Particles and Fields

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1. **Avogadro's law**

   The pressure and temperature of ideal gasses are related by

$$\frac{pV}{T} = Rn, \tag{1}$$

   where n is the number of moles of particles, and R a *universal* constant. It follows that at equal pressure and temperature a fixed volume of gas always contains the same number of particles.

   The number of particles per mole is

$$N_A = 6.022 \times 10^{23}, \tag{2}$$

   Avogadro's number; n moles of gas then contain $N = nN_A$ particles. Hence we can write the law of ideal gasses as

$$\frac{pV}{T} = kN, \tag{3}$$

   where $k = R/N_A$ is Boltzmann's constant.

2. **Atomic masses**

   By definition the atomic mass unit (a.m.u.) is one 12th of the mass of a C^{12} atom. As the mass of one mole of C^{12} atoms is 12 g = 0.012 kg, we have

$$1\,\text{a.m.u.} = \frac{0.001\,\text{kg}}{N_A} = 1.661 \times 10^{-27}\,\text{kg} \tag{4}$$

   To this precision the mass of a hydrogen atom is

$$m_H = 1.008\,\text{a.m.u.} = 1.674 \times 10^{-27}\,\text{kg}. \tag{5}$$

   From a measurement of e/m_e of the electron, and an independent determination of the charge e it follows, that

$$e = 1.602 \times 10^{-19}\,\text{C}, \qquad m_e = 0.911 \times 10^{-30}\,\text{kg}. \tag{6}$$

   The mass of the hydrogen nucleus (the proton), then is

$$m_p = 1.673 \times 10^{-27}\,\text{kg} = 1836\,m_e. \tag{7}$$

   In relativistic units

$$m_p\,c^2 = 0.938\,\text{GeV}, \qquad m_e\,c^2 = 0.511\,\text{MeV}. \tag{8}$$

   The traditional way to determine N_A is by independent measurements of the gasconstant R (gasses, osmotic solutions), and Boltzmann's constant k (spectral distribution of black body radiation (Planck), or diffusion phenomena (Einstein)).

3. **The Bohr atom**

An estimate of the size of atoms can be obtained from the theory of the hydrogen atom. Actually it does not require more than the Bohr's old version of the quantum theory, based on the correspondence principle between classical and quantum mechanics. A more rigorous approach would start from the Schrödinger equation for an electron in a Coulomb potential; for our purpose this leads to the same results.

The spectral lines of hydrogen form series, such as the Balmer series, the Paschen series, and several others; from experiments it was found that the frequencies and wavelengths in vacuum of these series satisfy the simple formula

$$\nu_n = \frac{c}{\lambda_n} = cR_H \left(\frac{1}{n^2} - \frac{1}{m^2} \right), \tag{9}$$

where (n, m) are integers: $n = 1, 2, 3, ...$ and $m = n+1, n+2, ...$; R_H is the Rydberg constant:

$$R_H = 1.097 \times 10^{-2}\,\mathrm{nm}^{-1}. \tag{10}$$

Bohr constructed a relation between these frequencies and the energy absorbed or emitted by an electron changing its orbit. In his model the electron energy in the n-th orbit is

$$E_n = -\frac{hcR_H}{n^2}, \qquad n = 1, 2, ..., \tag{11}$$

where the zero point of energy has been set to correspond to the ionization energy: $E_\infty = 0$. This rule was motivated by the quantum hypothesis of Planck and Einstein, which implies that the frequency of the photon emitted during a transition between adjacent orbits is

$$h\nu_n = \Delta E_n = E_{n+1} - E_n; \tag{12}$$

with Bohr's assumption (11) this equation returns the frequencies eq. (9) with $m = n + 1$.

Now consider an electron in a Coulomb potential. Classically, the electron could be in any Kepler-like orbit, but we consider only circular orbits here. The potential energy in such an orbit is

$$V_n = -\frac{e^2}{4\pi\varepsilon_0 r_n} = -\frac{\alpha\hbar c}{r_n}, \tag{13}$$

with α a dimensionless number, the *fine-structure constant*:

$$\alpha = \frac{e^2}{4\pi\varepsilon_0\hbar c} \approx \frac{1}{137}. \tag{14}$$

According to classical mechanics, the radial acceleration in this orbit is

$$\frac{m_e v_n^2}{r_n} = \frac{e^2}{4\pi\varepsilon_o r_n^2} \quad \Rightarrow \quad \frac{v_n^2}{c^2} = \frac{\alpha\hbar}{m_e c}\frac{1}{r_n}. \tag{15}$$

It follows, that the classical kinetic energy is

$$T_n = \frac{1}{2} m_e v_n^2 = \frac{\alpha \hbar c}{2r_n}, \tag{16}$$

and the total energy is given by

$$E_n = T_n + V_n = -\frac{\alpha \hbar c}{2r_n}. \tag{17}$$

Again, the zero point of energy here is taken to correspond to the fully ionized state $r_n = \infty$. By comparison with the expression (11) we then find for the radius r_n

$$r_n = a_0 \, n^2, \tag{18}$$

with a_0 the Bohr radius:

$$a_0 = r_1 = \frac{\alpha}{4\pi R_H} \approx 0.053 \, \mathrm{nm}. \tag{19}$$

Of course, according to quantum mechanics one can not speak about the precise position of the electron, and the above value of r_n represents only a statistical average. But the energy levels (11) are still correct in quantum mechanics; moreover, in the limit of large n quantum mechanics should go over into classical mechanics. As a result (19) is still a good estimate for the size of the hydrogen atom: its diameter is about 0.1 nm.

One can even use the Bohr model to derive the value of the Rydberg constant. In the classical limit $n \to \infty$ we can equate the spectral frequency ν_n:

$$\nu_n = \frac{\Delta E_n}{h} \approx \frac{2cR_H}{n^3}, \tag{20}$$

with the orbital frequency:

$$\nu_n = \frac{v_n}{2\pi r_n} = \frac{2c}{\alpha n^3} \sqrt{\frac{2h}{m_e c}} \, R_H^{3/2}. \tag{21}$$

It then follows that

$$R_H = \frac{\alpha^2 m_e c}{2h} \approx 1.097 \times 10^{-2} \, \mathrm{nm}^{-1}, \tag{22}$$

in agreement with the experimental value (10).

4. **Thomson scattering**

When a charged particle is hit by electromagnetic radiation, it oscillates under the influence of the wave fields, and as a result it will itself emit radiation. In this way electromagnetic energy and momentum are taken out of the original radiation field and scattered into different directions; see fig. 1.

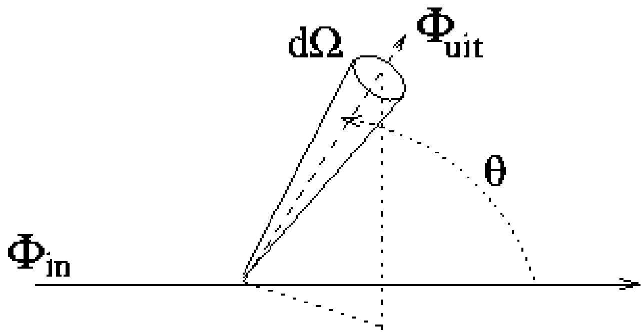

Figure 1: Thomson scattering

Thomson was the first to calculate the amount of radiation scattered as a function of the incident flux, and of the charge and mass of the particle. This quantity is expressed in terms of a *scattering cross section*, defined as follows. Let the incident radiation have an intensity (energy per unit of area per unit of time) Φ_{in}. If the intensity of the radiation scattered in the direction (θ, φ) is $\Phi_{out}(\theta, \varphi)$, then the total amount of energy scattered through a spherical surface element with area dA at distance R is $\Phi_{out}(\theta, \varphi)\, dA$. This amount must be proportional to the intensity of the incident radiation; therefore

$$\Phi_{out}(\theta, \varphi) dA = \Phi_{in}\, d\sigma(\theta, \varphi). \tag{23}$$

The quantity $d\sigma(\theta, \varphi)$ determines which fraction of the incident flux contributes to the scattered flux in the direction (θ, φ). It has the dimensions of an area. and can be interpreted as the size of the area in the total incident beam which accounts for the radiation flux scattered through dA.

Now consider a cone of rays with angles in the range $(\theta, \theta + \theta)$ and $(\varphi, \varphi + d\varphi)$. This cone cuts a spherical area out of the unit sphere of size

$$d\Omega = \sin\theta\, d\theta d\varphi. \tag{24}$$

At distance R the size of the spherical area element enclosed by the cone then is

$$dA = R^2 \sin\theta\, d\theta d\varphi = R^2 d\Omega. \tag{25}$$

Combination of eqs. (9) and (10) gives

$$\frac{d\sigma}{d\Omega} = \frac{R^2 \Phi_{out}(\theta, \varphi)}{\Phi_{in}}. \tag{26}$$

This quantity, which can be obtained directly from flux measurements, is called the *differential scattering cross section*. For the scattering of radiation by free charges Thomson found

$$\frac{d\sigma}{d\Omega} = \frac{r_e^2}{2} \left(1 + \cos^2\theta\right), \tag{27}$$

where r_e is the classical electron radius, defined by:

$$r_e = \frac{e^2}{4\pi\varepsilon_0\, m_e c^2} = 2.817\, \times 10^{-15}\,\mathrm{m}. \tag{28}$$

The *total* amount of radiation scattered per free charge and per unit of incident flux then is

$$\begin{aligned} \sigma_T &= \int \frac{d\sigma}{d\Omega}\, d\Omega = \frac{r_e^2}{2} \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta\, \sin\theta\, \left(1 + \cos^2\theta\right) \\ &= \frac{8\pi}{3}\, r_e^2 = 0.665 \times 10^{-28}\,\mathrm{m}^2. \end{aligned} \tag{29}$$

The characteristic value of this scattering cross section is often used in subatomic physics as a reference value; this unit of cross section is called a *barn*:

$$1\,\mathrm{barn} = 10^{-28}\,\mathrm{m}^2. \tag{30}$$

Eqs. (27) and (28) show, that the cross section is inversely proportional to the square of the particle mass. Therefore the cross section of the electron is much larger than that of the proton. Hence almost all radiation scattered by an atom is due to scattering by electrons. If the photon energy is much larger than the the electron binding energy, we can treat the electrons as free particles and use the Thomson formula for the scattering cross section.

Now consider a gas with a density of N atoms per unit of volume, and n electrons per atom. Then there are Nn electrons per unit of volume. If a light beam with intensity Φ runs through the gas, the amount of flux scattered out of the beam Per element of length dl is

$$d\Phi = -Nn\,\Phi\,\sigma_T dl. \tag{31}$$

Thomson compared this formula with the results of X-ray scattering in gasses, as measured by Barkla, and concluded that the number of electrons in a neutral atom is of the same order of magnitude as the atomic number of the element. In particular he concluded that the hydrogen atom contains only a single electron.

5. **Particle scattering**

Scattering of radiation and particles is a general technique for studying the structure and properties of matter at various levels. Rutherford, Geiger and Marsden were the first to systematically apply the scattering of charged particles to the study of atomic structure.

As in Thomson scattering, the crucial concept in all such experiments is that of the differential scattering cross section. It is defined as follows. For a homogeneous beam of incident particles, let the number of incident particles per unit of area and per unit of time be I_{in}; the number of particles scattered into the direction (θ, φ) per unit of area and time is denoted by $I_{out}(\theta, \varphi)$. Then the number of particles leaving per unit

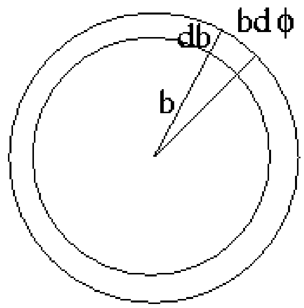

Figure 2: Impact parameter and differential cross section

of time through the spherical area element $dA = R^2 d\Omega$ (at a distance R from the scattering center) is proportional to the incident particle flux:

$$I_{out}(\theta, \varphi)\, R^2 d\Omega = I_{in}\, d\sigma(\theta, \varphi), \tag{32}$$

where $d\sigma(\theta, \varphi)$ is the effective area of the incident beam through which as many particles pass as are scattered into the cone with spherical opening angle $d\Omega$. The differential scatering cross section then is defined empirically by

$$\frac{d\sigma}{d\Omega} = \frac{R^2 I_{out}(\theta, \varphi)}{I_{in}}. \tag{33}$$

When integrated over a full solid angle 4π one obtains the total scattering cross section:

$$\sigma = \int d\Omega\, \frac{d\sigma}{d\Omega} = \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta\, \sin\theta\, \frac{d\sigma}{d\Omega}. \tag{34}$$

Fig. 2 provides a view of the scattering geometry in the direction of the incident beam. All particles entering at a distance in the range $(b, b + db)$ from the central axis are found in a ring of width db; for these particles b is the impact parameter. Let the section of the ring within the wedge with angular range $(\varphi, \varphi + d\varphi)$ represent the area $d\sigma(\theta, \varphi)$ accounting for all particles scattered into a cone with opening angle $d\Omega$ in the direction (θ, φ). Then by definition

$$d\sigma(\theta, \varphi) = |b\, db\, d\varphi| = \left| \frac{b}{\sin\theta} \frac{db}{d\theta} \right| \sin\theta\, d\theta\, d\varphi; \tag{35}$$

equivalently,

$$\frac{d\sigma}{d\Omega} = \left| \frac{b}{\sin\theta} \frac{db}{d\theta} \right|. \tag{36}$$

To relate this quantity to a theoretical model, one needs to specify the relation between impact parameter and scattering angles $b(\theta, \varphi)$.

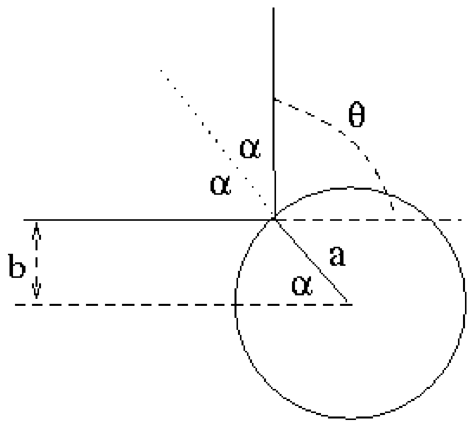

Figure 3: Scattering by a hard sphere.

a. Scattering by hard spheres

As an example of this description of scattering, we consider elastic scattering of point masses by a hard sphere of radius a, fig. 3. In this case, as in many others, the scattering kinematics is the same for all the polar angles φ, so $b(\theta)$ is a function of θ only. We calculate this quantity as follows. Let α be the angle between the direction of motion of the incident particle and the normal to the sphere at the point of impact; as we consider elastic scattering, the angles of impact and reflection w.r.t. the normal are equal, and therefore the scattering angle $\theta = \pi - 2\alpha$. As shown in fig. 3 the impact parameter then is

$$b = a \sin\alpha = a \sin\left(\frac{\pi - \theta}{2}\right) = a \cos\frac{\theta}{2}. \tag{37}$$

As a result

$$\frac{db}{d\theta} = -\frac{a}{2} \cos\frac{\theta}{2}, \tag{38}$$

and by substitution into eq. (36) we obtain for the differential and total cross section the expressions

$$\frac{d\sigma}{d\Omega} = \frac{a^2}{4}, \qquad \sigma = \pi a^2. \tag{39}$$

Thus, for elastic scattering by a hard sphere the differential cross section is constant (independent of the angles), and the total cross section equals the geometric cross section.

b. Rutherford scattering

The scattering of classical point charges via the Coulomb interaction was first worked out theoretically by Rutherford, with the purpose to understand the scattering of α-particles from massive atomic charges. The starting point is the Coulomb interaction of two particles with masses (m_1, m_2) and charges (q_1, q_2); using Newtons third law of motion, the force acting on the particles is

$$\mathbf{F} = -m_1 \ddot{\mathbf{r}}_1 = m_2 \ddot{\mathbf{r}}_2 = \frac{\kappa}{r^2}\,\hat{\mathbf{r}}, \qquad \kappa = \frac{q_1 q_2}{4\pi\epsilon_0}, \tag{40}$$

where

$$\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1, \tag{41}$$

and $\hat{\mathbf{r}}$ is the unit vector in the direction of $\mathbf{r}$. It is therefore useful to work in the CM frame, defined by

$$M\mathbf{R} = m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2, \qquad M = m_1 + m_2. \tag{42}$$

The center of mass moves uniformly:

$$\ddot{\mathbf{R}} = 0 \quad \Rightarrow \quad \dot{\mathbf{R}} = \text{constant}. \tag{43}$$

The special CM frame is the one in which $\mathbf{R} = 0$.

The relative acceleration is that of a single particle of reduced mass μ:

$$\mu \ddot{\mathbf{r}} = \frac{\kappa}{r^2}\,\hat{\mathbf{r}}, \qquad \mu = \frac{m_1 m_2}{M}. \tag{44}$$

The solution of this equation is a standard problem of classical mechanics. We use the conservation laws for energy and angular momentum in terms of the polar co-ordinates in the CM frame:

$$x = r\sin\theta\cos\varphi, \quad y = r\sin\theta\sin\varphi, \quad z = r\cos\theta. \tag{45}$$

The energy in the center of mass system is

$$\varepsilon = \frac{\mu}{2}\,\dot{\mathbf{r}}^2 + \frac{\kappa}{r} = \frac{\mu}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\varphi}^2\right) + \frac{\kappa}{r}, \tag{46}$$

whilst the angular momentum is

$$\mathbf{l} = \mu \mathbf{r} \times \dot{\mathbf{r}} \tag{47}$$

Conservation of angular momentum implies that both the size and direction of $\mathbf{l}$ are constant. Therefore the motion takes place in the plane perpendicular to $\mathbf{l}$; we choose our co-ordinates such that this plane is the x-y-plane. Then

$$\theta = \frac{\pi}{2}, \qquad \dot{\theta} = 0. \tag{48}$$

In that frame the angular momentum is

$$\mathbf{l} = (0, 0, l), \qquad l = \mu(x\dot{y} - y\dot{x}) = \mu r^2 \dot{\varphi}, \tag{49}$$

whilst the energy is

$$\varepsilon = \frac{\mu}{2}\,\dot{r}^2 + \frac{l^2}{2\mu r^2} + \frac{\kappa}{r}. \tag{50}$$

It follows directly, that

$$\dot{r}^2 = \frac{2\varepsilon}{\mu} - \frac{2\kappa}{\mu r} - \frac{l^2}{\mu^2 r^2}. \tag{51}$$

From these two equations we can determine the shape of the orbit, by taking

$$\left(\frac{dr}{d\varphi}\right)^2 = \left(\frac{\dot{r}}{\dot{\varphi}}\right)^2 = \frac{2\mu\varepsilon}{l^2}\,r^4 - \frac{2\mu\kappa}{l^2}\,r^3 - r^2. \tag{52}$$

Now it is convenient to switch to a new variable

$$s = \frac{1}{r} + \frac{\mu\kappa}{l^2}. \tag{53}$$

Then eq. (52) reduces to

$$\left(\frac{ds}{d\varphi}\right)^2 = \lambda^2 - s^2, \qquad \lambda^2 = \frac{\mu^2\kappa^2}{l^4}\left(1 + \frac{2\varepsilon l^2}{\mu\kappa^2}\right). \tag{54}$$

It follows, that

$$s = \lambda \cos(\varphi - \varphi_0), \tag{55}$$

and for $\kappa > 0$ (charges of equal sign) the solution for r is

$$r(\varphi) = \frac{l^2/\mu\kappa}{e\cos(\varphi - \varphi_0) - 1}, \qquad e = \sqrt{1 + \frac{2\varepsilon l^2}{\mu\kappa^2}} \geq 1. \tag{56}$$

This is the equation for a hyperbola with one focal point in $r = 0$. Indeed, when $\varphi_0 = 0$, the orbital equation in the x-y-plane in cartesian co-ordinates takes the form

$$\frac{(x - x_0)^2}{a^2} - \frac{y^2}{b^2} \equiv \frac{4\varepsilon^2}{\kappa^2}\left(x - \frac{e\kappa}{2\varepsilon}\right)^2 - \frac{2\varepsilon\mu y^2}{l^2} = 1. \tag{57}$$

For other values of φ_0 the shape is the same, but rotated clockwise over an angle φ_0.

The impact parameter in this case is the perpendicular distance to the line which is parellel to the asymptotic part of the particle orbit and which passes through the center of force (here repulsive), i.e. the focal point $r = 0$; see fig. 4. The angular momentum evaluated at $t \to -\infty$, $r \to \infty$ is

$$l = \mu b v_\infty, \qquad v_\infty = \lim_{r\to\infty} \sqrt{\dot{r}^2} = \sqrt{\frac{2\varepsilon}{\mu}}. \tag{58}$$

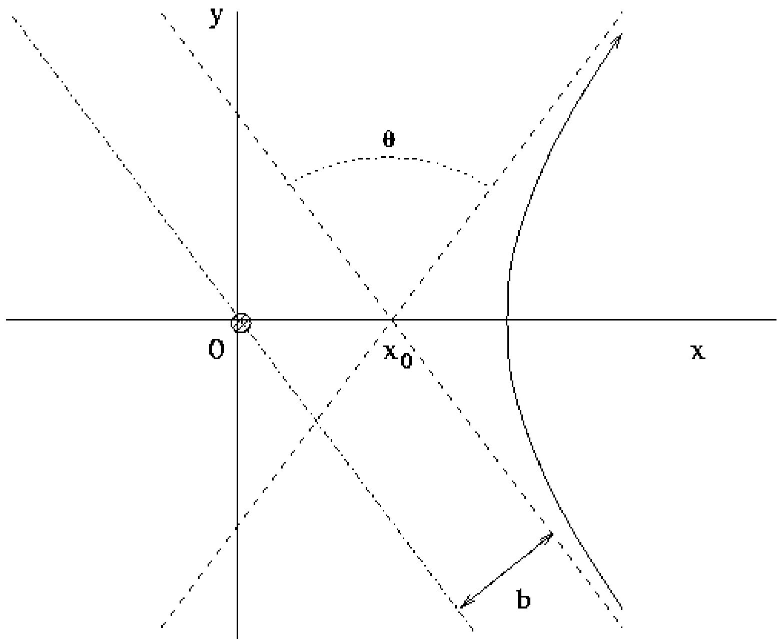

Figure 4: Hyperbolic orbit, with midpoint at $x_0 = \kappa/2e\varepsilon$.

Equivalently,

$$b = \frac{l}{\sqrt{2\mu\varepsilon}}, \tag{59}$$

which is the same value as that of b in eq. (57).

Now in both asymptotic regions of the orbit (initial and final), we have $r \to \infty$, and therefore at $t \to \pm\infty$:

$$\cos(\varphi_\infty - \varphi_0) = \cos(\varphi_{-\infty} - \varphi_0) = \frac{1}{e}, \tag{60}$$

where φ_0 is the angle between the symmetry axis of the hyperbola and the x-axis. It follows, that

$$\pi - \theta_{CM} = \varphi_\infty - \varphi_{-\infty} = 2(\varphi_\infty - \varphi_0), \tag{61}$$

and

$$\begin{aligned} \operatorname{cotan}\frac{\theta_{CM}}{2} &= \tan\left(\frac{\pi}{2} - \frac{\theta_{CM}}{2}\right) = \tan\left(\varphi_\infty - \varphi_0\right) \\ &= \sqrt{e^2 - 1} = \frac{2b\varepsilon}{\kappa}. \end{aligned} \tag{62}$$

It is now straightforward to evaluate the differential cross section

$$d^2\sigma = b\,|db|\,d\varphi = \frac{\kappa^2}{8\varepsilon^2}\,\frac{\cos\theta_{CM}/2}{\sin^3\theta_{CM}/2}\,d\theta_{CM}\,d\varphi. \tag{63}$$

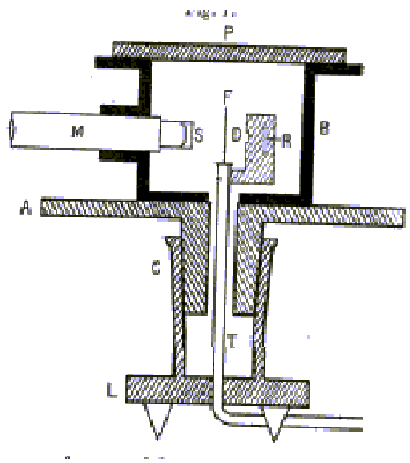

Figure 5: Sketch of Rutherford and Geiger's experiment.

Finally, using the trigonometric identity

$$2 \sin \frac{\theta}{2} \cos \frac{\theta}{2} = \sin \theta, \tag{64}$$

the expression for the differential cross section becomes

$$\left. \frac{d\sigma}{d\Omega} \right|_{CM} = \left(\frac{\kappa}{4\varepsilon} \right)^2 \frac{1}{\sin^4 \theta_{CM}/2}. \tag{65}$$

With κ given by eq. (40) this is the Rutherford differential cross section for Coulomb scattering. Observe, that in contrast to the case of scattering by hard spheres, the Rutherford cross section is not constant as a function of θ_{CM}, but decreases rapidly towards the value $(\kappa/4\varepsilon)^2$ for $\theta_{CM} \rightarrow \pi$, whilst it diverges for small angles in the forward direction. If we integrate the differential cross section over all angles θ_{CM} in a range (δ, π) and over all values of φ, we obtain

$$\sigma_{CM}(\delta) = \int_0^{2\pi} d\varphi \int_\delta^\pi d\theta \, \sin \theta \, \frac{d\sigma}{d\Omega} = \frac{\pi \kappa^2}{4\varepsilon^2} \frac{1 + \cos \delta}{1 - \cos \delta}. \tag{66}$$

From this formula it is clear, that the number of particles scattered from a beam per unit of time is finite if we require the scattering angle to be larger than some $\delta > 0$. However, if we count particles scattered over arbitrarily small angles, then even particles with a very large (infinite) impact parameter are scattered; in other words, if the beam has infinite width, an infinite number of particles up to arbitrarily large impact parameters is scattered by the Coulomb potential. For this reason the Coulomb potential is said to have an *infinite range*.

In 1909 Rutherford and his assistant Geiger, in part with help of their student Marsden, studied the structure of atoms by scattering 5 MeV α-particles on a target consisting of gold atoms; see fig. 5. In this experiment a source of α particles R is placed in a vacuum chamber with a target consisting of a gold foil F. The scattered α particles are observed by their impact on a ZnS scintillation screen S in front of the microscope M.

During the experiment they counted the numbers of α-particles scattered into different directions, and compared the results with the prediction (65) for scattering of point charges by a Coulomb interaction. From this analysis Rutherford concluded that the gold atoms contain a small massive nucleus behaving as a point charge up to distances 10,000 times smaller than the atom itself.

c. Generalizations

The Rutherford formula (65) describes elastic scattering of charged particles in the limit where the Coulomb force dominates. This is the limit of slow moving charges, where radiative effects can be neglected. Moreover, we treat the particles as point charges, with no internal structure or spin. At high energies –when relativity and radiation can not be neglected– and for particles of finite size and/or with spin the Rutherford formula is modified. In this paragraph we anticipate some of these generalizations.

First observe, that in the CM frame elastic scattering implies that the momenta of both the incoming and the outgoing particles are opposite and equal in magnitude:

$$\mathbf{p}_{1\,in} = -\mathbf{p}_{2\,in} = \mathbf{p}_{in}, \qquad \mathbf{p}_{1\,out} = -\mathbf{p}_{2\,out} = \mathbf{p}_{out}. \tag{67}$$

Moreover, energy conservation implies that the initial and final relative momentum are equal; indeed, for $r \to \infty$

$$\varepsilon = \frac{\mathbf{p}_{in}^2}{2\mu} = \frac{\mathbf{p}_{out}^2}{2\mu}. \tag{68}$$

Therefore the only effect of elastic scattering in the CM frame is the change of direction of the incoming and outgoing particles. This change in momentum of the incoming particles is the momentum transfer:

$$\mathbf{q} = \mathbf{p}_{out} - \mathbf{p}_{in}. \tag{69}$$

It follows using (68) and fig. 5, that

$$\begin{aligned} \mathbf{q}^2 &= \mathbf{p}_{in}^2 + \mathbf{p}_{out}^2 - 2\mathbf{p}_{in} \cdot \mathbf{p}_{out} \\ &= 4\mu\varepsilon\,(1 - \cos\theta_{CM}) = 8\mu\varepsilon \sin^2 \frac{\theta_{CM}}{2}. \end{aligned} \tag{70}$$

As a result we can rewrite the Rutherford scattering cross section in the equivalent form

$$\left.\frac{d\sigma}{d\Omega}\right|_{CM} = 4 Z_1^2 Z_2^2 \alpha^2 \, \frac{(\mu\hbar c)^2}{|\mathbf{q}|^4}. \tag{71}$$

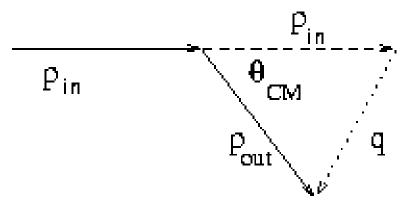

Figure 6: Momentum transfer in elastic scattering

In this form the explicit dependence on the energy ε has disappeared; moreover, the differential cross section decreases with increasing momentum transfer, as it corresponds to increasing scattering angle.

A direct extension of the Rutherford cross section for point charges is the scattering of a point charge by a finite charge distribution, e.g. an atomic nucleus of finite size. Let the total charge be Ze; then the charge distribution $Q(\mathbf{r})$ can be described by a normalized positive density function $\rho(\mathbf{r})$:

$$Q(\mathbf{r}) = Ze\rho(\mathbf{r}), \qquad \int d^3\mathbf{r}\,\rho(\mathbf{r}) = 1. \tag{72}$$

The scattering cross section then becomes a superposition of the cross section for each charge element in the distribution, with the result that in the CM frame

$$\left(\frac{d\sigma}{d\Omega}\right)_\rho = \left(\frac{d\sigma}{d\Omega}\right)_{Ruth} |F(\mathbf{q})|^2, \tag{73}$$

where the multiplicative form factor $F(\mathbf{q})$ is the Fourier transform of the distribution function ρ:

$$F(\mathbf{q}) = \int d^3\mathbf{r}\, e^{i\mathbf{q}\cdot\mathbf{r}/\hbar}\rho(\mathbf{r}), \tag{74}$$

where $\mathbf{q}$ is the momentum transfer (69). The factor $\hbar$ is included to make the exponential dimensionless. Therefore it is useful to define the wave vector $\boldsymbol{\xi}$ by $\mathbf{q} = \hbar\boldsymbol{\xi}$. In the case of a spherically symmetric distribution we then have a form factor $F(\xi)$ depending only on the wave number ξ, given by the single integral

$$F(\xi) = \int_0^\infty dr\, r^2\, \frac{\sin \xi r}{\xi r}\, \rho(r). \tag{75}$$

The cross section then takes the form

$$\left(\frac{d\sigma}{d\Omega}\right)_\rho = 4\, Z_1^2 Z_2^2\, \alpha^2\, \frac{\mu^2 c^2}{\hbar^2}\, \frac{F^2(\xi)}{\xi^4}. \tag{76}$$

A derivation of this result will be given later.

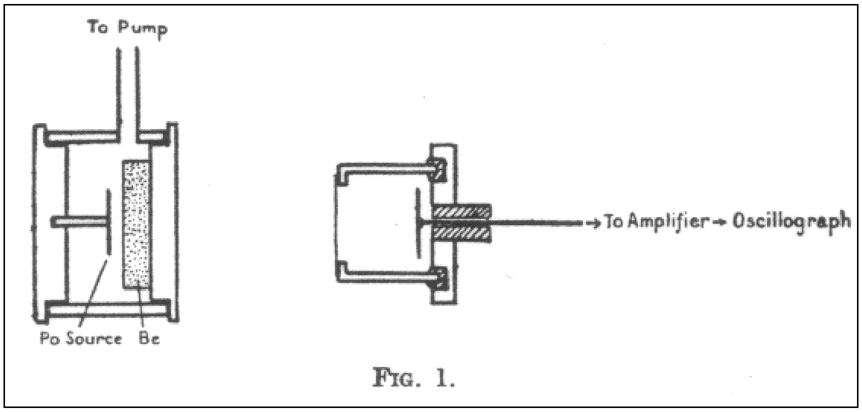

Figure 7: Chadwicks experiment used in the discovery of the neutron

6. **The neutron**

Rutherford established his atomic model in the period 1909-1911; it was completed by Niels Bohr's explanation of the spectrum of hydrogen, based on quantization of the electron orbits in the Rutherford model. Next it was infered by van den Broek[1] that the atomic number of an element equals the charge of the nucleus in elementary charges. This was also in agreement with the results from radioactive decay processes. However, the masses of atomic nuclei did not increase at the same rate as their charges; in fact, the masses increase faster than the charges in a somewhat irregular fashion. In view of the known properties of radioactive decay processes Rutherford conjectured that the nucleus consisted of a number of protons and electrons, with the total number of protons determining the mass, and the net number of protons minus electrons the charge. He proposed the existence of a special bound state of proton and electron which he termed the *neutron*. Of course, the existence of such a bound state contradicted Bohr's quantum theory, according to which the lowest energy bound state was the hydrogen atom[2], with a radius 4-5 orders of magnitude larger than the nucleus.

The puzzle was finally solved by Chadwick[3], who established the existence of neutral particle with almost the same mass as the proton being emitted by atomic nuclei. Following earlier experiments by Bothe and Becker, and by I. Joliot-Curie, he bombarded a beryllium target with α particles, which results in some form of neutral radiation ('rays'). These rays were able to knock protons out of a hydrogen-rich target, such as paraffin. As we now know the nuclear reaction involved was

$$ {}^4_2\mathrm{He} + {}^9_4\,\mathrm{Be} \rightarrow {}^{12}_6\,\mathrm{C} + n, \tag{77}$$

[1]A. van den Broek, *Nature* 87 (1911), 78
[2]Provided the binding results from Coulomb forces.
[3]J. Chadwick, *Nature* 129 (1932), 312; *Proc. Roy. Soc.* A 136 (1932), 692

but at the time it was not clear what n was; in fact the most likely explanation was that the neutral rays were γ-rays. Chadwick directed the rays at a paraffin window in front of an ionisation chamber, and measured the energy of the protons knocked out by the neutral radiation. He found that the protons carried away almost all the energy of the incident rays. This could not be explained if the protons were scattered by gamma rays (as in Compton scattering), as gamma rays would have carried away a substantial fraction of the energy themselves. The conclusion was that the neutral particles were *massive* particles scattering elastically with the protons, and that their mass was similar to that of the proton. Indeed, the mass of the neutron has been established to be

$$m_n = 1.675 \times 10^{-27}\ \text{kg} = 1839\, m_e, \tag{78}$$

and therefore

$$\frac{m_n - m_p}{m_p} = 1.4 \times 10^{-3}. \tag{79}$$

As nuclear particles of similar mass the proton and neutron are often refered to as *nucleons* (N), which can be either charged (p) or neutral (n).

7. Nuclear binding

After Chadwick's experiment the basic structure of the atom was clear: a charged nucleus consists of A nucleons: Z protons and $A - Z$ neutrons, with a total mass

$$M(A, Z)c^2 = Z\, m_p c^2 + (A - Z)\, m_n c^2 - A\Delta E_N. \tag{80}$$

Here the last term represents the binding energy, expressed in terms of an average binding energy per nucleon ΔE_N. At the time the nature of the binding force was unclear. It is not an electromagnetic force, as is clear form the existence of the deuteron, a stable bound state of a single proton and a single neutron. Also, at nuclear distances the binding force must be stronger than the Coulomb repulsion of the protons in nuclei with charge $Z \geq 2$.

In neutral atoms, the nucleus is surrounded by Z electrons, balancing the charge of the nucleus and bound to it by electric forces. The stability of the complete structure can only be understood in terms of quantum theory, as first proposed by Bohr. The applies to the stability of the nucleus as well.

The average binding energy per nucleon ΔE_N is different in different nuclei. In fig. 7 this quantity is plotted as a function of the total number of nucleons A. As a general trend it increases fast with A for light nuclei, although the exceptionally large binding energy of ${}^4_2\text{He}$ (α-particles) stands out. The maximum among the main isotopes is reached at ${}^{56}_{26}\text{Fe}$; after this element the average binding energy per nucleon decreases again. The absolute maximal value of ΔE_N is reached for the rare isotope ${}^{62}_{28}\text{Ni}$. For larger A values the nucleonic binding energy decreases again.

It follows from these figures, that energy can be released either by fusion of light nuclei, or by fission of heavy ones. Controled fission takes place in nuclear reactors

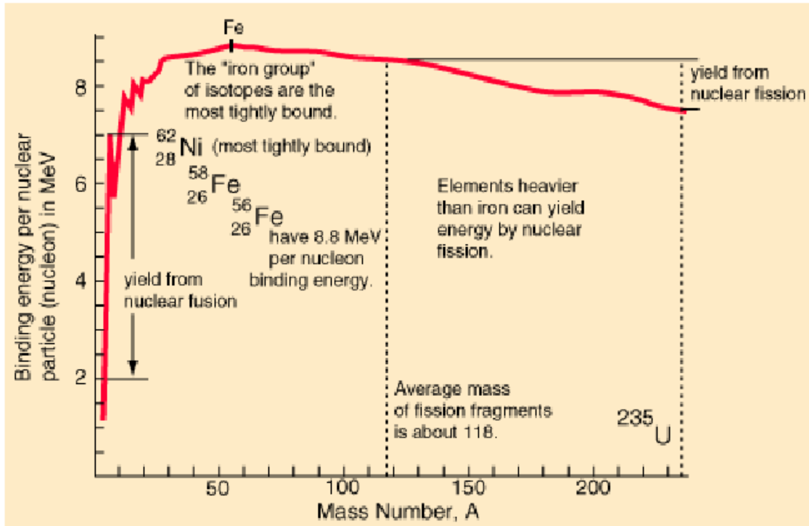

Figure 8: Binding energy per nucleon as a function of mass number A.

using heavy elements like $^{235}_{92}$U, which becomes unstable after capture of a (slow) neutron:

$$^{235}_{92}\mathrm{U} + n \rightarrow \mathrm{X} + \mathrm{Y},$$

where X and Y can be various elements, e.g. $_{52}$Te and $_{40}$Zr, or $_{56}$Ba and $_{36}$Kr. In addition neutrons are released, which can be used to split more $^{235}_{92}$U atoms in a chain reaction.

Fusion of elements takes place abundantly in the central cores of ordinary stars. In stars like the sun H is fused into He, in more massive stars He is fused to form heavier elements. If the star is massive enough, such fusion can continue until the core of a star consists of Fe. Then a point can be reached where the forces of self-gravity can no longer be balanced by thermal pressure from nuclear fusion processes, and the central core of the star implodes to nuclear density. In this very violent process the outer layers of the star are blown away, and a neutron star is left over. Such an event is known as a supernova. As far as we know all elements heavier than He have been made in this way by stellar fusion processes, and ejected into interstellar space by explosive events like novae and supernovae.

Most of the helium in the universe was formed in a much earlier epoch, when it was still hot and dense enough for nuclear fusion to take place spontaneously. The theory of the hot and dense early universe *(Big Bang)* predicts that He should represent about 25 % of all nucleonic mass in the universe; this is in very good agreement with the observed abundance of He. It follows, that practically all neutrons in the universe are part of primordial He nuclei: the total number of neutrons in heavier nuclei and neutron stars is only a fraction of the number of He atoms in the universe.

8. **Radioactivity**

Many nuclei are instable and transform into a nucleus with different values of A and/or Z by emitting particles and radiation. If this transmutation happens spontaneously it is called *radioactivity*. The two main forms of natural radioactivity are the processes known as α- and β-decay. The α-decay process occurs if a heavy nucleus emits an α-particle, changing both A and Z:

$$ {}^{A}_{Z}\mathrm{N} \rightarrow {}^{A-4}_{Z-2}\mathrm{N}' + {}^{4}_{2}\mathrm{He}. \tag{81} $$

An example is the natural decay

$$ {}^{238}_{92}\mathrm{U} \rightarrow {}^{234}_{90}\mathrm{Th} + {}^{4}_{2}\mathrm{He}. \tag{82} $$

The great stability of α-particles due to their large binding energy makes such decay more likely than the emission of single nucleons (protons or neutrons).

The β-decay process does not change A, but is does change the nuclear charge Z by one unit. In the process an electron or anti-electron is emitted, accompanied by an (anti-)neutrino:

$$ {}^{A}_{Z}\mathrm{N} \rightarrow {}_{Z+1}^{\;\;\;A}\mathrm{N}' + e^{-} + \bar{\nu}, \qquad {}^{A}_{Z}\mathrm{N} \rightarrow {}_{Z-1}^{\;\;\;A}\mathrm{N}' + e^{+} + \nu. \tag{83} $$

In such a process a neutron is converted into a proton, or the other way around. The most elementary form of nuclear β-decay is that of the free neutron:

$$ n \rightarrow p^{+} + e^{-} + \bar{\nu}. \tag{84} $$

This process is possible because the mass of the neutron is larger than the sum of the proton, electron and neutrino masses. The instability of the neutron explains why no free neutrons are present in the universe. On the other hand, neutrons can exist as stable particles inside nuclei when the binding energy exceeds the energy released in the emission of an electron and a neutrino. Indeed, energy conservation implies that the neutron decay can take place in a nucleus (A, Z) only if

$$ Zm_p c^2 + (A-Z)m_n c^2 - A\Delta E_N = (Z+1)m_p c^2 + (A-Z-1)m_n c^2 - A\Delta E_{N'} + E_e + E_\nu, \tag{85} $$

which gives

$$ E_e + E_\nu = \Delta m_N c^2 + A\left(\Delta E_{N'} - \Delta E_N\right). \tag{86} $$

Here $\Delta m_N = m_n - m_p$ is the nucleon mass difference. Now the energy necessary to make an electron and a neutrino is at least the sum of their rest energy:

$$ E_e + E_\nu \geq m_e c^2 + m_\nu c^2. \tag{87} $$

Therefore the process can occur only if there is a sufficient change in the binding energy per nucleon:

$$ \Delta E_{N'} - \Delta E_N \geq \frac{1}{A}\left(m_e + m_\nu - \Delta m_N\right) c^2. \tag{88} $$

In stable nuclei this condition is not satisfied, and the neutron can not decay.

Radioactive decay processes, like the α- and β-decays above, are random events: the decay of instable nuclei or particles are independent processes, the time at which an individual particle decays can not be predicted, even though for any type of decay process there is a well-defined *average* time after which a decay process takes place. Clearly, such a characteristic life time of an instable particle is a *statistical* concept, which can be tested experimentally only by observing large numbers of particles.

Consider particles of type A transforming into particles of type B. The total number of particles is conserved: $N = N_A + N_B =$ constant. Then the changes in number of particles A and B in a time interval Δt are equal and opposite:

$$\Delta N_A(t) + \Delta N_B(t) = 0. \tag{89}$$

Operationally the decay probability of particles A at time t is the fraction of particles decaying in the time interval $(t, t + \Delta t)$:

$$P_{A \to B}(t)\, \Delta t = -\frac{\Delta N_A(t)}{N_A(t)} = \frac{\Delta N_B}{N - N_B(t)}. \tag{90}$$

The statistical nature of the decay process has its origin in quantum theory, and the decay probability can be calculated in principle for each process in a specific quantum mechanical model. As proposed first by Marie Curie the decay probability is an inherent property of the instable particles, independent of the concentration, environment and history of the instable particles. Then the above probability $P_{A \to B}(t)$ is a constant depending only on the type of particles A and B:

$$P_{A \to B}(t) = \lambda \quad \Rightarrow \quad \frac{dN_A}{N_A} = -\lambda\, dt. \tag{91}$$

Starting with a pure sample of particles A, this leads to a law for radioactive transformations first derived by Rutherford:

$$N_A(t) = N\, e^{-\lambda t}, \qquad N_B(t) = N \left(1 - e^{-\lambda t}\right). \tag{92}$$

From Rutherford's law it follows, that the probability for a particle A to survive until time t is

$$P_A(t) = \frac{N_A(t)}{N} = e^{-\lambda t}. \tag{93}$$

Then the probablity for a particle to decay during the interval $(t, t + dt)$ is the probability to survive until time t minus the probability to survive until time $t + dt$:

$$W_A(t) dt = P_A(t) - P_A(t + dt) = -\frac{dP_A}{dt}\, dt = \lambda\, e^{-\lambda t} dt. \tag{94}$$

The average life time of a particle is the integrated survival time weighted with $W_A(t)$:

$$\tau = \int_0^\infty t\, W_A(t) dt = \lambda \int_0^\infty t\, e^{-\lambda t} dt = \frac{1}{\lambda}. \tag{95}$$

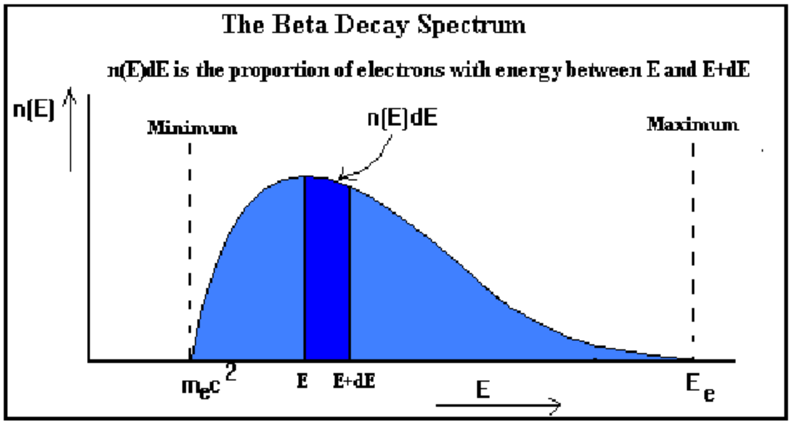

Figure 9: Electron spectrum in β-decay

The above result can be generalized to give the probability for a fraction n/N of particles A to have decayed at time t:

$$P_A(n;t) = \frac{(\lambda t)^n}{n!}\, e^{-\lambda t}. \tag{96}$$

This statistical law is the well-known Poisson distribution. This probability reaches its maximum at $t = n\tau$, as might be expected. The derivation of the Poisson distribution is sketched in exercise 4.2.

9. **The neutrino**

In nuclear β-decay processes an electron or positron is emitted, to balance the change of the nuclear charge. If this would be the only particle emitted by the nucleus, it would have a unique energy in the nuclear rest frame; in particular, in the case of neutron decay

$$E_e = \sqrt{m_e^2 c^4 + \mathbf{p}_e^2 c^2} = m_n c^2 - \sqrt{m_p^2\, c^4 + \mathbf{p}_p^2\, c^2}, \tag{97}$$

with

$$\mathbf{p}_n = \mathbf{p}_e + \mathbf{p}_p = 0. \tag{98}$$

As a result of these equations the kinetic energy of the electron (i.e. $\mathbf{p}_e^2$) is completely determined, fixing E_e:

$$E_e = \frac{(m_n^2 - m_p^2 + m_e^2)c^2}{2m_n} \tag{99}$$

This predicts a mono-energetic electron spectrum. However, in actual experiments the result was a continuum of electron energies, as shown in fig. 9.

The experimental result implies, that another particle is emitted, carrying away part of the energy. This particle can not have electric charge and must be a fermion with spin ($s = 1/2$) in order to account for conservation of the total angular momentum. It must also be very light, as implied by eq. (86) with $m_\nu c^2 \leq E_\nu$. The particle was postulated to exist by W. Pauli in 1930, but is very difficult to detect directly, as it is insensitive to either the electromagnetic or the strong nuclear forces. The only way to establish its existence directly is to observe an inverse β-decay event, of the type

$$\nu + n \rightarrow e^- + p^+, \qquad \bar{\nu} + p^+ \rightarrow e^+ + n. \tag{100}$$

These processes occur only very rarely, indicating that the interactions of neutrinos are very weak. They were finally observed by Reines and Cowan in 1956, using the intense anti-neutrino flux[4] of the Savannah River nuclear reactor. Subsequently, Davis and Harmer showed, that the process

$$\bar{\nu} + n \rightarrow e^- + p^+ \tag{101}$$

does *not* occur, which proves that the neutrino and anti-neutrino are different particles. This result implies that one can assign a new quantum number to particles, the lepton number L, which is conserved in β-decay and by the interactions (100), and which is opposite for the *leptons* (ν, e^-) and the *anti-leptons* $(\bar{\nu}, e^+)$:

$$L_{e^-} = L_\nu = +1, \qquad L_{e^+} = L_{\bar{\nu}} = -1, \tag{102}$$

whilst for the nucleons $L_p = L_n = 0$. The neutrino has spin $s = 1/2$, but its mass is presently still unknown. Most likely it is at least a million times less than the electron mass, but not zero.

10. The photon

The photon was among the first elementary particles known in physics, but its recognition as a particle took quite a long time. It started in 1900 with Planck's discovery of the law of black body radiation and his derivation of this law, based on discretizing the energy of monochromatic radiation emitted by a perfect black body at temperature T in quanta

$$E = h\nu, \tag{103}$$

where ν is the frequency. The next step was taken by Einstein in 1905. He argued that quanta represent the actual state of free radiation, and that this could explain the photoelectric effect: if light falls on a conducting surface, electrons are set free with an energy which depends only on the frequency of the light, not its intensity:

$$E_e = h\nu - P, \tag{104}$$

where P is the fixed energy the electron needs to escape form the surface of the conductor. In contrast, the number of electrons emitted *does* depend on the intensity.

[4] $\Phi_{\bar{\nu}} \simeq 5 \times 10^{13}/\mathrm{cm}^2$.

This effect could be explained simply if each electron absorbs a single photon, such that it gains a fixed energy $h\nu$; moreover, the number of electrons emitted is then proportional to the number of incident photons per unit of area.

It took Einstein till 1917 to take the next step, assigning photons a momentum of size $q = h/\lambda$ in the direction of motion:

$$\mathbf{q} = \frac{h\mathbf{k}}{2\pi} \equiv \hbar\mathbf{k}, \tag{105}$$

with $\mathbf{k}$ the wave vector, of magnitude

$$|\mathbf{k}| = \frac{\omega}{c} = \frac{2\pi\nu}{c} = \frac{2\pi}{\lambda}. \tag{106}$$

This implies that the photon is a particle with rest mass zero, always moving at the speed of light:

$$E^2 = h^2\nu^2 = \hbar^2\mathbf{k}^2c^2 = \mathbf{q}^2c^2. \tag{107}$$

This is to be compared with the energy-momentum relation for a massive particle like the electron:

$$E^2 = m^2c^4 + \mathbf{p}^2c^2. \tag{108}$$

The relations become identical if one takes $m = 0$.

The final step in the story of the photon was taken by Compton, and independently by Debije, who analyzed elastic scattering of a photon with an electron in purely kinematical terms, using energy and momentum conservation. If the electron is originally at rest, and we denote the momenta and energies after scattering by a prime, we have

$$\mathbf{q} = \mathbf{q}' + \mathbf{p}', \qquad E_\gamma + m_ec^2 = E'_\gamma + \sqrt{m_e^2c^4 + \mathbf{p}'^2c^2}. \tag{109}$$

If θ denotes the scattering angle of the photon:

$$\mathbf{q} \cdot \mathbf{q}' = |\mathbf{q}||\mathbf{q}'| \cos\theta, \tag{110}$$

then we find from eqs. (105) - (110) that in vacuum

$$\lambda' - \lambda = \frac{c}{\nu} - \frac{c}{\nu'} = \frac{h}{m_ec}\left(1 - \cos\theta\right). \tag{111}$$

This relation was verified in an experiment by Compton; the quantity

$$\lambda_e = \frac{h}{m_ec} \approx 2.426 \times 10^{-12}\,\mathrm{m}, \tag{112}$$

is called the *Compton wave length* of the electron. The Compton experiment definitely etablished the status of the photon as a particle. By conservation of angular momentum in atomic transitions, where photons are emitted or absorbed, it is easily established that the photon has spin $s = 1$.

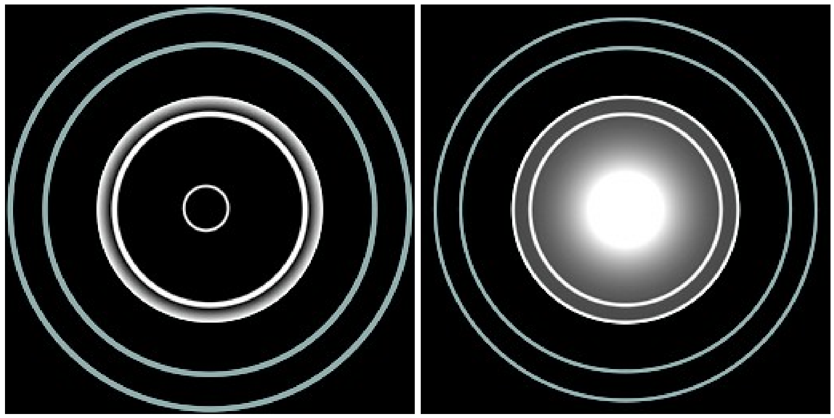

Figure 10: X-ray and electron diffraction patterns from scattering by aluminum

11. **Matter waves**

The quantum theory of light assigns particle properties (energy and momentum) to light waves. In 1923 De Broglie proposed that matter possesses wave properties. The correspondence is basically the same as for the photon:

$$E = \hbar\omega, \qquad \mathbf{p} = \hbar\mathbf{k}, \tag{113}$$

with

$$\omega = 2\pi\nu, \qquad |\mathbf{k}| = \frac{\omega}{c} = \frac{2\pi}{\lambda}. \tag{114}$$

Here λ is the wave length in empty space. The relativistic energy momentum relation between particles then takes the form

$$E^2 = m^2c^4 + \mathbf{p}^2c^2 \quad \Leftrightarrow \quad \omega^2 = \frac{m^2c^4}{\hbar^2} + \mathbf{k}^2c^2. \tag{115}$$

The characteristic wave length

$$\lambda\!\!\!^{-}_c = \frac{\hbar}{mc}, \tag{116}$$

is the reduced Compton wave length.

De Broglies proposal was confirmed in 1927 by the experiments of Davisson and Germer, who scattered electrons from a crystal lattice and found a diffraction pattern as expected for waves; fig. 10 shows a comparison of x-ray and electron diffraction patterns. The wave-particle duality is at the heart of quantum mechanics and plays a major role in particle physics. Indeed, according to quantum field theory all particles can be interpreted as quanta associated with some type of wave field. To a large extend particles can therefore be distinguished by the kind of wave equation their corresponding fields satisfy.

12. **Quantum mechanical scattering theory**

The relation between particles and fields in quantum theory is quite subtle. The simplest situation is that of a single non-relativistic particle without spin in an external field described by a potential $V(\mathbf{r})$. Its dynamics is described by the Schrödinger equation

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\,\Delta\psi + V(\mathbf{r})\psi. \tag{117}$$

The interpretation of the wave function $\psi(\mathbf{r},t)$ is, that it defines a probability density and a related probability current:

$$\rho = |\psi|^2, \qquad \mathbf{j} = -\frac{i\hbar}{2m}\left(\psi^*\nabla\psi - \psi\nabla\psi^*\right) \equiv -\frac{i\hbar}{2m}\,\psi^*\overleftrightarrow{\nabla}\,\psi. \tag{118}$$

If dV is a volume element centered on the point $\mathbf{r}$, then $\rho(\mathbf{r},t)\,dV$ defines the probability to find the particle in the volume dV at the time t. As the total probability to find the particle anywhere is unity, we need to normalize the wave function:

$$\int d^3\mathbf{r}\,|\psi|^2 = \int dV\rho = 1. \tag{119}$$

The density and current (118) satisfy an equation of continuity:

$$\frac{\partial\rho}{\partial t} + \nabla\cdot\mathbf{j} = 0. \tag{120}$$

This equation implies that the probability to find the particle in a volume V can change only if there is a non-zero probability for the particle to move accross the surface $\Sigma = \partial V$ which is the boundary of the volume; indeed, by Gauss' divergence theorem

$$\frac{d}{dt}\int_V dV\,\rho = -\int_V dV\,\nabla\cdot\mathbf{j} = -\oint_\Sigma d^2\sigma\,j_n, \tag{121}$$

where j_n is the normal component of the current accross the surface element $d^2\sigma$. Hence $\mathbf{j}$ is the probability current, describing the flow of probability between different points in space.

The stationary states are energy eigenstates:

$$i\hbar\frac{\partial\psi}{\partial t} = E\psi \quad \Rightarrow \quad \frac{\partial\rho}{\partial t} = 0, \tag{122}$$

provided E is real. Then

$$\left(-\frac{\hbar^2}{2m}\,\Delta + V(\mathbf{r})\right)\psi = E\psi. \tag{123}$$

Hence the allowed values of E are the eigenvalues of the hamiltonian operator

$$\mathcal{H} = -\frac{\hbar^2}{2m}\,\Delta + V(\mathbf{r}). \tag{124}$$

The Schrödinger equation (117), or its time-independent version (123), gives a good description of quantum theory of a single particle when the effects of spin and relativity can be neglected. For example, in the case of the Coulomb potential

$$V(\mathbf{r}) = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r}, \tag{125}$$

the discrete energy eigenvalues of the bound states are precisely the Bohr values (11).

In particle physics the scattering states, belonging to the continuous energy spectrum, are of special relevance. In particular the following results are important:
a. For scattering of particles with momentum $\mathbf{p} = \hbar\mathbf{k}$, incident parallel to the z-axis, by a potential of finite range centered at the origin, the energy eigenstates for large r take the asymptotic form

$$\psi(\mathbf{r}) \simeq C\left(e^{ikz} + f_{\mathbf{k}}(\theta,\varphi)\,\frac{e^{ikr}}{r}\right). \tag{126}$$

Here θ is the angle w.r.t. the z-axis, φ is the azimuthal angle in the transverse (x, y) plane, and C is a normalization constant.
b. To first approximation, the scattering amplitude $f_{\mathbf{k}}(\theta,\varphi)$ is given by

$$f_{\mathbf{k}}(\theta,\varphi) = -\frac{m}{2\pi\hbar^2}\int d^3\mathbf{r}'\, e^{i(\mathbf{k}-k\hat{\mathbf{r}})\cdot\mathbf{r}'}\, V(\mathbf{r}'), \tag{127}$$

where for particles incident in the z-direction $\mathbf{k} = (0, 0, k)$, and $\hat{\mathbf{r}}$ is the radial unit vector:

$$\hat{\mathbf{r}} = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)\,. \tag{128}$$

c. The differential cross section for scattering of the particle in the radial direction $\hat{\mathbf{r}}$ is given by the simple formula

$$\frac{d\sigma}{d\Omega} = \left|f_{\mathbf{k}}(\theta,\varphi)\right|^2. \tag{129}$$

We now present the main line of argument to derive these results; then we apply them to some relevant physical examples.

First we elucidate the meaning of the two terms in expression (126) for the wave function. The first term, e^{ikz}, is an incoming plane wave representing a free particle of momentum $\mathbf{p}_{in} = \hbar\mathbf{k}$ in the positive z-direction. The second term, proportional to e^{ikr}/r, is a radial momentum eigenstate, for an outgoing particle with momentum

$$\mathbf{p}_{out} = \hbar k\hat{\mathbf{r}}. \tag{130}$$

The magnitude of these momenta (and the corresponding wave vectors) are equal because of energy conservation:

$$2mE = \mathbf{p}_{in}^2 = \mathbf{p}_{out}^2 = \hbar^2 k^2. \tag{131}$$

To see why a radial momentum eigenstate takes this particular form, it is necessary to realize, that the hermitean operator for radial momentum is

$$p_r = -i\hbar \left(\frac{\partial}{\partial r} + \frac{1}{r} \right). \tag{132}$$

This is the correct form, because
(i) it satisfies the commutation relation

$$[r, p_r] = i\hbar; \tag{133}$$

(ii) it is hermitean w.r.t. the inner product

$$(\phi, \psi) = \int d^3\mathbf{r}\, \phi^*(\mathbf{r})\psi(\mathbf{r}) = \int_0^{2\pi} d\varphi \int_{-1}^{+1} d\cos\theta \int_0^\infty dr\, r^2\, \phi^*(r,\theta,\varphi)\psi(r,\theta,\varphi). \tag{134}$$

Indeed, it is straighforward to check that

$$(\phi, p_r\psi) = (p_r\phi, \psi)\,. \tag{135}$$

Having established the form (132), it follows that the eigenfunctions are given by:

$$p_r \left(\frac{e^{ikr}}{r} \right) = \hbar k \left(\frac{e^{ikr}}{r} \right). \tag{136}$$

Now the flux of incoming particles with wave vector $\mathbf{k} = (0, 0, k)$ is

$$j_{z\,in} = -\frac{i\hbar}{2m}|C|^2\, e^{-ikz}\, \overleftrightarrow{\nabla}_z\, e^{ikz} = |C|^2\, \frac{\hbar k}{m}; \tag{137}$$

Similarly, the flux of outgoing particles with wave vector $k\hat{\mathbf{r}}$ in the radial direction at angles (θ, φ) is given by

$$j_{r\,out} = -\frac{i\hbar}{2m}|C|^2\, |f_{\mathbf{k}}(\theta,\varphi)|^2 \left(\frac{e^{-ikr}}{r}\, \overleftrightarrow{\nabla}_r\, \frac{e^{ikr}}{r} \right) = |C|^2\, \frac{\hbar k}{mr^2}\, |f_{\mathbf{k}}(\theta,\varphi)|^2 + \mathcal{O}[1/r^3], \tag{138}$$

where terms that vanish faster than $1/r^2$ have not been written out. The differential scattering cross section is defined as in the classical theory by the effective area $d\sigma$ through which as many particles enter as leave at large r through the area $dA = r^2 d\Omega$ at angles (θ, φ):

$$j_{z\,in}\, d\sigma = j_{r\,out}\, r^2 d\Omega. \tag{139}$$

Comparison with the results (137) and (138) then gives:

$$\frac{d\sigma}{d\Omega} = \lim_{r\to\infty} \frac{r^2 j_{r\,out}}{j_{z\,in}} = |f_{\mathbf{k}}(\theta,\varphi)|^2\,. \tag{140}$$

This establishes the result (129).

It remains to derive the asymptotic form (126) of the wave function with the specific form (127) for $f_{\mathbf{k}}(\theta, \varphi)$ as the appropriate solution of the scattering problem defined by eq. (123). This can be achieved using Born's method.

The starting point is again the time-independent Schrödinger equation (123). Using (131) we can cast it into the form

$$-\left(\Delta + k^2\right)\psi = -\frac{2m}{\hbar^2}\,V\psi. \tag{141}$$

Now the inhomogeneous partial differential equation

$$-\left(\Delta + k^2\right)\phi = \rho, \tag{142}$$

is a generalization of Poisson's equation, with the special integral solution

$$\phi_s(\mathbf{r}) = \frac{1}{4\pi}\int d^3\mathbf{r}'\,\frac{e^{ik|\mathbf{r}'-\mathbf{r}|}}{|\mathbf{r}'-\mathbf{r}|}\,\rho(\mathbf{r}'). \tag{143}$$

Equivalently, defining the *Greens function*

$$G_{\mathbf{k}}(\mathbf{r}, \mathbf{r}') = \frac{1}{4\pi}\,\frac{e^{ik|\mathbf{r}'-\mathbf{r}|}}{|\mathbf{r}'-\mathbf{r}|}, \tag{144}$$

we have

$$-\left(\Delta + k^2\right)G_{\mathbf{k}}(\mathbf{r}, \mathbf{r}') = \delta^3(\mathbf{r}-\mathbf{r}'); \tag{145}$$

for a proof, see appendix 2. Solutions of the corresponding homogeneous equation

$$-\left(\Delta + k^2\right)\phi = 0, \tag{146}$$

are of the plane-wave form

$$\phi_{\mathbf{k}}^{(0)}(\mathbf{r}) = Ce^{i\mathbf{k}\cdot\mathbf{r}}, \qquad |\mathbf{k}| = k. \tag{147}$$

Therefore the general solution of eq. (142) is

$$\phi_{\mathbf{k}}(\mathbf{r}) = Ce^{i\mathbf{k}\cdot\mathbf{r}} + \frac{1}{4\pi}\int d^3\mathbf{r}'\,\frac{e^{ik|\mathbf{r}'-\mathbf{r}|}}{|\mathbf{r}'-\mathbf{r}|}\,\rho(\mathbf{r}'). \tag{148}$$

Next we use this result to transform eq. (141) into an integral equation:

$$\psi_{\mathbf{k}}(\mathbf{r}) = Ce^{i\mathbf{k}\cdot\mathbf{r}} - \frac{m}{2\pi\hbar^2}\int d^3\mathbf{r}'\,\frac{e^{ik|\mathbf{r}'-\mathbf{r}|}}{|\mathbf{r}'-\mathbf{r}|}\,V(\mathbf{r}')\,\psi_{\mathbf{k}}(\mathbf{r}'). \tag{149}$$

We recognize already a plane-wave solution and a scattering term due to the potential. Now following Born, we reinsert the equation for ψ under the integral sign:

$$\psi_{\mathbf{k}}(\mathbf{r}) = C\left[e^{i\mathbf{k}\cdot\mathbf{r}} - \frac{m}{2\pi\hbar^2}\int d^3\mathbf{r}'\,\frac{e^{ik|\mathbf{r}'-\mathbf{r}|}}{|\mathbf{r}'-\mathbf{r}|}\,V(\mathbf{r}')\,e^{i\mathbf{k}\cdot\mathbf{r}'} + \ldots\right] \tag{150}$$

The dots represent terms with two or more powers of the potential, which describe double and higher-order scattering processes. If the potential vanishes fast enough such that essentially all contributions to the integral come from a finite region $r' < a$, the higher-order scattering can be neglected to first approximation. Furthermore, we make the approximation

$$\begin{aligned} |\mathbf{r}' - \mathbf{r}| &= \sqrt{(\mathbf{r}' - \mathbf{r})^2} = r\sqrt{\left(\hat{\mathbf{r}} - \frac{\mathbf{r}'}{r}\right)^2} \\ &= r - \hat{\mathbf{r}} \cdot \mathbf{r}' + \mathcal{O}[1/r]. \end{aligned} \tag{151}$$

Then we obtain

$$\psi_{\mathbf{k}}(\mathbf{r}) = C\left[e^{i\mathbf{k}\cdot\mathbf{r}} - \frac{m}{2\pi\hbar^2}\,\frac{e^{ikr}}{r}\int d^3\mathbf{r}'\, e^{i(\mathbf{k}-k\hat{\mathbf{r}})\cdot\mathbf{r}'}\, V(\mathbf{r}') + \mathcal{O}[1/r^2]\right]. \tag{152}$$

In the limit of large r the first two terms reproduce the results (126) and (127), whilst the other terms become negligeable in comparison.

13. **Yukawa and Coulomb scattering**

We apply the formalism of the previous section to compute the quantum-mechanical cross section for a Yukawa potential

$$V(\mathbf{r}) = \kappa\,\frac{e^{-\lambda r}}{r}, \tag{153}$$

which reduces to the Coulomb potential in the limit $\lambda \to 0$. The corresponding scattering amplitude is

$$\begin{aligned} f_{\mathbf{k}}(\theta,\varphi) &= -\frac{m\kappa}{2\pi\hbar^2}\int d^3\mathbf{r}'\,\frac{1}{r'}\,e^{i(\mathbf{k}-k\hat{\mathbf{r}})\cdot\mathbf{r}'-\lambda r'}. \\ &= -\frac{m\kappa}{\hbar^2}\int_{-1}^{+1} d\cos\theta \int_0^\infty dr'\, r' e^{(i|\mathbf{k}-k\hat{\mathbf{r}}|\cos\theta-\lambda)r'} \\ &= \frac{im\kappa}{\hbar^2|\mathbf{k}-k\hat{\mathbf{r}}|}\int_0^\infty dr'\left(e^{-(\lambda-i|\mathbf{k}-l\hat{\mathbf{r}}|)r'} - e^{-(\lambda+i|\mathbf{k}-l\hat{\mathbf{r}}|)r'}\right) \\ &= -\frac{2m\kappa}{\hbar^2}\,\frac{1}{(\mathbf{k}-k\hat{\mathbf{r}})^2+\lambda^2}. \end{aligned} \tag{154}$$

Note, that $\hbar(\mathbf{k} - k\hat{\mathbf{r}})$ is the change of momentum of the particle, the momentum transfer $\mathbf{q}$ (69). Then the scattering cross section becomes

$$\frac{d\sigma}{d\Omega} = \frac{4m^2\kappa^2}{(\mathbf{q}^2 + \hbar^2\lambda^2)^2}. \tag{155}$$

In the limit $\lambda \to 0$ and with $\kappa = Z_1 Z_2 \alpha \hbar c$ this reproduces precisely the Rutherford cross section (71); indeed, as in eq. (70)

$$\mathbf{q}^2 = \hbar^2(\mathbf{k} - k\hat{\mathbf{r}})^2 = \hbar^2 \left(\mathbf{k}^2 - 2k\mathbf{k}\cdot\hat{\mathbf{r}} + k^2\right) = 2\hbar^2 k^2 \left(1 - \cos\theta)\right) = 4\hbar^2 k^2 \sin^2\frac{\theta}{2}. \quad (156)$$

we also get back the angular dependence as in eq. (65).

14. Form factors

The Rutherford formula and its Yukawa generalization (155) describe scattering by the potential of a single point charge. We now generalize this result to the case of an extended charge distribution. Consider a total charge Z, distributed over a finite volume, described by a density function $Ze\rho(\mathbf{r})$, where ρ is normalized to unity:

$$\int d^3\mathbf{r}\,\rho(\mathbf{r}) = 1. \quad (157)$$

The potential at point $\mathbf{r}$ due to charge elements $Ze\rho(\mathbf{r}')d^3\mathbf{r}'$ at the points $\mathbf{r}'$ is then given by the superposition

$$V(\mathbf{r}) = Z\kappa \int d^3\mathbf{r}'\,\rho(\mathbf{r}')\,\frac{e^{-\lambda|\mathbf{r}-\mathbf{r}'|}}{|\mathbf{r}-\mathbf{r}'|}. \quad (158)$$

By substitution into the Born formula (127) we can again compute the scattering amplitude

$$f_{\mathbf{k}}(\theta,\varphi) = -\frac{mZ\kappa}{2\pi\hbar^2}\int d^3\mathbf{r}'\,e^{i\mathbf{q}\cdot\mathbf{r}'/\hbar}\int d^3\mathbf{r}''\,\rho(\mathbf{r}'')\,\frac{e^{-\lambda|\mathbf{r}'-\mathbf{r}''|}}{|\mathbf{r}'-\mathbf{r}''|}, \quad (159)$$

where as before we use the momentum transfer $\mathbf{q} = \hbar(\mathbf{k} - k\hat{\mathbf{r}})$. By interchanging the order of integration (which is allowed here), this becomes

$$\begin{aligned} f_{\mathbf{k}}(\theta,\varphi) &= -\frac{mZ\kappa}{2\pi\hbar^2}\int d^3\mathbf{r}''\,\rho(\mathbf{r}'')\int d^3\mathbf{r}'\,e^{i\mathbf{q}\cdot\mathbf{r}'/\hbar}\,\frac{e^{-\lambda|\mathbf{r}'-\mathbf{r}''|}}{|\mathbf{r}'-\mathbf{r}''|} \\ &= -\frac{mZ\kappa}{2\pi\hbar^2}\int d^3\mathbf{r}''\,\rho(\mathbf{r}'')e^{i\mathbf{q}\cdot\mathbf{r}''/\hbar}\int d^3\mathbf{r}'\,e^{i\mathbf{q}\cdot(\mathbf{r}'-\mathbf{r}'')/\hbar}\,\frac{e^{-\lambda|\mathbf{r}'-\mathbf{r}''|}}{|\mathbf{r}'-\mathbf{r}''|}. \end{aligned} \quad (160)$$

Now we can shift the argument of the last integral by defining $\mathbf{R} = \mathbf{r}' - \mathbf{r}''$; clearly, the result no longer depends on $\mathbf{r}''$. Therefore the expression factorizes into two independent integrals. If we decompose $\mathbf{R}$ into spherical co-ordinates:

$$\begin{aligned} f_{\mathbf{k}}(\theta,\varphi) &= -\frac{mZ\kappa}{\hbar^2}\int d^3\mathbf{r}''\,\rho(\mathbf{r}'')e^{i\mathbf{q}\cdot\mathbf{r}''/\hbar}\int_{-1}^{+1} d\cos\theta\int_0^\infty dR\,R\,e^{-(\lambda - iqR\cos\theta)R} \\ &= -\frac{2mZ\kappa}{\mathbf{q}^2 + \hbar^2\lambda^2}\,F(\mathbf{q}), \end{aligned} \quad (161)$$

where the form factor $F(\mathbf{q})$ is the Fourier transform of the charge distribution:

$$F(\mathbf{q}) = \int d^3\mathbf{r}\, \rho(\mathbf{r})\, e^{i\mathbf{q}\cdot\mathbf{r}/\hbar}. \tag{162}$$

For the scattering cross section this result implies

$$\frac{d\sigma}{d\Omega} = \left.\frac{d\sigma}{d\Omega}\right|_{point} |F(\mathbf{q})|^2, \tag{163}$$

where the label *point* refers to the scattering cross section for a point-like charge, like the Rutherford cross section for Coulomb scattering. Actually, this is the result we used before in sect. 5, eqs. (73) and (74). It follows, that experimentally one can determine the charge distribution by measuring the differential cross section an comparing it with the point-like cross section:

$$|F(\mathbf{q})|^2 = \frac{(d\sigma/d\Omega)_{exp}}{(d\sigma/d\Omega)_{point}}. \tag{164}$$

As was discussed in sect. 5, for spherically symmetric charge distributions one finds the simplified expression

$$F(\xi) = 4\pi \int_0^\infty dr\, r^2\, \rho(r)\, \frac{\sin \xi r}{\xi r}, \tag{165}$$

with $\xi = |\mathbf{q}|/\hbar$, and hence by eq. (156)

$$\xi^2 = 4k^2 \sin^2 \frac{\theta}{2}.$$

By expanding the sine function, this can be rewritten as

$$F(\xi) = 4\pi \int_0^\infty dr\, r^2\, \rho(r) \left(1 - \frac{1}{3!}\, (\xi r)^2 + \ldots \right) = 1 - \frac{1}{6}\, \xi^2\, \overline{r^2} + \ldots, \tag{166}$$

where $\overline{r^2}$ is the expectation values of the square radius of the charge distribution:

$$\overline{r^2} = \int d^3\mathbf{r}\, r^2\, \rho(r). \tag{167}$$

This quantity can be extracted directly from the measured cross section by determining the slope of F at zero momentum transfer:

$$\overline{r^2} = -\,6 \left.\frac{dF}{d\xi^2}\right|_{\xi^2=0}. \tag{168}$$

For example, for a hard sphere of radius a (exercise 3.1.b) the form factor is

$$F_a(\xi) = \frac{3}{(a\xi)^3}\, (\sin a\xi - a\xi \cos a\xi) = 1 - \frac{1}{10}\, (a\xi)^2 + \ldots, \tag{169}$$

and the average distance of the charges to the origin is

$$\overline{r^2} = \frac{3}{5}\, a^2. \tag{170}$$

Fig. 11 shows the differential cross section of two Ca isotopes, as measured by electron scattering. The oscillations of the cross section as a function of angle (hence of ξ) support the hard-sphere model with $F(\xi)$ given by (169). From such measurements nuclear charge distributions can be reconstructed, as in fig. 12.

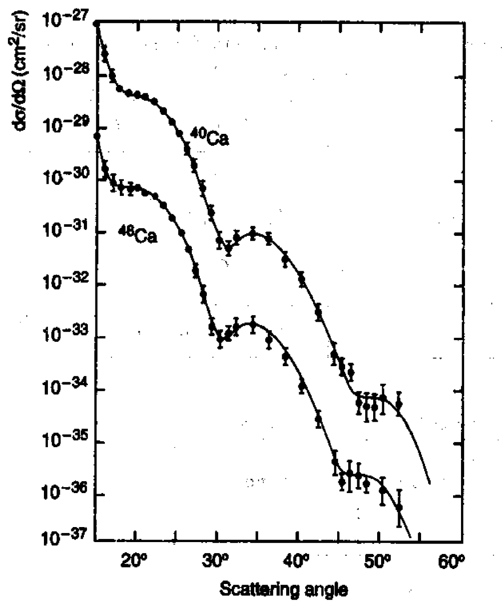

Figure 11: Form factors of Ca isotopes

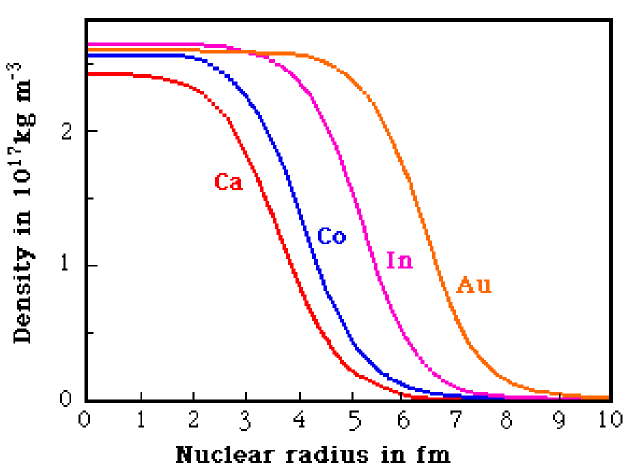

Figure 12: Nuclear density distributions

15. **Proton structure**

Studying elastic scattering of electrons on protons, Hofstadter showed in 1960 that the proton has a non-trivial form factor, which can be fitted quite well by an expontial charge distribution:

$$\rho(r) = \frac{1}{8\pi a^3}\, e^{-r/a} \quad \Rightarrow \quad F(\xi) = \frac{1}{(1+a^2\xi^2)^2}. \tag{171}$$

as in exc. 3.1.b. In fact, from fig. 13 the value of a can be determined to be

$$\frac{\hbar^2 c^2}{a^2} = 0.71\,\mathrm{GeV}^2 \quad \Rightarrow \quad a = 0.24\,\mathrm{fm}. \tag{172}$$

Thus the proton is not a point-like particle. More insight in the structure of the proton came from the study of the *inelastic* scattering of electrons on protons. In these experiments, pioneered by Friedman, Kendall and Taylor around 1970, some of the energy of the electrons is used to excite or break up the proton:

$$e + p \rightarrow e + X, \tag{173}$$

where X can be a proton state with higher internal energy (an excited state or resonance), or a variety of other particles. The production of new particles peaked at certain values of the energy and momentum transfer (specific scattering angles at fixed energy, or specific energy at fixed scattering angle), dictated by the value of the amount of mass created in the process; this is illustrated in fig. 14, as explained below. Moreover, the scattering angles were often surprisingly large. This indicated, that at very high energies the electrons scattered as if there were small point charges inside the proton. As the proton has spin $s = 1/2$ at least some of these constituents must have $s = 1/2$ as well. These fermionic constituents of the proton are the *quarks*, which were introduced as building blocks for a theory of nucleon structure for quite different theoretical reasons in the early 1960's by Gell-Mann.

The scattering experiments studying the structure of the proton used relativistic electrons. To understand the results we need to discuss the kinematics of relativistic scattering as sketched in fig. 15. The 4-momentum of the incoming electron is p, that of the outgoing electron p', and the difference is the 4-momentum $q = p - p'$ transfered to the proton. The relativistic expression for the momentum and energy transfer is given by

$$q^2 = (p-p')^2 = p^2 + p'^2 - 2p\cdot p' = (\mathbf{p} - \mathbf{p}')^2 - \frac{1}{c^2}\left(E_e - E'_e\right)^2, \tag{174}$$

with

$$p^2 = p'^2 = \mathbf{p}^2 - \frac{E_e^2}{c^2} = -m^2c^2, \tag{175}$$

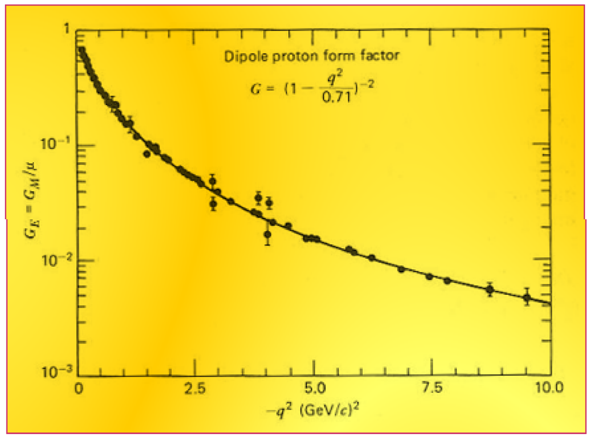

Figure 13: Proton form factor

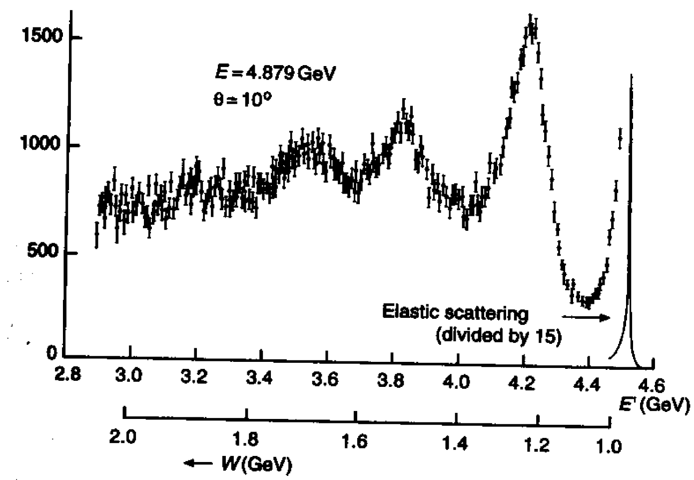

Figure 14: Inelastic ep cross section

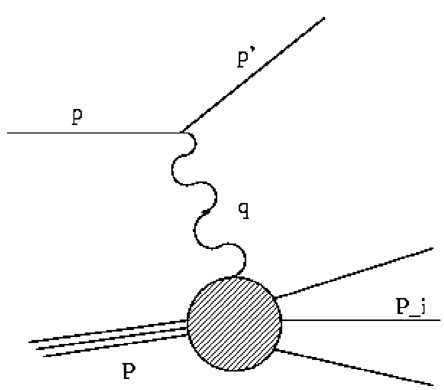

Figure 15: kinematics of electron-proton scattering

and

$$p \cdot p' = \mathbf{p} \cdot \mathbf{p}' - \frac{E_e E_e'}{c^2} = |\mathbf{p}||\mathbf{p}'| \cos\theta - \frac{E_e E_e'}{c^2}. \tag{176}$$

Let us consider separately the non-relativistic and the relativistic limit of this expression. In the non-relativistic case the energy difference is negligeable compared to the momentum difference. Indeed, for $(\mathbf{p}^2, \mathbf{p}'^2) \ll m^2c^2$ we obtain

$$E_e = \sqrt{m^2c^4 + \mathbf{p}^2c^2} \approx mc^2 + \frac{\mathbf{p}^2}{2m}, \qquad E_e' = \sqrt{m^2c^4 + \mathbf{p}'^2c^2} \approx mc^2 + \frac{\mathbf{p}'^2}{2m}, \tag{177}$$

and

$$\frac{1}{c}(E_e - E_e') \approx \frac{\mathbf{p}^2 - \mathbf{p}'^2}{2mc} = \frac{(\mathbf{p} + \mathbf{p}') \cdot (\mathbf{p} - \mathbf{p}')}{2mc} \ll |\mathbf{p} - \mathbf{p}'|. \tag{178}$$

Then combining the above equations we find for the non-relativistic 4-momentum transfer

$$\begin{aligned} q^2 &\approx -2m^2c^2 + 2\left(m^2c^2 + \frac{\mathbf{p}^2}{2} + \frac{\mathbf{p}'^2}{2}\right) - 2|\mathbf{p}||\mathbf{p}'| \cos\theta = (\mathbf{p} - \mathbf{p}')^2 \\ &= \mathbf{q}^2 = 4\mathbf{p}^2 \sin^2\frac{\theta}{2}. \end{aligned} \tag{179}$$

The other limiting case is that of highly relativistic electrons $(\mathbf{p}^2, \mathbf{p}'^2) \gg m^2c^2$. In that case

$$E_e \approx |\mathbf{p}|c, \qquad E_e' = |\mathbf{p}'|c. \tag{180}$$

Substitution of these results into eq. (174) then gives

$$q^2 \approx \frac{2E_eE_e'}{c^2}(1 - \cos\theta) = \frac{4E_eE_e'}{c^2} \sin^2\frac{\theta}{2}. \tag{181}$$

In the case of elastic scattering, the momentum transfer is the only relevant quantity determining the scattering cross section. This follows from the energy-momentum relation of the proton:

$$E_p^2 = M^2c^4 + \mathbf{P}^2c^2, \tag{182}$$

which holds both for the incoming and outgoing proton state. Now elastic scattering only changes the 4-momentum of the proton:

$$P' = P + q, \tag{183}$$

with

$$P^2 = -M^2c^2 = P'^2 = P^2 + 2P \cdot q + q^2. \tag{184}$$

Therefore

$$2P \cdot q + q^2 = 0. \tag{185}$$

If the proton is initially at rest: $P_\mu = (Mc, 0, 0, 0)$, then

$$P \cdot q = Mc\, q^0 = -M(E_e - E'_e), \tag{186}$$

where the expression in parenthesis is the energy change of the electron. We see, that this quantity can be identified with a Lorentz-invariant quantity

$$\nu = -\frac{P \cdot q}{M} = (E_e - E'_e)_{p-rest\, frame}\,. \tag{187}$$

For elastic scattering this quantity equals the 4-momentum transfer:

$$\nu = \frac{q^2}{2M}. \tag{188}$$

Therefore in elastic scattering the energy transfered to the proton is determined directly by the momentum transfered. This is no longer the case in inelastic scattering. Deviations from this relation characterize inelastic scattering: if the scattering breaks up the proton into a number of particles with masses M_i and 4-momenta P_i, such that $P_i^2 = M_i^2 c^2$, then

$$-(P+q)^2 = -\left(\sum_i P_i\right)^2 = \sum_i M_i^2 c^2 - 2\sum_{i<j} P_i \cdot P_j \equiv W. \tag{189}$$

For elastic scattering $W = M^2c^2$, but in all other cases W takes a different value. The difference is parametrized by a dimensionless quantity x:

$$x = \frac{q^2}{2M\nu} = 1 - \frac{(W - M^2c^2)}{2M\nu}. \tag{190}$$

For elastic scattering, $x = 1$, whilst the creation of mass ($W > M^2c^2$) implies $x < 1$. Of course, the maximal amount of energy that can be transfered to the proton is all of the initial electron energy, in which case $E'_e \to 0$; in that case $x = 0$. Hence the natural range of x is $0 < x \leq 1$. Fig. 14 shows the cross section at fixed initial energy E_e and fixed scattering angle θ as a function of W. The peaks correspond to energy transfers where excited states of the proton are created. The fact that such excited states exist proves that the proton has an internal structure. Similar results can be shown for the neutron.

16. Quantum fields and anti-matter

One of the most important implications of relativity is the possibility to convert matter into energy, and energy into matter. As we have seen, one aspect of this is the negative contribution of the binding energy of nucleons to the mass of nuclei. Much more dramatic illustrations are found in high-energy physics, where particles are created or destroyed in all kinds of dynamical processes. The first manifestations of such processes were observed in cosmic rays, which provide a natural source of high-energy particles.

The proper theoretical frame work to describe high-energy physics is relativistic quantum theory, which takes the form of *quantum field theory*. Quantum field theory is a generalization of the quantum theory of photons: particles are identified with the quanta of a field, with specific momentum and energy related to the wave vector and frequency by the Einstein-De Broglie relations (113):

$$E = \hbar\omega, \qquad \mathbf{p} = \hbar\mathbf{k}.$$

These quantities are related by the usual energy-momentum relation for relativistic particles, which now take the form of a *dispersion relation* for the frequency and wave number:

$$E^2 = m^2c^4 + \mathbf{p}^2c^2 \quad \Leftrightarrow \quad \omega^2 = \left(k_c^2 + \mathbf{k}^2\right) c^2, \tag{191}$$

where k_c is the Compton wave number:

$$k_c = \frac{mc}{\hbar}.$$

There are many ways to get such a dispersion relation from a relativistic wave equation. The simplest one is the Klein-Gordon equation

$$\left(\Box - k_c^2\right) \varphi \equiv \left(-\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \Delta - k_c^2 \right) \varphi = 0. \tag{192}$$

Indeed, this equation has relativistic plane wave solutions

$$\varphi(x; k) = a(k) e^{ik\cdot x} = a(k) e^{i(\mathbf{k}\cdot\mathbf{r} + k_0\, x^0)}, \tag{193}$$

provided

$$k^2 = k_\mu k^\mu = \mathbf{k}^2 - k_0^2 = -k_c^2, \qquad \omega^2 = k_0^2 c^2. \tag{194}$$

According to the Fourier theorem, the general solution of the Klein-Gordon equation can be written as a superposition of such waves:

$$\varphi(x) = \int \frac{d^4k}{(2\pi)^2}\, \delta(k^2 + k_c^2)\, a(k)\, e^{ik\cdot x}, \tag{195}$$

where the δ-function enforces the constraint (194). Just as in the case of the electromagnetic field, quantum field theory associates the plane-wave solutions (193) with quanta of fixed momentum and energy. The main difference between the Klein-Gordon and Maxwell equation is, that the field $\varphi(x)$ is a scalar with only one component, invariant under Lorentz tranformations, whereas the Maxwell field is a vector field $A_\mu = (A_0, \mathbf{A})$ which can have different polarizations. In quantum language this implies that the photon has a non-zero spin ($s = 1$), whilst the quanta of the Klein-Gordon field are spinless ($s = 0$).

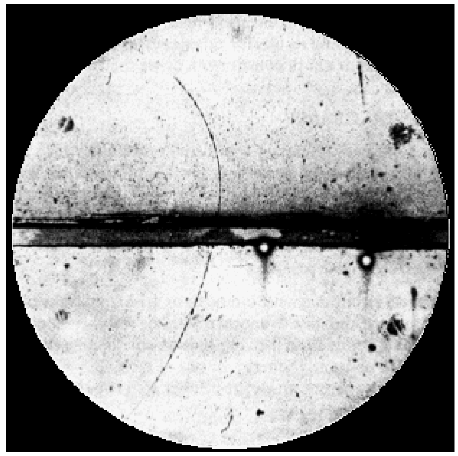

Figure 16: Discovery of the positron

A serious complication of the quantum field theory picture, is the presence of negative-frequency solutions:

$$k^0 = \pm\frac{\omega}{c} = \pm\sqrt{k_c^2 + \mathbf{k}^2}. \qquad (196)$$

The negative-frequency solutions have to be taken seriously, as they are necessary for the Fourier decomposition (195). However, the negative-frequency solutions naively represent negative-energy quanta in the particle interpretation, and such quanta seem unphysical. Dirac proposed, that addition of a negative energy-particle was to be interpreted as the annihilation of a positive-energy particle. This implies in particular that negative-energy electrons should be associated with positive charge, as annihilation of an electron *increases* the charge by $+e$. An alternative way to think of the negative-frequency solutions is to interpret them as a positive-energy quantum propagating backward in time. This picture also agrees with the reversal of the sign of charges, as propagating a positive charge forward in time is equivalent to propagating a negative charge backward in time.

From such ideas emerged the concept of anti-particles: for any kind of particle of given mass m there exists a complementary kind of particle with the same mass such that it can annihilate with the original particle without violating any conservation law. The latter statement implies, that the anti-particle has opposite charge or any other conserved quantum number associated with the original particle. The only remnant of the annihilation process consists of the energy-momentum and angular momentum associated with the original particle/anti-particle pair; these kinematical quantities are then used in the creation of new particles in a reversed process, where the total charge and other quantum numbers of the particles and anti-particles created have to add up to zero.

Dirac proposed the existence of anti-particles in 1931; in 1932 the positron was discovered experimentally by Anderson in cosmic rays. Fig. 16 shows the photographic plate on which the track of a positron was discovered. The track was made by a charged particle in a cloud chamber, placed in a magnetic field. The curvature of the track is due to Lorentz force on the electron; if the plate is the x-y-plane, and the B-field is perpendicular to it in the z-direction, then

$$m\frac{du_x}{d\tau} = qBu_y, \qquad m\frac{du_y}{d\tau} = -qBu_x, \tag{197}$$

where τ is the proper time and $u^\mu = dx^\mu/d\tau$ is the 4-velocity. The standard solution is

$$x(\tau) = R\sin\omega_B\tau, \qquad y(\tau) = R\cos\omega_B\tau, \tag{198}$$

with ω_B is the Larmor frequency:

$$\omega_B = \frac{qB}{m}. \tag{199}$$

By measuring the Larmor frequency the magnitude of q/m can be determined. In fig. 16 this quantity equals e/m of the electron. Now the orientation of the curvature of the track depends on the sign of the charge. To determine this, one has to know the orientation of the track: on which side did the particle enter and on which side did it leave? Anderson solved this question by placing a lead slab in the middle of the cloud chamber. Passing through the slab, the particle loses energy and the radius of curvature decreases. With this information it is clear, that the particle in fig. 16 entered from below; therefore the particle had a positive charge: it is a positron.

Anti-protons exist as well, but are much more difficult to identify in cosmic rays. They were first discovered in 1955, in a laboratory experiment using an accelerator, the Bevatron in Berkeley (USA). It should be noted, that some particles can be their own anti-particles. Of course this can only happen if they are electrically neutral; an example is the photon, but other examples exist. For the neutrinos the situation is as yet unclear.

17. Strongly interacting particles

The forces binding the nucleons (proton and neutron) in the nucleus are not electromagnetic, and at the scale of the nucleus they are much stronger than the repulsive Coulomb forces acting between the protons. There are good reasons to believe, that these forces are almost equal for the proton and the neutron, as shown by the fact that many isotopes with the same value of atomic number A but different charge Z have almost the same mass, i.e. the same binding energy per nucleon. Moreover, many excites states of isotopic nuclei are also close in energy, showing that the excitation mechanism is the same for both types of nucleon. This is illustrated in fig. 17 for some excited levels of nuclei of B,C and N with $A = 12$. Apparently, nuclear interactions are independent of the electric charge of a particle.

The first theory of nuclear forces was developed by Yukawa in 1934. It was based on the Klein-Gordon equation (192) as the equation for a relativistic field with Compton

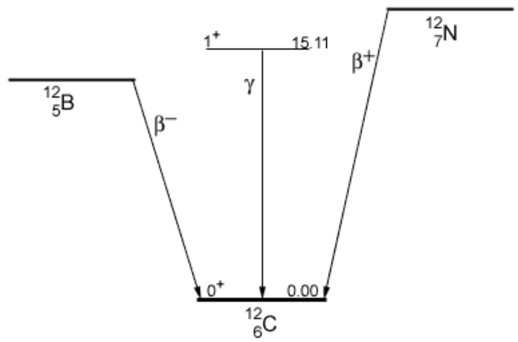

Figure 17: Energy levels of nuclei with $A = 12$

wave number k_c. Yukawa took the Compton wave length to be of order of the typical size of atomic nuclei, which is ~ 1 fm $= 10^{-15}$ m. The corresponding particle mass is

$$mc = \hbar k_c \approx 200\,\mathrm{MeV/c}. \tag{200}$$

Equivalently, we can compute the static potential of a point source for the Klein-Gordon field:

$$\frac{\partial \varphi}{\partial t} = 0 \quad \Rightarrow \quad \left(\Delta - k_c^2\right)\varphi = 4\pi\delta^3(\mathbf{r}). \tag{201}$$

This is the same equation as (144) for the Greens function in scattering theory if we replace $k \to ik_c$. Then the solution becomes

$$\varphi(\mathbf{r}) = \frac{e^{-k_c r}}{r}, \tag{202}$$

which is the Yukawa potential discussed before. This derivation establishes the direct relation between the mass of the φ-quanta and the range of the classical potential, both determined by the Compton wave number k_c. Based on these arguments Yukawa conjectured the existence of a particle with a mass in the range 100-200 MeV, interacting strongly with the proton and neutron. This particle was later identified with the pion, found in 1947 by Powell and Occhialini in cosmic rays.

Besides the nucleons and pions, there are also particles which are insensitive to the nuclear forces, such as the electron, the neutrino and the photon. Particles which do interact strongly form a separate class, the *hadrons*. Hadrons can be either fermions with odd half-integer spin $s = 1/2, 3/2, ...$, such as the nucleons $N = (p^+, n^0)$, or bosons with integer spin $s = 0, 1, 2, ...$ The fermionic hadrons are called *baryons*, whilst the bosonic hadrons are called *mesons*. There is an important distinction between baryons and mesons: whereas mesons can be created and destroyed in various interactions, the net number of baryons is conserved in all observed processes. More precisely, we can assign to all particles a quantum number, the *baryon number*, which

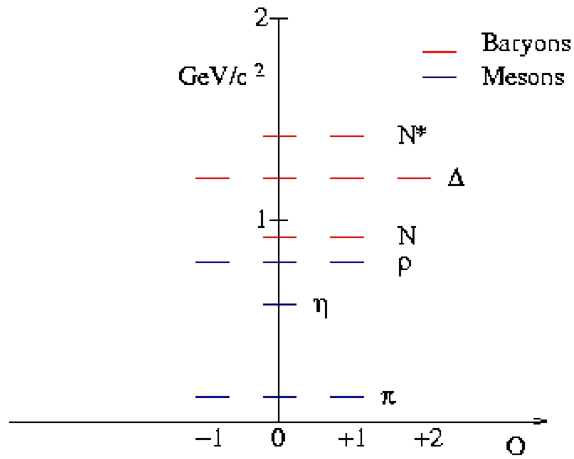

Figure 18: Spectrum of low-mass hadrons

takes the value $B = +1$ for known baryons like the nucleons, $B = -1$ for anti-baryons like the anti-proton and anti-neutron, and $B = 0$ for all other particles, mesons as well as non-hadronic particles. Then empirically it is observed, that during any of the known interaction processes the total baryon number of the initial and final states are equal.

The spectrum of hadrons is very rich; some of the lightest states are sketched in fig. 18. There are excited states of the nucleons, such as the doublet of spin-1/2 particles $N^* = (N^{*+}, N^{*0})$ with the same quantum numbers as the nucleons, but masses around 1440 MeV/c^2. One also finds baryons with different charge multiplicities, such as a 4-plet of spin-3/2 Δ-particles $(\Delta^{++}, \Delta^+, \Delta^0, \Delta^-)$ with masses around $m_\Delta = 1232$ MeV/c^2. Then there are the mesons, of which the lightest ones are the spin-0 pions (π^+, π^0, π^-) with masses

$$m_{\pi^\pm} = 139.6\,\mathrm{MeV/c}^2, \qquad m_{\pi^0} = 135.0\,\mathrm{MeV/c}^2, \tag{203}$$

and the neutral spin-0 η-meson with a mass of $m_\eta = 547.5$ MeV/c^2. Finally there exist higher-spin mesons, such as a triplet of spin-1 ρ-mesons (ρ^+, ρ^0, ρ^-) with masses $m_\rho = 769$ MeV/c^2.

Such a spectrum of states is easy to explain from a simple constituent quark picture of hadron structure. All strongly interacting particles mentioned in the previous paragraph can be constructed out of two kinds of spin-1/2 quarks: the up quark u with charge $Q = 2/3$ in units of e, and the down quark d with charge $Q = -1/3$, and their anti-particles $\bar{u}$ and $\bar{d}$ with the opposite charges $-2/3$ and $1/3$. In terms of such building blocks the nucleons have the structure

$$p = uud, \qquad n = udd, \tag{204}$$

with net spin 1/2. How this spin arises out of the consituent particles is a complicated story that must be postponed till later. Similarly the Δ-particles are represented by a full set of 3-quark states

$$(\Delta^{++}, \Delta^{+}, \Delta^{0}, \Delta^{-}) = (uuu, uud, udd, ddd). \tag{205}$$

On the other hand, the mesons consist of a quark and an anti-quark. The pions are spinless combinations

$$\pi^{+} = u\bar{d}, \qquad \pi^{0} = u\bar{u} - d\bar{d}, \qquad \pi^{-} = d\bar{u}, \tag{206}$$

whilst the ρ-meson has the same quark content, but with one net unit of angular momentum. Similarly, the η-particle has a quark content similar to, but not identical with, the π^0:

$$\eta = u\bar{u} + d\bar{d}. \tag{207}$$

In these last examples it can be seen that the $\pi^{\pm}$ are each others' anti-particle, whilst the π^0 and η are their own anti-particle: $\bar{\pi}^{+} = \pi^{-}$, $\bar{\pi}^{0} = \pi^{0}$, $\bar{\eta} = \eta$. The anti-particles of the nucleons are of course

$$\bar{p}^{-} = \bar{u}\bar{u}\bar{d}, \qquad \bar{n}^{0} = \bar{u}\bar{d}\bar{d}. \tag{208}$$

It is to be noted that in contrast to the neutral π^0 the neutron is *not* its own anti-particle, even though it is electrically neutral.

The average life time of the baryons N^* and Δ is extremely short, between 2 and 5×10^{-24} sec. These particle mainly decay into ordinary nucleons plus pions:

$$(N^*, \Delta) \rightarrow N + \pi. \tag{209}$$

Similarly, the ρ-mesons decay into 2 pions

$$\rho \rightarrow \pi + \pi. \tag{210}$$

with a comparable average life-time of 4×10^{-24} sec. As discussed, these decays conserve baryon number. We observe, that in these processes the pions seem to play a role similar to the photon in electromagnetic decays of excited nuclei $A^* \rightarrow A + \gamma$. Indeed, many strong interactions can be interpreted in terms of the emission and absorption of mesons like the pions. The very short life-time of hadrons like N^*, Δ and ρ connected with π-emission, as compared to those of excited states of nuclei decaying by γ-emission (taking 10^{-10} second or more), is another manifestation of the strength of the nuclear forces. In view of the analogy with nuclei excited by γ-interactions, unstable hadrons decaying by strong interactions are often refered to as *hadron resonances*.

In the simple quark-constituent picture of hadrons sketched here, the only sources of hadron spin can be internal spin and orbital angular momentum of quarks. In such a picture the existence of a spin-3/2 particles like the Δ^{++} or the Δ^{-} is quite a puzzle: these particles apparently consist of three identical quarks in an S-state, with their spin polarized all in the same direction. The existence of such a state seems to contradict the Pauli principle for fermions. Although the origin of hadron-spin is more complicated than this naive quark picture suggests, the problem is a serious one. It is solved by the fact that quarks possess additional quantum numbers, associated with the very strong forces that bind them to form hadrons. Indeed, in connection with these forces quarks carry a new kind of charge which can appear in 3 different varieties; for convenience these charge states are labeled by colors: (r, g, b) for red, green and blue. Similarly, the charge of anti-quarks is labeled by the anti-colors $(\bar{r}, \bar{g}, \bar{b})$. For this reason the strong interactions responsible for the existence of hadrons are often called the *color interactions*. An important rule for the construction of physical hadrons obeyed by the color forces is, that any hadron must be colorless ('white'): either a color charge is canceled by a anti-color charge, or the colors appear in triplets (rgb), c.q. anti-triplets $(\bar{r}\bar{g}\bar{b})$. This solves the problem if the existence of the Δ-particles: Δ^{++} does indeed consist of 3 u-quarks in an S-state with identical spin-polarizations, but each carries a different color charge (r, g, b) to make the net charge of the Δ vanish. Thus, nucleons, pions and ρ-particles are actually constructed out of 6 types of quarks:

$$u^a = (u^r, u^g, u^b), \qquad d^a = (d^r, d^g, d^b), \tag{211}$$

and the corresponding anti-quarks. The reason that electrons and neutrinos do not participate in the strong interactions is now quite obvious: they do not possess a color charge, and are therefore insensitive to the strong color force.

Finally, it follows from the quark picture that the conservation of baryons, counted by the baryon number B, is equivalent to the conservation of quarks, and can be counted by the same quantity if we assign quarks and anti-quarks a baryon number

$$B_q = \frac{1}{3}, \qquad B_{\bar{q}} = -\frac{1}{3}. \tag{212}$$

It then follows automatically that baryons have $B = +1$, anti-baryons have $B = -1$, and mesons have $B = 0$.

18. Leptons

In contrast to the quarks, the electron and neutrino are spin-1/2 fermions which do not carry a color charge. Therefore they are insensitive to the strong color interactions. Electrons do possess one unit of negative electric charge, and have electromagnetic interactions, but the neutrino is also electrically neutral. Thus neutrinos are insensitive to electro-magnetic as well as color interactions. This, in combination with their very small mass, explains why neutrinos are so difficult to observe.

In spite of the difference in electric charge, the electron and neutrino are related in their physical properties. This is most clear in the weak interactions in which anti-neutrinos are produced together with electrons, e.g. in the β-decay of the neutron

$$n \rightarrow p + e + \bar{\nu},$$

or in which neutrinos are produced after capture of an electron by a proton, as in the process ${}^{7}_{4}\mathrm{Be} + e \;\rightarrow\; {}^{7}_{3}\mathrm{Li} + \nu$, in which a proton is converted into a neutron:

$$p + e \;\rightarrow\; n + \nu. \tag{213}$$

In these processes the number of electrons plus neutrinos is conserved, if we count anti-particles as negative. Therefore the electron and neutrino are classified as a separate type of particles, the *leptons*, which in weak interactions of the type above obey the rule of conservation of *lepton number* L, defined such that

$$L_e = L_\nu = +1, \qquad L_{\bar{e}} = L_{\bar{\nu}} = -1. \tag{214}$$

The above examples show that there are some striking analogies between quarks and leptons: we have 2 types of spin-1/2 quarks, u and d, which differ by one unit of electric charge:

$$Q_u - Q_d = \frac{2}{3} - \left(-\frac{1}{3} \right) = 1, \tag{215}$$

and we also have two types of spin-1/2 leptons, ν and e, differing by one unit of electric charge:

$$Q_\nu - Q_e = 0 - (-1) = 1. \tag{216}$$

Moreover, it is an empirical rule, that the total number of quarks is conserved (baryon number), and that the total number of leptons is conserved (lepton number). In fact, we can also view neutron decay as β-decay of a d-quark:

$$d \;\rightarrow\; u + e + \bar{\nu}. \tag{217}$$

and electron capture as simple charge exchange between lepton and quarks:

$$u + e \;\rightarrow\; d + \nu. \tag{218}$$

Also, in some (rare) cases the charged pion π^+ decays by β-decay into a positron and a neutrino, which at the quark level is equivalent to

$$u + \bar{d} \;\rightarrow \nu + \bar{e}. \tag{219}$$

Hence a quark and anti-quark can transform into a lepton plus anti-lepton, provided charge is conserved. This suggest a deeper relation between quarks and leptons.

Of course, there are also differences, the most important of which is the color charge of the quarks, which is absent for the leptons. Furthermore the masses of the leptons are at least an order of magnitude smaller than those of the quarks. Some properties of quarks and leptons are summarized in table 1.

particle	spin	electric charge	color multiplicity	baryon number	lepton number
u	1/2	2/3	3	1/3	0
d	1/2	-1/3	3	1/3	0
ν	1/2	0	1	0	1
e	1/2	-1	1	0	1
$\bar{u}$	1/2	-2/3	$\bar{3}$	-1/3	0
$\bar{d}$	1/2	1/3	$\bar{3}$	-1/3	0
$\bar{\nu}$	1/2	0	1	0	-1
$\bar{e}$	1/2	1	1	0	-1

Table 1: Quantum numbers of stable quarks and leptons

19. **New particle families**

The positron and the pion were first discovered in cosmic rays. Such studies also revealed the existence of several other types of particles, in particular the muon μ^- and its anti-particle μ^+, and the kaons (K^0, K^+) and $(\bar{K}^0, \bar{K}^-)$.

a. The muon

The muon was discovered as a constituent of cosmic ray showers by Neddermeyer and Anderson in 1937. It has a mass

$$m_\mu = 105.6\,\mathrm{MeV/c^2}, \tag{220}$$

and turns out to be a lepton with the same spin and charge as the electron. It is instable, and decays after an average life-time of 2.2×10^{-6} seconds into an electron, a neutrino and an anti-neutrino. However, these two neutrinos are not of the same type: the muon has its own associated type of neutrino ν_μ, different from the electron neutrino ν_e, such that the precise decay process is

$$\mu^- \rightarrow e^- + \bar{\nu}_e + \nu_\mu, \qquad \mu^+ \rightarrow e^+ + \nu_e + \bar{\nu}_\mu. \tag{221}$$

These processes conserve lepton number; they are again examples of β-decay, like the β-decay of the d quark in (217). Hence β-decay is not only possible for hadrons, but also for leptons. In addition, the muon can also be *produced* in β-decay, just like the electron itself. The prime example of such a process is the muonic β-decay of the charged pion, first observed by Powell in 1947 (fig. 19):

$$\pi^+ \rightarrow \mu^+ + \nu_\mu, \qquad \pi^- \rightarrow \mu^- + \bar{\nu}_\mu. \tag{222}$$

In fact, this is by far the dominant decay mode of charged pions: more than 99.9% of all charged pions decay by this process, which takes on average 2.6×10^{-8} s. Therefore the charged pion lives much shorter than its decay product, the muon, and one finds many more muons in cosmic ray showers than pions themselves. In addition, neutrinos are abundant in cosmic ray showers as well, but they are much more difficult to observe.

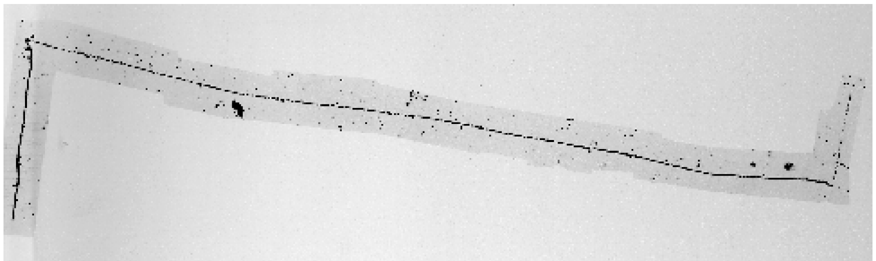

Figure 19: Discovery of the pion and its muon decay by Powell (1947)

In contrast to the charged pions, the neutral pion can decay directly by annihilation of a quark and its anti-quark into photons:

$$\pi^0 \to 2\,\gamma, \tag{223}$$

which takes only 0.8×10^{-16} s, much faster than the weak interaction process of β-decay, but much slower than typical strong decay processes such as $\rho \to 2\pi$, or $\Delta \to N + \pi$. Of course, as there are no lighter hadrons, the pions can not decay by strong interactions and would be stable if it were not for the electromagnetic and weak interactions of the quarks.

Note, that the pion decay processes (222) are direct analogues of (219); for example, the decay of π^+ can be interpreted as

$$u + \bar{d} \to \nu_\mu + \bar{\mu}, \tag{224}$$

which only replaces the positron and electron neutrino by the anti-muon and muon neutrino. Hence also in this respect the muon and electron behave in identical ways.

b. Kaons

The kaons are mesons, with spin $s = 0$ and a mass

$$m_{K^+} = 493.7\,\mathrm{MeV/c^2}, \qquad m_{K^0} = 497.7\,\mathrm{MeV/c^2}. \tag{225}$$

Like the pion, the charged kaon decays predominantly (2/3 of the events) by muonic β-decay:

$$K^+ \to \mu^+ + \nu_\mu. \tag{226}$$

Most other K^+-decays produce pions, e.g.

$$K^+ \to \pi^+ + \pi^0, \tag{227}$$

In addition to the main decay mode, also the average life time of the charged kaon is comparable to that of the pion: $\tau_{K^+} = 1.24 \times 10^{-8}$ s. Hence the K^+ behaves more like a π^+, rather than like more massive mesons such as the ρ^+, which decays into pions by strong interactions on a much shorter time scale of $\sim 10^{-24}$ s.

As mentioned, the life-time of the π^+ is long for a strongly interacting particle, because it is metastable: it is the lightest charged meson, and it can not decay into lighter particles by strong interactions. Therefore it can only decay by weak interactions such as β-decay, which are much slower. Now the K^+ seems to behave in a similar way, even though the pions are much lighter. This indicates that also the K^+ is metastable.

The explanation of the metastability of the charged kaons is that it contains a new type of quark, the s-quark (for *strange*), which has the same spin ($s = 1/2$) and charge ($Q = -1/3$), and also the same color charge states (r, g, b), as a d-quark, but which is about 20 times more massive. Therefore it can only decay into pions if the s-quark transforms into a u- or d-quark. This is possible, in particular the s-quark can transform by muonic β-decay

$$s \;\rightarrow\; u + \mu^- + \bar{\nu}_\mu, \qquad \bar{s} \;\rightarrow\; \bar{u} + \mu^+ + \nu_\mu. \tag{228}$$

In a directly related process, the s-quark can annihilate with a $\bar{u}$ anti-quark (or vice-versa) by producing a lepton pair:

$$s + \bar{u} \;\rightarrow\; \mu^- + \bar{\nu}_\mu, \qquad \bar{s} + u \;\rightarrow\; \mu^+ + \nu_\mu. \tag{229}$$

This is the mechanism for the dominant decay mode of the K^+ and K^-, which have the quark structure

$$K^+ = u\bar{s}, \qquad K^- = \bar{u}s, \tag{230}$$

and it explains the properties of the $K^\pm$, in particular its long life time: it is a weak-interaction process, taking a much longer time than the typical strong interaction. In contrast, the $K \rightarrow \pi\pi$ decay mode can be understood by a new kind of β-like decay in which a quark/anti-quark pair is produced, in stead of a lepton/anti-lepton pair:

$$s \;\rightarrow\; u + d + \bar{u}. \tag{231}$$

Whilst (230) represents the quark structure of the charged kaons, the neutral kaons are bound states of s- and d-quarks:

$$K^0 = d\bar{s}, \qquad \bar{K}^0 = s\bar{d}. \tag{232}$$

The characteristic feature is, that all kaons contain a single s- or $\bar{s}$-quark.

The upshot of these discoveries is, that the electron, neutrino and d-quark all have a more massive companion, which differs only in mass from the lighter particles, but not in spin, charge, color charge, lepton number or baryon number. Therefore it is not surpising that there also exists a more massive companion particle of the u-quark, with spin $s = 1/2$ and charge $Q = 2/3$. This is the c-quark (for *charm*), with a mass close to 1500 MeV/c^2, well above the proton mass. It was first discovered in 1974 independently by Richter in Stanford, and Ting in Brookhaven, both of whom found a new spin $s = 1$ meson

$$J/\psi = c\bar{c}, \qquad m_{J/\psi} = 3097\,\mathrm{MeV/c^2}, \tag{233}$$

with a life-time of about 10^{-21} s. Other mesons containing a single c-quark are the charm-analogues of the kaons:

$$D^+ = c\bar{d}, \quad D^0 = c\bar{u}, \quad \bar{D}^0 = u\bar{c}, \quad D^- = d\bar{c}, \tag{234}$$

and their strange companions

$$D_s^+ = c\bar{s}, \qquad D_s^- = s\bar{c}. \tag{235}$$

The c-quark has several decay modes, such as the muonic β-decay modes

$$c \rightarrow s + \mu^+ + \nu_\mu, \qquad c \rightarrow d + \mu^+ + \nu_\mu, \tag{236}$$

and the corresponding electron β-decays.

After the discovery of a complete second family of quarks and leptons, in the last quarter of the 20th century a third such family was found to exist. Again, these particles have all the properties of the ordinary stable quarks and leptons, except for the mass. The members of this family are the τ-lepton, a charged lepton with mass $m_\tau = 1.777$ GeV/c^2, and its associated neutrino ν_τ; and two quarks, the b-quark (for *bottom* or *beauty*), which is of the d-type with electric charge $Q = -1/3$; and the the t-quark (for *top* or *truth*) of the u-type, with charge $Q = 2/3$. Their masses are approximately

$$m_b = 4.2\,\mathrm{GeV/c^2}, \qquad m_t = 175\,\mathrm{GeV/c^2}. \tag{237}$$

Note that the t-quark is almost 200 times more massive than the hydrogen atom!

Now the natural question arises if there are still more particle families to be discovered. On the basis of experimental evidence obtained with the electron-positron collider LEP at CERN in the 1980's and 1990's it is known that there exist no more than 3 types of light neutrinos. Therefore the existence of a fourth family of quarks and leptons is considered unlikely. Another intrigueing question is, why there are three families of quarks and leptons at all, which are identical up to their masses, and not just one. It has been conjectured, that quarks and leptons may consist of yet more fundamental constituents. However, up to distances probed of the order of 10^{-18} m, quarks and leptons seem to be pointlike; there are no indications of non-trivial form factors. Therefore the Compton wave length of these new constituents is limited to scales below 10^{-3} fm, which implies masses of the order of a few hundred GeV or more. If this is true, the constituents are 10^6 times heavier than the electron, and at least 1000 times heavier than the muon or the s-quark, of which they are to be part. This seems very unnatural.

Finally, there is the question whether the second and third quark-lepton families play any role in nature as we observe it. Maybe these particles were important to processes in the early universe, but as yet this is not more than a conjecture. The (approximate) mass and charge spectrum of the quarks and leptons is summarized in table 2.

particle	mass (MeV/c^2)	particle	mass (MeV/c^2)	particle	mass (MeV/c^2)	charge $Q(e)$
quarks						
u	5	c	1.5×10^3	t	175×10^3	2/3
d	12	s	0.3×10^3	b	5×10^3	-1/3
leptons						
ν_e	$< 10^{-6}$	ν_μ	$< 10^{-6}$	ν_τ	$< 10^{-6}$	0
e	0.511	μ	105.7	τ	1.78×10^3	-1

Table 2: Approximate mass and charge of known quarks and leptons.

20. **Fermions**

So far we have paid little attention to the effects of spin as a degree of freedom of subatomic particles. However, spin degrees of freedom have to be taken into account to explain the spectrum of particle states, and polarization effects provide important tools for the analysis of the internal dynamics of composite systems such as hadrons. As we will discuss later, spin degrees of freedom are also essential to explain the properties of the weak interactions of quarks and leptons: the quantum numbers, cross sections and interaction mechanisms of weakly interacting fermions are qualitatively different for states with different spin polarization. To provide a basis for a discussion of such dynamical effects, this section explains the properties and propagation of free fermions, such as quarks and leptons.

We begin with a discussion of massless fermions, with 4-momentum p_μ satisfying

$$p_\mu p^\mu = 0 \quad \Leftrightarrow \quad \mathbf{p}^2 = p_0^2. \tag{238}$$

For fermions, the spin angular momentum can be described by the Pauli-matrices $\boldsymbol{\sigma}$ acting on a 2-component field, a *spinor*

$$\psi = \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}. \tag{239}$$

As a result, the energy-momentum relation can be rewritten as a linear eigenvalue equation:

$$\boldsymbol{\sigma} \cdot \mathbf{p}\, \psi = p_0\, \psi. \tag{240}$$

To show this, we apply the operator again:

$$(\boldsymbol{\sigma} \cdot \mathbf{p})^2 \, \psi = p_0^2 \, \psi, \tag{241}$$

and use the result that

$$(\boldsymbol{\sigma} \cdot \mathbf{p})^2 = \frac{1}{2} \sum_{i,j} (\sigma_i \sigma_j + \sigma_j \sigma_i) \, p_i p_j = \frac{1}{2} \sum_{i,j} 2 \, \delta_{ij} \, p_i p_j = \mathbf{p}^2. \tag{242}$$

By definition we take the energy $E = c|p_0|$ of a free particle to be positive. Then we can classify spinors according to the sign of the eigenvalues:

$$E \, \psi_\pm = \pm \, c \, \boldsymbol{\sigma} \cdot \mathbf{p} \, \psi_\pm. \tag{243}$$

The upper and lower sign distinguish between states which have the spin parallel $(+)$ or anti-parallel $(-)$ to the momentum. Such massless fermions are called *right-handed* and *left-handed*, respectively. The eigenvalue $\lambda = \pm 1$ such that

$$\frac{c \, \boldsymbol{\sigma} \cdot \mathbf{p}}{E} \, \psi_\pm = \lambda \, \psi_\pm = \pm \, \psi_\pm, \tag{244}$$

is called the *helicity* of the particle. For massless particles the distinction between positive and negative helicity is fundamental: massless particles always move at the speed of light w.r.t. any observer, and the relative direction of their momentum and spin are the same for all observers. In contrast, for a massive particle the direction of spin relative to momentum is observer-dependent. In particular, for an observer in the rest frame of the particle the momentum vanishes, and it is not even possible to speak of the handedness or spin polarization relative to the momentum.

For massive particles the energy momentum relation is

$$p_0^2 - \mathbf{p}^2 = m^2 c^2. \tag{245}$$

Therefore p_0 can not be an eigenvalue of $\boldsymbol{\sigma} \cdot \mathbf{p}$. However, for $m \neq 0$ we can define a new spinor χ by taking the difference

$$(p_0 - \boldsymbol{\sigma} \cdot \mathbf{p}) \, \psi = mc \, \chi. \tag{246}$$

By an argument similar to (242) it then follows that

$$mc \, (p_0 + \boldsymbol{\sigma} \cdot \mathbf{p}) \, \chi = (p_0 + \boldsymbol{\sigma} \cdot \mathbf{p}) \, (p_0 - \boldsymbol{\sigma} \cdot \mathbf{p}) \, \psi = \left(p_0^2 - \mathbf{p}^2 \right) \psi. \tag{247}$$

Hence ψ satisfies the correct energy-momentum relation (245) if and only if

$$(p_0 + \boldsymbol{\sigma} \cdot \mathbf{p}) \, \chi = mc \, \psi \quad \Leftrightarrow \quad \left(p_0^2 - \mathbf{p}^2 \right) \psi = m^2 c^2 \, \psi. \tag{248}$$

Note, that it automatically follows, that

$$\left(p_0^2 - \mathbf{p}^2 \right) \chi = mc \, (p_0 - \boldsymbol{\sigma} \cdot \mathbf{p}) \, \psi = m^2 c^2 \, \chi, \tag{249}$$

and χ is a second spinor satisfying the same energy-momentum relation. To understand the physical relation between ψ and χ, let us assume that $cp_0 = E > 0$. Then for very large energy $E \gg mc^2$ it follows, that

$$E\,\psi \approx c\,\boldsymbol{\sigma} \cdot \mathbf{p}\,\psi, \qquad E\,\chi \approx -c\,\boldsymbol{\sigma} \cdot \mathbf{p}\,\chi. \tag{250}$$

Therefore in the limit of very high energy, when the mass can be neglected, ψ is a positive-helicity spinor describing a right-handed particle, whilst χ is a negative-helicity spinor describing a left-handed particle. A massive fermion is therefore described by a combination of both positive and negative helicities, whilst massless fermions can have a single well-defined helicity, either positive or negative.

The dynamical equations (246) and (248) for free fermions ψ and χ have been written in the momentum representation. We can also rewrite them as field equations in configuration space[5]:

$$i\hbar\,(\partial_0 + \boldsymbol{\sigma} \cdot \nabla)\,\Psi(x) = mc\,\Phi(x), \qquad i\hbar\,(\partial_0 - \boldsymbol{\sigma} \cdot \nabla)\,\Phi(x) = mc\,\Psi(x). \tag{251}$$

These equations constitute the Dirac theory of free fermions. Obviously the original equations (246) and (248) are reobtained by taking plane-wave solutions

$$\Psi(x) = \psi\,e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t}, \qquad \Phi(x) = \chi\,e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t}, \tag{252}$$

provided one makes the standard identification

$$\mathbf{p} = \hbar\mathbf{k}, \qquad E = \hbar\omega.$$

The Dirac equations (251) imply, that the fields (Ψ, Φ) also satisfy the Klein-Gordon equation. Indeed, applying the second equation to the first one we find

$$\Box\,\Psi = -\,(\partial_0 - \boldsymbol{\sigma} \cdot \nabla)\,(\partial_0 + \boldsymbol{\sigma} \cdot \nabla)\,\Psi = \frac{m^2c^2}{\hbar^2}\,\Psi. \tag{253}$$

In this sense the Dirac equations can be interpreted as a 'square root' of the Klein-Gordon equation.

Often the equations for two 2-component spinors are rewritten in terms of a one equation for a single 4-component spinor which combines ψ and χ; this is described in sect. 3 of the appendix. The content of the theory is not changed by such a reformulation.

21. Gauge theories: I. Electrodynamics

The interactions of charged particles are described by Maxwell's theory of the electromagnetic field. In the context of quantum theory the equations and their solutions

[5] In the literature it is customary to use the notation Ψ_R and Ψ_L for right-and left-handed spinors; to avoid writing too many indices we will continue to distinguish them by using different letters: Ψ and Φ, respectively.

are interpreted in terms of photons, as discussed in sect. 10. The resulting quantum field theory is called Quantum Electrodynamics (QED).

The interactions between color charges, as carried by the quarks, are described by a field theory which is a generalization of Maxwell's theory, known as a Yang-Mills theory. The particular Yang-Mills theory for the color interactions is called Quantum Chromodynamics (QCD). But a Yang-Mills theory is also the basis for describing the weak interactions, which explain the various β-decay and -capture processes discussed in the preceeding sections.

The most important common aspect of Maxwell-Yang-Mills theories is the principle of gauge invariance; therefore these theories are also commonly known as gauge theories. In this section we explain the basic aspects of gauge theories in the context of electrodynamics. Full Yang-Mills theory is discussed in the next section.

Electrodynamics
Gauge invariance is a well-known property of Maxwell's theory of electrodynamics, which is summarized in the following field equations

$$\begin{aligned} \nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}, \quad & \nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} = 0, \\ \nabla \cdot \mathbf{B} = 0, \quad & \nabla \times \mathbf{B} - \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} = \frac{1}{\epsilon_0 c^2}\,\mathbf{j}. \end{aligned} \tag{254}$$

Here $(\mathbf{E}, \mathbf{B})$ are the electric and magnetic field strengths, and ρ and $\mathbf{j}$ are the electric charge and current densities. As there are apparently no magnetic charges, the magnetic field strength is divergence-free (first equation on second line). It implies, that the magnetic field strength can be expressed as the curl of a vector field $\mathbf{A}$:

$$\mathbf{B} = \nabla \times \mathbf{A}. \tag{255}$$

However, the vector field $\mathbf{A}$ for given $\mathbf{B}$ is not unique: any two fields differing by a pure gradient $\nabla\Lambda$ give rise to the same $\mathbf{B}$-field:

$$\mathbf{A}' = \mathbf{A} + \nabla\Lambda \quad \Rightarrow \quad \mathbf{B}' = \nabla \times \mathbf{A}' = \nabla \times \mathbf{A} = \mathbf{B}, \tag{256}$$

which holds because of the identity $\nabla \times \nabla\Lambda = 0$. The transformation $\mathbf{A} \rightarrow \mathbf{A}'$ is called a *gauge transformation*, and the invariance of $\mathbf{B}$ under these transformations is called *gauge invariance*. The reason for the existence of gauge invariance is simple: a general vector $\mathbf{B}(\mathbf{r})$ would have three independent components at any point in space; but the constraint $\nabla \cdot \mathbf{B} = 0$ fixes one relation between the three components, and therefore the physical $\mathbf{B}$-field has only *two* independent components at every point. If we express $\mathbf{B}$ in terms of a new vector $\mathbf{A}$, only two combinations of the $\mathbf{A}$-components can be relevant in the definition of $\mathbf{B}$. This is precisely what gauge invariance implies: the component of $\mathbf{A}$ which can be written as a pure gradient $\nabla\Lambda$ does not contribute to the magnetic field strength.

Note, that (255) can be used in Maxwell's equation for $\nabla \times \mathbf{E}$ to give

$$\nabla \times \left(\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} \right) = 0 \quad \Rightarrow \quad \mathbf{E} = \nabla \phi - \frac{\partial \mathbf{A}}{\partial t}, \tag{257}$$

where ϕ is the electric or scalar potential. Of course, not only the magnetic, but also the electric field strength should be gauge invariant; in other words, a gauge transformation of $\mathbf{A}$ should not change the electric field strength $\mathbf{E}$. This is achieved by defining a compensating gauge transformation of the scalar potential:

$$\phi' = \phi + \frac{\partial \Lambda}{\partial t}. \tag{258}$$

Then

$$\mathbf{E}' = \nabla \phi' - \frac{\partial \mathbf{A}'}{\partial t} = \nabla \phi - \frac{\partial \mathbf{A}}{\partial t} = \mathbf{E}. \tag{259}$$

The full set of gauge transformations becomes more transparent in a fully relativistic formulation of Maxwell's theory. To write Maxwell's equations in a covariant form, we define the 4-dimensional gradient

$$\partial_\mu = (\partial_0, \nabla) = \left(\frac{1}{c} \frac{\partial}{\partial t}, \nabla \right), \tag{260}$$

and the covariant 4-vector potential[6]

$$A_\mu = (A_0, \mathbf{A}) = \left(\frac{\phi}{c}, \mathbf{A} \right). \tag{261}$$

Then the electric and magnetic field strength arise as the components of a covariant anti-symmetric tensor

$$F_{\mu\nu} = -F_{\nu\mu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{262}$$

which reads explicitly:

$$F_{\mu\nu} = \begin{pmatrix} 0 & -\frac{1}{c}E_x & -\frac{1}{c}E_y & -\frac{1}{c}E_z \\ \frac{1}{c}E_x & 0 & B_z & -B_y \\ \frac{1}{c}E_y & -B_z & 0 & B_x \\ \frac{1}{c}E_z & B_y & -B_x & 0 \end{pmatrix}. \tag{263}$$

Now if we also introduce the 4-current

$$j^\mu = \left(j^0 . \mathbf{j} \right) = (\rho c, \mathbf{j}), \tag{264}$$

[6]The covariant notation is explained in sect. 2 of the appendix.

then in terms of the field-strength tensor and the 4-current the Maxwell equations take the simple covariant form

$$\partial^\mu F_{\mu\nu} = -\frac{1}{\epsilon_0 c^2}\, j_\nu, \tag{265}$$

and

$$\partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} + \partial_\lambda F_{\mu\nu} = 0. \tag{266}$$

From its definition it is clear, that the tensor $F_{\mu\nu}$ does not change under a gauge transformation of the 4-vector potential

$$A'_\mu = A_\mu + \partial_\mu \Lambda, \tag{267}$$

because of the identity

$$(\partial_\mu \partial_\nu - \partial_\nu \partial_\mu)\, \Lambda = 0. \tag{268}$$

Indeed, the covariant gauge transformations (267) encapsulate both type of gauge transformations (256) and (258). In the literature, the covariant vector field A_μ is known under various names: the *gauge field*, the *gauge potential*, or the *connection*.

Charges and currents

As the field strength tensor $F_{\mu\nu}$ is gauge invariant, the Maxwell eqs. (265) imply that the 4-current j_μ must be gauge invariant as well. Moreover, the 4-divergence of the current must vanish:

$$\partial_\mu j^\mu = -\epsilon_0 c^2\, \partial^\mu \partial^\nu F_{\mu\nu} = 0. \tag{269}$$

The last step follows, as the field strength tensor is anti-symmetric: $F_{\mu\nu} = -F_{\nu\mu}$, whilst the derivatives commute: $\partial_\mu \partial_\nu = \partial_\nu \partial_\mu$. Eq. (269) implies the conservation of electric charge. Indeed, the total charge in a volume V is

$$Q = \int_V d^3r\, \rho = \frac{1}{c} \int_V d^3r\, j^0. \tag{270}$$

Then as in eq. (121)

$$\frac{dQ}{dt} = \int_V d^3r\, \partial_0 j^0 = -\int_V d^3r\, \nabla \cdot \mathbf{j}, \tag{271}$$

by eq. (269). Finally, by Gauss' divergence theorem

$$\frac{dQ}{dt} = -\oint_\Sigma d^2\sigma\, j_n, \tag{272}$$

where Σ is the boundary surface of the volume V, and j_n is the normal component of the current $\mathbf{j}$ across the surface element $d^2\sigma$. Thus the charge Q in a volume V can change only by a current flowing across the boundary. If there is no net current across the boundary the total charge is constant, as a result of the vanishing of the 4-divergence of j^μ.

Gauge invariance and phase transformations
A different, complementary way to introduce gauge fields is to start from the quantum field of a charged particle. To take the simplest case, consider a spin-0 particle represented by a complex scalar field φ. For free, non-interacting particles this field satisfies the Klein-Gordon equation (192):

$$\left(\Box - \mu^2\right)\varphi = 0,$$

where μ equals the Compton wave number:

$$\mu = k_c = \frac{mc}{\hbar}. \tag{273}$$

This field equation is invariant under the transformation

$$\varphi' = e^{i\alpha}\,\varphi, \tag{274}$$

where the real number α represents a shift of the phase of the complex field φ, provided α is constant: we have to make the same phase shift in all space-time points. Indeed, if α would depend on the space-time point, then

$$(\partial_\mu \varphi)' = e^{i\alpha}\left(\partial_\mu + i\partial_\mu \alpha\right)\varphi, \tag{275}$$

and therefore invariance under phase transformations (274) is possible only if

$$(\partial_\mu \varphi)' = e^{i\alpha}\partial_\mu \varphi \quad \Leftrightarrow \quad \partial_\mu \alpha = 0. \tag{276}$$

However, with the help of the gauge field A_μ of the Maxwell theory we can make the equation for the field φ invariant under *local* phase shifts. The key is to replace the ordinary derivative ∂_μ by the *gauge-covariant derivative*

$$D_\mu \varphi = \left(\partial_\mu - ieA_\mu\right)\varphi, \tag{277}$$

and to combine the phase shift with a gauge transformation such that $\alpha = e\Lambda$:

$$A' = A_\mu + \partial_\mu \Lambda = A_\mu + \frac{1}{e}\,\partial_\mu \alpha. \tag{278}$$

Indeed, applying this rule

$$\begin{aligned} (D_\mu \varphi)' &= (\partial_\mu \varphi)' - ieA'_\mu \varphi' \\ &= e^{i\alpha}\left(\partial_\mu + i\partial_\mu \alpha\right)\varphi - e^{i\alpha}\left(ieA_\mu + i\partial_\mu \alpha\right)\varphi = e^{i\alpha} D_\mu \varphi. \end{aligned} \tag{279}$$

In the same way, it follows that

$$(D^\mu D_\mu \varphi)' = e^{i\alpha} D^\mu D_\mu \varphi. \tag{280}$$

Therefore, if one replaces in the Klein-Gordon equation all derivatives by gauge-covariant derivatives:

$$\left(D^\mu D_\mu - \mu^2\right)\varphi = 0, \tag{281}$$

then the equation becomes invariant under *local* phase shifts:

$$\left[\left(D^\mu D_\mu - \mu^2\right)\varphi\right]' = e^{i\alpha(x)}\left(D^\mu D_\mu - \mu^2\right)\varphi = 0. \tag{282}$$

After this replacement the Klein-Gordon equation no longer describes free particles, but charged particles interacting with the gauge field A_μ of electrodynamics. The strength of the interaction is determined by the coefficient e, which is therefore called the *coupling constant*. This coupling constant can be identified with the electric charge of free particles, if the current for the particles is defined in the proper way.

First note, that the complex conjugate field φ^* transforms by the conjugate phase transformation

$$(\varphi^*)' = e^{-i\alpha}\,\varphi^*, \tag{283}$$

and its gauge-covariant derivative becomes

$$D_\mu\varphi^* \;\equiv\; (D_\mu\varphi)^* = (\partial_\mu + ieA_\mu)\,\varphi^*, \tag{284}$$

such that

$$(D_\mu\varphi^*)' = e^{-i\alpha}\,D_\mu\varphi^*. \tag{285}$$

Expression (284) shows, that the precise definition of the gauge-covariant derivative D_μ acting on some field depends on how the field transforms under local gauge transformations; indeed, for the complex scalar field and its conjugate we have

$$\varphi' = e^{ie\Lambda}\varphi, \qquad \varphi^{*\prime} = e^{-ie\Lambda}\varphi^*, \tag{286}$$

and the coefficient of Λ in the phase transformations is the same as the coefficient of A_μ in the covariant derivative.

Now a gauge-invariant current 4-vector can be defined as

$$j_\mu = -ie\left(\varphi^* D_\mu\varphi - \varphi D_\mu\varphi^*\right) = -ie\,\varphi^*\overset{\leftrightarrow}{D}_\mu\,\varphi. \tag{287}$$

It is gauge invariant because of the rules (279) and (285). Moreover, its 4-divergence vanishes because of the field equation (281) and its complex conjugate:

$$\begin{aligned}\partial^\mu j_\mu &= -ie\partial_\mu\left(\varphi^* D_\mu\varphi - \varphi D_\mu\varphi^*\right) = -ie\left(\varphi^* D^2\varphi - \varphi D^2\varphi^*\right)\\ &= -ie\left(\varphi^*\mu^2\varphi - \mu^2\varphi^*\varphi\right) = 0\end{aligned} \tag{288}$$

Finally, consider a free particle enclosed in a volume V; the corresponding plane wave solution for a single particle with momentum $\mathbf{p} = \hbar\mathbf{k}$ and energy $E = \hbar\omega$ in the absence of external fields ($A_\mu = 0$) is

$$\varphi = \frac{1}{\sqrt{2\omega V}}\,e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t}. \tag{289}$$

For these solutions we then find

$$Q = ie \int_V d^3 r \left(\varphi^* \overleftrightarrow{D}_0 \varphi \right) = e, \tag{290}$$

i.e., e represent the charge of the single-particle states of the field φ.

In summary, the full theory of the electrodynamics of charged spin-0 particles is defined by the gauge-covariant Klein-Gordon equation (281) and the Maxwell equations (265), (266), with the current defined by (287).

The electrodynamics of fermions is constructed in a similar way. For massive charged fermions, like electrons, the Dirac equations (251) can be made invariant under local phase transformations

$$\Psi' = e^{ie\Lambda(x)}\, \Psi, \qquad \Phi' = e^{ie\Lambda(x)}\, \Phi. \tag{291}$$

This is achieved by extending the partial derivatives to gauge-covariant derivatives:

$$i\hbar \left(D_0 + \boldsymbol{\sigma} \cdot \mathbf{D} \right) \Psi = mc\Phi, \qquad i\hbar \left(D_0 - \boldsymbol{\sigma} \cdot \mathbf{D} \right) \Phi = mc\Psi, \tag{292}$$

with the same gauge-covariant derivatives as for the complex scalar field:

$$D_\mu = \partial_\mu - ieA_\mu.$$

Here e is the charge of the particles represented by the spinor fields (Ψ, Φ). If we identify this with the elementary charge of the electron, then the equations (292) describe charged leptons like the electron, the muon and the τ-lepton. For other particles, like the u- and d-quarks, the electric charge is different, and the gauge-covariant derivatives are modified accordingly:

$$D_\mu \Psi_u = \left(\partial_\mu + \frac{2ie}{3}\, A_\mu \right) \Psi_u, \qquad D_\mu \Psi_d = \left(\partial_\mu - \frac{ie}{3}\, A_\mu \right) \Psi_d. \tag{293}$$

This implies, that the field equations which describe quarks and leptons must be invariant under local phase transformations

$$\Psi'_u = e^{-2ie\Lambda/3}\, \Psi_u, \quad \Psi'_d = e^{ie\Lambda/3}\, \Psi_d, \quad \Psi'_\nu = \Psi_\nu, \quad \Psi'_e = e^{ie\Lambda}\, \Psi_e, \tag{294}$$

for u- and d-type quarks, neutrinos and charged leptons, respectively. It follows, that an explanation of the values of the electric charges of quarks and leptons is equivalent to an explanation of the invariance of the laws of nature under the local gauge and phase transformations (294). An explanation for these particular phase transformations is provided for example by field theories known as gauge-unification theories (GUTs), which attempt to construct a single gauge theory (a Yang-Mills theory) unifying the electromagnetic, color and weak interactions.

22. **Gauge theories: II. Chromodynamics**

Like the electromagnetic interactions, the interactions between color charges are mediated by vector fields. The quanta of these fields are called *gluons*, with properties similar to photons: transversely polarized and propagating at the speed of light, at least over short distances. Over larger distances gluons and color charges are tightly bound, with the result that no free particles carry a net color charge. This phenomenon is called *color confinement*.

The theoretical basis of the theory of color interactions is gauge invariance, but of a somewhat more general kind than in electrodynamics. As we have discussed, color charges appear in triplets; therefore they are carried by particles which must be quanta of a triplet of complex fields

$$\Psi = \begin{bmatrix} \Psi_r \\ \Psi_g \\ \Psi_b \end{bmatrix}. \tag{295}$$

Ignoring spin for the moment, we can assume that any (hypothetical) free fields satisfy the Klein-Gordon equation

$$\left(\Box - \mu^2\right) \Psi = 0. \tag{296}$$

To take into account the fermionic nature of quarks, replace this equation by the Dirac equations. In both cases the equations are invariant under a 3×3 unitary transformation rotating the fields:

$$\Psi' = \mathbf{U}\Psi, \qquad \mathbf{U}^{-1} = \mathbf{U}^\dagger. \tag{297}$$

The transformation is unitary so as to preserve the scalar product

$$\Psi'^\dagger \Psi' = \Psi^\dagger \mathbf{U}^{-1}\mathbf{U}\Psi = \Psi^\dagger \Psi. \tag{298}$$

The free field equation (296) is invariant only under constant transformations, with the same $\mathbf{U}$ at every point in space-time. However, the theory of color interactions actually has a larger symmetry: it is invariant under *local* transformations $\mathbf{U}(x)$ provided that $\det \mathbf{U}(x) = 1$. Such transformations are called *special unitary transformations*, and the full set of such transformations in an N-dimensional space of complex vectors is known as the group $SU(N)$. In the present case, chromodynamics possesses a local invariance of the type $SU(3)$.

To realize this invariance in the field equation, we proceed as in the case of electrodynamics. We introduce a gauge-covariant derivative

$$D_\mu \Psi = \left(\partial_\mu - ig\mathbf{A}_\mu\right) \Psi, \tag{299}$$

with a hermitean 3×3 matrix of gauge fields $\mathbf{A}_\mu$, such that

$$\left(D_\mu \Psi\right)' = \mathbf{U}\, D_\mu \Psi, \qquad \mathbf{A}_\mu^\dagger = \mathbf{A}_\mu. \tag{300}$$

This is possible only if we let $\mathbf{A}_\mu$ transform in the appropriate way:

$$\begin{aligned}(D_\mu \Psi)' &= \mathbf{U}\partial_\mu \Psi + (\partial_\mu \mathbf{U})\,\Psi - ig\mathbf{A}'_\mu \mathbf{U}\Psi \\ &= \mathbf{U}\left(\partial_\mu + \mathbf{U}^{-1}\partial_\mu \mathbf{U} - ig\mathbf{U}^{-1}\mathbf{A}'_\mu \mathbf{U}\right)\Psi,\end{aligned} \tag{301}$$

and therefore eq. (300) holds if and only if

$$\mathbf{A}'_\mu = \mathbf{U}\mathbf{A}_\mu \mathbf{U}^{-1} - \frac{i}{g}\,(\partial_\mu \mathbf{U})\mathbf{U}^{-1}. \tag{302}$$

Note, that if we would replace $\mathbf{U}$ by an ordinary phase factor, we get back the old result:

$$\mathbf{U} \rightarrow e^{ig\Lambda} \quad \Rightarrow \quad \mathbf{A}'_\mu \rightarrow \mathbf{A}_\mu + \partial_\mu \Lambda. \tag{303}$$

Because of the constraint $\det \mathbf{U} = 1$ we can take $\mathbf{A}_\mu$ to be traceless; more precisely

$$\mathrm{Tr}\,\mathbf{U}^{-1}\partial_\mu \mathbf{U} = 0 \quad \Rightarrow \quad \mathrm{Tr}\,\mathbf{A}'_\mu = \mathrm{Tr}\,\mathbf{A}_\mu. \tag{304}$$

Therefore the condition $\mathrm{Tr}\,\mathbf{A}_\mu = 0$ is invariant under gauge transformations. The first equation (304) can be derived in various ways; an easy way is to use the identity

$$\mathrm{Tr}\,\ln \mathbf{U} = \ln \det \mathbf{U} = 0, \tag{305}$$

where in the last step we have used $\det \mathbf{U} = 1$. The first step is easy to prove for a diagonal matrix; if the diagonal elements are λ_a, then

$$\mathrm{Tr}\,\ln \mathbf{U} = \sum_a \ln \lambda_a = \ln \prod_a \lambda_a = \ln \det \mathbf{U}, \tag{306}$$

but it holds generally for non-singular matrices $\mathbf{U}$. Variation of the first expression in (305) then gives

$$\delta\,\mathrm{Tr}\,\ln \mathbf{U} = \mathrm{Tr}\,\mathbf{U}^{-1}\delta \mathbf{U} = 0. \tag{307}$$

Now a hermitean 3×3 matrix which is traceless has 8 independent components. Therefore the hermitean traceless gauge field $\mathbf{A}_\mu$ represents 8 vector degrees of freedom at every space-time point. This explains why the interactions between 3 color charges, as carried by the quarks, are mediated by 8 vector gluons.

Next we construct the QCD field equations as a generalization of Maxwell's equations. First we define the field strength tensor

$$\mathbf{F}_{\mu\nu} = \partial_\mu \mathbf{A}_\nu - \partial_\nu \mathbf{A}_\mu - ig\,[\mathbf{A}_\mu, \mathbf{A}_\nu]\,, \tag{308}$$

where the last term represents the commutator of two matrix-valued gauge fields. This matrix-valued field strength tensor satisfies

$$\mathbf{F}_{\mu\nu} = -\mathbf{F}_{\nu\mu} = \mathbf{F}^\dagger_{\mu\nu}, \tag{309}$$

and transforms homogeneously under gauge transformations (302) as

$$\mathbf{F}'_{\mu\nu} = \mathbf{U}\mathbf{F}_{\mu\nu}\mathbf{U}^{-1}. \tag{310}$$

The gauge-covariant derivative of $\mathbf{F}_{\mu\nu}$ is defined again by requiring the gauge transformations of this derivative to be like those of constant transformations:

$$D_\mu \mathbf{F}_{\nu\lambda} = \partial_\mu \mathbf{F}_{\nu\lambda} - ig\left[\mathbf{A}_\mu, \mathbf{F}_{\nu\lambda}\right] \quad \Rightarrow \quad \left(D_\mu \mathbf{F}_{\nu\lambda}\right)' = \mathbf{U}\left(D_\mu \mathbf{F}_{\nu\lambda}\right)\mathbf{U}^{-1}. \tag{311}$$

The gauge-covariant generalization of the Maxwell equations are

$$D^\mu \mathbf{F}_{\mu\nu} = -\mathbf{J}_\nu, \qquad D_\mu \mathbf{F}_{\nu\lambda} + D_\nu \mathbf{F}_{\lambda\mu} + D_\lambda \mathbf{F}_{\mu\nu} = 0. \tag{312}$$

After the physicists who first proposed them (in a different context) these equations are known as the *Yang-Mills* equations. In our particular example the gauge-covariant current for a triplet scalar field Ψ is

$$\left(\mathbf{J}_\mu\right)_{ab} = -ig\left[\left(D_\mu \Psi\right)_a \Psi_b^\dagger - \Psi_a \left(D_\mu \Psi^\dagger\right)_b\right], \tag{313}$$

and the covariant Klein-Gordon equation for Ψ is

$$\left(D^2 - \mu^2\right)\Psi = 0. \tag{314}$$

The Klein-Gordon equation guarantees the vanishing of the gauge-covariant 4-divergence of the current:

$$D_\mu \mathbf{J}^\mu = 0 \quad \Leftrightarrow \quad \partial_\mu \mathbf{J}^\mu = ig\left[\mathbf{A}_\mu, \mathbf{J}^\mu\right]. \tag{315}$$

Equations (312)-(314) summarize the basic equations for the chromodynamics of spin-0 particles. The theory of QCD for quarks and gluons is defined by exactly such a set of equations for spin-1/2 particles, rather than spin-0 particles.

Although mathematically electrodynamics and chromodynamics look quite similar, physically they are very different. First, in contrast to the electrically neutral gauge field in Maxwell's theory, the Yang-Mills gauge fields carry color charges themselves. Indeed, we can rewrite the inhomogeneous Yang-Mills equation in the non-manifestly gauge-covariant form

$$\partial^\mu \mathbf{F}_{\mu\nu} = -\mathbf{I}_\nu, \qquad \mathbf{I}_\nu = \mathbf{J}_\nu - ig\left[\mathbf{A}^\mu, \mathbf{F}_{\mu\nu}\right]. \tag{316}$$

Therefore even in the absence of triplet fields ($J_\mu = 0$) there is a contribution to the current from the gauge fields themselves. Also note that, in contrast to $\mathbf{J}_\mu$, the total (gauge variant) current $\mathbf{I}_\mu$ is divergence free:

$$\partial^\mu \mathbf{I}_\mu = -\partial^\mu \partial^\nu \mathbf{F}_{\mu\nu} = 0. \tag{317}$$

Then the gauge-variant charge

$$\mathbf{Q} = \int_V d^3r \, \mathbf{I}^0, \tag{318}$$

is conserved: $d\mathbf{Q}/dt = 0$, provided there is no net flow of color charge through the bounding surface of V. Another, more mathematical argument is that eq. (310) shows, that the Yang-Mills field-strength tensor $\mathbf{F}_{\mu\nu}$ is not gauge invariant, but gauge covariant:

$$F'_{\mu\nu} = \mathbf{U}\mathbf{F}_{\mu\nu}\mathbf{U}^{-1}.$$

Therefore $(F_{\mu\nu})_{ab}$ transforms under gauge transformations in the same way as a triplet/anti-triplet combination of fields $\Psi_a \Psi_b^\dagger$. Indeed, the gauge field carries a combination of a color charge and a color anti-charge:

$$\mathbf{A} = \left(\begin{array}{ccc} A_{r\bar{r}} & A_{g\bar{g}} & A_{b\bar{b}} \\ A_{g\bar{r}} & A_{g\bar{g}} & A_{g\bar{b}} \\ A_{b\bar{r}} & A_{b\bar{g}} & A_{b\bar{b}} \end{array} \right), \tag{319}$$

with the condition that $A_{r\bar{r}} + A_{g\bar{g}} + A_{b\bar{b}} = 0$. Therefore one can also think of the Yang-Mills gauge fields as a kind of color dipoles (with an orientation in color space, rather than real space).

The physical result of this picture of the color-charged gauge field is that its self-interactions lead to an enhancement of the color field at large distances: virtual gluons in the vacuum align in such a way that the field of a color charge grows stronger when probed from afar than from close by. This anti-screening effect is known as *asymptotic freedom*: interactions between quarks become weaker when the quarks are closer to one another. This is quite the opposite of what happens in electrodynamics. It is likely that the explanation of color confinement is also related to this effect.

23. Spontaneous symmetry breaking

Let us return to the theory of a single gauge field A_μ coupled to a complex scalar field φ, as in scalar electrodynamics (sect. 20). The field equations (265) and (282) and the definition of the current (287) can be derived from an action principle, where the action is

$$S = \int d^4x \left(-\frac{\epsilon_0 c^2}{4} F_{\mu\nu}^2 - D^\mu \varphi^* D_\mu \varphi - \mu^2 \varphi^* \varphi \right). \tag{320}$$

Variation of the fields φ and A_μ changes the action by

$$\begin{aligned} \delta S \; = \; & \int d^4x \, \left[\delta A^\nu \left(\epsilon_0 c^2 \partial^\mu F_{\mu\nu} - ie(\varphi^* D_\nu \varphi - \varphi D_\nu \varphi^*) \right) \right. \\ & \left. + \, \delta\varphi^* \left(D^2 - \mu^2 \right) \varphi + \delta\varphi \left(D^2 - \mu^2 \right) \varphi^* \right], \end{aligned} \tag{321}$$

modulo partial integrations. When the field equations are satisfied, the action is stationary under such variations: $\delta S = 0$; in particular, this variational principle automatically provides the correct definition of the current j_μ.

The action (320) is also invariant under the local gauge transformations (267) and (286):

$$A'_\mu = A_\mu + \partial_\mu \Lambda, \quad \varphi' = e^{ie\Lambda}\,\varphi, \quad \varphi^{*\,\prime} = e^{-ie\Lambda}\,\varphi^*,$$

even if the field equations are *not* satisfied. Now the action can be extended with self-interactions of the complex scalar field while preserving gauge invariance, as follows:

$$S_{ext} = \int d^4x \left(-\frac{\epsilon_0 c^2}{4}\, F_{\mu\nu}^2 - D^\mu \varphi^* D_\mu \varphi - \mu^2 \varphi^* \varphi - \frac{\lambda}{4}\, (\varphi^* \varphi)^2 \right). \tag{322}$$

Adding these terms to the action amounts to modification of the field equations by extra terms. Varying the action S_{ext} like in (321) it is actually seen, that the Maxwell equations for the gauge field A_μ are not changed; however, the Klein-Gordon equation for the scalar field is modified:

$$D^2 \varphi - \mu^2 \varphi - \frac{\lambda}{2}\, \varphi^* \varphi^2 = 0. \tag{323}$$

Clearly, when $D_\mu \varphi = 0$, the only solution of this equation for $\lambda > 0$ is $\varphi = 0$. However, this is no longer true if we replace $\mu^2 \rightarrow -\nu^2$, which switches the sign of the $\varphi^*\varphi$-term in the action S_{ext}. In that case eq. (323) admits a solution

$$D_\mu \varphi = 0, \quad \varphi^* \varphi = \frac{2\nu^2}{\lambda} \equiv v^2. \tag{324}$$

This argument only serves to show, that it is possible for scalar fields φ to have a constant non-zero value in empty space (where $A_\mu = 0$).

Now consider the Maxwell equations (265), and write them more explicitly

$$\epsilon_0 c^2 (\Box A_\nu - \partial_\nu\, \partial^\mu A_\mu) = ie\, (\varphi^* \overleftrightarrow{\partial_\nu}\, \varphi) + 2e^2 \varphi^* \varphi A_\nu. \tag{325}$$

If $\varphi = v =$ constant, then $\partial_\nu \varphi = 0$ and

$$\Box A_\nu - \partial_\nu \partial \cdot A - \kappa^2 A_\nu = 0, \tag{326}$$

where

$$\kappa^2 = \frac{2e^2 v^2}{\epsilon_0 c^2} = \frac{8\pi\hbar}{c}\, \alpha v^2. \tag{327}$$

Next we can use the gauge freedom to eliminate the term $\partial \cdot A'$; indeed, after a gauge transformation

$$\partial \cdot A' = \partial \cdot A + \Box \Lambda = 0, \tag{328}$$

provided we choose

$$\Box \Lambda = -\partial \cdot A. \tag{329}$$

The eq. (326) becomes

$$\left(\Box - \kappa^2\right) A'_\mu = 0. \tag{330}$$

This is a Klein-Gordon equation for the vector field A'_μ, and implies that the free wave solutions have the property

$$A'_\mu = a_\mu \, e^{i\mathbf{k}\cdot\mathbf{r} - i\omega t}, \qquad \omega^2 = \mathbf{k}^2 c^2 + \kappa^2 c^2. \tag{331}$$

Thus κ plays the role of a Compton wave number, and the quanta of the gauge field behave as massive particles with a mass

$$m_A^2 = \frac{\hbar^2 \kappa^2}{c^2} = \frac{8\pi\hbar^3}{c^3} \, \alpha v^2. \tag{332}$$

The first lesson from this example is, that if a scalar field for charged particles acquires a vacuum value v, then the quanta of the vector field (here the photons) acquire a mass. In fact, this is roughly what happens in a superconductor. In a superconductor electrons with opposite spin and momentum interact via phonons to form Cooper pairs, which are like scalar particles of charge $2e$; the corresponding scalar field Φ condenses at low temperature, with the result that $|\Phi|^2 > 0$. Then electromagnetic fields can no longer penetrate the superconductor: photons behave like pions with characteristic range $2\pi/\kappa$. This is the basis for the explanation of the Meissner effect, the impossibility for a magnetic field to penetrate a superconductor.

The condensed state in which the scalar field possesses a vacuum value is not manifestly gauge invariant. Indeed, in this state we must have

$$\varphi = v \, e^{i\alpha}, \qquad \varphi^* = v \, e^{-i\alpha}, \tag{333}$$

where α is arbitrary; it can be changed by a gauge transformation: $\alpha' = \alpha + e\Lambda$. Thus there are infinitely many field configurations which realize the condensed state, all related by gauge transformations. This shows that an invariance of the action and the field equations is not necessarily an invariance of the solutions describing the field configuration. This phenomenon is known as *spontaneous symmetry breaking*.

Another lesson we can draw from this example of spontaneous symmetry breaking by a scalar condensate is the existence of a massive neutral spin-0 particle in the condensed state. To see this, parametrize the scalar field as

$$\varphi(x) = (v + h(x)) \, e^{-ie\theta(x)}. \tag{334}$$

Then the covariant derivative becomes

$$D_\mu \varphi = e^{-ie\theta} \left(\partial_\mu h - ie(v + h)(A_\mu + \partial_\mu \theta)\right). \tag{335}$$

Clearly by a gauge transformation (267) and (286) with $\Lambda = \theta$ we can eliminate all dependence on θ, and we are only left with

$$\varphi'(x) = v + h(x), \qquad (D_\mu \varphi)' = \left(\partial_\mu - ieA'_\mu\right)(v + h) = -ievA'_\mu + D'_\mu h. \tag{336}$$

Then the field equation (323) for the scalar field in the vacuum ($A_\mu = 0$) becomes

$$\left(\Box - \lambda v^2\right) h = \frac{3\lambda v}{2}\, h^2 + \frac{\lambda}{2}\, h^3. \tag{337}$$

For small h, when the terms $\mathcal{O}(h^2)$ on the right-hand side can be neglected, this is the Klein-Gordon equation for free neutral spin-0 particles with mass

$$m_h^2 = \frac{\hbar^2 \lambda v^2}{c^2} = \frac{2\hbar^2 \nu^2}{c^2}. \tag{338}$$

The mechanism of spontaneous symmetry breaking can also be applied to Yang-Mills type gauge theories. The main conclusions are the same: if a scalar condensate generates spontaneous symmetry breaking, the gauge fields become massive, and in the spectrum of scalar fields there remains a neutral massive spin-0 particle. In the context of particle physics, the mechanism of spontaneous symmetry breaking was first proposed by Brout and Englert; the appearance of a neutral spin-0 particle as a remnant of the scalar condensate was first argued by Higgs; therefore this particle is called the *Higgs particle*.

24. Weak interactions

Neutrinos are electrically and color neutral, but they are produced in weak interactions like β-decay processes. An important experiment was performed in 1957 by C.S. Wu and co-workers at the National Bureau of Standards in Washington. She took a sample of radioactive ^{60}Co and cooled it to very low temperature in a magnetic field $\mathbf{B}$, such that the nuclear spins were strongly polarized (65 %). The cobalt decays by β-decay to ^{60}Ni by converting a neutron to a proton:

$$^{60}\mathrm{Co} \rightarrow {}^{60}\mathrm{Ni} + e + \bar{\nu}.$$

Wu found, that the majority of electrons came out on one side, polarized with negative helicity: they were predominantly left-handed. By conservation of momentum and angular momentum, the anti-neutrino comes out predominantly right-handed. This is shown schematically in fig. 20. Now the electron is massive and mostly non-relativistic; therefore electrons can be produced with both helicities, and there is no *a priori* reason to expect them to be produced with preference for left-handed spin polarization in the rest frame of the Co nucleus. On the other hand, the anti-neutrinos are always highly relativistic as the mass scale of neutrinos is too small to be measured directly even today[7]. Therefore the anti-neutrino can be assigned a definite handedness, and at least it is apparently *produced* in weak interactions with a definite handedness.

This conclusion holds quite generally: in all β-decay type processes, like

$$u \rightarrow d + e^+ + \nu_e, \qquad s \rightarrow u + \mu^- + \bar{\nu}_\mu, \qquad \mu^- \rightarrow e^- + \bar{\nu}_e + \nu_\mu, \tag{339}$$

[7] At the time of the experiment, neutrinos were generally assumed to be massless.

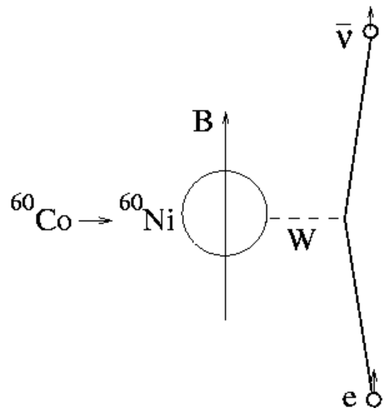

Figure 20: Parity violation in β-decay

particles (quarks and leptons) are always produced with left-handed polarization, whilst the anti-particles produced are right-handed. This implies that weak interactions violate mirror symmetry, or *parity*. Indeed, under a mirror transformation a vector representing linear motion is reflected in the opposite direction if the motion is perpendicular to the mirror plane, and in the same direction if the motion is parallel to the plane. However, a vector associated with circular motion, like angular momentum, has the opposite behaviour: its reflection points in the same direction if the vector is directed perpendicular to the mirror plane, and in the opposite direction if it is directed parallel to the plane; such a vector which is mirrored differently is called an *axial* vector. The point is illustrated in fig. 21. If right-handed neutrinos (and left-handed anti-neutrinos) do not exist, or are not produced in weak interactions, then the mirror process of these decays is not possible, and nature is apparently not invariant under parity transformations. This idea was first conceived by T.D. Lee and C.N. Yang, and taken up as an experimental challenge by C.S. Wu.

It was suggested early on, that weak interactions might be mediated by massive vector bosons, creating a short-range Yukawa-type of potential in the static limit. Moreover, the vector bosons mediating β-decay should carry electric charge, as they change the charge of the original quark or lepton by one unit; c.f. eq. (339). These charged intermediate vector bosons are called W^+ and W^-. The extreme short range of the weak interactions indicates that their mass is substantial. Since the W-bosons were discovered in experiments in 1983, their mass has been determined to be

$$m_W = 80.3\,\mathrm{GeV/c^2}. \tag{340}$$

At the same time, parity-violation indicates that these W-bosons couple only to the left-handed polarization states of quarks and leptons, and right-handed states of anti-quarks and anti-leptons. Now the Dirac equations (292) or (452) show, that for massive fermions the spinor field representing the right-handed states is proportional to the gauge-covariant derivatives of the spinor field representing the left-handed ones,

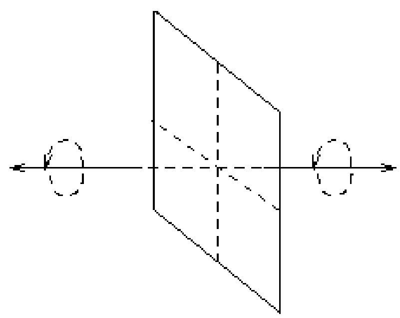

Figure 21: Parity operation

and vice-versa. If one of these fields carries a gauge charge, the other one must carry the same charge. The only consistent way to have left- and right-handed polarization states with different gauge charges, is to have $m = 0$, i.e. strictly massless quarks and leptons.

Obviously, real quarks and leptons are *not* massless[8]. Actually, this situation parallels that of the gauge bosons themselves: the mathematical description starts with massless gauge fields, but the true particle states in the spectrum correspond to massive vector bosons via spontaneous symmetry breaking and the Higgs mechanism as in sect. 23. This suggests how the weak interactions of quarks and leptons can be described consistently; in the Dirac equations the mass is to be replaced by a scalar field, which can take a non-zero constant value $|\langle\varphi\rangle| = v$ in the vacuum:

$$i\hbar\,(D_0 + \boldsymbol{\sigma}\cdot\mathbf{D})\,\Psi = f\,\varphi\,\Phi, \qquad i\hbar\,(D_0 - \boldsymbol{\sigma}\cdot\mathbf{D})\,\Phi = f\,\varphi^*\,\Psi, \tag{341}$$

where f is some coupling parameter, then spontaneous symmetry breaking will generate a mass

$$m_\Psi c = fv. \tag{342}$$

However, the gauge charges of the 2-component spinor fields Ψ and Φ can now be different, as the gauge transformations of Ψ must be the same as those of the combination $(\varphi\,\Phi)$, and those of Φ must be the same as those of $(\varphi^*\,\Psi)$. If φ itself changes non-trivially under gauge transformations, then the gauge charge of the field combination $(\varphi\,\Phi)$ –equal to that of Ψ– is different from that of Φ itself.

Although it is generally accepted that spontaneous symmetry breaking is the mechanism by which weak gauge bosons and weakly interacting fermions get a mass, it is not known whether these masses are all generated by a single weakly interacting scalar field, which would predict the existence of a single neutral Higgs particle, or whether there is more than one weakly interacting scalar field involved. In the latter case many new spin-0 particles, charged as well as neutral ones, are expected to exist.

[8] At least all charged ones are massive.

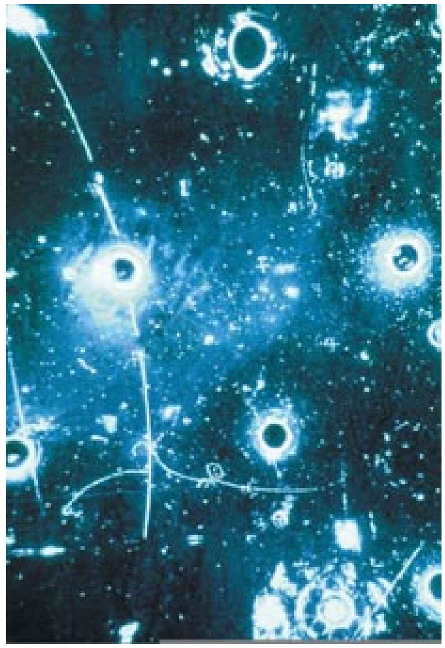

Figure 22: Neutral current event in the Gargamelle bubble chamber

In any case, spontaneous symmetry breaking solves two problems at once: how the gauge bosons become massive, and how massless fermions of opposite helicity with different weak charges can effectively combine into massive fermion states. Of course, these massive fermions do not have specific gauge charges anymore, but this is not a problem when the gauge symmetry is broken: charges can appear and disappear into the vacuum because it is filled with a non-zero charged scalar field.

Yang-Mills theories with spontaneously broken gauge invariance describing charged vector bosons coupled to leptons and hadrons were constructed and discussed[9] during the 1960's. It was found that such theories of the weak interactions also contain at least one neutral gauge boson; in principle, this could have been the photon, but it turns out that in nature there actually is another *massive* neutral vector boson as well, the Z-boson. Its mass even exceeds that of the W-bosons:

$$m_Z = 91.2\,\mathrm{GeV/c^2}. \tag{343}$$

The first indication for the existence of a massive neutral gauge boson came from the observation of elastic neutrino scattering at CERN in 1973, using a very large bubble chamber known as Gargamelle; see fig. 22. In this picture an electron is being projected forward after scattering with an unseen anti-neutrino of the muon-type:

$$\bar{\nu}_\mu + e^- \rightarrow \bar{\nu}_\mu + e^-. \tag{344}$$

[9]The correct gauge symmetry and much of the dynamics were developed in particular by Glashow, Brout and Englert, Higgs, Salam and Weinberg.

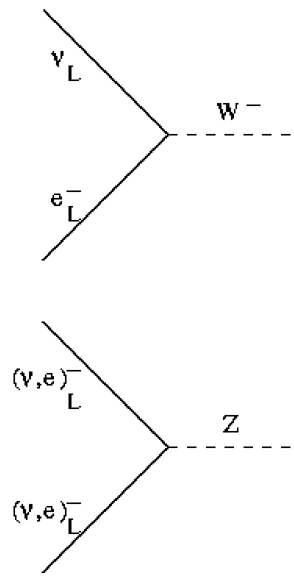

Figure 23: Leptonic weak interactions

The energy is transferred through exchange of a neutral vector boson which –as the neutrino has no electric charge– can not be the photon. Before the existence of this Z-boson was confirmed directly, such events were refered to as *neutral current interactions*. Their observation was an important indication in favor of the Yang-Mills models with spontaneously broken symmetry. Fig. 23 shows the basic building blocks of the weak interactions for leptons. In addition, it is also possible for a *right-handed* electron e_R (or a left-handed positron) to emit a Z-boson, but *not* a W-boson. For quarks the basic weak interactions are similar. In essence they are obtained by the replacements $\nu \longrightarrow u$ and $e \longrightarrow d$; there are however some small but important modifications due to mixing between different generations of quarks (d, s, b) in the coupling to W-bosons.

To avoid these complications in the quark sector, we first consider the weak and electromagnetic interactions of charged leptons and neutrinos. Denoting the spinor fields of the left-handed neutrino and the electron by Φ_ν and Φ_e, respectively, the Dirac equation for these field in the massless limit (before spontaneous symmetry breaking) must read

$$\sum_{f'=(\nu,e)} (D_0 - \boldsymbol{\sigma} \cdot \mathbf{D})_{ff'} \Phi_f = 0, \tag{345}$$

Here the gauge field, and therefore the complete differential operator, is a 2×2 matrix:

$$D_\mu = \partial_\mu - i\mathbf{Z}_\mu, \qquad \mathbf{Z} = \begin{pmatrix} g_\nu Z & g_w W^+ \\ g_w W^- & -g_e Z + eA \end{pmatrix}. \tag{346}$$

For the time being, we have kept the coupling constants g_i of the weak vector fields

$(W^{\pm}, Z)$ arbitrary. In contrast, in the absence of mass terms the right-handed electron represented by Ψ_e satisfies an equation

$$(D_0 + \boldsymbol{\sigma} \cdot \mathbf{D})\, \Psi_e = 0, \tag{347}$$

with

$$D_\mu = \partial_\mu - ieA_\mu - i\tilde{g}_e Z_\mu. \tag{348}$$

If the right-handed neutrino exists, it has no weak or electric gauge charges; therefore it does not couple to any gauge fields; in the limit of vanishing neutrino masses:

$$(\partial_0 + \boldsymbol{\sigma} \cdot \nabla)\, \Psi_\nu = 0. \tag{349}$$

The above couplings suggest how a unified gauge theory can be constructed for the weak and electromagnetic interactions which, after spontaneous symmetry breaking, can explain the presence of three massive and one massless vector particles $(W^{\pm}, Z, A)$ and which exhibits different couplings of left- and righthanded fermions to the weak bosons.

For the right-handed fermions the starting point is to have couplings to a single neutral gauge field B_μ, which is like the Maxwell field and represented by the combination of photon and Z-boson in eq. (348) (assuming all gauge fields to be massless)

$$D_\mu \Psi_e = (\partial_\mu - ig_1 B_\mu)\, \Psi_e, \qquad D_\mu \Psi_\nu = \partial_\mu \Psi_\nu, \tag{350}$$

with

$$g_1 B_\mu = eA_\mu + \tilde{g}_e Z_\mu. \tag{351}$$

The left-handed spinor fields (Φ_ν, Φ_e) are taken to couple to this field B_μ as well, but with a different charge, and in addition they couple to gauge fields which allow to mix the neutrino and electron fields by a local unitary 2×2 transformation[10]:

$$D_\mu \begin{bmatrix} \Phi_\nu \\ \Phi_e \end{bmatrix} = \left(\partial_\mu - \frac{ig_1}{2}\, B_\mu - \frac{ig_2}{2}\, \mathbf{W}_\mu \right) \begin{bmatrix} \Phi_\nu \\ \Phi_e \end{bmatrix}, \tag{352}$$

where

$$\mathbf{W} = \begin{pmatrix} W_3 & W_1 - iW_2 \\ W_1 + iW_2 & -W_3 \end{pmatrix}. \tag{353}$$

$\mathbf{W}$ is a traceless hermitean 2×2 matrix of gauge fields, just like the gluons $\mathbf{A}$ form a traceless hermitean 3×3 matrix of gauge fields. The B-field couples with equal strength to both spinor fields, i.e. we may interpret B as a diagonal matrix of gauge fields which is proportional to the unit matrix:

$$\mathbf{B} = \begin{pmatrix} B & 0 \\ 0 & B \end{pmatrix}. \tag{354}$$

[10] More precisely: an $SU(2)$ transformation.

By comparing eqs. (346) and (352) we see, that we must have

$$g_1\mathbf{B} + g_2\mathbf{W} = \begin{pmatrix} g_1 B + g_2 W_3 & g_2(W_1 - iW_2) \\ g_2(W_1 + iW_2) & g_1 B - g_2 W_3 \end{pmatrix} = 2\mathbf{Z}. \tag{355}$$

It follows, that we can make the identifications

$$B = \cos\theta_w\, A + \sin\theta_w\, Z, \qquad W_3 = \cos\theta_w\, Z - \sin\theta_w\, A, \tag{356}$$

where the angle θ_w and the coupling constants are related by

$$e = g_1 \cos\theta_w = g_2 \sin\theta_w, \qquad \tilde{g}_e = e\tan\theta_w, \quad g_e = e\,\mathrm{cotan}2\theta_w, \quad g_\nu = \frac{e}{\sin 2\theta_w}. \tag{357}$$

Furthermore,

$$W^\pm = \frac{W_1 \mp iW_2}{\sqrt{2}}, \qquad g_w\sqrt{2} = \frac{e}{\sin\theta_w}. \tag{358}$$

Therefore the coupling of all particles, left- and righthanded, to the photon and the weak gauge bosons can be expressed in terms of only 2 parameters: the elementary electric charge e and the weak mixing angle θ_w. In particular, there are strict relations between the strength of weak interactions by W-exchange (like β-decay), weak interactions by Z-exchange (e.g., neutrino scattering) and electromagnetic interactions by photon exchange (such as electron scattering). These relations are well-verified, and the experimental value of θ_w is given by

$$\sin^2\theta_w = 0.2315. \tag{359}$$

It follows, that the weak and electromagnetic interactions indeed have a common origin in terms of massless gauge fields $(B, \mathbf{W})$ such that after spontaneous symmetry breaking the combination (356) corresponding to the photon remains massless, whilst the orthogonal combination corresponding to the Z, and the W's, become massive by the interaction with a set of non-vanishing vacuum scalar fields. Such a mechanism also generates masses for the leptons, as illustrated in eqs. (341) and (342).

For the quarks, the weak interactions are described in an analogous way. First, the left-handed quark fields (Φ_u, Φ_d) couple to both the W- and B-gauge fields:

$$D_\mu \begin{bmatrix} \Phi_u \\ \Phi_d \end{bmatrix} = \left(\partial_\mu + \frac{ig_1}{6}\, B_\mu - \frac{ig_2}{2}\, \mathbf{W}_\mu \right) \begin{bmatrix} \Phi_u \\ \Phi_d \end{bmatrix}. \tag{360}$$

Rewriting this in terms of A- and Z-fields, this becomes

$$\frac{ig_1}{6}\, B - \frac{ig_2}{2}\, \mathbf{W} = \frac{e}{3} \begin{pmatrix} 2A - \frac{1+2\cos 2\theta_w}{\sin 2\theta_w}\, Z & g_w W^+ \\ g_w W^- & -A + \frac{2+\cos 2\theta_w}{\sin 2\theta_w}\, Z \end{pmatrix}. \tag{361}$$

The coupling to the charged W-bosons are the same as for the leptons: g_w is a universal coupling parameter for all left-handed quarks and leptons. Furthermore, the electric charges of the u- and d-quarks then come out correctly as $2e/3$ and $-e/3$, whilst the couplings to the Z-boson more complicated functions of θ_w times e.

As expected, right-handed quarks do not couple to the W-fields, but only to the B-field, in such a way that the electric charges come out right:

$$D_\mu \Psi_u = \left(\partial_\mu + \frac{2ig_1}{3}\, B_\mu\right)\Psi_u, \qquad D_\mu \Psi_d = \left(\partial_\mu - \frac{ig_1}{3}\, B_\mu\right)\Psi_d. \tag{362}$$

In terms of photon and Z-boson fields:

$$\frac{2ig_1}{3}\, B = \frac{2e}{3}\left(A + \tan\theta_w\, Z\right), \qquad -\frac{ig_1}{3}\, B = -\frac{e}{3}\left(A + \tan\theta_w\, Z\right). \tag{363}$$

Again, given the electric charges and the value of θ_w, the coupling to the Z-boson is predicted, and turns out to be in full agreement with experimental values.

However, in the next step the quarks differ from the leptons: the symmetry breaking which gives rise to the masses of weak vector bosons and fermions now mixes the d-type quarks. Equivalently, we can take the mass-eigenstates of the quarks to be well-defined, but then the d-type quarks with which the u-type quarks interact via exchange of $W^\pm$-bosons (as in β-decay) are superpositions of the mass eigenstates.

In mathematical terms we note, that all leptons of the same type (neutrino or charged) and all quarks of the same type (u- or d-type) have the same gauge charges, and therefore the same gauge-covariant derivatives. This implies, that the gauge-covariant derivative of any linear combination of fermion fields of the same type is identical to that of any of these fermion fields; for example, for the left-handed quarks, let

$$\begin{bmatrix} \Phi'_u \\ \Phi'_d \end{bmatrix} = c_1 \begin{bmatrix} \Phi_u \\ \Phi_d \end{bmatrix} + c_2 \begin{bmatrix} \Phi_c \\ \Phi_s \end{bmatrix} + c_3 \begin{bmatrix} \Phi_t \\ \Phi_b \end{bmatrix}, \tag{364}$$

with $|c_1|^2 + |c_2|^2 + |c_3|^2 = 1$; then

$$D_\mu \begin{bmatrix} \Phi'_u \\ \Phi'_d \end{bmatrix} = \left(\partial_\mu + \frac{ig_1}{6}\, B_\mu - \frac{ig_2}{2}\, \mathbf{W}_\mu\right) \begin{bmatrix} \Phi'_u \\ \Phi'_d \end{bmatrix}, \tag{365}$$

exactly as for the original left-handed u- and d-quark fields. Similarly, we can take linear combinations of the right-handed quark fields. Now suppose, that we take linear combinations of the left- and right handed quark fields such, that after spontaneous symmetry breaking the u-quark fields are mass eigenstates:

$$i\hbar\left(D_0 - \boldsymbol{\sigma}\cdot\mathbf{D}\right) \begin{bmatrix} \Phi'_u \\ \Phi'_d \end{bmatrix} = c \begin{bmatrix} m_u \Psi'_u \\ \mu_1 \Psi'_d + \mu_2 \Psi'_s + \mu_3 \Psi'_b \end{bmatrix} + ..., \tag{366}$$

where the dots denote terms containing non-linear couplings to the remaining scalar fields (e.g., the Higgs field). The point is, that a linear transformation which diagonalizes the mass terms of the u-type quarks does not necessarily diagonalize those of the d-type quarks. Of course, we could decide to define the d-type quark fields $\Phi''_{d,s,b}$ and $\Psi''_{d,s,b}$ as the mass-eigenstates as well; re-expressing the $\Phi'_{d,s,b}$ and $\Psi'_{d,s,b}$ in terms of these we then get general expression

$$\mu_1 \Psi'_d + \mu_2 \Psi'_s + \mu_3 \Psi'_b \equiv V_{ud}\, m_d \Psi''_d + V_{us}\, m_s \Psi''_s + V_{ub}\, m_b \Psi''_b, \tag{367}$$

such that eq. (366) becomes

$$\begin{aligned} i\hbar\,(D_0 - \boldsymbol{\sigma}\cdot\mathbf{D}) &\left[\begin{array}{c} \Phi'_u \\ V_{ud}\,\Phi''_d + V_{us}\,\Phi''_s + V_{ub}\,\Phi'' \end{array} \right] \\ &= c \left[\begin{array}{c} m_u \Psi'_u \\ V_{ud}\, m_d \Psi'_d + V_{us}\, m_s \Psi''_s + V_{ub}\, m_b \Psi''_b \end{array} \right], \end{aligned} \tag{368}$$

and similarly for the quark fields $\Phi'_{c,t}$ and $\Psi'_{c,t}$, combining with their own linear combinations of $\Phi''_{d,s,b}$ and $\Psi''_{d,s,b}$ defined by coefficients (V_{cd}, V_{cs}, V_{cb}) and (V_{td}, V_{ts}, V_{tb}), respectively. As far as the diagonal terms in the gauge-covariant derivatives are concerned, one could split the expression in the lower components of the field vector in the separate terms for each of the d-type fields $\Phi''_{d,s,b}$; however, there are also off-diagonal terms, which describe the coupling of u-type with d-type quarks through exchange of the charged vector bosons $W^{\pm}$. Hence in terms of the mass eigenstates, the charged-boson couplings become off-diagonal; for example, a c-quark can now decay by emission of a W^+-boson into a d, s or b quark, with probabilities given by $|V_{cd}|^2$, $|V_{cs}|^2$ and $V_{cb}|^2$. Of course, as probabilities must add up to one, the coefficients satisfy

$$|V_{id}|^2 + |V_{is}|^2 + V_{ib}|^2 = 1, \qquad i = (u, c, t), \tag{369}$$

and similarly

$$|V_{ua}|^2 + |V_{ca}|^2 + V_{ta}|^2 = 1, \qquad a = (d, s, b). \tag{370}$$

As these relations show, V_{ia} is a unitary matrix, known as the CKM-matrix[11]. The matrix elements V are measured in a variety of experiments. As an example, consider the inelastic scattering of neutrino's on protons. If the neutrino changes into a charged lepton by W-exchange, a u-quark in the proton can be transformed into a d-quark or into an s-quark:

$$\nu_\mu + p^+ \rightarrow \mu^- + n^0, \quad \text{or} \quad \nu_\mu + p^+ \rightarrow \mu^- + \Lambda^0, \tag{371}$$

where Λ^0 is a baryon with the quark content (uds). All other factors being equal, the ratio of the cross sections for these reactions is given by

$$\frac{\sigma(\nu_\mu + p \rightarrow \mu + \Lambda)}{\sigma(\nu_\mu + p \rightarrow \mu + n)} = \frac{|V_{us}|^2}{|V_{ud}|^2}. \tag{372}$$

Presently, measuring the CKM matrix is a very active and lively field of research in particle physics.

[11]Its construction was realized for 2 quark families by Cabibbo, and for 3 quark families by Kobayashi and Maskawa.

Appendix

1. The Thomson cross section

In this appendix we derive the scattering of electromagnetic radiation —such as light— by charged particles. This scattering process was first analysed in classical theory by Thomson as discussed in par. 4.

In the classical context, the charges are treated as non-relativistic particles. Let $\boldsymbol{\xi}(t)$ be the position of a point charge q at time t; the scalar and vector potential of the charge moving at velocity $\mathbf{v} = d\boldsymbol{\xi}/dt$ are

$$\phi(\mathbf{x},t) = \frac{q}{R - \mathbf{u}\cdot\mathbf{R}}, \qquad \mathbf{A}(\mathbf{x},t) = \frac{q\mathbf{u}/c}{R - \mathbf{u}\cdot\mathbf{R}}, \tag{373}$$

where

$$q = \frac{e}{4\pi\epsilon_0}, \qquad \mathbf{u} = \frac{\mathbf{v}}{c}, \qquad \mathbf{R} = \mathbf{x} - \boldsymbol{\xi}(t'), \qquad R = |\mathbf{R}|\,, \tag{374}$$

with t' the retarded time

$$t' = t - \frac{R}{c}. \tag{375}$$

Observe that these expressions reduce to the Coulomb potential in the limit $\mathbf{u} \to 0$. The electric and magnetic field strength at point $\mathbf{x}$ at time t are computed from

$$\mathbf{E} = -\nabla\phi - \frac{\partial\mathbf{A}}{\partial t}, \qquad \mathbf{B} = \nabla\times\mathbf{A}, \tag{376}$$

with the result

$$\begin{aligned} \mathbf{E} &= \frac{q}{(R - \mathbf{u}\cdot\mathbf{R})^3}\left[(\mathbf{R} - R\mathbf{u})\left(1 - \mathbf{u}^2\right) + \mathbf{R}\times((\mathbf{R} - R\mathbf{u})\times\dot{\mathbf{u}}/c)\right]. \\ c\mathbf{B} &= \hat{\mathbf{R}}\times\mathbf{E}, \qquad \hat{\mathbf{R}} = \mathbf{R}/R. \end{aligned} \tag{377}$$

Here $\dot{\mathbf{u}} = d^2\boldsymbol{\xi}/dt'^2$ is the retarded acceleration. Observe that

$$c^2\mathbf{B}^2 = \mathbf{E}^2 = \frac{1}{2}\left(\mathbf{E}^2 + c^2\mathbf{B}^2\right) = \frac{\mathcal{H}}{\epsilon_0}, \tag{378}$$

with $\mathcal{H}$ the energy density in the field.

If we go to a very large distance from the point charge, the first term in the expression for $\mathbf{E}$ becomes negligeable w.r.t. the second one, as it contains one less power of R. Then if the velocity is small ($u \ll 1$) the expression simplifies to

$$\mathbf{E} = \frac{q}{cR^3}\,\mathbf{R}\times(\mathbf{R}\times\dot{\mathbf{u}})\,, \qquad c\mathbf{B} = \frac{q}{cR^2}\,\dot{\mathbf{u}}\times\mathbf{R}. \tag{379}$$

The Poynting vector $\mathbf{N}$ is given by

$$c\mathbf{N} = \epsilon_0 c\,\mathbf{E} \times \mathbf{B} = -\epsilon_0\,\mathbf{E} \times \left(\mathbf{E} \times \hat{\mathbf{R}}\right) = \epsilon_0 \mathbf{E}^2\,\hat{\mathbf{R}} = \mathcal{H}\,\hat{\mathbf{R}}, \tag{380}$$

and represents the flux of field energy, directed perpendicular the electric and magnetic field strength. To see this, recall that Maxwell's equations in free space imply

$$H = \frac{\epsilon_0}{2} \int d^3x \left(\mathbf{E}^2 + c^2\mathbf{B}^2\right), \qquad \frac{dH}{dt} = -c \int d^3x\, \nabla \cdot \mathbf{N}. \tag{381}$$

This shows directly that $\mathbf{N}$ represents the momentum density in the field, i.e. the energy flux across the surface bounding the volume of integration. For a large sphere centered at distance R from the position of the point charge this becomes

$$\Phi = -\frac{dH}{dAdt} = c^2\,|\mathbf{N}| = c\mathcal{H}, \tag{382}$$

where we have taken into account that the Poynting vector is directly centrally, i.e. outward from and perpendicular to the spherical surface. To evaluate this flux it is now sufficient to calculate the magnetic field $\mathbf{B}$, eq. (379).

We now calculate $\mathbf{B}$ for motion of the charge under influence of a plane electromagnetic wave incident along the z-axis. We consider a charge moving under the influence of an electromagnetic wave moving in the z-direction:

$$\mathbf{E}_{inc} = E_0 \cos\left(\omega t - kz\right)\mathbf{i}, \qquad c\mathbf{B}_{inc} = \mathbf{k} \times \mathbf{E} = E_0 \cos\left(\omega t - kz\right)\mathbf{j}. \tag{383}$$

Then the Lorentz force on the point charge located at $z = 0$ is

$$\mathbf{F} = e\left(\mathbf{E}_{inc} + \mathbf{v} \times \mathbf{B}_{inc}\right) \approx eE_0 \cos\omega t\,\mathbf{i}. \tag{384}$$

In the last expression we have neglected the magnetic force, which is smaller by a factor $u = v/c \ll 1$. The magnetic field emitted by the oscillating point charge is given by (379):

$$c\mathbf{B} = \frac{q}{c^2 R}\,\dot{\mathbf{v}} \times \hat{\mathbf{R}} = \frac{q}{mc^2 R}\,\mathbf{F} \times \hat{\mathbf{R}}. \tag{385}$$

Now $\mathbf{F}$ points in the x-direction, and

$$\left|\mathbf{i} \times \hat{\mathbf{R}}\right| = \sin\psi, \tag{386}$$

where ψ is the angle between the centrally directed vector $\hat{\mathbf{R}}$ and the direction of polarization $\mathbf{i}$ of the electric field of the incident waves. Thus the outward energy flux induced by this polarized radiation is

$$\Phi_{pol} = \frac{\epsilon_0 q^2}{m^2 c^3 R^2}\, e^2 E_0^2 \cos^2\omega t\, \sin^2\psi. \tag{387}$$

Taking the time average:

$$\overline{\cos^2 \omega t} = \frac{1}{2}, \tag{388}$$

the time averaged energy flux becomes

$$\overline{\Phi}_{pol} = \frac{e^4}{4\pi\epsilon_0} \frac{E_0^2 \sin^2 \psi}{8\pi m^2 c^3 R^2}. \tag{389}$$

Now the flux of incident radiation is

$$\Phi_{inc} = \frac{c\epsilon_0}{2} \left(\mathbf{E}_{inc}^2 + c^2 \mathbf{B}_{inc}^2 \right) = c\,\epsilon_0 E_0^2 \cos^2 \omega t \quad \Rightarrow \quad \overline{\Phi}_{inc} = \frac{c\epsilon_0}{2} E_0^2. \tag{390}$$

The fraction of incident radiation scattered into the area element $dA = R^2 \sin\theta d\theta d\varphi$ then is

$$d^2\sigma_{pol} = \frac{\overline{\Phi}_{pol}}{\overline{\Phi}_{inc}} dA = \left(\frac{e^2}{4\pi\epsilon_0 mc^2} \right)^2 \sin^2 \psi \, d\Omega. \tag{391}$$

The classical radius of the electron is defined such that

$$\frac{e^2}{4\pi\epsilon_0 \, r_e} = mc^2 \quad \Leftrightarrow \quad r_e = \frac{e^2}{4\pi\epsilon_0 \, mc^2}; \tag{392}$$

with this definition

$$d^2\sigma_{pol} = r_e^2 \, \sin^2 \psi \, d\Omega. \tag{393}$$

The above computation holds for incident radiation with the electric field polarized along the x-axis. To average over all directions of polarization in the x-y-plane, we must re-express the angle ψ in terms of new polar angles θ and ϕ, defined by the direction of $\mathbf{E}_{inc}$ rather than that of the x-axis. This is done as follows; if $\mathbf{n}$ represents the unit vector in the direction of the electric field $\mathbf{E}_{inc}$, then

$$\sin^2 \psi = \left| \mathbf{n} \times \hat{\mathbf{R}} \right|^2 = \mathbf{n}^2 \hat{\mathbf{R}}^2 - \left(\mathbf{n} \cdot \hat{\mathbf{R}} \right)^2 = 1 - \sin^2 \theta \, \cos^2 \phi, \tag{394}$$

where θ still is the angle of outgoing radiation with the z-axis and can therefore be identified with the earlier θ appearing in $d\Omega$. Then the average over all polarizations of the electric field strength in the x-y-plane is equivalent to an average over all directions ϕ (which in general is not necessarily the same as the angle φ of the outgoing radiation w.r.t. the x-axis):

$$\frac{1}{2\pi} \int_0^{2\pi} d\phi \, \sin^2 \psi = \frac{1}{2} \left(1 + \cos^2 \theta \right). \tag{395}$$

It follows that the fraction of incident flux of unpolarized radiation scattered into the spherial angle $d\Omega$ is:

$$d^2\sigma = \frac{1}{2\pi} \int_0^{2\pi} d\phi \, d^2\sigma_{pol} = \frac{r_e^2}{2} \left(1 + \cos^2 \theta \right) d\Omega. \tag{396}$$

The quantity $d\sigma/d\Omega$ is the *differential scattering cross section* for scattering of electromagnetic waves by a free electron in the non-relativistic limit ($u = v/c \ll 1$).

Finally, the total cross section for this process is obtained by integrating over all directions represented by $d\Omega$, and is called the Thomson cross section:

$$\sigma_T = \int_{unit\ sphere} d^2\sigma = \frac{8\pi}{3}\, r_e^2. \tag{397}$$

2. **The scattering Green's function**

In this appendix we prove: the Green's function

$$G_k(\mathbf{r}) = \frac{1}{4\pi}\,\frac{e^{ikr}}{r} \tag{398}$$

satisfies the inhomogeneous linear partial differential equation

$$-\left(\Delta + k^2\right) G_k(\mathbf{r}) = \delta^3(\mathbf{r}). \tag{399}$$

First

$$\boldsymbol{\nabla}\,\frac{e^{ikr}}{r} = (ikr-1)\,\frac{e^{ikr}}{r^2}\,\hat{\mathbf{r}}, \tag{400}$$

where $\hat{\mathbf{r}}$ is the radial unit vector. Then, taking the divergence of this result, it immediately follows that for $\mathbf{r} \neq 0$

$$\left(\Delta + k^2\right) G_k(\mathbf{r}) = 0. \tag{401}$$

Hence we concentrate on the region near the origin $\mathbf{r} = 0$. We prove: for a spherical volume V_R with radius R

$$\lim_{R\to 0}\int_{V_R}\left(\Delta + k^2\right) G_k(\mathbf{r})\, d^3V = -1, \tag{402}$$

showing that the integral remains finite even if the sphere is contracted to a point. First, we prove that the second term does not contribute:

$$\int_{V_R}\frac{e^{ikr}}{r}\, d^3V = \int_0^{2\pi} d\varphi \int_{-1}^{+1} d\cos\theta \int_0^R dr\, r^2\,\frac{e^{ikr}}{r} = \frac{4\pi}{k^2}\,\left(e^{ikR}-1\right), \tag{403}$$

and therefore

$$\lim_{R\to 0} k^2 \int_{V_R}\frac{e^{ikr}}{r}\, d^3V = \lim_{R\to 0} 4\pi\left(e^{ikR}-1\right) = 0. \tag{404}$$

Finally we consider the first term; we use Gauss' theorem to get

$$\int_{V_R}\Delta\,\frac{e^{ikr}}{r}\, d^3V = \int_{V_R}\boldsymbol{\nabla}\cdot\left(\boldsymbol{\nabla}\,\frac{e^{ikr}}{r}\right) d^3V = \int_{\Sigma_R}\left(\nabla\,\frac{e^{ikr}}{r}\right)_n d^2\Sigma, \tag{405}$$

where Σ_R is the surface of the sphere V_R, n denotes the normal (radial) component of the gradient on the surface, and $d^2\Sigma$ is the two-dimensional integration element. Now using the result (400) evaluated at $r = R$ one finds

$$\left. \left(\nabla \frac{e^{ikr}}{r} \right)_n \right|_{r=R} = (ikR - 1)\, \frac{e^{ikR}}{R^2}. \tag{406}$$

The spherical surface element is

$$d^2\Sigma = R^2 \sin\theta \, d\theta d\varphi. \tag{407}$$

Substitution in eq. (405) gives

$$\int_{V_R} \Delta \, \frac{e^{ikr}}{r} \, d^3V = 4\pi \, (ikR - 1)\, e^{ikR}. \tag{408}$$

Combining the results (402), (404) and (408) we finally get

$$\lim_{R\to 0} \int_{V_R} \left(\Delta + k^2 \right) G_k(\mathbf{r}) \, d^3V = \lim_{R\to 0} \, (ikR - 1)\, e^{ikR} = -1. \tag{409}$$

This proves the result (402).

3. **Special relativity**

Special relativity is based on two important empirical observations:
1. the existence of a special class of co-ordinate systems, in which all free particles are in rest or move uniformly in a straight line;
2. the universality and invariance of the speed of light, which is the same for observers in all inertial systems.

Newton's first law states, that free particles are at rest or move with constant velocity on straight lines. This is only true for observers who are not accelerated by external forces themeselves. The co-ordinate systems associated with such obervers are called *inertial frames*.

If a particle moves with constant velocity on a straight line in one such frame, it will also move on a straight line in any shifted, rotated or moving frame, provided the translation or change in orientation is constant, or the velocity of one system w.r.t. the others is constant in a fixed direction. Therefore frames connected to an inertial frame by a constant translation, constant rotation or constant linear motion are also inertial frames. Examples:
a. *translation:*

$$x' = x + a, \qquad y' = y, \qquad z' = z; \tag{410}$$

b. *rotation:*

$$x' = x\cos\alpha - y\sin\alpha, \qquad y' = x\sin\alpha + y\cos\alpha, \qquad z' = z; \tag{411}$$

c. *linear motion:*

$$x' = \gamma(x - vt), \qquad y' = y, \qquad z' = z, \tag{412}$$

with v the relative velocity and γ a proportionality constant to be determined.

According to the special theory of relativity Minkowski space-time intervals are the same in all inertial frames:

$$dx'^2 + dy'^2 + dz'^2 - c^2 dt'^2 = dx^2 + dy^2 + dz^2 - c^2 dt^2, \tag{413}$$

where the light velocity c has the same value on both sides. Indeed, for an observer in the frame (x, y, z, t) a light ray moves with velocity $\mathbf{v}$ given by

$$dx^2 + dy^2 + dz^2 - c^2 dt^2 = 0 \quad \Rightarrow \quad \mathbf{v}^2 = \left(\frac{dx}{dt} \right)^2 + \left(\frac{dy}{dt} \right)^2 + \left(\frac{dz}{dt} \right)^2 = c^2. \tag{414}$$

By eq. (413) this is then also true in any other inertial frame:

$$dx'^2 + dy'^2 + dz'^2 - c^2 dt'^2 = 0 \quad \Rightarrow \quad \mathbf{v}'^2 = c^2. \tag{415}$$

Therefore the velocity of light is a universal constant, taking the same value in all inertial frames.

It is easy to see that the space-time interval (413) is invariant under the translations (410)

$$dx' = d(x + a) = dx, \qquad dy' = dy, \qquad dz' = dz, \tag{416}$$

and under the rotations (411):

$$dx' = dx \cos\alpha - dy \sin\alpha, \qquad dy' = dx \sin\alpha + dy \cos\alpha, \qquad dz' = dz. \tag{417}$$

It is also invariant for frames connected by linear motion, provided the constant γ takes the special value:

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}, \tag{418}$$

and the clock time is adjusted in a similar way:

$$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}}, \qquad ct' = \frac{ct - vx/c}{\sqrt{1 - v^2/c^2}}, \tag{419}$$

such that

$$dx' = \frac{dx - vdt}{\sqrt{1 - v^2/c^2}}, \qquad cdt' = \frac{cdt - vdx/c}{\sqrt{1 - v^2/c^2}}. \tag{420}$$

It is then straightforward to establish that

$$dx'^2 - c^2 dt'^2 = dx^2 - c^2 dt^2. \tag{421}$$

The Minkoswki interval (413) can be combining the *contravariant* space-time interval

$$dx^\mu = (dx^0, dx^1, dx^2, dx^3) = (cdt, dx, dy, dz) \tag{422}$$

with the *covariant* space-time interval

$$dx_\mu = (dx_0, dx_1, dx_2, dx_3) = (-cdt, dx, dy, dz). \tag{423}$$

Then

$$\sum_{\mu=0}^{3} dx^\mu dx_\mu = -c^2dt^2 + dx^2 + dy^2 + dz^2. \tag{424}$$

Any set of quantities $a^\mu = (a^0, a^1, a^2, a^3)$ of which the components in different inertial frames are related in the same way as the contravariant intervals dx^μ is called a *contravariant four-vector*. For example, for two frames moving with relative velocity v in the x-direction the components of a' must be related to those of a by (420)

$$a'^1 = \frac{a^1 - va^0/c}{\sqrt{1 - v^2/c^2}}, \qquad a'^0 = \frac{a^0 - va^1/c}{\sqrt{1 - v^2/c^2}}. \tag{425}$$

Similarly, a set of quantities $a_\mu = (a_0, a_1, a_2, a_3)$ transforming between inertial frames as the covariant intervals dx_μ is called a *contravariant four-vector*. They must be related to the contravariant components like the covariant and contravariant differentials:

$$(a_0, a_1, a_2, a_3) = (-a^0, a^1, a^2, a^3). \tag{426}$$

Therefore

$$\sum_{\mu=0}^{3} a^\mu a_\mu = -(a^0)^2 + (a^1)^2 + (a^2)^2 + (a^3)^2. \tag{427}$$

It is standard practice in such multipication of covariant and contravariant vectors to omit the summation sign; this is known as the Einstein summation convention:

$$a^\mu a_\mu \equiv \sum_{\mu=0}^{3} a^\mu a_\mu. \tag{428}$$

The switch from contravariant to covariant components is achieved by a similar multiplication with the Minkowski metric or its inverse:

$$\eta_{\mu\nu} = \eta^{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{429}$$

Then

$$a_\mu = \sum_{\nu=0}^{3} \eta_{\mu\nu} a^\nu \equiv \eta_{\mu\nu} a^\nu, \quad a^\mu = \sum_{\nu=0}^{3} \eta^{\mu\nu} a_\nu \equiv \eta^{\mu\nu} a_\nu, \tag{430}$$

whilst similar matrix multiplication gives

$$\eta^{\mu\lambda}\eta_{\lambda\nu} = \delta^{\mu}_{\nu}. \tag{431}$$

In all these products we always sum over a common upper and lower index. We can also use the Minkowski metric to define the Minkowski space-time interval

$$\begin{aligned} \eta_{\mu\nu}\, dx^{\mu}dx^{\nu} &= \eta_{00}(dx^0)^2 + \eta_{11}(dx^1)^2 + \eta_{22}(dx^2)^2 + \eta_{33}(dx^3)^2 \\ &= -c^2dt^2 + dx^2 + dy^2 + dz^2. \end{aligned} \tag{432}$$

One can distinguish three kinds of space-time intervals: space-like, light-like and time-like. The distinction depends on the sign of the expression (432):

$$-c^2dt^2 + dx^2 + dy^2 + dz^2 = \begin{cases} ds^2 > 0 & \Rightarrow \quad \text{space-like;} \\ 0 & \Rightarrow \quad \text{light-like;} \\ -c^2d\tau^2 < 0 & \Rightarrow \quad \text{time-like.} \end{cases} \tag{433}$$

Note that intervals between two points on the worldline of a single particle are time-like; in particular

$$d\tau = dt\sqrt{1 - \frac{1}{c^2}\left(\frac{d\mathbf{r}}{dt}\right)^2} = dt\sqrt{1 - \frac{\mathbf{v}^2}{c^2}}, \tag{434}$$

where $\mathbf{v}$ is the velocity of the particle. Therefore $d\tau$ is the time interval measured on a clock in the rest frame of the particle (in which $\mathbf{v} = 0$); this is known as the *proper time.*

Similar to the Lorentz-invariant line element (432), we can define the Lorentz-invariant Laplace operator:

$$\Box = \eta^{\mu\nu}\frac{\partial^2}{\partial x^{\mu}\partial x^{\nu}} = -\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}. \tag{435}$$

It is customary to use the abbreviated notation

$$\partial_{\mu} = \frac{\partial}{\partial x^{\mu}}, \qquad \partial^{\mu} = \eta^{\mu\nu}\partial_{\nu} = \frac{\partial}{\partial x_{\mu}}, \qquad \Box = \partial^{\mu}\partial_{\mu}. \tag{436}$$

An example of another 4-vector is the 4-velocity of a particle, obtained as the derivative of the space-time position w.r.t. proper time:

$$u^{\mu} = \frac{dx^{\mu}}{d\tau}. \tag{437}$$

Using relation (434) between proper time and observer time, it follows that

$$u^{\mu} = (\gamma c, \gamma v_x, \gamma v_y, \gamma v_z), \tag{438}$$

where γ is the relativistic time-dilation factor, and $\mathbf{v}$ is the ordinary velocity in observer co-ordinates:

$$\gamma = \frac{1}{\sqrt{1 - \mathbf{v}^2/c^2}}, \quad v_i = \frac{dx_i}{dt}. \tag{439}$$

As proper time is oberver-invariant, it is easy to check that u^μ transforms under Lorents transformations as the co-ordinate differentials themselves, cf. eq. (425). Also note, that we can construct an invariant

$$u^\mu u_\mu = -c^2. \tag{440}$$

Similarly we define the 4-momentum

$$p^\mu = m u^\mu = (\gamma mc, \gamma m v_x, \gamma m v_y, \gamma m v_z), \tag{441}$$

which is a 4-vector because u^μ is, whilst m is invariant. The 4-momentum satisfies

$$p^\mu p_\mu = -m^2 c^2. \tag{442}$$

Explicitly, the space- and time-components read

$$\mathbf{p} = \frac{m\mathbf{v}}{\sqrt{1 - \mathbf{v}^2/c^2}}, \qquad p^0 = \frac{mc}{\sqrt{1 - \mathbf{v}^2/c^2}} = \frac{E}{c}, \tag{443}$$

where E is the total relativistic energy of the particle. Eq. (442) can then be written in the more familiar form

$$E^2 = \mathbf{p}^2 c^2 + m^2 c^4. \tag{444}$$

The interest in the momentum 4-vector derives from the fact, that the total 4-momentum is conserved in elastic collisions:

$$P_f^\mu = P_i^\mu, \qquad P_{i,f}^\mu = \sum p_{i,f}^\mu, \tag{445}$$

where the labels (i, f) refer to the initial and final states of motion.

4. **The covariant Dirac equation**

The Dirac theory of a free massive spin-1/2 particle is defined by equations (251):

$$i\hbar \left(\partial_0 + \boldsymbol{\sigma} \cdot \nabla \right) \Psi(x) = mc\, \Phi(x), \qquad i\hbar \left(\partial_0 - \boldsymbol{\sigma} \cdot \nabla \right) \Phi(x) = mc\, \Psi(x). \tag{446}$$

In the massless case, these equations decouple and one can use either one of them, depending on the handedness (chirality) of the particle. By construction, the solutions of the Dirac equations describe relativistic particles.

It is often convenient to rewrite the two equations (446) for two 2-component spinors as a single equation for a 4-component spinor, as follows. Define a 4-component spinor by a direct sum of the left- and right-handed ones

$$\Theta = \left[\begin{array}{c} \Psi \\ \Phi \end{array} \right] = \left[\begin{array}{c} \left(\begin{array}{c} \Psi_1 \\ \Psi_2 \end{array} \right) \\ --- \\ \left(\begin{array}{c} \Phi_1 \\ \Phi_2 \end{array} \right) \end{array} \right]. \tag{447}$$

Furthermore, introduce a set of 4×4 Dirac matrices

$$\gamma^\mu = \left(\gamma^0, \boldsymbol{\gamma} \right); \qquad \gamma^0 = \left(\begin{array}{cc} 0 & 1_2 \\ 1_2 & 0 \end{array} \right), \quad \boldsymbol{\gamma} = \left(\begin{array}{cc} 0 & -\boldsymbol{\sigma} \\ \boldsymbol{\sigma} & 0 \end{array} \right), \tag{448}$$

where 1_d stands for the d-dimensional unit matrix[12]. Then the pair of Dirac equations (446) can be written as a single equation

$$(-i\hbar\gamma^\mu \partial_\mu + mc)\,\Theta = 0 \quad \Leftrightarrow \quad \left(\begin{array}{c|c} mc & -i\hbar(\partial_0 - \boldsymbol{\sigma}\cdot\nabla) \\ \hline -i\hbar(\partial_0 + \boldsymbol{\sigma}\cdot\nabla) & mc \end{array} \right) \left[\begin{array}{c} \Psi \\ \Phi \end{array} \right] = 0. \tag{449}$$

It is straightforward to check, that the Dirac matrices γ^μ satisfy an anti-commutation relation

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = -2\,\eta^{\mu\nu}\,1_4. \tag{450}$$

It is then easy to verify, that the Dirac equation implies the Klein-Gordon equation:

$$\left(-\Box + \mu^2 \right)\Theta = (i\gamma\cdot\partial + \mu)\,(-i\gamma\cdot\partial + \mu)\,\Theta = 0, \tag{451}$$

with $\mu = mc/\hbar$ the Compton wave number.

To describe the interactions of fermions with gauge fields of the Maxwell-Yang-Mills type, it is now sufficient to replace the particles derivatives in the Dirac equation by gauge-covariant derivatives:

$$\partial_\mu \to D_\mu = \partial_\mu - ig\mathbf{A}_\mu \quad \Rightarrow \quad (-i\gamma^\mu D_\mu + \mu)\,\Theta = 0. \tag{452}$$

This way one constructs e.g. the theories of Quantum Electrodynamics (QED) and Quantum Chromodynamics (QCD) of quarks and leptons.

[12]Often the unit matrices are not written explicitly, as they just multiply every component by 1.

Exercises 1

1. *Finestructure constant*
   Define
   $$\alpha = \frac{e^2}{4\pi\varepsilon_0\hbar c}$$
   a. Look up the values of $(e, \varepsilon_0, c, \hbar)$ in MKS units.
   b. Show that α is dimensionless.
   c. Compute the numerical value of α.

2. *Lorentz transformations*
   An inertial system Σ' moves w.r.t. an inertial system Σ with (constant) velocity $\mathbf{v} = (v, 0, 0)$. The co-ordinates in Σ' are related to those in Σ by the special Lorentz-transformations
   $$\begin{pmatrix} ct' \\ x' \end{pmatrix} = \frac{1}{\sqrt{1 - \mathbf{v}^2/c^2}} \begin{pmatrix} 1 & -v/c \\ -v/c & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix}.$$
   a. A light-ray moves in system Σ with velocity $dx/dt = c$. Show that in Σ' its velocity dx'/dt' is the same.
   b. If the observer O' is at rest in the origin of Σ', show his position in Σ is given by $x = vt$.
   c. O' measures a time interval $\Delta t'$ on a clock at rest in his system; show that in Σ the corresponding interval is
   $$\Delta t = \frac{\Delta t'}{\sqrt{1 - \mathbf{v}^2/c^2}}.$$
   d. A muon is a charged particle with mass m_μ s.t.
   $$m_\mu c^2 = 105 \text{ MeV}.$$
   It is instable, and decays on average after a time
   $$\tau_\mu = 2.2\ \mu\text{s}.$$
   If the muon has an energy of $E = 1$ GeV, how far can it travel on average before it decays?

Exercises 2

1. *Center of Mass (CM) system*

Consider the interaction of 2 particles with masses (m_1, m_2) and positions $(\mathbf{r}_1, \mathbf{r}_2)$. In (non-relativistic) classical mechanics their momenta are

$$\mathbf{p}_1 = m_1\dot{\mathbf{r}}_1, \qquad \mathbf{p}_2 = m_2\dot{\mathbf{r}}_2,$$

and the energy is

$$E = \frac{1}{2}m_1\dot{\mathbf{r}}_1^2 + \frac{1}{2}m_2\dot{\mathbf{r}}_2^2 + V(\mathbf{r}_1, \mathbf{r}_2),$$

with V the potential energy of the interaction. For central forces the potential depends only on the relative particle separation $\mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$:

$$V(\mathbf{r}_1, \mathbf{r}_2) = V(r), \qquad r = |\mathbf{r}| = \sqrt{(\mathbf{r}_2 - \mathbf{r}_1)^2}.$$

a. Define the center of mass $\mathbf{R}$ by

$$M\mathbf{R} = m_1\mathbf{r}_1 + m_2\mathbf{r}_2, \qquad M = m_1 + m_2.$$

Show that the energy can be written as

$$E = \frac{1}{2}\,M\dot{\mathbf{R}}^2 + \frac{1}{2}\,\mu\dot{\mathbf{r}}^2 + V(r),$$

where the reduced mass is given by

$$\mu = \frac{m_1 m_2}{m_1 + m_2}.$$

b. The CM system is the inertial sytem in which the center of mass is at rest in the origin of co-ordinates: $\mathbf{R} = 0$. Check that in this system

$$\mathbf{P} = \mathbf{p}_1 + \mathbf{p}_2 = 0,$$

and that in this system the energy is

$$E_{CM} \equiv \varepsilon = \frac{1}{2}\,\mu\dot{\mathbf{r}}^2 + V(r).$$

c. Introduce spherical co-ordinates

$$x = r\sin\theta\cos\varphi, \quad y = r\sin\theta\sin\varphi, \quad z = r\cos\theta.$$

Derive that in these co-ordinates

$$\varepsilon = \frac{1}{2}\,\mu\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\,\dot{\varphi}^2\right) + V(r).$$

2. *Angular momentum*

The total angular momentum of a system is defined as

$$\mathbf{L} = \sum_i \mathbf{r}_i \times \mathbf{p}_i = \sum_i m_i \mathbf{r}_i \times \dot{\mathbf{r}}_i.$$

a. Show that for a 2-particle system with a central force (as in the first exercise) the total angular momentum is conserved:

$$\frac{d\mathbf{L}}{dt} = 0.$$

b. Explain why this result implies that the relative motion of the 2-particle system is confined to a plane.

c. Consider the CM-system in polar co-ordinates. Choose the plane of motion to be the equatorial plane $\theta = \pi/2$. Show that in these co-ordinates

$$\mathbf{L} = (0, 0, l), \qquad l = \mu r^2 \dot{\varphi},$$

and

$$\varepsilon = \frac{\mu}{2} \left(\dot{r}^2 + r^2 \dot{\varphi}^2 \right) + V(r) = \frac{\mu}{2}\, \dot{r}^2 + \frac{l^2}{2\mu r^2} + V(r).$$

Exercises 3

1. Let $\rho(\mathbf{r})$ be a charge distribution normalized to unity:

$$\int d^3r\,\rho(\mathbf{r}) = 1.$$

The form factor $F(\boldsymbol{\kappa})$ is the Fourier transform

$$F(\boldsymbol{\kappa}) = \int d^3r\, e^{i\boldsymbol{\kappa}\cdot\mathbf{r}}\rho(\mathbf{r}).$$

a. Show, that for a spherically symmetric charge distribution $\rho(r)$ the form factor is given by

$$F(\kappa) = 4\pi \int_0^\infty dr\, r^2\, \frac{\sin\kappa r}{\kappa r}\,\rho(r) = \frac{4\pi}{\kappa}\int_0^\infty dr\, r\, \sin\kappa r\,\rho(r)$$

b. Compute $F(\kappa)$ for the cases

$$(i) \quad \rho(r) = \frac{1}{8\pi a^3}\, e^{-r/a},$$

$$(ii) \quad \rho(r) = \begin{cases} \dfrac{3}{4\pi a^3}, & r < a; \\ 0, & r > a, \end{cases}$$

where a is a constant.

2. The masses of the nuclei ${}^7_4$Be and ${}^7_3$Li are

$$m_{Be} = 7.016\,929 \text{ amu}, \qquad m_{Li} = 7.016\,004 \text{ amu}.$$

a. Explain the reaction (electron capture)

$${}^7_4\text{Be} + e \rightarrow {}^7_3\text{Li} + \nu_e.$$

b. Compute the energy and momentum of the neutrino ν_e in the limit $m_\nu = 0$ if the Be-atom is initially at rest.

Exercises 4

1. *Radioactive chains*
A radioactive element A decays into element B with a probability per atom per unit of time λ_A. The element B is also radioactive and decays to a stable element C with a probability λ_B per atom and per unit of time.
a. Explain that the change in time of the average number of atoms of kind A and B is given by
$$dN_A(t) = -\lambda_A N_A(t) dt,$$
$$dN_B(t) = -\lambda_B N_B(t) dt + \lambda_A N_A(t) dt.$$
b. Derive an expression for the change $dN_C(t)$ of the average number of atoms C in a time interval dt.
c. At time $t = 0$ we start out with a pure sample of N_0 atoms of type A; calculate the number of atoms of each kind (A, B, C) as a function of time.
d. Sketch the solutions $N_A(t)$, $N_B(t)$ and $N_C(t)$.

2. *Poisson statistics*
Radioactive decay processes are statistically independent events: the probability of a single atom decaying in a time interval dt is proportional to the length of the interval, independent of its history and independent of the concentration of atoms.

Show:
For a sufficiently large number N of a radioactive atoms the probability $P_n(t)$ of a fraction n/N decaying in a time t depends only on the decay probability λ of a single atom, and on the length of the time interval t:
$$P_n(t) = \frac{(\lambda t)^n}{n!} \, e^{-\lambda t}. \qquad (A)$$

Hints:
(i) The statistical independence of the decay processes implies that the probability for n_1 decays in a time interval Δ_1 and n_2 decays in a non-overlapping time interval Δ_2 is
$$P(n_1, \Delta_1; n_2, \Delta_2) = P_{n_1}(\Delta_1) P_{n_2}(\Delta_2).$$
(ii) The probability for a single decay in a short time interval Δ is
$$P_1(\Delta) = \lambda \Delta + \mathcal{O}(\Delta^2).$$
(iii) The probabiblity per unit of time for two or more decays in a short time interval Δ vanishes in the limit $\Delta \to 0$:
$$P_n(\Delta) = \mathcal{O}(\Delta^2), \qquad n \geq 2.$$

From these properties, first show that

$$P_0(t+\Delta) = P_0(t)P_0(\Delta) = P_0(t)\left(1 - \lambda\Delta + \mathcal{O}(\Delta^2)\right).$$

Now prove that in the limit $\Delta \to 0$, it follows that

$$\frac{dP_0(t)}{dt} = -\lambda P_0(t).$$

Next consider the probability for one or more decays to take place in an interval $t + \Delta$; show that

$$P_n(t+\Delta) = \sum_{k=0}^{n} P_k(t)P_{n-k}(\Delta) = P_n(t)\,(1-\lambda\Delta) + P_{n-1}(t)\,\lambda\Delta + \mathcal{O}(\Delta^2).$$

By taking the limit $\Delta \to 0$, derive

$$\frac{dP_n(t)}{dt} = -\lambda P_n(t) + \lambda P_{n-1}(t), \qquad n \geq 1.$$

Finally, with the boundary conditions $P_0(0) = 1$, $P_n(0) = 0$ for $n \geq 1$, solve the differential equations for $P_n(t)$ to derive the result (A):

$$P_0(t) = e^{-\lambda t}, \qquad P_1(t) = \lambda t\, e^{-\lambda t}, \qquad P_2(t) = \frac{(\lambda t)^2}{2}\, e^{-\lambda t}, \qquad \ldots$$

Verify that at any time t the sum of all probabilities is unity:

$$\sum_{n=0}^{\infty} P_n(t) = 1.$$

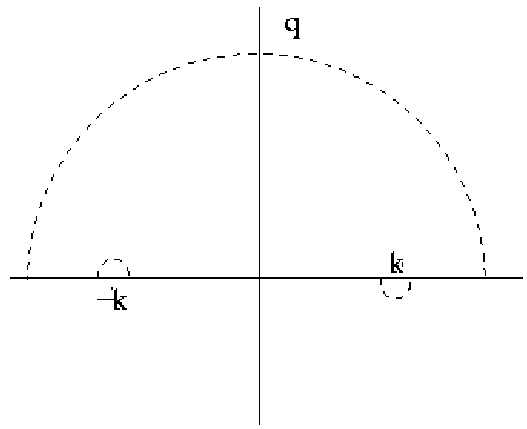

Figure 24: Contour voor G_k

Excercises 5

1. Show that the function

$$G_k(\mathbf{r}) = \frac{e^{ikr}}{r}$$

is a solution of the inhomogeneous partial differential equation

$$\left(\Delta + k^2\right) G_k(\mathbf{r}) = -4\pi\, \delta^3(\mathbf{r}),$$

where $\delta^3(\mathbf{r})$ is the 3-dimensional Dirac δ-function, with the Fourier representation

$$\delta^3(\mathbf{r}) = \int \frac{d^3q}{(2\pi)^3}\, e^{i\mathbf{q}\cdot\mathbf{r}}.$$

Hint:
Show that the Fouriertransform of $G_k(\mathbf{r})$ is given by

$$\tilde{G}_k(\mathbf{q}) = \int d^3r\, e^{-i\mathbf{q}\cdot\mathbf{r}} G_k(\mathbf{r}) = \frac{4\pi}{q^2 - k^2}.$$

Next perform the inverse Fouriertransform to compute $G_k(\mathbf{r})$. To compute the integral over $q = |\mathbf{q}|$, use the contour of fig. 24.

2. Compute the average radius squared $(\overline{r^2})$ of a charge distribution which is a hard sphere of radius a (see exercise 3.1).

Exercises 6

1. Consider two inertial co-ordinate systems x^μ and x'^μ; we suppose that at the moment the origins of the two systems coincide the clocks are synchronized.
   a. Argue that these systems are related by a homogeneous transformation
   $$x'^\mu = x^\nu \, \Lambda_\nu^{\ \mu}.$$
   b. Let the two systems are related by a rotation in the y-z-plane:
   $$\Lambda = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\theta & -\sin\theta \\ 0 & 0 & \sin\theta & \cos\theta \end{pmatrix}.$$
   Construct the inverse transformation
   $$x^\mu = x'^\nu \, \left(\Lambda^{-1}\right)_\nu^{\ \mu}.$$
   Show that Λ and Λ^{-1} satisfy the property:
   $$\left(\Lambda^{-1}\right)_\nu^{\ \mu} = \eta^{\mu\kappa}\Lambda_\kappa^{\ \lambda}\eta_{\lambda\nu}. \tag{A}$$
   c. Alternatively, let the systems be related by a Lorentz boost along the x-axis:
   $$\Lambda = \begin{pmatrix} \frac{1}{\sqrt{1-v^2/c^2}} & \frac{v/c}{\sqrt{1-v^2/c^2}} & 0 & 0 \\ \frac{v/c}{\sqrt{1-v^2/c^2}} & \frac{1}{\sqrt{1-v^2/c^2}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$
   Again, construct the inverse transformation and show that Λ and Λ^{-1} satisfy the above property (A).

2. The Pauli matrices are used to represent the spin operators for non-relativistic fermions like electrons:
   $$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$
   a. Check that the operators σ_i satisfy the commutation relations:
   $$[\sigma_i, \sigma_j] = \sigma_i\sigma_j - \sigma_j\sigma_i = 2i\,\epsilon_{ijk}\sigma_k,$$

where the completely anti-symmetric ϵ-symbol is:

$$\epsilon_{ijk} = \begin{cases} +1, & \text{if } (ijk) = \text{even permutation of (123);} \\ -1, & \text{if } (ijk) = \text{odd permutation of (123);} \\ 0, & \text{in all other cases.} \end{cases} .$$

Show from this result, that the operators

$$s_i = \frac{\hbar}{2}\,\sigma_i$$

satisfy the standard commutation relations of angular momentum.
b. Show that the Pauli matrices satisfy the following *anti-commutation* relations:

$$\{\sigma_i, \sigma_j\} = \sigma_i\sigma_j + \sigma_j\sigma_i = 2\,\delta_{ij}\,e_2,$$

where e_2 is the 2-dimensional unit matrix:

$$e_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$