Fields and symmetries

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Scalar fields

Free spinless particles are quanta of scalar fields satisfying the Klein-Gordon equation ($\hbar = c = 1$):

$$(\Box + m^2)\varphi = 0, \qquad \Box = \partial^\mu \partial_\mu = \partial_0^2 - \Delta$$

In momentum space this implies the energy-momemtum dispersion relation

$$-p^\mu p_\mu + m^2 = 0 \quad \Leftrightarrow \quad E^2 = \vec{p}^{\,2} + m^2$$

Charged scalar fields:

$$\left[(\partial - ieA)^2 + m^2\right]\varphi = 0$$

Charge conjugation: $\varphi^c = \varphi^* \longrightarrow$

$$\left[(\partial + ieA)^2 + m^2\right]\varphi^c = 0$$

Conserved electro-magnetic current:

$$j_\mu = ie\,\varphi^* \overset{\leftrightarrow}{\partial}_\mu \varphi + 2e^2 A_\mu \varphi^* \varphi, \qquad \partial^\mu j_\mu = 0$$

Real scalar fields: $\varphi^c = \varphi \longrightarrow$ can not carry electric charge.

The KG equation for a charged scalar is invariant under local gauge tranformations

$$\varphi \to e^{ie\alpha}\varphi, \qquad A_\mu \to A_\mu + \partial_\mu \alpha$$

Pauli spinors

The Pauli matrices σ_i, $i = (1, 2, 3)$, are defined by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

They are hermitean: $\sigma_i^\dagger = \sigma_i$, and satisfy the anti-commutation relations

$$\{\sigma_i, \sigma_j\} = \sigma_i \sigma_j + \sigma_j \sigma_i = 2\delta_{ij}\, 1_2$$

The commutator also has a simple structure:

$$[\sigma_i, \sigma_j] = \sigma_i \sigma_j - \sigma_j \sigma_i = 2i\varepsilon_{ijk}\, \sigma_k$$

Cf. orbital angular momentum:

$$L_i = -i\varepsilon_{ijk} x_j \partial_k \quad \to \quad [L_i, L_j] = i\varepsilon_{ijk} L_k$$

$\longrightarrow$ Total angular momentum of non-relativistic fermions:

$$J_i = L_i + \frac{\sigma_i}{2}, \qquad [J_i, J_j] = i\varepsilon_{ijk} J_k$$

Observe:

$$\sigma_2 \sigma_i \sigma_2 = -\sigma_i^T$$

Dirac spinors

In 4-dimensional Minkowski space we can find a set of 4 matrices γ_μ $(\mu = 0, 1, 2, 3)$, with the property

$$\{\gamma_\mu, \gamma_\nu\} = \gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = -2\eta_{\mu\nu}\, 1_4$$

Here $\eta_{\mu\nu} = \text{diag}(+1, -1, -1, -1)$ is the Minkowski metric.

An explicit representation is:

$$\gamma_0 = -\gamma^0 = \begin{pmatrix} -i1_2 & 0 \\ 0 & i1_2 \end{pmatrix}, \qquad \gamma_i = \begin{pmatrix} 0 & -i\sigma_i \\ i\sigma_i & 0 \end{pmatrix}$$

It can be conventient to define a set of hermitean gamma matrices

$$\gamma_4 = i\gamma_0 = \begin{pmatrix} 1_2 & 0 \\ 0 & -1_2 \end{pmatrix}$$

Then for $m = (1, 2, 3, 4)$ we have

$$\{\gamma_m, \gamma_n\} = 2\delta_{mn}$$

The hermiticity $\gamma_m^\dagger = \gamma_m$ is equivalent with $\gamma_\mu^\dagger = \gamma_0\gamma_\mu\gamma_0$.

Charge conjugation matrix: $C = C^\dagger = C^{-1} = -C^T$ such that

$$C^{-1}\gamma_\mu C = -\gamma_\mu^T$$

In our representation: $C = \gamma_2\gamma_0$.

Dirac equation

Free relativistic fermions are the quanta of a spinor field satisfying

$$(\partial\!\!\!/ + m)\psi = 0, \qquad \partial\!\!\!/ = \gamma^\mu \partial_\mu$$

Hermitean conjugation:

$$\bar{\psi}(-\overleftarrow{\partial\!\!\!/} + m) = 0$$

with $\bar{\psi} = \psi^\dagger \gamma_0$.

Charge conjugation: $\psi^c = C\bar{\psi}^T \longrightarrow$

$$(\partial\!\!\!/ + m)\psi^c = 0$$

Majorana spinors: $\psi = \psi^c$.

Every Dirac spinor can be decomposed into two Majorana spinors:

$$\psi = \varphi_1 + i\varphi_2,$$

$$\varphi_1 = \varphi_1^c = \tfrac{1}{2}\left(\psi + \psi^c\right), \qquad \varphi_2 = \varphi_2^c = \tfrac{1}{2i}\left(\psi - \psi^c\right)$$

Charged fermions

Dirac equation for spinors coupled to the e.m. field

$$(\not{\partial} - ie\not{A} + m)\psi = 0$$

Then the charge-conjugate spinor satisfies

$$(\not{\partial} + ie\not{A} + m)\psi^c = 0$$

Conserved electro-magnetic current:

$$j_\mu = -ie\bar{\psi}\gamma_\mu\psi, \qquad \partial^\mu j_\mu = 0$$

$\longrightarrow$ Majorana fermions can not carry electric charge.

Gauge invariance:
the coupled Dirac equation is invariant under the combined transformations

$$\psi \to e^{ie\alpha}\psi, \qquad A_\mu \to \partial_\mu\alpha$$

Majorana spinors can not be transformed in this way, because

$$\psi^c \to e^{-ie\alpha}\psi^c$$

Observe the analogy:

complex scalar field $\leftrightarrow$ Dirac spinor
real scalar field $\leftrightarrow$ Majorana spinor

Chiral spinors

Define

$$\gamma_5 = \frac{i}{4!}\,\varepsilon^{\kappa\lambda\mu\nu}\,\gamma_\kappa\gamma_\lambda\gamma_\mu\gamma_\nu = \frac{1}{4!}\,\varepsilon^{klmn}\gamma_k\gamma_l\gamma_m\gamma_n = \gamma_1\gamma_2\gamma_3\gamma_4$$

It has the following properties

$$\gamma_5 = \gamma_5^\dagger, \qquad \gamma_5^2 = 1_4, \qquad \gamma_5\gamma_\mu + \gamma_\mu\gamma_5 = 0$$

In our standard representation:

$$\gamma_5 = \begin{pmatrix} 0 & -1_2 \\ -1_2 & 0 \end{pmatrix}$$

It can be diagonalized by a unitary transformation to the form

$$\tilde{\gamma}_5 = \begin{pmatrix} 1_2 & 0 \\ 0 & -1_2 \end{pmatrix}$$

The eigenstates of γ_5 are called chiral or Weyl spinors; notation:

$$\gamma_5\psi_L = \psi_L, \qquad \gamma_5\psi_R = -\psi_R$$

A Dirac spinor can be decomposed into two chiral spinors:

$$\psi = \psi_L + \psi_R, \qquad \psi_{L,R} = \frac{1}{2}\,(1 \pm \gamma_5)\psi$$

Weyl fermions

Dirac equation in terms of free chiral spinors:

$$-\partial\!\!\!/ \psi_L = m\psi_R, \qquad -\partial\!\!\!/ \psi_R = m\psi_L$$

A Weyl particle is a fermion described by only a left-handed spinor. Weyl particles must be massless:

$$\psi_R = 0 \quad \rightarrow \quad \partial\!\!\!/ \psi_L = 0, \qquad m = 0$$

The charge conjugate of a left-handed spinor is right-handed and vice versa:

$$\gamma_5 (\psi_L)^c = -(\psi_L)^c, \qquad \gamma_5 (\psi_R)^c = (\psi_R)^c$$

$\longrightarrow$ a chiral spinor can not be a Majorana spinor.

For a Majorana spinor we have

$$\psi_L = (\psi_R)^c = C(\overline{\psi_R})^T, \qquad \psi_R = (\psi_L)^c = C(\overline{\psi_L})^T$$

Action principles

The KG equation for a real scalar field can be derived from an action

$$S_b = \int d^4x \, \frac{1}{2} \left(\partial_\mu \varphi \, \partial^\mu \varphi - m^2 \varphi^2 \right)$$

Modulo boundary terms, its variation is

$$\delta S_b = - \int d^4x \, \delta\varphi (\Box + m^2) \varphi = 0 \quad \Leftrightarrow \quad (\Box + m^2)\varphi = 0$$

The Dirac equation for a Majorana spinor can be derived from an action

$$S_f = \frac{i}{2} \int d^4x \, \bar{\psi} (\not{\partial} + m) \psi$$

with the variation

$$\delta S_f = i \int d^4x \, \delta\bar{\psi} (\not{\partial} + m)\psi = 0 \quad \Leftrightarrow \quad (\not{\partial} + m)\psi = 0$$

The action exists only if the components of ψ anti-commute among themselves:

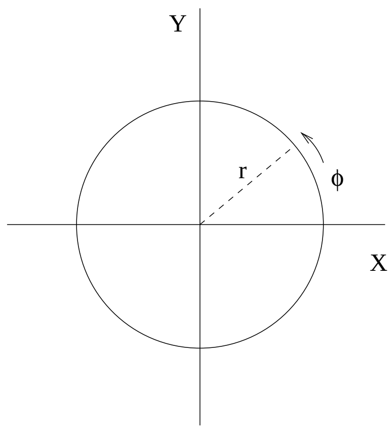
$$\psi_a \psi_b = -\psi_b \psi_a$$

Observe: $\bar{\psi}\psi = \bar{\psi} C \bar{\psi}^T$, and $C = -C^T$.

Rotations

$$x = r\cos\varphi, \qquad y = r\sin\varphi$$

$$L_z = -i\partial_\varphi = -i(x\partial_y - y\partial_x)$$



$\longrightarrow$ Infinitesimal rotation in x-y-plane:

$$\begin{aligned}\psi[\varphi+\delta\varphi] &= \psi[\varphi] + \delta\varphi\,\partial_\varphi\psi[\varphi] + \ldots. \\ &= \psi[\varphi] + i\delta\varphi\,L_z\psi[\varphi] + \ldots\end{aligned}$$

Similarly, L_x and L_y act to generate rotations in the y-z- and z-x-planes, respectively.

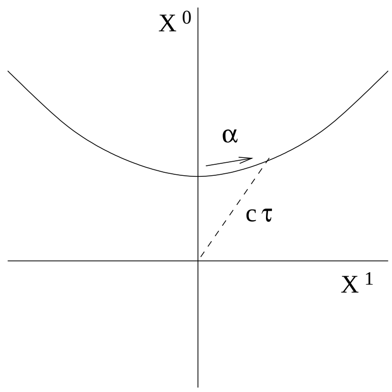
$$[L_i, L_j] = i\varepsilon_{ijk}L_k$$

$\longleftrightarrow$ rotations in different planes do not commute.
But: the difference is again a rotation.

Lorentz transformations

$$x^0 = c\tau\cosh\alpha = \frac{c\tau}{\sqrt{1-v^2/c^2}}, \qquad x^1 = c\tau\sinh\alpha = \frac{v\tau}{\sqrt{1-v^2/c^2}}$$

$$K_x = -i\partial_\alpha = i(x_0\partial_1 - x_1\partial_0)$$



$\longrightarrow$ Infinitesimal boost in x-t-plane:

$$\begin{aligned} \psi[\alpha+\delta\alpha] &= \psi[\alpha] + \delta\alpha\,\partial_\alpha\psi[\alpha] + \ldots. \\ &= \psi[\alpha] + i\delta\alpha\,K_x\psi[\alpha] + \ldots \end{aligned}$$

Similarly, K_y and K_z act to generate boosts in the y-t- and z-t-planes, respectively.

$$[K_i, K_j] = -i\varepsilon_{ijk}L_k$$

$\longleftrightarrow$ boosts in different planes do not commute.
And:
the difference is a rotation $\longrightarrow$ rotations and Lorentz boosts mix.

The algebra of Lorentztransformations

Covariant generators of Lorentz transformations (rotations and boosts):

$$M_{\mu\nu} = -i\,(x_\mu \partial_\nu - x_\nu \partial_\mu)$$

with the commutation relations

$$[M_{\mu\nu}, M_{\kappa\lambda}] = i\,(\eta_{\nu\lambda} M_{\mu\kappa} - \eta_{\nu\kappa} M_{\mu\lambda} - \eta_{\mu\lambda} M_{\nu\kappa} + \eta_{\mu\kappa} M_{\nu\lambda})$$

Fermions:

Lorentz transformations not only effect the co-ordinates, but also mix the components of a spinor; let

$$\sigma_{\mu\nu} = -\frac{i}{4}\,[\gamma_\mu, \gamma_\nu]$$

then

$$M_{\mu\nu} = -i\,(x_\mu \partial_\nu - x_\nu \partial_\mu) + \sigma_{\mu\nu}$$

satisfies the same commutation relations; the first term generates the orbital part of the Lorentz transformation, the last term the spin part.

Translations and momentum:

$$P_\mu = -i\partial_\mu \quad \rightarrow \quad \psi(x + \delta x) = \psi(x) + i\delta x^\mu\, P_\mu \psi(x) + \ldots$$

They obey non-trivial commutation relations

$$[M_{\mu\nu}, P_\lambda] = i\,(\eta_{\mu\lambda} P_\nu - \eta_{\nu\lambda} P_\mu)$$

Internal symmetries

Symmetries exist when the equations governing a set of fields can be transformed into each other without changing the parameters (masses and couplings).

Example:
Consider two complex scalar fields φ_a, $a = (1, 2)$ satisfying the equations

$$(\Box + m^2)\varphi_a = 0$$

As the two fields have the same mass, the equations remain satisfied under a transformation

$$\varphi_a \longrightarrow \tilde{\varphi}_a = U_a{}^b \varphi_b,$$

where U is a unitary 2×2 matrix:

$$U^{-1} = U^\dagger$$

The unitarity guarantees, that $\varphi^\dagger\varphi$ is invariant:

$$\tilde{\varphi}^\dagger\tilde{\varphi} = \varphi^\dagger U^\dagger U \varphi = \varphi^\dagger\varphi$$

Thus the transformation is still a symmetry in the presence of interactions of the form:

$$(\Box + m^2)\varphi + \lambda(\varphi^\dagger\varphi)\varphi = 0$$

Groups

The set of all unitary 2×2 transformations forms a group:

1. The product of two matrices is again a unitary 2×2 matrix
2. The unit matrix is one of the elements of the group
3. The inverse of a unitary matrix is again a unitary matrix; in fact $U^{-1} = U^\dagger$

This group is called $U(2)$.

The subset of all unitary matrices with unit determinant

$$\det U = 1$$

is a special subset: it is a group by itself.
This group is called $SU(2)$.

An infinitesimal $SU(2)$ tranformation is of the form

$$U \approx 1 + \frac{i}{2}\alpha_i\sigma_i + \ldots$$

Proof:

$$U^\dagger = 1 - \frac{i}{2}\alpha_i\sigma_i + \ldots = U^{-1}$$

For a doublet of fields this implies:

$$\tilde{\varphi} = \varphi + \delta\varphi$$

$$\delta\varphi = \frac{i}{2}\alpha_i\sigma_i\,\varphi = \frac{i}{2}\begin{pmatrix} \alpha_3 & \alpha_1 - i\alpha_2 \\ \alpha_1 + i\alpha_2 & -\alpha_3 \end{pmatrix}\begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}$$

The matrices $T_i = \sigma_i/2$ are called the generators of the group $SU(2)$ and obey the commutation relations (Lie algebra)

$$[T_i, T_j] = i\varepsilon_{ijk}T_k$$

Because of the similarity with spin, $SU(2)$ as an internal symmetry group is often called isospin.

The fields φ_a are eigenvectors of T_3, the third component of isospin, with eigenvalues $\pm 1/2$.
The other two generators mix the components, i.e. they change the eigenvalue of T_3. To see this, define

$$T_+ = (T_1 + iT_2) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$T_- = (T_1 - iT_2) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

T_+ changes an isospin $-1/2$ state into isospin $+1/2$,
T_- changes an isospin $+1/2$ state into isospin $-1/2$.

Broken symmetries

If the field equations are symmetric except for one parameter, this parameter is said to break the symmetry; for example:

$$(\Box + m_1^2)\,\varphi_1 + \lambda(\varphi^\dagger\varphi)\,\varphi_1 = 0$$

$$(\Box + m_2^2)\,\varphi_2 + \lambda(\varphi^\dagger\varphi)\,\varphi_2 = 0$$

There is an exact $U(2)$ symmetry if $m_1 = m_2$; but the symmetry is broken by the masses if $m_1 \neq m_2$.

Spontaneous symmetry breaking
Let $m_1^2 = m_2^2 \;\rightarrow\; -\mu^2 < 0$; then there exist constant solutions

$$\varphi^\dagger\varphi = \frac{\mu^2}{\lambda}$$

At least one of the components gets a vacuum expectation value; we can always make a $U(2)$-transformation such that this component is φ_2. Now write

$$\varphi_1 = \varphi, \qquad \varphi_2 = \frac{\mu}{\sqrt{\lambda}} + \chi_1 + i\chi_2$$

The complex field φ and the real field χ_2 then behave like interacting massless fields (Goldstone fields):

$$\Box\,\varphi = 0 + (\text{interaction terms})$$

$$\Box\,\chi_2 = 0 + (\text{interaction terms})$$

but χ_1 behaves like a massive field (Higgs field):

$$(\Box + 2\mu^2)\,\chi_1 = 0 + (\text{interaction terms})$$

$\longrightarrow$ mass $m_H^2 = 2\mu^2$

Supersymmetry

Supersymmetry extends the group of Lorentz transformations and translations with transformations between bosons and fermions $\longrightarrow$ supersymmetry tranformations must change the spin of a field/particle state.

The generators of infinitesimal supersymmetry transformations form a Majorana spinor Q_a s.t. $Q^c = C\bar{Q}^T = Q$. They satisfy (anti-)commutation relations

$$\{Q_a, \bar{Q}_b\} = Q_a\bar{Q}_b + \bar{Q}_b Q_a = \tfrac{1}{2}\gamma^{\mu}_{ab} P_{\mu},$$

$$[M_{\mu\nu}, Q] = \sigma_{\mu\nu} Q, \qquad [P_{\mu}, Q] = 0.$$

Consider a state Φ which is an eigenstate of momentum for a particle at rest: $P_{\mu} = (m, 0, 0, 0)$; acting on such a state the supersymmetry relations give

$$\{Q, \bar{Q}\} = \frac{m}{2}\gamma^0 = -\frac{im}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

In terms of two-component spinors we can write

$$Q = \begin{pmatrix} u \\ v \end{pmatrix} \quad \rightarrow \quad \bar{Q} = Q^{\dagger}\gamma_0 = (-iu^{\dagger}, iv^{\dagger})$$

Then

$$Q = Q^c \quad \Leftrightarrow \quad v = -i\sigma_2 u^*$$

It follows, that

$$\left\{u, u^\dagger\right\} = \left\{v, v^\dagger\right\} = \frac{m}{2}, \qquad \left\{u, v^\dagger\right\} = \left\{v, u^\dagger\right\} = 0$$

The last relation is equivalent with

$$\{u, u\} = \{v, v\} = 0$$

The components of $(u, u^\dagger)$behave like fermion creation/annihilation operators. Indeed, they add/remove $1/2$ unit of spin:

$$J_3 = M_{12} + \sigma_{12} \quad \text{where} \quad \sigma_{12} = \frac{1}{2}\begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}$$

and hence

$$[J_3, Q] = \sigma_{12}\, Q \quad \Leftrightarrow \quad [J_3, u] = \frac{1}{2}\,\sigma_3 u$$

If the state Φ has spin $J_3\Phi = j_3\Phi$, then the states $\Psi_i = u_i\Phi$ obtained by acting with a supersymmetry operator on Φ have spin

$$\begin{aligned} J_3\Psi &= J_3 u\Phi = [J_3, u]\,\Phi + uJ_3\Phi \\ &= \left(j_3 + \frac{1}{2}\,\sigma_3\right) u\Phi = \left(j_3 + \frac{1}{2}\,\sigma_3\right)\Psi \end{aligned}$$

Therefore Ψ_i correspond to states with spin $j_3 \pm 1/2$.

In addition there is a state $\Phi_{+-} = u_+ u_- \Phi = u_+ \Psi_-$ which has spin

$$
\begin{aligned}
J_3 \Phi_{+-} &= [J_3, u_+]\, u_- \Phi + u_+ [J_3, u_-]\, \Phi + u_+ u_- J_3 \Phi \\
&= \left(j_3 + \frac{1}{2} - \frac{1}{2} \right) \Phi_{+-} = j_3\, \Phi_{+-}
\end{aligned}
$$

No other states can be constructed $\longrightarrow$ multiplets of states connected by supersymmetry transformations consist of states with 3rd component of spin

$$
J_3 \longrightarrow \left(j_3 - \frac{1}{2}, j_3, j_3, j_3 + \frac{1}{2} \right)
$$

Important cases are:

- $j_3 = 0$ (massive scalar multiplet)
- $j_3 = 1/2$ (massive vector multiplet)

For massless particles the multiplets are shorter, consisting only of half the number of states:

- $j_3 = (0, 1/2)$ (chiral multiplet)
- $j_3 = (1/2, 1)$ (massless vector multiplet)
- $j_3 = (3/2, 2)$ (supergravity multiplet)