Extending the subtraction method to NNLO

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The need for NNLO calculations

Hunting for the Higgs and other yet-to-be-discovered particles requires a better knowledge of the theoretical cross section.

The strong coupling constant α_s is one fundamental parameter of the theory and its precise value affects the magnitude of many (background) processes.

The next generation of colliders will increase the experimental precision. This has to be matched by an improvement in the accuracy of theoretical predictions.

Theoretical predictions are calculated as a power expansion in the coupling. Higher precision is reached by including the next higher term in the perturbative expansion.

What is necessary:

NNLO calculations

Perturbative NNLO calculations

The experimental needs are numerical programs which yield predictions for a wide range of observables.

Fully differential NNLO programs for

- Bhabha scattering
- $pp \to 2$ jets
- $e^+e^- \to 3$ jets

which allow the calculation of any infrared safe observable.

Necessary ingredients for a NNLO calculation

- Calculation of the (two-loop) amplitudes.

  Requires: Two-loop integrals and tensor reduction.

- Cancellation of IR divergences has to be done before any Monte Carlo integration.

  Requires: Extension of the subtraction or slicing method to NNLO.

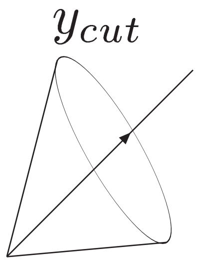
- The final numerical computer program.

  Requires: Stable and efficient numerical methods.

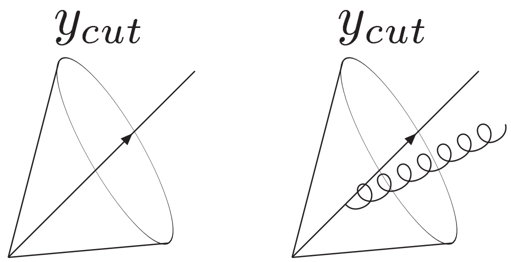
Modeling of jets:

In a perturbative calculation jets are modeled by only a few partons. This improves with the order to which the calculation is done.

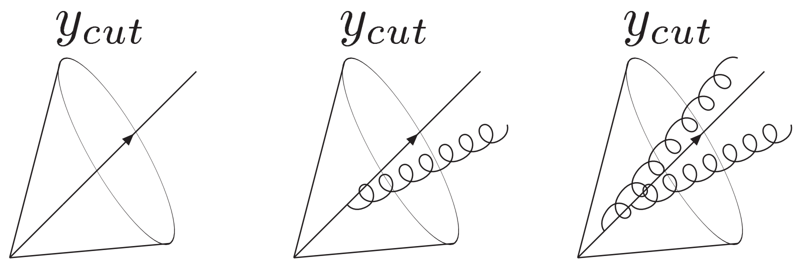
At leading order:



At next-to-leading order:



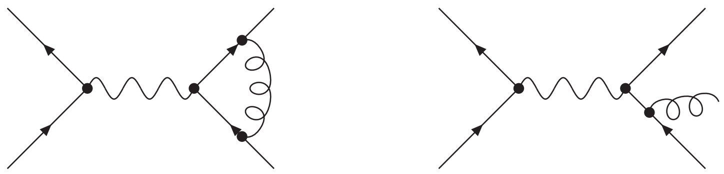
At next-to-next-to-leading order:



Infrared divergences and the Kinoshita-Lee-Nauenberg theorem

In addition to ultraviolet divergences, loop integrals can have infrared divergences.

For each IR divergence there is a corresponding divergence with the opposite sign in the real emission amplitude, when particles becomes soft or collinear (e.g. unresolved).



The Kinoshita-Lee-Nauenberg theorem: Any observable, summed over all states degenerate according to some resolution criteria, will be finite.

Infrared-safe observables and event shapes

Observables which do not depend on long-distance behaviour, are called infrared-safe observables and can reliably be calculated in perturbation theory.

Example: Thrust

$$T = \max_{\hat{n}} \frac{\sum_i |\vec{p}_i \cdot \hat{n}|}{\sum_i |\vec{p}_i|}$$

In particular, it is required that they do not change value, if infinitessimal soft or collinear particles are added.

At NNLO:

$$\begin{aligned} \text{Single unresolved}: \quad & \mathcal{O}_{n+1}(p_1, ..., p_{n+1}) & \rightarrow & \quad \mathcal{O}_n(p_1', ..., p_n'), \\ \text{Double unresolved}: \quad & \mathcal{O}_{n+2}(p_1, ..., p_{n+2}) & \rightarrow & \quad \mathcal{O}_n(p_1', ..., p_n'). \end{aligned}$$

General methods at NLO

Fully differential NLO Monte Carlo programs need a general method to handle the cancelation of infrared divergencies.

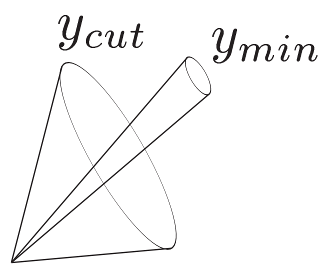
- Phase space slicing
  - e^+e^-: W. Giele and N. Glover, (1992)
  - initial hadrons: W. Giele, N. Glover and D.A. Kosower, (1993)
  - massive partons, fragmentation: S. Keller and E. Laenen, (1999)

- Subtraction method
  - residue approach: S. Frixione, Z. Kunzst and A. Signer, (1995)
  - dipole formalism: S. Catani and M. Seymour, (1996)
  - massive partons: L. Phaf and S.W. (2001),
    S. Catani, S. Dittmaier, M. Seymour and Z. Trócsányi, (2002)

Phase space slicing

Splits the integration of the real emission contribution into a region $y > y_{min}$ and a region $y < y_{min}$.

The former is free of singularities and the integration can be performed numerically there.
In the latter the matrix element is approximated and the integration over the one-parton phase space is performed analytically.



- Introduces an error of order y_{min}.
- The first region gives a contribution of the form

$$a \ln^2 y_{min} + b \ln y_{min} + c$$

  The logarithms $\ln^2 y_{min}$ and $\ln y_{min}$ cancel against the contribution from the second region.
- But: Cancelation happens only numerically!

The dipole formalism

The dipole formalism is based on the subtraction method. The NLO cross section is rewritten as

$$\begin{aligned}\sigma^{NLO} &= \int_{n+1} d\sigma^R + \int_n d\sigma^V \\ &= \int_{n+1} \left(d\sigma^R - d\sigma^A\right) + \int_n \left(d\sigma^V + \int_1 d\sigma^A\right)\end{aligned}$$

The approximation $d\sigma^A$ has to fulfill the following requirements:

- $d\sigma^A$ must be a proper approximation of $d\sigma^R$ such as to have the same pointwise singular behaviour in D dimensions as $d\sigma^R$ itself. Thus, $d\sigma^A$ acts as a local counterterm for $d\sigma^R$ and one can safely perform the limit $\varepsilon \to 0$.

- Analytic integrability in D dimensions over the one-parton subspace leading to soft and collinear divergences.

The subtraction method at NNLO

Contributions at NNLO:

$$
\begin{aligned}
d\sigma^{(0)}_{n+2} &= \left({\mathcal{A}^{(0)}_{n+2}}^{*} \mathcal{A}^{(0)}_{n+2} \right) d\phi_{n+2}, \\
d\sigma^{(1)}_{n+1} &= \left({\mathcal{A}^{(0)}_{n+1}}^{*} \mathcal{A}^{(1)}_{n+1} + {\mathcal{A}^{(1)}_{n+1}}^{*} \mathcal{A}^{(0)}_{n+1} \right) d\phi_{n+1}, \\
d\sigma^{(2)}_{n} &= \left({\mathcal{A}^{(0)}_{n}}^{*} \mathcal{A}^{(2)}_{n} + {\mathcal{A}^{(2)}_{n}}^{*} \mathcal{A}^{(0)}_{n} + {\mathcal{A}^{(1)}_{n}}^{*} \mathcal{A}^{(1)}_{n} \right) d\phi_{n},
\end{aligned}
$$

Adding and subtracting:

$$
\begin{aligned}
\langle \mathcal{O} \rangle^{NNLO}_{n} &= \int \left(\mathcal{O}_{n+2}\, d\sigma^{(0)}_{n+2} - \mathcal{O}_{n+1} \circ d\alpha^{(0,1)}_{n+1} - \mathcal{O}_{n} \circ d\alpha^{(0,2)}_{n} \right) \\
&+ \int \left(\mathcal{O}_{n+1}\, d\sigma^{(1)}_{n+1} + \mathcal{O}_{n+1} \circ d\alpha^{(0,1)}_{n+1} - \mathcal{O}_{n} \circ d\alpha^{(1,1)}_{n} \right) \\
&+ \int \left(\mathcal{O}_{n}\, d\sigma^{(2)}_{n} + \mathcal{O}_{n} \circ d\alpha^{(0,2)}_{n} + \mathcal{O}_{n} \circ d\alpha^{(1,1)}_{n} \right).
\end{aligned}
$$

The leading singular behaviour

Factorization of one-loop amplitudes in single unresolved limits:

$$A_n^{(1)} \quad = \quad \text{Sing}^{(0,1)} \cdot A_{n-1}^{(1)} + \text{Sing}^{(1,1)} \cdot A_{n-1}^{(0)},$$

Z. Bern, Del Duca, W. Kilgore, C. Schmidt, D. Kosower, P. Uwer, S. Catani, M. Grazzini

Factorization of tree amplitudes in double unresolved limits:

$$A_n^{(0)} \quad = \quad \text{Sing}^{(0,2)} \cdot A_{n-2}^{(0)},$$

F. Berends, W. Giele, J. Cambell, N. Glover, S. Catani, M. Grazzini, V. Del Duca, A. Frizzo, F. Maltoni, D. Kosower

For the subtraction method one needs also the subleading singular behaviour !

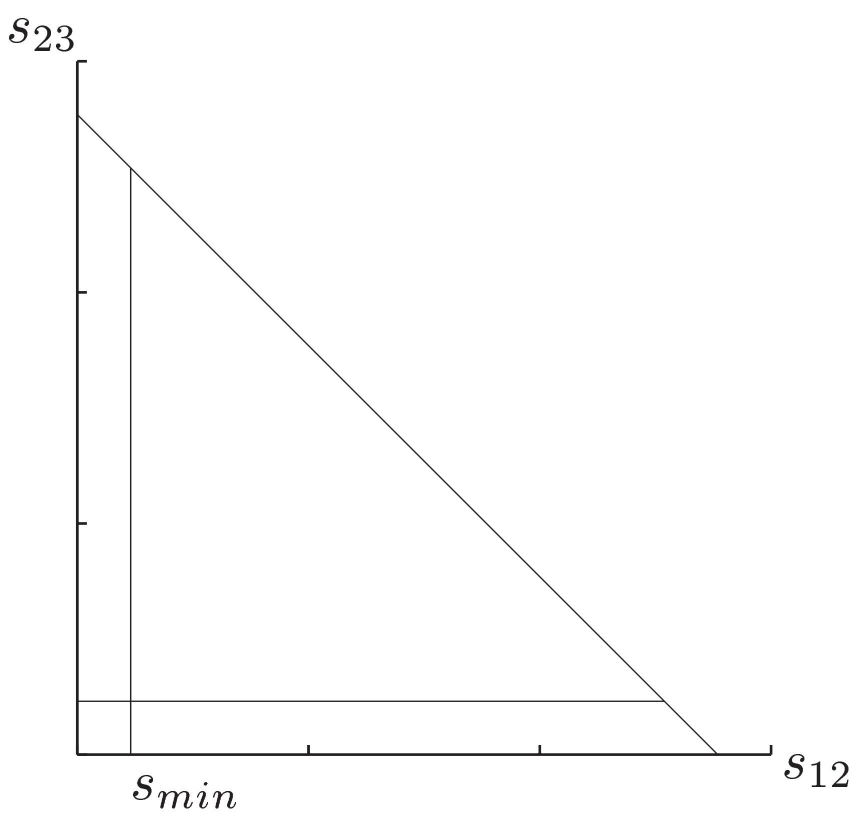
An example: $e^+e^- \to$ 2 jets at NLO

The matrix element squared for $\gamma^* \to qg\bar{q}$:

$$M_3 = 8(1-\varepsilon)\left[2\frac{s_{123}^2}{s_{12}s_{23}} - 2\frac{s_{123}}{s_{12}} - 2\frac{s_{123}}{s_{23}} + (1-\varepsilon)\frac{s_{23}}{s_{12}} + (1-\varepsilon)\frac{s_{12}}{s_{23}} - 2\varepsilon\right]$$

The subtraction terms:

$$\mathcal{D}_{12,3} + \mathcal{D}_{32,1} = 8(1-\varepsilon)$$
$$\left[2\frac{s_{123}^2}{s_{12}(s_{12}+s_{23})} - 2\frac{s_{123}}{s_{12}} + (1-\varepsilon)\frac{s_{23}}{s_{12}}\right]$$
$$+\left[2\frac{s_{123}^2}{s_{23}(s_{12}+s_{23})} - 2\frac{s_{123}}{s_{23}} + (1-\varepsilon)\frac{s_{12}}{s_{23}}\right]$$

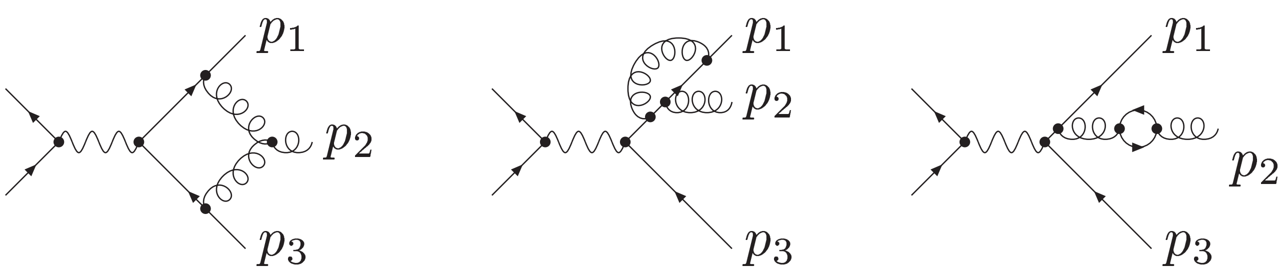


Colour disentanglement

Full QCD amplitudes do not factorize in the singular limits, but primitive amplitudes do.

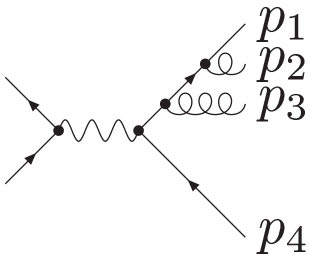
Primitive amplitudes distinguished by:

- fixed cyclic ordering of the QCD partons
- definite routing of the external fermion lines through the diagram
- particle content circulating in the loop



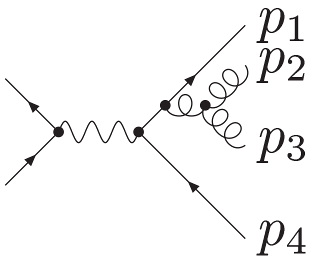
An example involving double unresolved configurations

The leading-colour contributions to $e^+e^- \to qgg\bar{q}$.



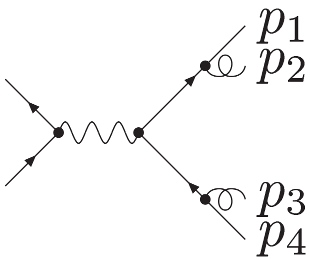
Double unresolved configurations:

- Two pairs of separately collinear particles
- Three particles collinear
- Two particles collinear and a third soft particle
- Two soft particles
- Coplanar degeneracy



Single unresolved configurations:

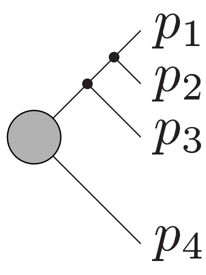
- Two collinear particles
- One soft particle



NNLO subtraction terms

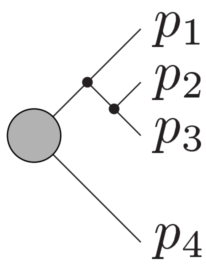
The $(n+2)$-parton contribution:

$$\int \left(\mathcal{O}_{n+2}\, d\sigma^{(0)}_{n+2} - \mathcal{O}_{n+1} \circ d\alpha^{(0,1)}_{n+1} - \mathcal{O}_{n} \circ d\alpha^{(0,2)}_{n} \right)$$

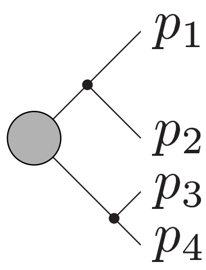


The NNLO subtraction term is written as

$$d\alpha^{(0,2)}_{n} \quad = \quad \sum_{T} \left(\mathcal{D}^{(0,2)}_{(0,0)n} - \mathcal{D}^{(0,2)}_{(0,1)n} \right),$$

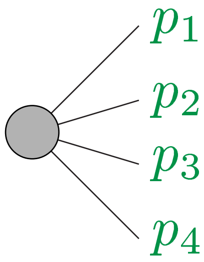


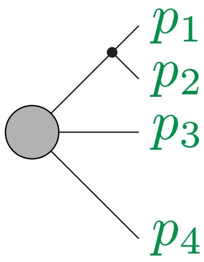
where $\sum\limits_{T} \mathcal{D}^{(0,2)}_{(0,0)n}$ approximates $d\sigma^{(0)}_{n+2}$ and
$\sum\limits_{T} \mathcal{D}^{(0,2)}_{(0,1)n}$ approximates $d\alpha^{(0,1)}_{n+1}$.

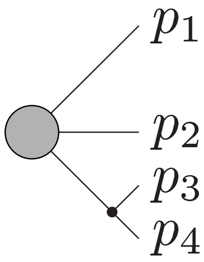


Single unresolved contributions

For an observable, which vanishes on n-parton configurations:

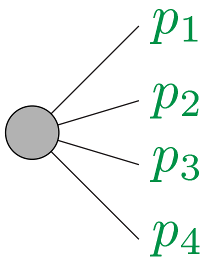
$\lim_{s_{12}\to 0}$ 

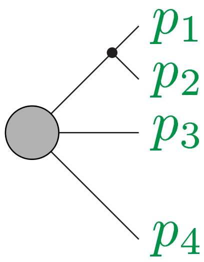
$-$ 

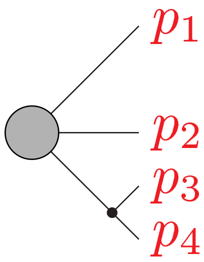
$-$ 

$=$ integrable

For an observable, which does not vanish on n-parton configurations:

$\lim_{s_{12}\to 0}$ 

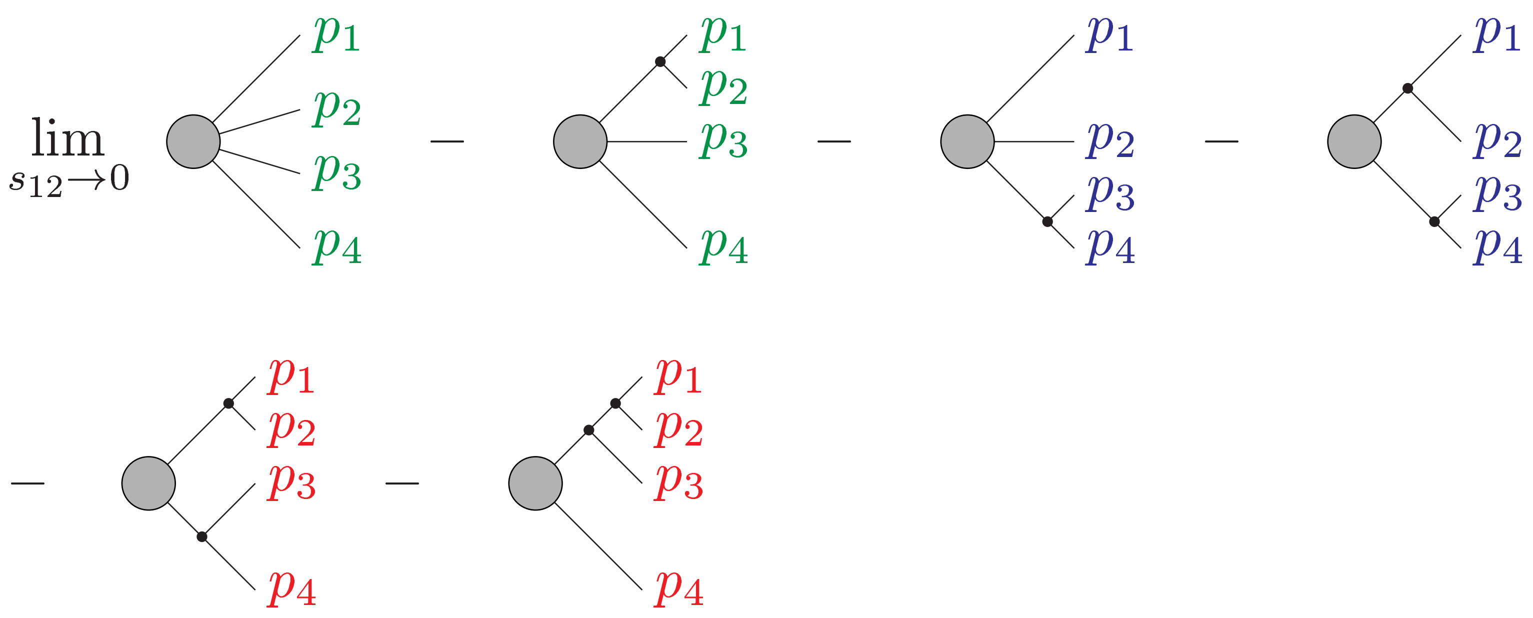
$-$ 

$-$ 

$\neq$ integrable

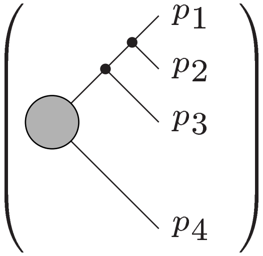
Single unresolved contributions

The subtraction terms are subtracted over the complete phase space. The integrand has to be finite also in single unresolved limits !



Results

$\mathcal{P}^{(0,2)}_{(0,0)\; q\to qgg}$



$$= \not{p}_e^{(2)} \frac{4}{s_{1234}^2}\frac{1}{x_1^2}\left\{\frac{1}{y_1\left(y_1+\bar{y}_1\bar{z}_1\right)}\left[\frac{2}{\left(y_1+\bar{y}_1\bar{z}_1\right)x_1+\bar{y}_1 x_2}+\frac{2}{y_1 x_1+u}\right.\right.$$

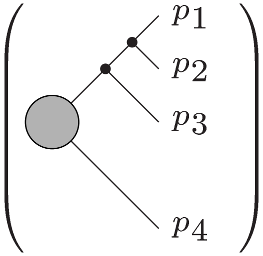
$$\left. + \frac{u}{\left(y_1+\bar{y}_1\bar{z}_1\right)x_1+\bar{y}_1 x_2}\left(-2+(1-\varepsilon)u\right)+\frac{\bar{y}_1 x_2}{u}\left(-2+(1-\varepsilon)\bar{y}_1 x_2\right)\right]$$

$$+\frac{1}{y_1}\left[\frac{2}{u\left[\left(y_1+\bar{y}_1\bar{z}_1\right)x_1+\bar{y}_1 x_2\right]}+\frac{1}{x_1+\bar{y}_1 x_2}\left(-2+(1-\varepsilon)u\right)-\frac{4}{u}+(1-\varepsilon)\frac{\bar{y}_1 x_2}{u}-\frac{\bar{y}_1 x_2}{u}\left(-2+(1-\varepsilon)\bar{y}_1 x_2\right)\right.$$

$$\left.+\varepsilon(1-\varepsilon)u-\varepsilon(1-\varepsilon)\bar{y}_1 x_2-1+3\varepsilon\right]+(1-\varepsilon)^2\frac{\bar{y}_1\bar{z}_1}{y_1}+3-4\varepsilon+\varepsilon^2+\frac{x_1}{y_1}\left[-\varepsilon\left(\frac{2}{y_1+\bar{y}_1\bar{z}_1}-2+(1-\varepsilon)\bar{y}_1\bar{z}_1\right)\right.$$

$$\left.\left.+\frac{2}{u}-(1-\varepsilon)\left(\frac{x_1}{u}-2\bar{y}_1 z_1+(1-\varepsilon)\bar{y}_1 z_1 x_1\right)+\frac{1}{u\left(y_1+\bar{y}_1\bar{z}_1\right)}\left(-2+(1-\varepsilon)x_1\right)\right]\right\}$$

$\mathcal{P}^{(0,2)}_{(0,1)\; q\to qgg}$



$$= \not{p}_e^{(2)} \frac{4}{s_{1234}^2}\frac{1}{x_1^2}\left[\frac{2}{x_1+x_2}-2+(1-\varepsilon)\,x_2-\varepsilon x_1\right]$$

$$\times\frac{1}{y_1}\left[\frac{2}{1-z_1(1-y_1)}-2+(1-\varepsilon)\,(1-y_1)(1-z_1)\right]$$

Outlook

Extension of the subtraction method to NNLO:

- Construction of the subtraction terms

$\longrightarrow$ Subtracted matrix elements can be integrated numerically !

To be done:

- Analytic integration over the unresolved phase space