this is indeed so. Incidentally, we thereby

[3] Caveat: $\hbar$ small does not mean $\hbar = 0$, and therefore instanton contributions can be significant if the local minimum is only $\mathcal{O}(\hbar)$ than the global one. An example is the case of a theory in which the symmetry is only very slightly broken by a weak source.

prove that $\Gamma''(\phi) > 0$, in other words: the effectivce action is a *concave* function of the classical field.

The effective action has a diagrammatic interpretation. Let us call *one-particle irreducible* (1PI) any diagram that can *not* be made disconnected by the cutting of any one internal line, and denote complete sets of 1PI diagrams with a given number of external lines by cross-hatched blobs. Single vertices also count as 1PI. The SD equation can then be rephrased by concentrating on the 1PI nature of th first encountered vertex, as follows:



(26)

This proves that the n-point vertex of the effective theory is determined by the set of all 1PI diagrams with precisely n external legs[4]. Once the effective action is given, all Green's functions are given by (effective) tree diagrams.

2 Euclidean and Minkowskian Field Theory

2.1 The free one-dimensional field

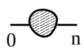
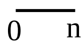
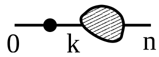
The treatment of a single zero-dimensional field extends easily to that of more fields all located at the same zero-dimensional spacetime point. An interaction term $\lambda\varphi_1^{n_1}\varphi_2^{n_2}$ corresponds to a Feynman vertex $-\lambda$, etctera. Let us consider an infinite set of fields φ_n, $\infty < n < \infty$. The simplest action in which the fields exhibit mutual correlations is

$$S(\{\varphi_n\}) = \sum_n \left(\frac{\mu^2}{2}\varphi_n^2 - \gamma\varphi_n\varphi_{n-1} - J_n\varphi_n\right) \quad , \tag{27}$$

where a set of sources J_n has been added. The object of interest is the *two-point* correlator $\Pi_{n,m} = \langle\varphi_n\varphi_m\rangle$. Because of symmetry and (discrete) translational invariance we have $\Pi_{n,m} = \Pi_{|n-m|}$ so that we may restrict ourselves

[4]Note that this presupposes that $\phi(J)$ is perturbative, that is, instanton effects and such can be neglected.

to $\Pi_{0,n} = \Pi(n)$. The SD equation (for vanishing source) for the correlator reads

$$\Pi(n) \;=\;$$  $$=$$  $$\delta_{0,n} + \sum_{k=\pm1}$$ 

$$= \frac{1}{\mu^2}\delta_{0,n} + \frac{\gamma}{\mu^2}\left(\Pi(n-1) + \Pi(n+1)\right) \quad . \tag{28}$$

This is most easily solved by Fourier transform:

$$R(z) = \sum_n \Pi(n)\, e^{-izn} \quad \to \quad R(z) = \frac{1}{\mu^2 - 2\gamma\cos(z)} \quad , \tag{29}$$

leading to

$$\Pi(n) = \frac{1}{2\pi}\int\limits_{-\pi}^{\pi} dz \frac{e^{izn}}{\mu^2 - 2\gamma\cos(z)} \quad . \tag{30}$$

Rather than viewing this theory as one of many fields at the same spacetime point, we may reiterpret is as a theory of single field values at different spacetime points. This requires the introduction of an (infinitesimal!) spacetime separation Δ between the succesive points, so that we may replace the label by a position variable x, and z by a wave-vector (momentum) variable k. We then take a limit $\Delta \to 0$ so that both x and k remain regular:

$$x = n\Delta \quad , \quad k = z/\Delta \quad , \quad \Delta \to 0 \,,\; n \to \infty \,,\; z \to 0 \quad . \tag{31}$$

The only sensible way to arrive at a limit for the correlator is to take, in this *continuum limit*,

$$\gamma \sim \frac{1}{\Delta} \quad , \quad \mu^2 \sim \frac{2}{\Delta} + m^2\Delta \quad , \tag{32}$$

with some number m having the dimension of inverse distance[5]. The correlator then becomes

$$\Pi(n) \quad \to \quad R(x) = \frac{1}{2\pi}\int\limits_{-\infty}^{\infty} dk\; \frac{e^{ikx}}{k^2 + m^2} = \frac{1}{2m}\exp(-m|x|) \quad . \tag{33}$$

The action is then that of a free scalar Euclidean quantum field:

$$S(\{\varphi_n\}) \quad \to \quad S[\varphi] = \int\limits_{-\infty}^{\infty} dx \left(\frac{1}{2}\varphi'(x)^2 + m^2\varphi(x)^2 - J(x)\varphi(x)\right) \quad . \tag{34}$$

[5]This means that m is not the mass of the particle, but rather its inverse Compton wavelength. The term 'classical field equation' is therefore only partly appropriate.

The set $\{\varphi_n\}$ has been replaced by the field $\phi(x) = \varphi_{x/\Delta}$. We have also used the continuum limit $J_n \to \Delta J(x)$. The classical equation of motion is also obtained by taking the continuum limit of the discrete case:

$$\begin{aligned}\frac{\partial}{\partial\varphi_n}S(\{f\}) &= \mu^2\varphi_n - \gamma(\varphi_{n-1}+\varphi_{n+1}) - J_n = 0 \\ &\Rightarrow \frac{\partial^2}{(\partial x)^2}\varphi(x) - m^2\varphi(x) - J(x) = 0 \ , \qquad (35)\end{aligned}$$

which is the Euler-Lagrange equation for the continuum action.

2.2 The interacting field in one dimension – Feynman rules in momentum space

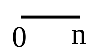
The free-field action can be provided with self-interaction terms. For local φ^4 self-interaction, the discrete version reads

$$S(\{\varphi_n\}) = \sum_n \left(\frac{\mu^2}{2}\varphi_n^2 - \gamma\varphi_n\varphi_{n-1} + \frac{\lambda_4}{4!}\varphi_n^4 - J_n\varphi_n\right) \ , \qquad (36)$$

of which the continuum version reads (upon $\lambda_4 \to \Delta\lambda$):

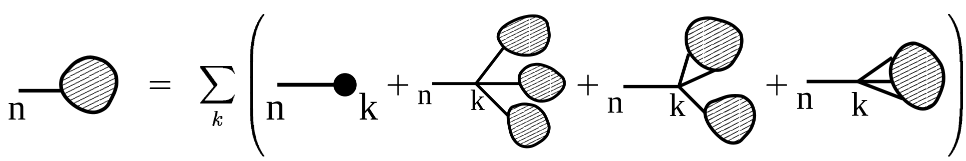
$$S[\varphi] = \int\limits_{-\infty}^{\infty} dx \ \left(\frac{1}{2}\varphi'(x)^2 + \frac{m^2}{2}\varphi(x)^2 + \frac{\lambda}{4!}\varphi(x)^4 - J(x)\varphi(x)\right) \ . \qquad (37)$$

From now on, we shall use the Dyson-summed two-point correlator:



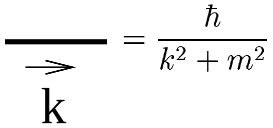
$$= \Pi(n) \ . \qquad (38)$$

The SD equation then becomes

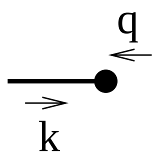


$$\begin{aligned}\phi_n(\{J\}) &= \sum_k \Pi(n-k)\left(J_k - \frac{\lambda}{6}\phi_k(\{J\})^3\right. \\ &\left. -\frac{\hbar\lambda}{2}\phi(\{J\})\frac{\partial}{\partial J_k}\phi(\{J\}) - \frac{\hbar^2\lambda}{6}\frac{\partial^2}{(\partial J_k)^2}\phi(\{J\})\right) \qquad (39)\end{aligned}$$

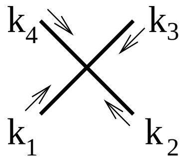
Here, the fact has been taken into accunt that the $\lambda\varphi^4$ interaction may take place at any point in space. The Feynman rules can now also given in terms of the wave vectors k than in terms of posititions x. This is on the one hand due to the fact that translational invariance implies momentum conservation (while there is no 'conservation of position'), and also because of the fact that experiment is more adequate at fixing and measuring particles' momenta than their positions. We have



$$= \frac{\hbar}{k^2 + m^2} \quad ,$$



$$= \frac{2\pi}{\hbar}\delta(k+q) \quad ,$$



$$= -\frac{2\pi}{\hbar}\delta(k_1 + k_2 + k_3 + k_4) \quad ; \tag{40}$$

moreover, there is an integral $\int dk/2\pi$ to be performed over every momentum k. Quite often, the source $J(q)$ emits or absorbs only a single momentum mode, so that effectively the momenta of the external legs are fixed. It is easily seen that in a diagram with L closed loops there are precisely L momentum integrals that are not resolved by then Dirac delta functions of momentum conservation, and in addition each connected diagram contains precisely one delta function imposing overall momentum conservation.

2.3 Field theory in more dimensions

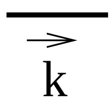
The one-dimensional case can easily be extended to higher dimensions. For instance, the free, sourceless action in two dimensions can be chosen most simply to be

$$S(\{\varphi\}) = \sum_{n_1,n_2} \left(\frac{\mu^2}{2}\varphi_{n_1,n_2}^2 - \gamma\varphi_{n_1,n_2}\varphi_{n_1,n_2-1} - \gamma\varphi_{n_1,n_2}\varphi_{n_1-1,n_2} \right) \quad . \tag{41}$$

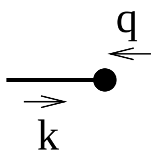
This theory is most simply viewed as one residing on a two-dimensional square lattice. In fact, it is the γ terms, that is, the notion of *nearest neighbours*, that determines the dimensionality of the theory[6]. The continuum

[6]Since a pair of integers (n_1, n_2) can be mapped one-to-one into a single integer n, a two-dimensional lattice model can be forced into a one-dimensional representation. The con-

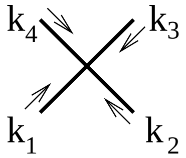
limit of the mass and interaction terms, Eq.(32), have to be modified so as to lead to a sensible correlator, but again this is unambiguous. The resulting Feynman rules, for d dimensions, will read



$$= \frac{\hbar}{\vec{k}^2 + m^2} \quad ,$$



$$= \frac{(2\pi)^d}{\hbar}\delta^d(\vec{k} + \vec{q}) \quad ,$$



$$= -\frac{(2\pi)^d}{\hbar}\lambda\delta^d(\vec{k}_1 + \vec{k}_2 + \vec{k}_3 + \vec{k}_4) \quad , \tag{42}$$

with an integral $\int d^d k/(2\pi)^d$ over every momentum $\vec{k}$. The continuum action (with self-interaction and source) reads

$$S[\varphi] = \int d^d x \ \left(\frac{1}{2}\left(\vec{\nabla}\varphi(\vec{x})\right)^2 + \frac{m^2}{2}\varphi(\vec{x})^2 + \frac{\lambda}{4!}\varphi(\vec{x})^4 - J(\vec{x})\varphi(\vec{x}) \right) \quad , \tag{43}$$

and the classical field equation is seen to be

$$-\vec{\nabla}^2\varphi(\vec{x}) + m^2\varphi(\vec{x}) + \frac{\lambda}{3!}\varphi(\vec{x})^3 - J(\vec{x}) = 0 \quad . \tag{44}$$

2.4 Minkowski space and the $i\epsilon$ prescription

So far, the theory has been formulated in Euclidean space, where the propagator $R(\vec{x})$:

$$R(\vec{x}) = \frac{1}{(2\pi)^d}\int\limits_{-\infty}^{\infty} d^d k \frac{\exp(i\vec{x}\cdot\vec{k})}{\vec{k}^2 + m^2} \quad , \tag{45}$$

is invariant under rotations. That is, the *physically relevant distance* between two points in spacetime $\vec{x}_1$ and $\vec{x}_2$ is given by

$$d(\vec{x}_1, \vec{x}_2)^2 = \sum_{j=1}^{4}\left(x_1^{(j)} - x_2^{(j)}\right)^2 \quad . \tag{46}$$

---

tinuum limit, however, will be different, and give rise to a weird-looking one-dimensional theory.

In our world, however, the time coordinate t plays an exceptional rôle, and the physically relevant distance between two spacetime points $(t_1, \vec{x}_1)$ and $(t_2, \vec{x}_2)$ is rather

$$d(\vec{x}_1, \vec{x}_2)^2 = \left(x_1^{(0)} - x_2^{(0)}\right)^2 - \sum_{j=1}^{3} \left(x_1^{(j)} - x_2^{(j)}\right)^2 \quad , \tag{47}$$

where $x_n^{(0)} = ct_n$, c the constant speed of light. In four-dimensional quantum field theory, we implement this in two steps. First, one assumes that the path integral and the Green's functions remain well-defined if the fourth position coordinate, x^4, is taken to run along the imaginary axis rather than along the real one (a *physical* assumption, called the Euclidean postulate); and, secondly, we replace x^4 by ix^0, so that x^0 runs along the real axis again (a *mathematical* manipuliation). This operation is called the Wick rotation. Taking an overall factor $-i$ out of the integral, we then arrive at a path integral with integrand $\exp(iS/\hbar)$, where

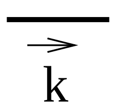
$$S[\varphi] = \int d^4x \, \left(\frac{1}{2}\partial^\mu\varphi\partial_\mu\varphi - \frac{1}{2}(m^2 - i\epsilon)\varphi^2 - \frac{1}{4!}\lambda\varphi^4\right) \quad , \tag{48}$$

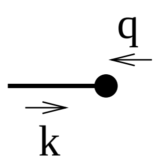
with $d^4x = d^3\vec{x}dx^0$. Here, we have added a term $i\epsilon\varphi^2$, with ϵ very small but positive, in order to ensure that the path integral remains well-behaved for very large φ values. This is the so-called $i\epsilon$ presecription (see below). In calculating, the limit $\epsilon \to 0$

In the expression for the propagator, we have to make a similar Wick rotation. Investigating the position of the poles of the propagator (45) after the replacement $m^2 \to m^2 - i\epsilon$, it is seen that the k^4 integral can be changed from $-\infty \to +\infty$ into $-i\infty \to +i\infty$, so that we likewise replace $k^4 \to ik^0$, and find after some trivial algebra:

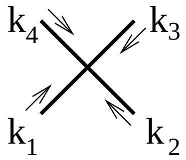
$$R(x) = \frac{1}{(2\pi)^4} \int d^4k \; e^{-ik\cdot x} \frac{i}{k^2 - m^2 + i\epsilon} \quad , \tag{49}$$

with $d^4k = d^3\vec{k}dk^0$. In momentum space, the Feynman rules for the Minkowski theory now become



$= \dfrac{i}{\vec{k}^2 - m^2 + i\epsilon}$,


$= -i(2\pi)^4\delta^d(\vec{k} + \vec{q})$,



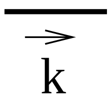
$$= -i\lambda(2\pi)^4\delta^d(\vec{k}_1 + \vec{k}_2 + \vec{k}_3 + \vec{k}_4) \quad , \tag{50}$$

Here and in the following, we shall adopt the usual convention of choosing units such that both $\hbar$ and c have the numerical value unity[7].

Before finishing, we wish to point out the physical (rather than the ad-hoc looking mathematical) meaning of the $i\epsilon$ prescription. Keeping ϵ fixed, we may compute the response of the field φ to a source that emits particles at a fixed instant $t = 0$ (and hence with all energies) and at all positions $\vec{x}$ but with zero momentum,that is, we take $J(x) = \delta(t)$. Writing $\omega = \sqrt{m^2 - i\epsilon} \approx m^2 - i\gamma/2m$, we can then compute

$$\phi(x) = \int d^4yR(x-y)J(y) = \frac{i}{2\omega}e^{-i\omega t} \quad \Rightarrow \quad |\phi(x)|^2 \propto \exp(-\gamma t/m) \quad . \tag{51}$$

We find a particle density that decays exponentially (and uniformly all over space) with a lifetime m/γ. This is precisely what is expected for an *unstable* particle. We hence find the (effective) propagator for a particle with large but finite lifetime:



$$= \frac{i}{\vec{k}^2 - m^2 + im\Gamma} \quad ,$$

where Γ is the total decay width (inverse lifetime) of the particle. The $i\epsilon$ prescription is simply the recipe that, to describe perfectly stable particles, we must take them as unstable first, and then let the lifetime approach infinity[8]. Later one, we shall employ this.

2.5 Time flow, classical kinematics and antiparticles

A number of corrollaries can be drawn from the form of the propagator (49). For given $k^\mu = (k^0, \vec{k})$, let us define $\omega(\vec{k}) = \sqrt{(\vec{k}^2 + m^2)} > 0$. The correlator

[7]This is often presented as a triviality since in principle the necessary factors of $\hbar$ and c can be put back in whenever desired. In practice, this may involve a good deal of puzzling and is thus "...left as an excercise to the interested reader..."

[8]Where this limit is defined, it simply means that, say, an electron with a lifetime of 4×10^{26} years should behave in the same manner as a perfectly stable one — and who knows which situation is the actual one?

for *positive* times $x^0 > 0$ can be written as[9]

$$
\begin{aligned}
R(x) &= \frac{1}{(2\pi)^3} \int\limits_{-\infty}^{\infty} \frac{d^3\vec{k}}{2\omega(\vec{k})} e^{-ix^0\omega(\vec{k})+\vec{x}\cdot\vec{k}} \\
&= \frac{1}{(2\pi)^4} \int\limits_{-\infty}^{\infty} d^4k\ \delta(k^2-m^2)\theta(k^0) e^{-ix^0\omega(\vec{k})+\vec{x}\cdot\vec{k}} \ , \qquad (52)
\end{aligned}
$$

while for *negative* times $x^0 < 0$ we find

$$
\begin{aligned}
R(x) &= \frac{1}{(2\pi)^3} \int\limits_{-\infty}^{\infty} \frac{d^3\vec{k}}{2\omega(\vec{k})} e^{+ix^0\omega(\vec{k})+\vec{x}\cdot\vec{k}} \\
&= \frac{1}{(2\pi)^4} \int\limits_{-\infty}^{\infty} d^4k\ \delta(k^2-m^2)\theta(-k^0) e^{+ix^0\omega(\vec{k})+\vec{x}\cdot\vec{k}} \ , \qquad (53)
\end{aligned}
$$

We conclude that *positive energies propagate forward in time, and negative energies propagate backwards in time.* Note that here the sign of ϵ becomes important: time is defined as that direction in which the density of unstable particles *decreases.*

A second point is the behaviour of the propagator of macroscopic distances and times. Let us take a source

$$
J(x) \propto \exp\left(-\frac{(x^0)^2}{4\sigma_0^2} - \frac{\vec{x}^2}{4\sigma^2} - iEx^0 + \vec{p}\cdot\vec{x}\right) \ , \qquad (54)
$$

which describes emission of particles in a space region of size σ^3 and a time interval of size σ^0 around the origin, with energies around some $E > 0$ and momenta around some $\vec{p}$. The field at position $\vec{x}$ and time $x^0 > 0$ is then given by

$$
\begin{aligned}
\phi(x) &\propto \int\limits_{-\infty}^{\infty} \frac{d^3\vec{k}}{2\omega(\vec{k})} \exp(A(\vec{k})) \ , \\
A(\vec{k}) &= -ix^0\omega(\vec{k}) + i\vec{k}\cdot\vec{x} - \sigma_0^2(E-\omega(\vec{k}))^2 - \sigma^2(\vec{p}-\vec{k})^2 \ . \qquad (55)
\end{aligned}
$$

[9]The two alternative expressions for the integration element are equivalent. The first one is familiar from quantum mechanics, while the second one has the advantage of explicit Lorentz invariance.

The real part of $A(\vec{k})$ shows that propagation will only be appreciable if there is a $\vec{k}$ such that $\vec{k} \approx \vec{p}$ and $\omega(\vec{k}) \approx E$, so that the emitted particles must be on their mass shell:

$$E^2 \approx \vec{p}^2 + m^2 \quad ; \tag{56}$$

and the points x in space where the propagator is then appreciable must be such that the phase is stationary:

$$\frac{\partial}{\partial k_j} A(\vec{k}) \approx 0 \quad \Rightarrow \quad x^j \approx x^0 \, \frac{k_j}{\omega(\vec{k})} = x^0 \frac{p_j}{E} \quad (j = 1, 2, 3) \ . \tag{57}$$

That is, to make it over macroscopic distances and times[10], free particles must be emitted on their mass shell, and will then move with uniform veloicity given by $\vec{p}/E$.

A last conclusion stems from the fact that we interpret only *positive-energy* modes as particles, moving forward in time. Suppose, now, that a negative-energy particle is excited. This will move from some point A to some point B in space-time, with $A^0 > B^0$, carrying negative energy. We may equally interpret this as *another* sort of particle, moving from B to A, with positive energy: the antiparticle. If the particles carry not only energy-momentum but also, say, positive electric charge, then the antiparticle is seen to carry negative charge from B to A. Hence: for every observed type of particle, there must be a corresponding antiparticle, with opposite quantum numbers such as charge (colour, baryon number...), but with the same mass en lifetime as the particle (after all, they stem from the same propagator) [11].

3 Scattering and Decay Processes, Unitarity

In particle phenomenology, among the most important quantities to be computed are 2-particle scattering cross sections and 1-particle decay rates. These involve the preparation of a 1- or 2-particle state, a macroscopic time interval for any interactions to take place, and the observatino of the time-evolved state. It is only the central part, where the interactions take place, that we discuss here.

[10] These conclusions do not hold for microscopic distances and times — this is just the uncertainty principle which forbids us to emit particles simultaneously at an exactly specified place and time and with exactly determined energy-momentum.

[11] Envisioning antiaprticles as particles moving backwards in time quickly becomes cumbersome as soon as causal sequences are considered, but it *is* a consistent world view.

3.1 Phase space elements

So far, we have only studied Green's functions in which particles interact with one another, and can therefore not be considered free. The propagator, on the other hand, describes the movement of particles without interactions, but it tends to diverge as these approach their mass shell. We must therefore find a recipe to turn a Green's function into a scattering matrix element while keeping track of the mass shell condition for particles that come in, or move out to, infinity. One of the necessary ingredients is a rule for counting (or normalizing) the external states. Note that this is to some extent arbitrary, since a change in the definition of the density of states can always be compensated by a change in the recipe for turning Green's functions into scattering amplitudes. What we shall therefore do is: (**a**) choose a density of states; (**b**) assume that there is a consistent way of transforming a Green's function into a scattering amplitude; (**c**) choose a physical requirement to find this correspondence by the requirement of consistency.

The first ingredient is the density of states for a final state consisting of n produced particles, moving off to infinity. These all have to have positive energy and must be on their mass shell. Denoting the momenta by p_j^μ and the masses by m_j ($j = 1, 2, \ldots, n$), we choose the density of states (the inifintesimal phase-space integration element) as

$$dV(P; p_1, \ldots, p_n) = \left(\prod_{j=1}^{n} \frac{1}{(2\pi)^3} d^4 p_j \; \delta(p_j^2 - m_j^2) \; \theta(p_j^0) \right) \times (2\pi)^4 \; \delta^4(P - p_1 - \cdots - p_n) \; . \tag{58}$$

Here, P^μ is the total momentum, which is conserved. A first rule, therefore, is to eliminate the overall $(2\pi)^4 \delta^4(\cdots)$ for energy-momentum conservation from the Green's function. The factors (2π) in dV are conventional. We shall see that the prescription for the normalization of the *incoming* particles will follow automatically.

By Fermi's Golden Rule, the *form* of partial decay rates and scattering cross sections must be, for the decay rate:

$$d\Gamma = F_\Gamma \; \left\langle |\mathcal{M}|^2 \right\rangle \; dV \; , \tag{59}$$

and, for the scattering cross section,

$$d\sigma = F_\sigma \; \left\langle |\mathcal{M}|^2 \right\rangle \; dV \; . \tag{60}$$

Here, the F are flux factors (involving the incoming-state normalizations) to be determined, and $\langle |\mathcal{M}|^2 \rangle$ is the absolute value squared of the scattering matrix element, suitably summed over spins, colours, etcetera for the final states, and averaged for the initial states.

3.2 The truncation argument

In the following, we shall identify particles by their momenta. We investigate a scattering process in which a number of particles $p_1, p_2, \ldots, p_n$ are produced from an initial state P. One of these, p_1, is assumed to be unstable but very long-lived. Some distance away from the scattering vertex, therefore, it will decay into a number of decay products $q_1, q_2, \ldots, q_m$. The total process is therefore

$$P \to p_2 + p_3 + \cdots + p_n + q_1 + q_2 + \cdots + q_m \quad .$$

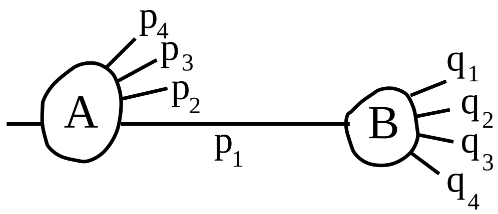
As the lifetime of the particle p_1 becomes very long, we expect that this can better be described as the two-step process

$$P \to p_1 + p_2 + \cdots + p_n \quad , \quad p_1 \to q_1 + q_2 + \cdots + q_m \quad .$$

If these descriptions are to be equivalent, we must have

$$\begin{aligned} & d\sigma(P \to p_2 + p_3 + \cdots + p_n + q_1 + q_2 + \cdots + q_m) \\ = \quad & d\sigma(P \to p_1 + p_2 + \cdots + p_n) \, \frac{1}{\Gamma} \, d\Gamma(p_1 \to q_1 + q_2 + \cdots + q_m) \quad , \end{aligned} \tag{61}$$

where Γ is the total decay width of particle p_1. The corresponding diagrammatic picture is



We now assume that the external lines $P, p_2, \ldots, p_n, q_1, \ldots, q_m$ have been properly treated such that the Green's function has been turned into a scattering matrix element. The total scattering amplitude $\mathcal{M}$ is therefore given by

$$\mathcal{M} = \mathcal{M}_A \, \frac{i}{p_1^2 - m_1^2 + i m_1 \Gamma} \, \mathcal{M}_B \quad , \tag{62}$$

where $\mathcal{M}_{A,B}$ stand for the two blobs labelled A and B. The total differential cross section must therefore be given by

$$\begin{aligned} & d\sigma(P \to p_2 + p_3 + \cdots + p_n + q_1 + q_2 + \cdots + q_m) \\ = \; & F|\mathcal{M}_A|^2 \frac{1}{(p_1^2 - m_1^2)^2 + m_1^2\Gamma^2} |\mathcal{M}_B|^2 dV(P; p_2, \ldots, p_n, q_1, \ldots, q_m) \end{aligned} \tag{63}$$

In the limit $\Gamma \to 0$, we may replace

$$\left((p_1^2 - m_1^2)^2 + m_1^2\Gamma^2\right)^{-1} \to \frac{\pi}{m_1\Gamma}\delta(p_1^2 - m_1^2) \ . \tag{64}$$

After some trivial algebra, we may write this as

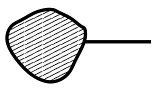
$$\begin{aligned} & d\sigma(P \to p_2 + p_3 + \cdots + p_n + q_1 + q_2 + \cdots + q_m) \\ = \; & F \, |\mathcal{M}_A|^2 \, dV(P; p_1, p_2, \ldots, p_n) \, \frac{1}{\Gamma} \, \frac{1}{2m_1} |\mathcal{M}_B|^2 \, dV(p_1, q_1, \ldots, q_m) \end{aligned} \tag{65}$$

Comparison with the physical requirement (61) shows that

- The fluxfactor in the decay of particle p_1 must be

$$F_\Gamma = 1/2m_1 \tag{66}$$

- The propagator of the particle p_1 disappeared into the Dirac delta for its mass shell. This *truncation* is therefore the following Feynman rule:

   $\to 1$,

  that is, a factor 1 for each external (on-shell) incoming or outgoing particle. In the above truncation argument, we must therefore assume that this prescription *had already been followed* for the external particles $P, p_2, \ldots, p_n, q_1, \ldots, q_m$.

Note that the flux factor F_Γ is given for the particle p_1 at rest, since the total width Γ is also by definition given for a particle at rest. For a moving particle, the total decay width is reduced by p_1^0/m_1 because of time dilatation, and the normalization factor for a *moving particle* is therefore not $1/2m_1$ but $1/2p_1^0$. For a two-particle initial state with incoming momenta p_1 and p_2, therefore, the combined normalization must be $1/4m_1p_2^0$ in the situation where p_1 is at rest and p_2 impinges onto it. This gives us the *transition rate*: in practice one is rather interested in the cross section, that is, the transition rate per unit flux. Since the flux is in this case simply the velocity $|\vec{p}_2|/p_2^0$, we find

$$F_\sigma = \frac{1}{4m_1|\vec{p}_2|} = \frac{1}{2\left[((p_1+p_2)^2 - m_1^2 - m_2^2)^2 - 4m_1^2 m_2^2\right]^{1/2}} \ , \qquad (67)$$

where the last form is explicitly Lorentz invariant. For small masses, we may approximate $F_\sigma = 1/(2(p_1+p_2)^2)$.

In summary, we have derived the *truncation* rule that an external (scalar) particle contributes a factor 1 rather than a divergent propagator, and we have found the forms for F_Γ and F_σ. The scattering amplitude is seen to equal the truncated Green's function with the overall $(2\pi)^4\delta^4(\cdots)$ removed.

3.3 Connected diagrams

Consider a one-particle state decaying into a number of particles. The diagrams describing this may be either connected or disconnected. A disconnected diagram must therefore contain at least one connected part from which one or more particles exit *without* any particle entering it. This is forbidden by energy conservation, and therefore the only contribution to a decay amplitude must come from the connected diagrams.

Also in two-particle scattering cross sections, disconnected diagrams that contain connected parts without incoming lines are forbidden. There may be disconnected diagrams consisting of two connected pieces each with one incoming particle. These will only contribute if both incoming particles are in fact unstable. If both incoming particles are stable, therefore, the only contributing graphs are therefore again the connected ones.

Although this is usually glossed over in the literature, it is therefore only energy conservation that motivated our concentrating on the connected Green's functions. In some circumstances, such as the scattering of unstable particles, the argument fails: but in this case the truncation argument itself fails as well since the incoming particles cannot have come from infinity, and the better treatment would involvce also the production of the unstable particles (this is especially relevant in muon-muon scattering). Antoher exceptional case may be scattering at very high temperatures, where the surrounding heat bath may occasionally excite particles 'from the vacuum'.

3.4 Dimensionality and unitarity

Let E be a unit of energy. Each d^4p in dV is then $\propto E^4$, and since a Dirac delta has the inverse dimension of its argument, the dimension of dV for an n-particle final state is $\propto E^{2n-4}$. From their definition, $F_\sigma \propto E^{-2}$ and $F_\gamma \propto E^{-1}$. It follows that any amplitude with n external legs (incoming and outgoing) has dimension E^{4-n}.

Suppose that we consider theories with only dimensionless coupling constants. If we keep all scattering angles in a given process fixed and increase the total energy $\sqrt{s}$, the amplitude will then scale as $s^{2-n/2}$ when $\sqrt{s}$ becomes much larger than any of the particle masses involved[12]. In that case, the cross section scales as $1/s$. If, on the other hand, there are dimensionful coupling constants, the behaviour of the amplitude is different. Suppose, for instance, that the product of all couplings in an amplitude is $1/\Lambda^2$, with $\Lambda \propto E$. Then the amplitude will be $\propto E^{6-n}/\Lambda^2$, and the cross section will, at large energies, behave as s/Λ^4. That this is not an academic issue becomes clear from the following.

Quantum-mechanically, the total set of all possible initial states (at time $-\infty$) is a complete orthonormal set, and so is the set of all possible final states (at time $+\infty$). The scattering amplitude for a given process is the overlap between the given initial state, time-evolved from time $-\infty$ to time $+\infty$, and the observed final state. The total set of all scattering amplitudes, therefore, must be a unitary matrix, called the S-matrix. Assuming, for simplicity, that the initial and final states are discrete sets[13], it follows that each element of the S-matrix must be smaller than 1 in absolute value. More concretely, the *total* cross section for two-particle inital state of fixed total angular momentum J to go into anything is bounded by

$$\sigma(J) \leq \frac{16\pi(2J+1)}{s} \quad . \tag{68}$$

Any theory will have to respect this *unitarity bound*: a violation implies a violation of the conservation of probability. A cross section such as above,

[12]This argument neglects terms like $\log(s/m^2)$ which may arise from loop corrections.

[13]If we label the states by the momenta of the particles, the labelling is of course continuous. However, discrete labellings are possible, for instance we may describe a two-particle initial state in terms of its total energy, its total angular momentum in the centre-of mass frame, and one component of the angular momentum. These last two take on discrete values.

that rises with s, is therefore inherently flawed and can only be viewed as the low-energy approximation to a more fundamental theory. Therefore, theories that contain couplings with negative energy dimension are suspect.

In the following we shall not study the unitarity bound in detail, but we shall aim at finding theories in which matrix elements do *not* exhibit dangerous high-energy behaviour. That is, $2 \to 2$ matrix elements should asymptotically be at most s^0, and $2 \to 3$ matrix elements should asymptotically decrease at least as fast as $1/\sqrt{s}$.

4 Spinning Particles

4.1 Non-minimal propagators

So far, we have studied particles that can only carry energy and momentum information[14]. The form of the propagator, $i/k^2 - m^2 + i\epsilon$, is dictated by very general considerations, and especially the form of the denominator can hardly be touched without endangering the usual particle picture. Particles that carry additional information such as spin, therefore, must be described by propagators with the same denominator, but a modified numerator:

$$\frac{\mathcal{H}(k,m)}{k^2 - m^2 + i\epsilon} \ ,$$

where $\mathcal{H}(k,m)$ is some nontrivial function of k^μ and m. At this point, note that a propagator must run between two vertices, and so far the vertices of a theory have not been fixed. If $\mathcal{H}(k,m)$ is just a scalar function, we may therefore always absorb, say, $\sqrt{\mathcal{H}}$ into each vertex, and the remaining propagator will be what we had before. It follows that $\mathcal{H}$ must carry *some* indices, that couple with corresponding indices in the vertices; and, moreover, it must *not* be possible to write $\mathcal{H}$ as a product of just two factors. An additional requirement is that the choice of $\mathcal{H}$ must lead to scattering amplitudes $\mathcal{M}$ that are Lorentz-invariant, *i.e.* there must not be any loose indices floating around. We may conclude that $\mathcal{H}$ must be a matrix-like object containing a sum over the degrees of freedom contained in the particle. That is, let e_i be the matrix representation of degree of freedom number i for an on-shell particle; the different degrees of freedom must be orthogonal, therefore $\bar{e}_i e_j = 0$

[14] Properties like electric charge only appear once we discuss interactions, and are therefore not described by the free-particle propagator.

for unequal i and j. Then,

$$\mathcal{H}(k,m)\rfloor_{k^2=m^2} = \sum_i e_i \bar{e}_i \ , \tag{69}$$

and it must have the form of a projection operator (up to a constant):

$$\mathcal{H}(k,m)\mathcal{H}(k,m)\rfloor_{k^2=m^2} \ \propto \ \mathcal{H}(k,m)\rfloor_{k^2=m^2} \ . \tag{70}$$

4.2 Spin-1/2 particles

4.2.1 Dirac matrices

The simplest choice for $\mathcal{H}$ is a linear function of k^μ and m:

$$\mathcal{H}(k,m) = k^\mu + am \ , \tag{71}$$

with a some constant (any prefactor in front of k^μ can always be absorbned into the vertices). Clearly, this choice is unsuitable because of Lorentz invariance, and moreover contains only a single index. We might *define* a fixed vector b^μ and take

$$\mathcal{H}(k,m) = k \cdot b + am \ , \tag{72}$$

but this would inject a fixed vector b^μ in all matrix elements, violating the isotropy of spacetime. In fact, a better approach is to define four matrices, the Dirac matrices:

$$(\gamma^\mu)^\alpha{}_\beta \ , \ \ \mu = 0,1,2,3 \ .$$

Here, the indices α and β are not Lorentz indices but just the matrix indices. We then propose

$$\mathcal{H}(k,m) = k_\mu \gamma^\mu + am1 = \not{k} + am1 \ , \tag{73}$$

where '1' denotes the unit matrix, and we have introduced the 'slash' notation. It may seem that we have now just replaced the fixed vector b by a set of fixed matrices, and have again violated Lorentz invariance. However, we now require that *it must be possible to eliminate all Dirac matrices inside the absolute square of any matrix element in a strictly covariant manner.* The Dirac matrices must therefore obey a special relation. The simplest possibility[15] is to require

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}1 \ . \tag{74}$$

[15]We do not want a relation involving more than two Dirac matrices since then not all matrices can be eliminated. Therefore, $g^{\mu\nu}$ must occur on the right-hand side, and therefore the left-hand side must also be *symmetric* in the indices.

This is the famous *anticommutation* relation between Dirac matrices. Note that this requirement cannot be satisfied if the γ's are simple numbers, and this is the reason to use matrices instead. This relation indeed suffices to remove any reference to the choice of particular Dirac matrices from all cross sections. The Lorentz indices of any vectors contracted with γ's will then be linked to indices of other vectors in a Lorentz-invariant manner, and the matrix elements will indeed be Lorentz-invariant.

4.2.2 Dirac algebra

Up to this point, the size of the Dirac matrices has not been specified. Let the dimensionality be N, that is, $\mathrm{Tr}\,[1] = N$, where '1' is the unit matrix, and $\mathrm{Tr}\,[\cdots]$ stands for the trace. We also define

$$\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 \tag{75}$$

from which we immediately have

$$\gamma^5\gamma^5 = 1 \quad , \quad \gamma^5\gamma^\mu = -\gamma^\mu\gamma^5 \quad . \tag{76}$$

We can now establish the trace identities without more ado: first,

$$\mathrm{Tr}\,[\gamma^\mu] = \mathrm{Tr}\left[\gamma^5\gamma^5\gamma^\mu\right] = \mathrm{Tr}\left[\gamma^5\gamma^\mu\gamma^5\right] = \mathrm{Tr}\,[-\gamma^\mu] = -\mathrm{Tr}\,[\gamma^\mu] = 0 \quad , \tag{77}$$

where we have used the cyclic property of the trace and the anticommutation with γ^5. Similarly, we have

$$\mathrm{Tr}\,[\gamma^\mu\gamma^\nu] = \mathrm{Tr}\,[\gamma^\nu\gamma^\mu] = \frac{1}{2}\mathrm{Tr}\,[\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu] = Ng^{\mu\nu} \quad , \tag{78}$$

$$\mathrm{Tr}\left[\gamma^5\gamma^\mu\right] = \mathrm{Tr}\left[\gamma^\mu\gamma^5\right] = -\mathrm{Tr}\left[\gamma^5\gamma^\mu\right] = 0 \quad , \tag{79}$$

and similarly a trace of any odd number of γ's vanishes; also,

$$\mathrm{Tr}\left[\gamma^5\right] = \frac{1}{4}\mathrm{Tr}\left[\gamma^5\gamma^\mu\gamma_\mu\right] = \frac{1}{4}\mathrm{Tr}\left[\gamma_\mu\gamma^5\gamma^\mu\right] = -\frac{1}{4}\mathrm{Tr}\left[\gamma^5\gamma_\mu\gamma^\mu\right] = -\mathrm{Tr}\left[\gamma^5\right] = 0 \quad . \tag{80}$$

Moreover, by repeated anticommutatio we find

$$\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta = 2g^{\mu\nu}\gamma^a l\gamma^\beta - 2g^{\mu\alpha}\gamma^n u\gamma^\beta + 2g^{\mu\beta}\gamma^\nu\gamma^\alpha - \gamma^\nu\gamma^\alpha\gamma^\beta\gamma^\mu \quad , \tag{81}$$

so that

$$\mathrm{Tr}\left[\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^b e\right] = N\left(g^{\mu\nu}g^{\alpha\beta} - g^{\mu\alpha}g^{\nu\beta} + g^{\mu\beta}\gamma^{\nu\alpha}\right) \quad . \tag{82}$$

The extension to a larger even number γ's is straightforward: and this also leads immediately to

$$\mathrm{Tr}\left[\gamma^5\gamma^\mu\gamma^\nu\gamma^\alpha\gamma^\beta\right] = iN\epsilon^{\mu\nu\alpha\beta} \tag{83}$$

and, as a corollary,

$$\mathrm{Tr}\left[\gamma^5\gamma^\mu\gamma^\nu\right] = \frac{i}{4}N\epsilon^{\mu\nu\alpha\beta}g_{\alpha\beta} = 0 \ . \tag{84}$$

Here we have introduced the Levi-Civita tensor $\epsilon^{\mu\nu\alpha\beta}$ which is antisymmetric in all indices, and $\epsilon^{0123} = -1$.

It can be shown that we may, without loss of generality, assume the following for the Hermitian character of the Dirac matrices:

$$(\gamma^0)^\dagger = \gamma^0 \quad , \quad (\gamma^k)^\dagger = -\gamma^k \ , \ \ k = 1,2,3 \ . \tag{85}$$

This leads us to define the Dirac conjugate of Γ, a string of γ's:

$$\overline{\Gamma} = \gamma^0(\Gamma)^\dagger\gamma^0 \ . \tag{86}$$

Immediately, this gives

$$\overline{1} = 1 \ , \ \overline{\gamma^\mu} = \gamma^\mu \ , \ \overline{[\gamma^\mu,\gamma^\nu]} = -[\gamma^\mu,\gamma^\nu] \ , \ \overline{\gamma^5\gamma^\mu} = \gamma^5\gamma^\mu \ , \ \overline{\gamma^5} = -\gamma^5 \ . \tag{87}$$

These results will, indeed, allow us to eliminate all Dirac matrices from cross sections and decay widths.

4.2.3 Spin-projection operators

We now pose the question what are the most general objects Π_j, built up from Dirac matrices and Lorentz vectors, that obey

$$\overline{\Pi_j} = \Pi_j \quad , \quad \Pi_j\Pi_k = \delta_{j,k}\Pi_j \ . \tag{88}$$

It is a straightforward (but somewhat cumbersome) excercise to show that there are four possibilities[16]:

$$\Pi(\lambda_1,\lambda_2) = \frac{1}{4m}(\lambda_1\not{p} + m)(1 + \lambda_2\gamma^5\not{s}) \ , \tag{89}$$

[16] Up to overall factors that can be absorbed into the vertices at either end of the propagator.

where $\lambda_{1,2}$ take the values $\pm$, and

$$p^2 = m^2 \quad , \quad s^2 = -1 \quad , \quad p \cdot s = 0 \ . \tag{90}$$

p^μ is of course the momentum of the state singled out by the projection operator, and s^μ is the *spin vector* of this state. We see that we can take the Dirac matrices to be 4-dimensional: $N = 4$. Also, we can choose

$$\mathcal{H}(p.m) = \not{p} + m \quad \Rightarrow \quad (\mathcal{H}(p,m))^2 = 2m\mathcal{H}(p,m) \ , \tag{91}$$

on the mass shell $p^2 = m^2$, and there we also have:

$$\begin{aligned} p^0 > 0 \ &: \ \not{p} + m = 2m(\Pi(+,+) + \Pi(+,-)) \ , \\ p^0 < 0 \ &: \ \not{p} + m = -2m(\Pi(-,+) + \Pi(-,-)) \ . \end{aligned} \tag{92}$$

In each case, then, precisely two degrees of freedom are present in the projectin operator $\mathcal{H}$, and therefore this propagator describes a spin-1/2 particle[17], hence a fermion. We may define *spinors* u and *anti-spinors* v by

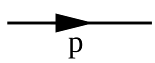
$$\begin{aligned} (\not{p} - m)u(p,s) = 0 \quad &, \quad (1 - \gamma^5\not{s})u(p,s) = 0 \\ (\not{p} + m)v(p,s) = 0 \quad &, \quad (1 - \gamma^5\not{s})v(p,s) = 0 \ . \end{aligned} \tag{93}$$

and define the Dirac conjugate by $\overline{u} = (u)^\dagger\gamma^0$ (similar for v). This leaves the normalization free: we use

$$\sum_{\pm s} u(p,s)\overline{u}(p,s) = \frac{1}{2}(\not{p} + m) \quad , \quad \sum_{\pm s} v(p,s)\overline{v}(p,s) = \frac{1}{2}(\not{p} - m) \ . \tag{94}$$

4.2.4 Feynman rules for fermions

At this point we have all the necessary ingredients. We have, for internal lines:

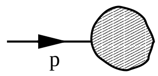


$$\rightarrow \quad \frac{i(\not{p} + m)}{p^2 - m^2 + i\epsilon} \ . \tag{95}$$

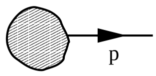
Since the sign of p matters here, the line is conventionally drawn oriented, and one counts the momentum in the direction of the arrow. For external

[17]Strictly speaking, we ought to study the influence of rotations on the projection operators to establish the spin: the conclusion remains.

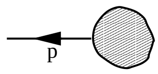
lines we can repeat the truncation argument above, and then we have:

imcoming particle : 

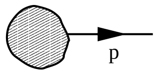
$\rightarrow \; u(p,s) \;\; ,$

outgoinging particle : 

$\rightarrow \; \overline{u}(p,s) \;\; ,$

imcoming antiparticle : 

$\rightarrow \; \overline{v}(p,s) \;\; ,$

outgoinging antiparticle : 

$\rightarrow \; -v(p,s) \;\; ,$

(96)

Note the minus sign in the last line. To put it there rather than with the incoming antiparticle is conventional. In the squared matrix element, of course, one would be tempted to discrard it altogether, but its presence implies the following additional Feynman rules for fermions:

- A minus sign for every closed fermion loop;
- Two diagrams in which *any* two external fermion lines interchange position changes its sign.

This is the (in)famous Fermi minus sign: amplitudes change sign under the interchange of any two fermions in the process. It should be remarked that any particle of *integer* spin can, if we so desire, be described by a collection of an *even* number of spin-1/2 spinors. Therefore, an interchange of two integer-spin particles leads to no sign change: this is the spin-statistics theorem.

4.3 Spin-1 particles

4.3.1 Massive spin-1 particles

The next-simplest extension of the scalar propagator is one that contains two powers of the momentum. In that case, no special matrices are necessary, and we may write

$$\mathcal{H}(p,m) = a g^{\mu\nu} + b p^\mu p^\nu \;\; , \tag{97}$$

with some constants a and b. It is straightforward to recognize the degrees of freedom inside this form: at the mass shell, with $p^2 = m^2$, there are precisely three real orthogonal vectors ε_j^μ, $j = 1,2,3$, that obey

$$\varepsilon_j \cdot \varepsilon_k = -\delta_{j,k} \quad , \quad \varepsilon_j \cdot p = 0 \;\; , \tag{98}$$

and[18]

$$\sum_{j=1,2,3} \varepsilon_j^\mu \varepsilon_j^\nu = -g^{\mu\nu} + \frac{1}{m^2} p^\mu p^\nu \quad . \tag{99}$$

This, then, is the form we choose for $\mathcal{H}$ is this case, and we see that it describes a massive spin-1 particle. Of course, we may also employ complex linear combinations of the *polarization vectors* ε (correspondong to circular or elliptic rather than linear polarization). Again using the truncation argument, the Feynman rules are seen to read

$$\begin{array}{rcl} \text{internal propagator} & : & \dfrac{i(-g^{\mu\nu} + p^\mu p^\nu)}{p^2 - m^2 + i\epsilon} \quad , \\ \text{incoming particle} & : & \varepsilon^\mu \quad , \\ \text{outgoing particle} & : & (\varepsilon^\mu)^* \quad . \end{array} \tag{100}$$

4.3.2 The high-energy limit

Our description of massive spin-1 particles incubates a serious problem. This is most easily seen by an explicit example. Let the on-shell particle have energy E and move in the positive z direction with momentum p. The four-momentum is then

$$p^\mu = (E, 0, 0, p) \quad . \tag{101}$$

Of the three polarizations, we may choose two *transverse* ones:

$$\varepsilon_{T,1}^\mu = (0, 1, 0, 0) \quad , \quad \varepsilon_{T,2} = (0, 0, 1, 0) \quad , \tag{102}$$

and then the remaining one, the *longitudinal* polarization, has the form

$$\varepsilon_L^\mu = (p/m, 0, 0, E/m) \quad . \tag{103}$$

As the particle's energy increases, the transverse polarization vectors are not affected, but the longitudinal one increases with energy: indeed, as $E \gg m$ we have

$$\varepsilon_L^\mu \approx \frac{1}{m} p^\mu + \mathcal{O}(m/E) \quad . \tag{104}$$

Since in a process with external spin-1 particles these polarization vectors occur as factors, longitudinally polarized particles potentially lead to unitarity violation. The solution is not to search for a better propagator, but rather

[18] This is most easily seen in the rest frame of the massive particle.

to impose on the **vertices** *of the theory the property that, for longitudinal external polarization, cancellations must occur that dampen the high-energy growth.* Consider a scattering process in which a longitudinally polarized spin-1 particle is produced at high energy. The amplitude can be written (with real polarization vectors) as

$$\mathcal{M} = J_\mu \varepsilon_L^\mu \quad , \quad \varepsilon_L^\mu \approx \frac{1}{m} p^\mu \ . \tag{105}$$

Here, J is the rest of the process that emits the particle: the source of the particle. We now simply require that

$$\mathcal{M}\rfloor_{\varepsilon_L \to p} = J_\mu p^\mu = \mathcal{O}(m) \ , \tag{106}$$

to be obtained by judicious choice of the vertices.

4.3.3 Massless spin-1 particles

The case $m = 0$ is especially urgent. For a process in which a source (an electromagnetic current!) that emits or absorbs a photon, we must require

$$J_\mu p^\mu = 0 \ . \tag{107}$$

Going back from momentum to position language, this implies

$$\partial^\mu J_\mu = 0 \ , \tag{108}$$

which we recognize as *current conservation.* In fact, we require more: in order to avoid the dangerous $p^\mu p^\nu / m^2$ in the propagator, we must require that current conservation holds also for off-shell photons. If this can be arranged, the Feynman rule for the photon propagator may be written simply as

$$-i\frac{g^{\mu\nu}}{p^2 + i\epsilon} \ ,$$

and all reference to m has disappeared.

Another way to derive the requirement of e.m. current conservation is the following. It is an empirical fact that a photon's polarization is always transverse. This is, strictly speaking, not a Lorentz-invariant statement: if the polarization vector $\varepsilon(p)$ is transverse in the frame where the photon has momentum p, a boost to a frame where the momentum is q leads, in general, to a polarization vector $\varepsilon(q)$ that is not purely transverse. The unwanted longitudinal part, however, is in that case proportional to q. If the *observable* part of the polarization vector is transverse, then the part proportional to q must drop out from all processes that produce or absorb a photon: current conservation.

4.3.4 A quick look back: helicity states for fermions

The algebraic requirements for polarization vectors ε are identical to those for the spin vector of spin-1/2 particles. Therefore, the longitudinal spin vector will also blow up at high energies. Why didn't that bother us before? The answer is that, although the spin vector indeed blows up, the projection operators Π do not: indeed, if $\vec{s}$ is in the same direction as $\vec{p}$ and we increase the energy p^0 (or reduce the mass m) we shall find:

$$\begin{aligned} u(p, \pm s)\overline{u}(p, \pm s) &= \frac{1}{2}(1 \pm \gamma^5 \not{s})(\not{p} + m) \\ &\approx \frac{1}{2}(1 \pm \gamma^5 \not{p}/m)(\not{p} + m) \approx \frac{1}{2}(1 \pm \gamma^5)\not{p} \;, \end{aligned} \tag{109}$$

where we have used $\not{p}\not{p} = m^2$. Longitudinally polarized spin-1/2 particles are therefore not bothersome[19], and their description reduces to one in which only the sign $\pm$ before the γ^5 is relevant. These are so-called *helicity states*.

5 The Electroweak Standard Model

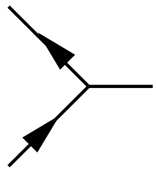
We shall now describe the structure of the electroweak standard model, not, as is usual, starting with imposing any symmetry, but rather by requesting reasonable behaviour of the theory's predicitons, especially in the light of the potential problems with unitarity at high energies.

5.1 Quantum electrodynamics

One of the most directly important types of interaction is that between photons and the fermions describing leptons and quarks. Since the photon is

[19] One of the many small miracle built into the Dirac algebra...

massless, current conservation is essential here. It turns out that essentially the only acceptable vertex coupling a phton with fermions is



$$\mu \quad \to \quad iq\gamma^\mu \ , \tag{110}$$

where μ is the Lorentz index carried by the photon, and q is the electric charge of the fermion. To see that this vertex can indeed ensure current conservation, suppose that an electron scatters of a nucleus with an interaction described by this vertex:

$$e^-(p) + N \to e^-(q) + N \ ,$$

where N stands for the nucleus. The momentum of the photon exchanged between the electron and the nucleus is then $k^\mu = p^\mu - q^\mu$, and the part of the diagram describing the electron scattering is

$$\mathcal{M}^\mu = -ie\ \overline{u}(q)\gamma^\mu u(p) \ \ . \tag{111}$$

The requirement of current conservation is that this object vanishes when we contract it with the photon momentum, as indeed is does:

$$\begin{aligned} \mathcal{M}^\mu\ k_\mu &= -ie\ \overline{u}(q)\not{k}u(p) = -ie\overline{u}(q)(-\not{q} + \not{p})u(p) \\ &= -ie\ \overline{u}(q)(-(\not{q} - m) + (\not{p} - m))u(p) = 0 \ \ , \end{aligned} \tag{112}$$

where we have used the Dirac equation (94). This is of course the very simplest example, but it can be shown that *any* process involving fermions and photons interacting by this vertex will exhibit current conservation, by manipulations of exactly the same type. This establishes QED as a viable theory of elementary particle interactions.

In the above, it was essential that the incoming and outgoing fermion had exactly the same mass: indeed, an interaction of the type $e^-\mu^+\gamma$ will *not* obey current conservation. This is an example of the more general statement that *flavour-changing neutral currents* are an unwanted feature.

Another feature is that in the above example we have not used the masslessness of the photon. Indeed, in QED there are no particular problems if the photon becomes massive. Therefore, a theory with massive spin-1 particles interacting with electrons is just as viable. However, if the interaction vertex contains an additonal $\gamma^5\gamma^\mu$ contribution, current conservation will again be endangered for massive fermions (but not for massless ones).

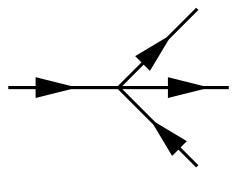
5.2 Muon decay and the W

5.2.1 The Fermi model

The archetypical weak-interaction process is that of muoun decay:

$$\mu^-(P) \ \rightarrow \ e^-(p)\, \nu_\mu(q)\, \overline{\nu}_e(k) \ \ ,$$

Phenomenology indicates that this decay is well described by the amplitude corresponding to the diagram



,

and given by the amplitude of the Fermi model:

$$\mathcal{M} = -i\frac{G_F}{\sqrt{2}}\overline{u}(q)(1+\gamma^5)\gamma^\mu u(P)\,\overline{u}(p)(1+\gamma^5)\gamma_\mu v(k) \ \ , \tag{113}$$

where G_F is to be given by experiment. From the expressions for $u\overline{u}$, etcetera, we see that each spinor carries a dimension $E^{1/2}$, and the Fermi coupling G_F must therefore have dimension E^{-2}: as we discussed above, this is potentially dangerous. The total decay width of the muon is given (for massless electrons) by

$$\Gamma(\mu) = \frac{G_F^2 m_\mu^5}{192\pi^3} \ \ , \tag{114}$$

and from the measured mass and lifetime of the muon we then find

$$G_F = 1.16\ 10^{-5}\ \mathrm{GeV}^{-2} \ \ . \tag{115}$$

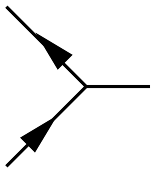
That, indeed, the Fermi coupling leads to problems is seen if we assume this vertex to be truly fundamental, and apply it to the related process of muon-neutrino scattering at high energy:

$$\sigma(\mu^-\overline{\nu}_\mu \ \rightarrow \ e^-\overline{\nu}_e) \propto G_F^2\ s \ \ , \tag{116}$$

which follows by elementary dimensional analysis (remember that cross sections must have dimension E^{-2}). Therefore, the Fermi vertex cannot be a consistent description of the weak interaction at the fundamental level.

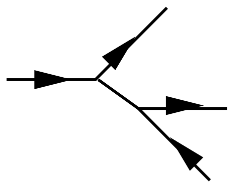
5.2.2 The W particle

The way out of the problems with the Fermi model is to assume that the four-fermion interaction is not truly fundamental, but rather mediated by another particle, the W, that couples to $\mu\overline{\nu}_\mu$ and $e\overline{\nu}_e$, with the vertex



$$\mu \quad \to \quad ig_w(1+\gamma^5)\gamma^\mu \quad , \tag{117}$$

where now μ is the index carried by the W particle, and g_w is the weak interaction coupling, to be determined. The diagram for muon decay is, in this more detailed picture, given by



and the scattering amplitude is now

$$\mathcal{M} = i\frac{g_w^2}{(p+k)^2 - m_w^2}\ \overline{u}(q)(1+\gamma^5)\gamma^\mu u(P)\ \overline{u}(p)(1+\gamma^5)\gamma_\mu v(k) \quad . \tag{118}$$

Here, the $p^\mu p^\nu$ part of the W propagator vanishes since we take the electron and its neutrino to be massless. We see that the phenomenological success of the Fermi model is reproduced if we require that $m_w^2 \gg (p+k)^2$, that is, $m_w \gg m_\mu$, and, moreover,

$$\frac{g_w^2}{m_w^2} = \frac{G_F}{\sqrt{2}} \quad . \tag{119}$$

The 'real' weak coupling constant, g_w, is nicely dimensionless. Note that, if we want perturbation theory to hold, we need g_w to be substantially smaller than 1, and hence m_w substantially smaller than 350 GeV.

5.3 Yang-Mills couplings and the Z

5.3.1 $WW\gamma$ coupling

The W particle, whose emission turns a charged lepton into its neutrino, must have unit charge (by electromagnetic current conservation!), and therefore we have both a W^+ and a W^-, each other's antiparticles. The W, therefore, couples to the photon. Imposing both current conservation, and good high-energy behaviour for longitudinal W in processes like

$$e^-\overline{\nu}_e \to W^-\gamma \quad , \quad u\overline{d} \to W^+\gamma \ ,$$

(where we use the same Dirac algebra as we did for QED), forces us to adopt the following three-point vertex coupling two W's and a photon:

$$W^+(p_1)^\mu W^-(p_2)^\nu \gamma(k)^\alpha \quad : \quad -ieV(p,\mu;q,\nu;k,\alpha) \ , \tag{120}$$

where V is the *Yang-Mills three-boson vertex*:

$$V(p_1,\mu_1;p_2,\mu_2;p_3,\mu_3) = (p_1-p_2)^{\mu_3}g^{\mu_1\mu_2} + (p_2-p_3)^{\mu_1}g^{\mu_2\mu_3} + (p_3-p_1)^{\mu_2}g^{\mu_3\mu_1} \ , \tag{121}$$

and we have counted all momenta as going out from the vertex (we shall always use this convention). Note that this vertex is antisymmetric under interchange of any two particles, and therefore it can only occur if all three bosons in the vertex are *different*. Therefore, this vertex cannot occur only photons are present, as in QED[20].

5.3.2 W pair production and the Z

Since we have assumed the W to be an actual, existing particle, we have to confront processes in which they can be produced. For instance, using the vertices available so far, we have $e^+e^- \to W^+W^-$ by the two diagrams, one with the exchange of a neutrino between the two electrons, and the other by their annihilating into a photon, which then decays into a W pair. It is found that, if one of the W's is longitudinal, the high-energy behaviour of the amplitude is unaccpetable. The solution is to introduce yet another, neutral, particle, the Z, that can be produced by e^+e^- annihilation, and couples, like to photon, to W^+W^-. We then have a third Feynman diagram available. This diagram will have to cancel, on the one hand, the purely γ^μ vertex of the photon diagram , and on the other hand the $(1+\gamma^5)\gamma^\mu$ vertices of the neutrino graph, and its coupling to the fermions must therefore be a mixture of the two. Moreover, since we assume it to be a real particle, it must be produceable in a process like $e^-\overline{\nu}_e \to W^-Z$. Requiring acceptable high-energy behaviour in the following processes:

$$e^+e^- \to W^+W^- \quad , \quad \nu_e\overline{\nu}_e \to W^+W^- \quad , \quad e^-\overline{\nu}_e \to W^-Z \ ,$$

[20] In QCD, all the gluons are electrically neutral, but they can occur in different *colours*.

leads to the following vertices for the fermions and the W's:

$$\begin{aligned}
e^+e^-Z^\mu &: \quad i(v_e + a_e\gamma^5)\gamma^\mu \ , \\
\nu\overline{\nu}Z^\mu &: \quad a_\nu(1+\gamma^5)\gamma^\mu \ , \\
v_e &= -\frac{e}{4s_w c_w}(1-4s_w^2) \quad , \quad a_e = -a_\nu \quad = \quad -\frac{e}{4s_w c_w} \ , \\
W^+(p)^\mu W^-(q)^\nu Z(k)^\alpha &: \quad ig_{wwz}V(p,\mu;q,\nu;k,\alpha) \ , \qquad (122)
\end{aligned}$$

Here, $s_w = \sin(\theta_w)$, $c_w = \cos(\theta_w)$, where, the *weak mixing angle* θ_w is given by the electric and weak couplings:

$$g_{wwz}^2 + e^2 = 8g_w^2 \quad , \quad e^2 \equiv g_w^2 s_w^2 \ . \qquad (123)$$

It follows that

$$g_{wwz} = -\frac{ec_w}{s_w} \quad , \quad g_w = \frac{e}{s_w\sqrt{8}} \ , \qquad (124)$$

so that the W mass (but not yet the Z mass!) is given by s_w as well:

$$m_W = \left(\frac{\pi\alpha}{G_F\sqrt{2}}\right)^{1/2}\frac{1}{s_w} = \frac{37.28}{s_w}\ \text{GeV} \ . \qquad (125)$$

Using up and down quarks rather than neutrinos and electrons, we would find

$$\begin{aligned}
a_u &= -a_d = \frac{e}{4s_w c_w} \ , \\
v_u &= a_u(1-\frac{8}{3}s_w^2) \quad , \quad v_d = a_d(1-\frac{4}{3}s_w^2) \ . \qquad (126)
\end{aligned}$$

At this point, then, we have established the three weak bosons, and found the *algebraic* structure of the coupling between the Z and the other particles. The electroweak mixing angle is only defined as a parameter relating the various couplings, and the Z mass is still undetermined.

5.3.3 Four-boson vertices

In addition to the $2 \to 2$ processes involving four fermions of two fermions and two bosons, we also have processes involving four electroweak bosons. First, we consider the process $W^+W^- \to ZZ$. The Yang-Mills three-boson couplings allow us to draw two Feynman diagrams, in which the two incoming

W's exchange another W. The two three-boson vertices involved each have a linear energy dependence, and the internal boson propagator effcectively goes as the inverse square of the energy[21]. The main energy dependence, here, is therefore contained in the polarization vectors of the external bosons. If one of them is longitudinal, we may again expect problems with unitarity, and indeed the two diagrams give rise to $\mathcal{M}$ proportional to the energy. To counter this, we introduce a *Yang-Mills four-point vertex*:

$$(W^+)^\mu (W^-)^\nu (Z)^\alpha (Z)^\beta \quad : \quad g_{wwzz} U(\mu,\nu;\alpha,\beta) \ , \tag{127}$$

with

$$U(\mu,\nu;\alpha,\beta) = 2g^{\mu\nu}g^{\alpha\beta} - g^{\mu\alpha}g^{\nu\beta} - g^{\mu\beta}g^{\nu\alpha} \ , \tag{128}$$

and

$$g_{wwzz} = -g_{wwz}^2 = -\frac{e^2 c_w^2}{s_w^2} \ . \tag{129}$$

Similarly, we can study the other four-boson processes:

$$W^+W^- \to Z\gamma \quad , \quad W^+W^- \to \gamma\gamma \quad , \quad W^+W^- \; to \; W^+W^- \quad ,$$

and from them determine the analogous four-boson couplings:

$$\begin{aligned} (W^+)^\mu (W^-)^\nu (Z)^\alpha (\gamma)^\beta \quad &: \quad g_{wwz\gamma} U(\mu,\nu;\alpha,\beta) \ , \\ (W^+)^\mu (W^-)^\nu (\gamma)^\alpha (\gamma)^\beta \quad &: \quad g_{ww\gamma\gamma} U(\mu,\nu;\alpha,\beta) \ , \\ (W^+)^\mu (W^+)^\nu (W^-)^\alpha (W^-)^\beta \quad &: \quad g_{wwww} U(\mu,\nu;\alpha,\beta) \ , \end{aligned} \tag{130}$$

with

$$g_{wwz\gamma} = -\frac{e^2 c_w}{s_w} \quad , \quad g_{ww\gamma\gamma} = -e^2 \quad , \quad g_{wwww} = -\frac{e^2}{s_w^2} \ . \tag{131}$$

The three- and four-boson couplings are, indeed, precisely those that can be derived in the standard manner from the imposition of gauge symmetry on the interactions: here, we see them arriving from the necessity of having acceptable high-energy behaviour of scattering amplitudes.

[21]Naively, the $p^\mu p^\nu$ part of the propagator indicates that the propagator at fixed scattering angle approaches a constant, but this part of the propagator, when coupled to the Yang-Mills vertices, gives contributions proportional to (the square of) the external boson masses rather than their energies, and therefore is not dangerous in this process.

5.4 The Higgs

Continuing the high-energy behaviour game, we may also study the process $W^+W^- \rightarrow ZZ$ in the situation where all four bosons are longitudinal. In that case, the two diagrams with three-boson couplings grow with the energy E as E^4, and so does the single diagram with the four-boson vertex. The cancellation built in to repair the case of a single longitudinally polarized boson still does its job in this case, and the sum of the three diagrams cancels down to an E^2 behaviour. Since this is not enough and all vertices so far have been fixed, we need to introduce yet another particle. The simplest solution is to assume a single, electrically neutral, spinless particle that couples to W^+W^- and to ZZ: this is the Higgs particle H. The couplings are then

$$(W^+)^\mu(W^-)^\nu H \;:\; ig_{wwh}g^{\mu\nu} \quad , \quad (Z)^\mu(Z)^\nu H \;:\; ig_{zzh}g^{\mu\nu} \;, \tag{132}$$

and the extra diagram in which the W^+W^- annihilate into a Higgs boson which then decays into ZZ reads, at $\sqrt{s}/2 = E \gg m_{W,Z}$:

$$-ig_{wwh}g_{wwz}\frac{E^4}{m_W^2 m_Z^2}\frac{1}{s - m_H^2} \quad . \tag{133}$$

By choosing the Higgs couplings, we can ensure the necessary cancellation of the $W^+W^- \rightarrow ZZ$ amplitude down to E^0 behaviour. In addition, there are the processes

$$W^+W^- \rightarrow W^+W^- \quad , \quad W^+W^- \rightarrow ZH$$

that also involve the exchange of a Higgs (processes with an external photon do not lead to problems since the photon cannot be longitudinally polarized once current conservation is ensured, and the Higgs is neutral). Combining the constraint from these processes, we arrive at

$$g_{wwh} = -\frac{em_W}{s_w} \quad , \quad g_{zzh} = -\frac{em_z}{s_w c_w} \quad , \tag{134}$$

and, more interestingly,

$$m_W = m_Z c_w \quad ! \tag{135}$$

We see that it is the assumption of a single neutral Higgs that fixes *a connection between the ratio of W and Z masses and that of electromagnetic and weak couplings.* On the other hand, just as the assumption of a Z particle to repair W pair production did not lead to a bound on the Z mass, the assumption of a Higgs does not lead to a constraint on the Higgs mass itself.

5.4.1 Other Higgs interactions

The Higgs particle interacts with the electroweak bosons in proportion to their masses (and therefore the Higgs does not react to photons, at least at the tree level). A similar behaviour occurs in its coupling to ferimons: if we re-examine the process $e^+e^- \rightarrow W^+W^-$ with both W's longitudinally polarized, and nonzero electron mass, we find that also in this process a Higgs has to be assumed, coupling not only to W^+W^- but also to e^+e^-, with coupling

$$e^+e^-H \ : \ i\frac{em_e}{2s_w m_W} \ ; \tag{136}$$

and analogously for the other fermions: massless fermions do not couple to the Higgs.

Two more couplings are relevant: by looking at

$$ZZ \rightarrow HH \quad , \quad W^+W^- \rightarrow HH$$

we find that there are 2-boson-2-Higgs four-point couplings:

$$ZZHH \ : \ i\frac{e^2}{2s_w^2 c_w^2} \quad , \quad W^+W^-HH \ : \ i\frac{e^2}{2s_w^2} \ . \tag{137}$$

The phenomenon that the Higgs couples in proportion to the mass of the coupling particle persists also if we consider the Higgs itself: to investigate this, we must conider $2 \rightarrow 3$ processes since the $2 \rightarrow 2$ ones have now been exhausted[22]. From requiring good high-energy behaviour in the following processes:

$$ZZ \rightarrow ZZH \quad , \quad ZZ \rightarrow HHH$$

we find that there must be coupling between two bosons and two Higgses, and three- and four-Higgs self-interactions:

$$HHH \ : \ i\frac{3em_H}{2s_w m_W} \quad , \quad HHHH \ : \ i\frac{3e^2 m_H^2}{4s_w^2 m_W^2} \ . \tag{138}$$

It might be assumed that the procedure of considering scattering processes and requiring good high-energy behaviour leading to yet more particles continues *ad nauseam*: but it can be shown that, in fact the above couplings suffice to make the high-energy behaviour of any process acceptable, at least at the tree level. And, since we have effectively derived the complete standard electroweak model, we may have confidence that all processes will be well-behaved at all orders.

[22] Apart from processes like $HH \rightarrow HH$ which are well-behaved without more ado.

5.4.2 Non-minimal Higgs models

We have seen that the assumption of a single neutral Higgs particle implies the mass relation $m_W = m_Z c_w$. Now, consider the process

$$u\bar{d} \;\to\; W^+ Z \;\;,$$

where we take both external bosons longitudinal, and keep the fermion (quark) masses. In this case, there are three diagrams, one of which contains the three-boson vertex. If we study the high-energy behaviour of this process, we find that it is acceptable *only* if the above relation between the W and Z mass holds. If we suppose, for the sake of argument, that it does *not* hold, no neutral Higgs can solve our problem since none will fit into the diagram (at the tree level). Therefore, we need *charged* Higgses in addition to the neutral ones. It is therefore important to study the mass relation: if it does not hold, the Higgs sector must contain charged Higgses and also *more than one* neutral Higgses, since it was the assumption of a single neutral Higgs that imposes the mass relation. This *non-minimal* Higgs sector is characteristic of, among others, supersymmetric electroweak models. It may be noted that the couplings of the charged Higgses to the fermions contain terms with 1 and γ^5, and are, indeed, precisely those of the supersymmetric ones.

5.4.3 A robust upper bound

It is instructive to finish these notes with a cogent argument due to Veltman. Let us look again at the Higgs-mediated amplitude (133). Since this diagram is tailored for cancelling the unwanted high-energy behaviour of the other three graphs in $WW \to ZZ$, we know that the *complete* amplitude must read

$$-ig_{wwh}g_{wwz}\frac{E^4}{m_W^2 m_Z^2}\left(\frac{1}{s-m_H^2}-\frac{1}{s}\right) = -ig_{wwh}g_{wwz}\frac{E^4}{m_W^2 m_Z^2}\frac{m_H^2}{s(s-m_H^2)} \;\;. \tag{139}$$

At energies much higher than the Higgs mass, this becomes

$$-ig_{wwh}g_{wwz}\frac{m_H^2}{4m_W^2 m_Z^2} \;\;.$$

As the Higgs mass increases, we shall therefore still violate the unitarity bound on the cross section, even if it approaches a constant: if the Higgs is to heavy, more than about 1 TeV, it ‘comes too late’ to save unitarity (at least at the tree level, but higher-order calculations do not indicate an improvement). We may therefore have some confidence in the *no-lose theorem*: at the LHC we shall either find the Higgs, and feel proud and clever that we confirm the electroweak standard model, or we shall find dramatic new physics!