The Standard Model and Beyond

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Chapter 1

Chiral fermions

1.1 Introduction

The majority of the elementary states of matter we encounter in nature at subatomic scales are chiral fermion states; they correspond to massless spin-1/2 particles with only one polarization state, the opposite polarization being carried by the corresponding anti-particle. The polarization is described by the chirality, to which we associate a quantum number taking by convention the value +1 for a left-handed particle, and -1 for a right-handed one.

The interactions of the fermionic states of matter are mediated by bosonic fields, which can have scalar (spin-0), vector (spin-1) or tensor (spin-2) character. The only long-range force with tensor character we are aware of is the gravitational force, which is extremely weak. As a result, in subatomic physics it can only play a role close to the Planck scale. In much of the standard phenomenology of particles and their interactions it can therefore be disregarded.

Vector fields are associated with conserved quantum numbers (generalized forms of charge), related to various types of symmetries between fields. In these notes much attention is devoted both to the character of the charges and the nature of the symmetries they represent.

Scalar fields, finally, are associated with symmetry breaking and the generation of particle masses. They can also be related to fermions by supersymmetry, a subject to be explored in some detail. As yet, no fundamental scalar fields have been observed in experiments. But intensive and dedicated searches are and will be undertaken at present and planned accelerators like those of CERN and FNAL.

Chiral fermions in four-dimensional space-time are described by Weyl spinors, i.e. eigenspinors of γ_5 with eigenvalue $+1$ for left-handed Weyl spinors, and -1 for right-handed Weyl spinors[1]:

$$\gamma_5 \, \psi_L = \psi_L, \qquad \gamma_5 \psi_R = -\psi_R. \tag{1.1}$$

[1]Details and conventions are collected in appendix A.

Weyl spinors can not be self-conjugate under charge conjugation; that is, a left-handed current:

$$j_L^\mu = \bar{\psi}_L \gamma^\mu \psi_L, \tag{1.2}$$

is physically distinct from a right-handed current

$$j_R^\mu = \bar{\psi}_R \gamma^\mu \psi_R. \tag{1.3}$$

However, it is straightforward to construct a self-conjugate spinor from a Weyl spinor, as follows. The charge conjugate of a left-handed spinor ψ_L is a right-handed spinor:

$$\psi_R \equiv (\psi_L)^c = \mathcal{C} \bar{\psi}_L^T. \tag{1.4}$$

For a proof and discussion see the appendix, in particular eqs.(A.72) and (A.77). The inverse property then holds as well:

$$\psi_L = (\psi_R)^c = \mathcal{C}\, \bar{\psi}_R^T. \tag{1.5}$$

Now the linear combination

$$\chi \equiv \psi_L + \psi_R, \tag{1.6}$$

is by construction self-conjugate:

$$\chi^c = \mathcal{C}\, \bar{\chi}^T = \psi_R + \psi_L = \chi. \tag{1.7}$$

In other words, χ is a Majorana spinor. This transformation of a Weyl spinor into a Majorana spinor is invertible; indeed, we can simply apply projection operators:

$$\psi_L = \frac{1}{2} \left(1 + \gamma_5\right) \chi, \qquad \psi_R = \frac{1}{2} \left(1 - \gamma_5\right) \chi. \tag{1.8}$$

We can also consider the other linear combination

$$\eta = i \left(\psi_L - \psi_R\right), \tag{1.9}$$

which is a Majorana spinor as well: $\eta^c = \eta$. From eq.(1.8) it is obvious, that η is not an independent spinor, but related to χ by

$$\eta = i\gamma_5\, \chi. \tag{1.10}$$

Note that the necessary presence of the imaginary factor i in this relation between Majorana spinors is precisely the reason why the Weyl spinors (1.8) can *not* be Majorana spinors at the same time. Finally, it can now be established that any Dirac spinor Ψ is a complex linear combination of two Majorana spinors:

$$\Psi = \chi_1 + i\chi_2. \tag{1.11}$$

The 'real' and 'imaginary' parts can be constructed by considering combinations of Ψ and its charge-conjugate Ψ^c:

$$\chi_1 \,=\, \frac{1}{2}\left(\Psi + \Psi^c\right), \qquad \chi_2 \,=\, \frac{1}{2i}\left(\Psi - \Psi^c\right). \tag{1.12}$$

Although Weyl and Majorana spinors are mathematically equivalent, their physical interpretations are quite different. In four-dimensional space-time Weyl spinors are complex, and carry a non-trivial phase factor. Therefore a single Weyl spinor can always be made to transform under vector-like $U(1)$-transformations, which act oppositely on a spinor field and its charge conjugate, such as associated e.g. with electro-magnetism. In contrast, a single Majorana spinor can not carry a vector-like $U(1)$ charge, as it is invariant under charge conjugation. For masses and Yukawa couplings the reverse is true: a single Majorana spinor can be coupled consistently to a real scalar field and carry a mass, whereas a single Weyl spinor satisfying a phase-preserving Dirac equation is necessarily massless.

The issue is, that a Dirac like operator $D\!\!\!/ = \gamma^\mu(\partial_\mu - ieA_\mu)$ changes the chirality:

$$\gamma_5\, D\!\!\!/ \,=\, -\, D\!\!\!/\, \gamma_5, \tag{1.13}$$

hence $D\!\!\!/ \psi_L$ is a right-handed spinor and can not be proportional to a real scalar field or mass times ψ_L itself. Moreover, the charge-conjugate $\psi_R = (\psi_L)^c$ transforms with the opposite phase under $U(1)$ transformations $\psi_L \rightarrow \psi_L^{\prime} = e^{i\alpha}\psi_L$. Therefore a consistent equation for a single Weyl spinor invariant under $U(1)$ transformations must be of type

$$D\!\!\!/ \psi_L \,=\, 0. \tag{1.14}$$

In contrast, the free Dirac operator $\partial\!\!\!/$ does respect the Majorana property: $\partial\!\!\!/ \psi$ has the same charge-conjugation properties as ψ itself, but the factor i in the minimal gauge-field coupling via the covariant derivative $D_\mu = \partial_\mu - ieA_\mu$ destroys the Majorana property of a spinor if A_μ is real or has a real part; hence a complex $U(1)$ gauge coupling is not allowed for a single Majorana spinor. Majorana spinor can be coupled consistently to real scalars by a Yukawa-type coupling, and can carry a mass; then

$$\partial\!\!\!/\, \chi \,=\, -\,(m + g\varphi)\, \chi, \tag{1.15}$$

both sides of this equation behaving similarly under charge conjugation. In terms of the Weyl spinors (1.8) this equation reads:

$$\partial\!\!\!/\, \psi_L \,=\, -\,(m + g\varphi)\, \psi_R. \tag{1.16}$$

Note however, that this eqaution is not invariant under $U(1)$ chiral phase transformations $\psi_L \rightarrow \psi_L^{\prime} = e^{i\alpha}\psi_L$.

1.2 Fermion masses

The gauge field theories of importance to particle physics are characterized by a compact symmetry group G, generally a product of semi-simple and abelian subgroups. In these theories fermionic matter is described by multiplets of chiral spinors building unitary representations of G. Consider a multiplet of left-handed spinors ψ_L^I in a unitary representation of G with anti-hermitean generators $(T_A)^I{}_J = -(T_A^\dagger)^I{}_J$. Then elements of G connected to the identity can be written in the form

$$U[\alpha^A] \,=\, e^{\alpha^A T_A}. \tag{1.17}$$

Thus infinitesimal transformations of the spinor fields take the form

$$\delta\psi_L^I \,=\, \alpha \cdot T^I{}_J \, \psi_L^J. \tag{1.18}$$

The right-handed spinors $\psi_{R\,I} = (\psi_L^I)^c$ transform according to the complex conjugate representation:

$$\delta\psi_{R\,I} \,=\, -\,\alpha \cdot \left(\psi_{R\,J} T^J{}_I \right). \tag{1.19}$$

A representation and its complex conjugate can be distinct; this distinction is manifest in our notation, as we write upper indices I for the representation to which the left-handed spinors belong, and lower indices for the complex conjugate representation. In general the representation is reducible, and can be decomposed into irreducible representations, some of which may actually be real. The latter happens if the generators T_A are real, and therefore anti-symmetric, matrices in some irreducible subspace of the representation space spanned by the ψ_L^I.

The most general linear Dirac equation preserving chirality which we can write down for these spinor fields ψ_L^I and their charge-conjugate fields in Minkowski space-time, is of the form

$$\partial\!\!\!/ \, \psi_L^I \,=\, i\gamma^\mu A_{\mu\,J}^I \, \psi_L^J \,+\, \left(\Phi^{IJ} + \sigma^{\mu\nu} H_{\mu\nu}^{IJ} \right) \, \psi_{R\,J} \tag{1.20}$$

All the external fields $(A_\mu, \Phi, H_{\mu\nu})$ on the right-hand side can be complex. The term involving the vector fields is of the form of minimal coupling through covariant derivatives, appropriate for Maxwell-Yang-Mills gauge fields. The scalar fields enter through the Yukawa couplings, and the anti-symmetric tensors couple by a Pauli-term.

If the scalar fields develop a vacuum expectation value:

$$\langle \Phi^{IJ} \rangle \,=\, M^{IJ}, \tag{1.21}$$

this induces mass terms for some of the fermions. If the components Φ^{IJ} involved are not singlets of G, the symmetry group is spontaneously broken. For the v.e.v.'s to respect the full symmetry group G, either the spinors ψ_L^I and $\psi_{R\,J}$ must both be singlets, in which case the corresponding $T_{A\,J}^I = 0$, for all A; or there must exist an invariant 2nd rank tensor under G. For example, in the case of an irreducible real representation ψ_L^i being included in the mutiplet ψ_L^I, we can have

$$M^{IJ} = \begin{cases} m\delta^{ij}, & I = i, J = j; \\ 0, & \text{other values of } (I, J). \end{cases} \tag{1.22}$$

Namely, for a real representation $t_A^{ij} = -t_A^{ji}$, and

$$\delta\langle\Phi^{ij}\rangle = m\,\alpha \cdot \left(t^{ik}\delta^{kj} + t^{jk}\delta^{ik}\right) = m\,\alpha \cdot \left(t^{ij} + t^{ji}\right) = 0. \tag{1.23}$$

In the case of the 2-component spinor representation of $SU(2)$ there also exists an invariant anti-symmetric tensor $\varepsilon^{\alpha\beta}$, allowing a mass term

$$\partial\!\!\!/\psi_L^\alpha = \mu\,\varepsilon^{\alpha\beta}\,\psi_{R\beta}. \tag{1.24}$$

However, repeated use of this equation and its charge conjugate show that

$$\left(\Box + |\mu|^2\right)\psi_L^\alpha = 0, \tag{1.25}$$

implying that the corresponding particles have an imaginary mass. Therefore the mass term (1.24) is not allowed. This problem appears in every subsector in which the mass matrix M^{IJ} is anti-symmetric. In consequence, the reality of the mass eigenvalues ($m_I^2 \geq 0$) requires M^{IJ} to be symmetric. Indeed, in the absence of external vector and tensor fields

$$\partial\!\!\!/\psi_L^I = M^{IJ}\psi_{R\,J}, \qquad \partial\!\!\!/\psi_{R\,I} = M^*_{IJ}\psi_L^J. \tag{1.26}$$

For a symmetric mass matrix, $M^*_{IJ} = M^\dagger_{JI} = M^\dagger_{IJ}$, and therefore

$$\Box\psi_L^I = M^{IJ}\,\partial\!\!\!/\psi_{R\,J} = (MM^\dagger)^I{}_J\,\psi_L^J, \qquad \Box\psi_{R\,I} = (M^\dagger M)_I{}^J\,\psi_{R\,J}. \tag{1.27}$$

As the non-zero eigenvalues of $MM^\dagger$ and $M^\dagger M$ are now positive and equal, we can construct pairs of chiral spinors $\chi_{L,R}^I$ which are eigenstates belonging to the same eigenvalue $m_I^2 > 0$ of $MM^\dagger$:

$$\left(-\Box + m_I^2\right)\chi_{L,R}^I = 0. \tag{1.28}$$

They can be combined into Majorana eigenspinors of the Dirac operator:

$$\chi^I = \chi_L^I + \chi_R^I, \qquad \left(\partial\!\!\!/ + m_I\right)\chi^I = 0. \tag{1.29}$$

Note that for anti-symmetric M^{IJ} the eigenvalues of the mass-squared matrix change sign compared to the kinetic operator $\Box$, and therefore become unphysical when non-zero. As this argument only depends on the symmetry properties of the mass matrix M, and not on its behaviour under G-transformations, this property holds both in symmetry-breaking and non-symmetry breaking sectors of M.

In general the Majorana mass eigenstates χ^I do not belong to a proper representation of G whenever the masses break G spontaneously. Only for the real representations and singlets are masses compatible with non-broken internal symmetry. However, there may exist residual symmetries, realized as a non-broken subgroup H of G, which still act on the mass eigestates in a non-trivial way. Then the spinors must form linear representations of this residual symmetry group, and the masses will be degenerate in the subspaces belonging to irreducible representations of H. Examples of various patterns of masses are discussed below.

1.3 The nature of matter

Although these lectures address the physical nature of matter and the forces acting on it, it should be stated clearly at the outset, that we do not know what the dominant form of matter in the universe is. We do of course know the dominant form of matter in our terrestrial environment and its close surroundings: baryonic matter, in the form of protons and neutrons in the nuclei of the chemical elements, provides by far the largest contribution to the mass-energy of the solar system. This can be inferred directly from the motion of planets and other solar satellites. The great accuracy with which Kepler laws of planetary motion are satisfied indicates that there can be no large quantitites of 'hidden' mass floating around in our solar system, as this would certainly make itself felt through its gravitational effects.

One might think that large quantities of exotic material could be piled up in the sun, as its only gravitational effect would be a contribution to the solar mass. This would not change the Kepler nature of the planetary orbits. However, for a sizable fraction of the solar mass to consist of non-baryonic matter would require a complete overhaul of the solar model, which relates the mass, composition, temperature and light output of the sun with the hydrodynamics of the solar plasma and its nuclear fuel cycles. Therefore it is unlikely that a significant amount of exotic non-baryonic matter is present in our cosmic neighborhood.

In addition to baryonic matter, there is also leptonic matter. Its local mass-energy density is small compared to that of baryonic matter, because leptons are generally light compared to baryons, with the exception of the instable charged τ-lepton. However, two remarks are in order at this point:

- Baryons and leptons are not completely independent forms of matter; for one thing, although baryon number and lepton number can be violated individually in physical processes, the *difference* between baryon and lepton number in the universe seems to be exactly conserved.

- If neutrinos have mass, it is conceivable that cosmic neutrinos carry a significant fraction of the energy-density of matter in the universe at large, as much as or more than there is in baryonic matter.

Going beyond our cosmic neighborhood, from the distribution of stellar rotational velocities in galaxies it is generally inferred that there exist large amounts of non-luminous matter in galaxies. Similar observations, though less precise, apply to the motions of individual galaxies in clusters. Theoretical arguments based on plausible reconstructions of the evolution of the universe suggest that a large amount of non-baryonic matter is required to fit the observations of cosmological parameters to the standard Big-Bang model of the history of the universe. Altogether this mix of models and observations strongly suggest that non-baryonic and non-leptonic forms of matter may play an important role in the life of the cosmos at large scales, even today.

The description of fermionic matter provided by the Standard Model of particle physics refers exclusively to baryonic and leptonic matter. The minimal Standard Model also requires the existence of a heavy neutral scalar Higgs boson, and a heavy neutral vector boson (the Z-boson). However, as these particles are not stable, they do not contribute directly to the mass-energy content of the universe. Therefore astrophysical and cosmological considerations provide a strong motivation for exploring models of physics beyond the minimal Standard Model.

1.4 The Standard Model

The Standard Model describes the electro-weak and strong color interactions of quarks and leptons. This theory is in agreement with all experimental data on subatomic phenomena collected so far. In outline the theory describes a number of generations (three at present) of chiral fermions —quarks and leptons— interacting with massless gauge bosons which mediate the electro-magnetic and color forces, and massive vector bosons responsible for the weak interactions. In addition there is at least one massive scalar field, the Higgs boson, a remnant of the sector of fields responsible for the breaking of the isospin-cum-hypercharge symmetry of the weak interactions.

The Standard Model describes gauge interactions of quarks and leptons based on the local symmetry group $G_{sm} = SU(3) \times SU(2) \times U(1)$. The known quarks and leptons build a specific 15-dimensional chiral representation of this group, which is reducible and 3-fold degenerate. Each 15-dimensional representation consists of the following left-handed Weyl spinors :

$$\Psi_L = \left(Q_L^{\alpha i}, U_{L\alpha}, D_{L\alpha}, L_L^i, E_L \right), \qquad \alpha = (1,2,3), \quad i = (1,2). \tag{1.30}$$

The left-handed spinors on the r.h.s. define irreducible representations of G_{sm}, with the content

$$\left\{ (3,2)_{1/6}, (\bar{3},1)_{-2/3}, (\bar{3},1)_{1/3}, (1,2)_{-1/2}, (1,1)_1 \right\}. \tag{1.31}$$

	$SU(3)$	$SU(2)$	Y	Q
$Q_L^{\alpha i}$	3	2	$\frac{1}{6}$	$(\frac{2}{3}, -\frac{1}{3})$
$U_{L\,\alpha}$	$\bar{3}$	1	$-\frac{2}{3}$	$-\frac{2}{3}$
$D_{L\,\alpha}$	$\bar{3}$	1	$\frac{1}{3}$	$\frac{1}{3}$
L_L^i	1	2	$-\frac{1}{2}$	$(0, -1)$
E_L	1	1	1	1
H^i	1	2	$\frac{1}{2}$	$(1, 0)$

Table 1: Family structure (15-plet) of quarks and leptons, and quantum numbers of Higgs doublet.

Here the numbers in the parentheses give the $SU(3)$ and $SU(2)$ representations, whilst the suffix gives the $U(1)$ hypercharge Y. The 15-plet Ψ_L is usually called a *generation* or *family* of quarks and leptons.

In table 1 we have once more summarized the representation content of a single fermion family, to which has been added the electric charge Q. This quantity is defined as a linear combination of the $U(1)$ hypercharge Y and the 3rd component T_3 of $SU(2)$:

$$Q \;=\; Y \,+\, T_3. \tag{1.32}$$

An irreducible representation of $SU(2)$ with Casimir eigenvalue $\mathbf{T}^2 = \tau(\tau+1)$ is said to have isospin τ, and in any isospin multiplet the 3rd component of isospin then takes $2\tau+1$ values $T_3 = -\tau, -\tau+1, ..., \tau$. As a result the electric charge Q in an isospin multiplet takes values $Q = Y - \tau, ..., Y + \tau$; thus Y is the average electric charge in the multiplet and T_3 is the difference between the actual charge and the average charge.

The symmetry group G_{sm} is a local gauge symmetry, hence the covariant derivative defining the Dirac operator for a family of fermions takes the form

$$D_\mu \Psi_L \;=\; (\partial_\mu - g_3 A_\mu \cdot G - g_2 W_\mu \cdot T - i g_1 B_\mu Y)\; \Psi_L, \tag{1.33}$$

where G, T and iY denote the anti-hermitean generators of $SU(3)$, $SU(2)$ and $U(1)$, respectively; g_3, g_2 and g_1 are the coupling constants of the corresponding factors of the gauge group. The Yang-Mills vector fields (A_μ, W_μ, B_μ) have as many components as there are generators, i.e. $(8, 3, 1)$.

In addition to gauge interactions, the fermions also interact with an $SU(2)$ doublet of complex scalar fields, the Higgs doublet H^i, with G_{sm}-representation content $(1,2)_{1/2}$. However, this interaction mixes the 3 families. Let us take N_f families, labeled by indices $A, B, C, ... = (1, .., N_f)$. In the minimal Standard Model $N_f = 3$. Following eq.(1.20) we write

$$D\!\!\!\!/\,\Psi_L^A = \Phi^{AB}\,\Psi_{RB}, \tag{1.34}$$

where ψ_R represents the charge-conjugate righthanded fermions. The covariant derivatives are given by eq.(1.33), and taken to be diagonal in the space of families:

$$D\!\!\!\!/\,^A{}_B = \delta^A_B\,D\!\!\!\!/. \tag{1.35}$$

The components of the symmetric scalar field Φ^{AB} are defined in terms of the Higgs-doublet H^i as

$$\Phi^{AB} = \begin{bmatrix} 0 & f_U^{AB}\varepsilon^{ij}H_j^*\delta^\alpha_\beta & f_D^{AB}H^i\delta^\alpha_\beta & 0 & 0 \\ f_U^{BA}\varepsilon^{ji}H_i^*\delta^\beta_\alpha & 0 & 0 & 0 & 0 \\ f_D^{BA}H^j\delta^\beta_\alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & f_L^{AB}H^i \\ 0 & 0 & 0 & f_L^{BA}H^j & 0 \end{bmatrix}. \tag{1.36}$$

Here f_I^{AB}, $I = (U, D, L)$ are the complex dimensionless Yukawa coupling constants between the Higgs doublet and the quarks and the leptons. In terms of the irreducible representations of the gauge group[2] $G_{sm} = SU(3) \times SU(2) \times U_Y(1)$ this becomes

$$\begin{aligned} D\!\!\!\!/\,Q_L^{A\alpha i} &= f_U^{AB}\varepsilon^{ij}H_j^*U^\alpha_{RB} + f_D^{AB}H^iD^\alpha_{RB}, \\ D\!\!\!\!/\,U^A_{L\alpha} &= f_U^{BA}\varepsilon^{ij}H_j^*\,Q_{RB\alpha i}, \\ D\!\!\!\!/\,D^A_{L\alpha} &= f_D^{BA}H^i\,Q_{RB\alpha i}, \\ D\!\!\!\!/\,L_L^{Ai} &= f_L^{AB}H^iE_{RB}, \\ D\!\!\!\!/\,E_L^A &= f_L^{BA}H^iL_{RBi}. \end{aligned} \tag{1.37}$$

It is a quite special feature of the minimal Standard Model that one can introduce Yukawa couplings for all fermions in this way, with the $U(1)$ hypercharges matching on both sides of every equation. Also, it is only because of the existence of the invariant anti-symmetric $SU(2)$-tensor $\varepsilon_{ij} = -\varepsilon^{ij}$ that a coupling to H^i and H_i^* can be present in the Yukawa coupling of a single family of chiral fermions.

[2]The label Y is used here to remind the reader that the corresponding charge is the hypercharge.

Now in the Standard Model masses can be induced for the fermions by a vacuum expectation value of one of the components of the Higgs doublet in such a way, that the $SU(2) \times U_Y(1)$ is broken, but $SU(3)$ and the $U_Q(1)$ generated by the electric charge are preserved. Consider an infinitesimal transformation of the Higgs doublet:

$$\delta H \,=\, i\left(\alpha_0 Y + \frac{1}{2}\, \boldsymbol{\alpha} \cdot \boldsymbol{\tau} \right)\, H. \tag{1.38}$$

With the hypercharge of the Higgs doublet $Y = 1/2$, we can redefine the generators and the parameteres as follows:

$$\begin{aligned} & Q = Y + T_3 = \frac{1}{2}\,(1 + \tau_3)\,, \qquad Z = Y - T_3 = \frac{1}{2}\,(1 - \tau_3)\,, \\ & T_{\pm} = \frac{1}{\sqrt{2}}\,(T_1 \pm iT_2) = \frac{1}{2\sqrt{2}}\,(\tau_1 \pm i\tau_2)\,; \\ & \alpha_Q = \frac{1}{2}\,(\alpha_0 + \alpha_3)\,, \qquad \alpha_Z = \frac{1}{2}\,(\alpha_0 - \alpha_3)\,, \qquad \alpha_{\pm} = \frac{1}{\sqrt{2}}\,(\alpha_1 \mp i\alpha_2)\,. \end{aligned} \tag{1.39}$$

Then the transformations (1.38) of the Higgs doublet take the form

$$\delta H \,=\, i\left(\alpha_Q Q \,+\, \alpha_Z Z \,+\, \frac{1}{2}\,(\alpha_+ \tau_+ + \alpha_- \tau_-) \right) H. \tag{1.40}$$

From the explicit expression for Q (1.39) we observe, that the lower component H^2 of the Higgs doublet carries no electric charge: $QH^2 = 0$. Therefore a vacuum expectation value $H^2 = v$ is invariant under the electric $U_Q(1)$ transformations as given in eq.(1.40). Also it is trivially invariant under $SU(3)$. Therefore our choice of representation of the conserved electric charge operator Q allows a non-zero vacuum expectation value[3]

$$\langle H \rangle \,=\, \left(\begin{array}{c} 0 \\ v \end{array} \right). \tag{1.41}$$

We note in passing, that the v.e.v. of the Higgs doublet is also inert under the operation T_-. However, the v.e.v. is not invariant under the operations Z and T_+, and therefore in the vacuum the full group G_{sm} is broken down to $SU(3) \times U_Q(1)$. This is manifest from the properties of the mass eigenstates. Consider the fermion fluctuation equation in the vacuum, when we replace the scalar fields by their v.e.v.:

$$\not{D} \Psi_L^A \,=\, M^{AB} \Psi_{RB}, \tag{1.42}$$

with $M^{AB} = \langle \Phi^{AB} \rangle$. These eigenstates do not transform covariantly under $SU(2) \times U_Y(1)$. Indeed, the mass matrix M^{AB} is given by insertion of the v.e.v.

[3] In fact one can always perform a local $SU(2) \times U_Y(1)$ transformations to realize the gauge condition $H^1 = \text{Im}\ H^2 = 0$; then one can expand Re $H^2 = v + h$, with real v. This gauge condition is called the *unitary gauge*.

(1.41) into equation (1.36):

$$M^{AB} = \begin{bmatrix} 0 & f_U^{AB}\delta^i_1 v^* \delta^\alpha_\beta & f_D^{AB}\delta^i_2 v \delta^\alpha_\beta & 0 & 0 \\ f_U^{BA}\delta^j_1 v^* \delta^\beta_\alpha & 0 & 0 & 0 & 0 \\ f_D^{BA}\delta^j_2 v \delta^\beta_\alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & f_L^{AB}\delta^i_2 v \\ 0 & 0 & 0 & f_L^{BA}\delta^j_2 v & 0 \end{bmatrix}. \tag{1.43}$$

In order to interpret the physical properties of the mass eigenstates we must diagonalize the mass matrix M^{AB}. As a full family of fermions defines a reducible representation of the gauge group G_{sm}, the diagonalization of the mass matrix is to be considered in irreducible blocks separately. To begin with, consider the leptons $(L_L^{A\,i}, E_L^A)$ and their charge conjugates $(L_{R\,Ai}, E_{R\,A})$. The Dirac equation for these fields become

$$\begin{aligned} \not{D} L_L^{A\,1} &= 0, \\ \not{D} L_L^{A\,2} &= v f_L^{AB} E_{R\,B}, \quad \not{D} E_{R\,A} = v^* f^*_{L\,AB} L_L^{B\,2}. \end{aligned} \tag{1.44}$$

We can now introduce the unitary matrices X^A_B and Y^{AB} making the Yukawa matrices $v f_L^{AB}$ diagonal and real:

$$v X^A{}_C f_L^{CD} Y^\dagger_{DB} = -m_L^A \, \delta^A_B, \tag{1.45}$$

where the m_L^A are real positive mass eigenvalues[4]. Now transform the doublet L_L^{Ai} by

$$L_L^{Ai} \to X^A{}_B \, L_L^{Bi}, \tag{1.46}$$

and redefine the $SU(2)$ singlet by

$$E_{R\,A} \to Y^{AB} E_{R\,B}. \tag{1.47}$$

The unitary $N_f \times N_f$ transformations $X^A{}_B$ and Y^{AB} commute with the $SU(2) \times U_Y(1)$ gauge transformations, and hence with the covariant derivatives $\not{D}$. As a result the Dirac spinors

$$E^A = X^A{}_C L_L^{C\,2} + Y^{AB} E_{R\,B}, \tag{1.48}$$

and the left-handed spinors

$$N_L^A = X^A{}_B L_L^{B\,1}, \tag{1.49}$$

[4]This can always be achieved by multiplying X (or Y) by a diagonal matrix diag $(e^{i\theta_1}, ..., e^{i\theta_{N_f}})$, which respects unitarity.

satisfy the Dirac equations

$$
\begin{aligned}
&\left(\not\partial + \frac{i}{4}\, g_2 \not W^3 (1+\gamma_5) - i g_1 \not B Y + m_L^A\right) E^A \,-\, \frac{i g_2}{\sqrt{2}} \not W^- N_L^A \,=\, 0, \\
&\left(\not\partial - \frac{i}{2}\, g_2 \not W^3 - i g_1 \not B Y\right) N_L^A \,-\, \frac{i g_2}{2\sqrt{2}} \not W^+ (1+\gamma_5)\, E^A \,=\, 0.
\end{aligned}
\tag{1.50}
$$

Thus we reach the conclusion that the lepton sector describes N_f distinct massive charged leptons with electric charge $Q = -1$, which for $N_f = 3$ we identify with e, μ and τ, and N_f massless chiral fermions with charge $Q = 0$, to be identified with the neutrinos. Note that $W^{\pm} = (W^1 \mp i W^2)/\sqrt{2}$ are the off-diagonal components of the weak gauge fields, corresponding to the decomposition of the $SU(2)$ algebra in terms of isospin raising and lowering operators as in eq.(1.39). As is clear from these equations, they transfer electric charge between charged leptons and neutrinos.

It is clear, that by choosing as our basis for the lepton families the chiral spinors defined in (1.46) and (1.47), after the spontaneous break-down of the full $SU(2) \times U_Y(1)$ gauge symmetry we directly obtain the mass eigenstates E^A of eq.(1.50). This means basically that the 'family mixing' in the lepton sector was an artifact of our choice of basis for the chiral $SU(2)$ doublets and singlets in the lepton sector. This can be undone, and with a judicious choice of basis no physically observable family mixing remains.

Turning now to the quark sector, we can perform similar $U(N_f)$ transformations $X^A_{(U,D)\,B}$ and $Y^{AB}_{(U,D)}$ to diagonalize the mass matrices for the U and D quarks:

$$
v X^A_{(U)\,C}\, f_U^{CD}\, Y^{\dagger}_{(U)\,DB} \,=\, -m_U^A\, \delta^A_B, \qquad v X^A_{(D)\,C}\, f_D^{CD}\, Y^{\dagger}_{(D)\,DB} \,=\, -m_D^A\, \delta^A_B. \tag{1.51}
$$

Then the Dirac spinors

$$
U^{A\alpha} \,=\, X^A_{(U)\,B}\, Q_L^{B\alpha 1} \,+\, Y^{AB}_{(U)}\, U^{\alpha}_{R\,B}, \qquad D^{A\alpha} \,=\, X^A_{(D)\,B}\, Q_L^{B\,\alpha 2} \,+\, Y^{AB}_{(D)}\, D^{\alpha}_{R\,B}, \tag{1.52}
$$

are mass eigenstates. They are not eigenstates of $SU(2) \times U_Y(1)$, but as is clear form table 1 they do carry a well-defined electric charge, with $Q = 2/3$ for the U-type quarks, and $Q = -1/3$ for the D-type quarks. Applying eqs.(1.37), we find that they satisfy the Dirac equations

$$
\begin{aligned}
&\left(\not\partial - i g_3 \not A \cdot \frac{\lambda}{2} - \frac{i}{4}\, g_2 \not W^3 (1+\gamma_5) - i g_1 Y \not B + m_U^A\right) U^A \\
&\qquad -\, \frac{i}{2\sqrt{2}}\, g_2 \not W^+\, (1+\gamma_5)\, \left(X_{(U)} X^{\dagger}_{(D)}\right)^A_{\ \ B}\, D^B \,=\, 0,
\end{aligned}
\tag{1.53}
$$

and

$$\left(\partial\!\!\!/ - ig_3 A\!\!\!/ \cdot \frac{\lambda}{2} + \frac{i}{4}\, g_2 W\!\!\!\!/^{\,3}(1+\gamma_5) - ig_1 Y B\!\!\!/ + m_D^A\right) D^A$$
$$- \frac{i}{2\sqrt{2}}\, g_2 W\!\!\!\!/^{\,-}\,(1+\gamma_5)\,\left(X_{(D)}X^{\dagger}_{(U)}\right)^A_{\ \ B} U^B \;=\; 0. \tag{1.54}$$

Here the Gell-Mann matrices λ^a are the generators of $SU(3)$ in the triplet representation.

As evident from these equations, the charge-changing interactions mediated by the $W^{\pm}$-bosons are the only ones which mix the families. The unitary matrix

$$V \;=\; X_{(D)}X^{\dagger}_{(U)}, \tag{1.55}$$

describing this mixing is known as the Cabibbo-Kobayashi-Maskawa (CKM) matrix. No family mixing occurs in the sector of diagonal $SU(2)\times U_Y(1)$ gauge fields W^3_μ and B_μ. Observe, that not all of the elements of V have physical significance. Indeed, one can always multiply each mass-eigenstate by a complex phase

$$U^A \;\rightarrow\; e^{i\theta^A_{(U)}}\,U^A, \qquad D^A \;\rightarrow\; e^{i\theta^A_{(D)}}\,D^A. \tag{1.56}$$

which preserves the equations (1.53) and (1.54) except for redefinition of the CKM-matrix:

$$V^A_{\ \ B} \;\rightarrow\; e^{i\theta^A_{(D)}}\,V^A_{\ \ B}\,e^{-i\theta^B_{(U)}}. \tag{1.57}$$

This redefinition involves $2N_f$ phases, but as only the differences between these phases enter in eq.(1.57), really only $2N_f - 1$ independent phase differences contribute to changing the CKM matrix V. Any unitary $N_f \times N_f$ transformation like V depends on N_f^2 independent parameters. It can be decomposed into a real $N_f \times N_f$ orthogonal transformation, depending on $N_f(N_f-1)/2$ rotation angles in the various orthogonal co-ordinate planes, and a number of phase transformations depending on the remaining $N_f(N_f+1)/2$ parameters. Of these phases, $2N_f-1$ can be removed by the redefinition (1.57). Therefore $(N_f-1)(N_f-2)/2$ independent physically observable phases are present in V. Note that for a single family ($N_f = 1$) there can be no family mixing, and there is no CKM matrix; for two families ($N_f = 2$) there is family mixing and a CKM matrix depending on one physically observable real rotation angle θ_C. For $N_f = 3$ we get three real rotation angles, and one complex phase factor. This is the situation presently in agreement with all experimental and observational evidence. If there would be a fourth family, the CKM matrix must be extended to have six real angles and three complex phases.

Note also, that if two quark masses would be degenerate (e.g., $m_s = m_b$), then one angle and one phase would become unobservable, and the CKM matrix becomes a function of two real angles only; then V can be cast in real orthogonal form: $V^{-1} = V^T$.

Having analysed this rather complicated family mixing in the quark sector, we observe that if neutrinos are not massless, but have Yukawa interactions with a set of right-handed neutrinos which are singlets under the gauge group G_{sm}, there could be a similar mixing matrix in the lepton sector as well.

1.5 Gauge interactions of quarks and leptons

The covariant derivatives in the Dirac equations (1.50) for the leptons, and (1.53) and (1.54) for the quarks, contain gauge couplings to the gauge bosons of G_{sm}. However, as the fermion mass eigenstates are not eigenstates of $SU(2) \times U_Y(1)$, the operators for hypercharge and third component of isospin act differently on the left- and righthanded components of the spinors. On the other hand, we know that the mass eigenstates do have a well-defined electric charge. Therefore we decompose the gauge fields of hypercharge and third component of isospin into combinations coupling to electric charge Q and weak charge Z, respectively. More precisely, we consider a rotation in the space of electrically neutral gauge fields

$$W_\mu^3 \,=\, \cos\theta_w Z_\mu \,+\, \sin\theta_w A_\mu, \qquad B_\mu \,=\, -\sin\theta_w Z_\mu \,+\, \cos\theta_w A_\mu, \tag{1.58}$$

such that the mass-eigenstates couple diagonally to the electro-magnetic $U_Q(1)$ gauge field A_μ.

First, consider the leptons. As they do not carry color charges, the covariant derivative only contains $SU(2) \times U_Y(1)$ gauge fields. The charged gauge fields $W^\pm$ mediate the transformations of neutrinos into charged leptons and vice versa. The explicit coupling of other two gauge fields W^3 and B to charged leptons is obtained by writing out the positive and negative chirality projections in the covariant derivatives, which read

$$\begin{aligned}
&\left(\partial\!\!\!/ + \frac{i}{4}\,g_2 W\!\!\!\!/^{\,3}(1+\gamma_5) + ig_1 B\!\!\!/ \left[\frac{1}{4}\,(1+\gamma_5) + \frac{1}{2}\,(1-\gamma_5)\right]\right)\, E^A \;= \\
&\left(\partial\!\!\!/ + \frac{i}{4}\,g_2 \left[\cos\theta_w Z\!\!\!/ + \sin\theta_w A\!\!\!/\right](1+\gamma_5) - \frac{i}{4}\,g_1 \left[\sin\theta_w Z\!\!\!/ - \cos\theta_w A\!\!\!/\right](3-\gamma_5)\right) E^A,
\end{aligned} \tag{1.59}$$

and

$$\begin{aligned}
&\left(\partial\!\!\!/ - \frac{i}{2}\,g_2 W\!\!\!\!/^{\,3} + \frac{i}{2}\,g_1 B\!\!\!/\right)\, N_L^A \;= \\
&\qquad \left(\partial\!\!\!/ - \frac{i}{2}\,g_2 \left[\cos\theta_w Z\!\!\!/ + \sin\theta_w A\!\!\!/\right] - \frac{i}{2}\,g_1 \left[\sin\theta_w Z\!\!\!/ - \cos\theta_w A\!\!\!/\right]\right)\, N_L^A.
\end{aligned} \tag{1.60}$$

Now by definition, we wish the Maxwell field to couple to the leptons through vector-like $U_Q(1)$ gauge couplings with electric charges $Q_E = -1$ and $Q_N = 0$ in units of the elementary charge e; therefore up to the coupling to the Z-field and the charged W-fields, the covariant derivatives must be of the form

$$(\not{\partial} + ie\not{A})\; E^A, \quad \text{and} \quad \not{\partial} N_L^A. \tag{1.61}$$

This is achieved by taking the weak mixing angle θ_w and the elementary unit of charge e to be related to the coupling constants g_1 and g_2 by

$$\tan\theta_w \;=\; \frac{g_1}{g_2}, \qquad e \;=\; g_1\cos\theta_w \;=\; g_2\sin\theta_w \;=\; \frac{g_1 g_2}{\sqrt{g_1^2+g_2^2}}. \tag{1.62}$$

Eqs.(1.50) with the full expressions for the leptonic covariant derivatives then become

$$\begin{aligned}&\left(\not{\partial} + ie\not{A} + \frac{ie}{4\sin\theta_w\cos\theta_w}\not{Z}\left[1-4\sin^2\theta_w+\gamma_5\right] + m_L^A\right) E^A \\ &\quad = \frac{ie}{\sqrt{2}\sin\theta_w}\not{W}^- N_L^A,\end{aligned} \tag{1.63}$$

and

$$\left(\not{\partial} - \frac{ie}{2\sin\theta_w\cos\theta_w}\not{Z}\right) N_L^A \;=\; \frac{ie}{2\sqrt{2}\sin\theta_w}\not{W}^+(1+\gamma_5)\,E^A. \tag{1.64}$$

These equations represent the field equations of the observable leptons (the mass eigenstates) up to coupling to dynamical scalar fields.

In the quark sector of the Standard Model the gauge couplings are treated in complete correspondence. Using the same definitions (1.58) and (1.62) the Dirac equations for the up- and down-type massive quarks in the absence of dynamical scalar fields become

$$\begin{aligned}&\left(\not{\partial} - ig_3\not{A}\cdot\frac{\lambda}{2} - \frac{2ie}{3}\not{A} - \frac{ie}{4\sin\theta_w\cos\theta_w}\not{Z}\left[1-\frac{8}{3}\sin^2\theta_w+\gamma_5\right] + m_U^A\right) U^A \\ &\quad = \frac{ie}{2\sqrt{2}\sin\theta_w}\not{W}^+\,(1+\gamma_5)\,V^{\dagger A}{}_B\,D^B,\end{aligned} \tag{1.65}$$

and

$$\begin{aligned}&\left(\not{\partial} - ig_3\not{A}\cdot\frac{\lambda}{2} + \frac{ie}{3}\not{A} + \frac{ie}{4\sin\theta_w\cos\theta_w}\not{Z}\left[1-\frac{4}{3}\sin^2\theta_w+\gamma_5\right] + m_D^A\right) D^A \\ &\quad = \frac{ie}{2\sqrt{2}\sin\theta_w}\not{W}^-\,(1+\gamma_5)\,V^A{}_B\,U^B.\end{aligned} \tag{1.66}$$

1.6 The CKM matrix and CP-violation

In section 1.4 we have found that the CKM mixing matrix between the D-quarks of different families can be parametrized in terms of three real rotation angles and one complex phase factor. One of several current parametrizations, given preference by the Particle Data Group, is

$$V = \left(\begin{array}{ccc} c_1 & -s_1 c_3 & -s_1 s_3 \\ s_1 c_2 & c_1 c_2 c_3 - s_2 s_3 e^{i\delta} & c_1 c_2 s_3 + s_2 c_3 e^{i\delta} \\ s_1 s_2 & c_1 s_2 c_3 + c_2 s_3 e^{i\delta} & c_1 s_2 s_3 - c_2 c_3 e^{i\delta} \end{array} \right), \tag{1.67}$$

Here $c_i = \cos\theta_i$, $s_i = \sin\theta_i$, with $(\theta_1, \theta_2, \theta_3)$ the three real rotation angles, and δ is the phase angle.

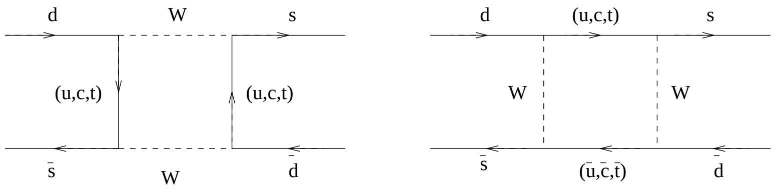

Fig. 1: Lowest order contributions to K^0-$\bar{K}^0$ mixing.

The presence of the phase allows processes to take place which violate CP-invariance. The simplest example is the transition between neutral kaons and anti-kaons. A neutral kaon K^0 can be characterized in terms of its quark compostion as a state $|d\bar{s}\rangle$, whilst an antikaon $\bar{K}^0$ has the opposite quark content: $|\bar{d}s\rangle$. The quarks in these states are the mass eigenstates which, as we have established, are a mixture of weak eigenstates. Therefore weak processes can change the quark content whilst conserving electric charge, so as to transform a K^0 into a $\bar{K}^0$ or vice versa.

At the quark level these processes are described by the interactions of the type shown in the box-diagrams of fig. 1. Because of the presence of the complex phase δ, the eigenstates of the effective hamiltonian for the K^0-$\bar{K}^0$ interaction resulting from the mixing are not eigenstates of CP. Therefore the decay processes of the observed kaons do not respect CP-symmetry.

Explicit parametrizations of the CKM matrix, like (1.67), have the advantage of not introducing any spurious parameters into the description of physical processes. However, as the parametrization is not unique, different parametrizations may be more or less well adapted to different physical situations. Therefore it is useful to extract results from properties of the mass-matrices and the CKM matrix which are independent of its representation [1].

The first step is to consider the *unitarity triangles*, which can be constructed for any 3×3 unitary matrix in the complex plane. The unitarity of V, as expressed by the condition

$$\left(V^{-1}\right)^A_{\ B} = \left(V^{\dagger}\right)^A_{\ B} = V^{*\,A}_{B}, \tag{1.68}$$

implies the standard bilinear relations $VV^{\dagger} = V^{\dagger}V = 1$, or in components

$$\sum_C V^A_{\ C} V^{*\,C}_{B} = \delta^A_B, \qquad \sum_C V^{*\,A}_{C} V^C_{\ B} = \delta^A_B, \qquad V^{*\,A}_{B} \equiv \left(V^B_{\ A}\right)^*. \tag{1.69}$$

Here and in the rest of this section all summations are written out explicitly; unitl the end of this section, repeated indices without summation sign are *not* to be summed over. Next we define two sets of complex numbers:

$$z_C(AB) = V^A_{\ C} V^{*\,C}_{B}, \qquad \zeta_C(AB) = V^{*\,A}_{C} V^C_{\ B}. \tag{1.70}$$

For any pair of unequal indices $A \neq B$ these complex numbers satisfy the relation

$$z_1(AB) + z_2(AB) + z_3(AB) = 0, \qquad \zeta_1(AB) + \zeta_2(AB) + \zeta_3(AB) = 0. \tag{1.71}$$

As each complex number defines a vector in the two-dimensional plane, any three complex numbers, associated with a fixed pair (AB) and adding up to zero, define a triangle beginning and ending in the origin. This is represented in fig.2.

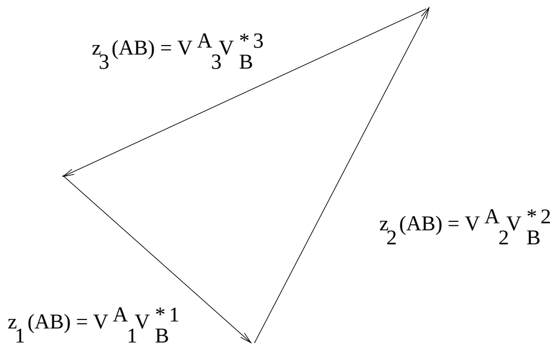

Fig. 2: Unitarity triangle in the complex plane.

Similar triangles can be drawn for the complex numbers $\zeta_C(AB)$ when $A \neq B$. The (oriented) area of such a triangle is given by

$$\begin{aligned} A(z_1, z_2, z_3) &= = \frac{1}{2}\,\mathrm{Im}\,(z_1^* z_2) = -\frac{1}{2}\,\mathrm{Im}\,(z_1^* z_3) = \frac{1}{2}\,\mathrm{Im}\,(z_2^* z_3) \\ &= \frac{1}{2}\sum_{j,k=1}^{3} \varepsilon_{ijk}\,\mathrm{Im}\,(z_j^* z_k). \end{aligned} \tag{1.72}$$

The last line holds for any choice of i, because relation (1.71) always allows to transform one expression into any other; observe, that in such a substitution $z_i^* z_i$

is real, i.e. $\mathrm{Im}\, z_i^* z_i = 0$. In other words, the area is cyclically symmetrical[5] in its arguments (z_1, z_2, z_3).

An important result we will derive is, that the area of the triangle is actually the same for all non-diagonal choices of (AB), and also independent of whether it is constructed from the numbers z or ζ:

$$A^{(AB)}(z_1, z_2, z_3) = A^{(A'B')}(\zeta_1, \zeta_2, \zeta_3). \tag{1.73}$$

These facts can be expressed concisely by the equation

$$\mathrm{Im}\,\left(V^A_{\ C} V^B_{\ D} V^{*D}_{\ A} V^{*C}_{\ B}\right) = \mathcal{J} \sum_{M,N} \varepsilon^{ABM} \varepsilon_{CDN}, \tag{1.74}$$

where $\mathcal{J}$ is independent of the choice of (A, B, C, D). We now present a proof. First note, that the left-hand side of eq.(1.74) can be written alternatively as

$$\mathrm{Im}\,\left(V^A_{\ C} V^B_{\ D} V^{*D}_{\ A} V^{*C}_{\ B}\right) = \mathrm{Im}\,\left[z_C(AB) z_D^*(AB)\right] = \mathrm{Im}\,\left[\zeta_A^*(CD)\zeta_B(CD)\right]. \tag{1.75}$$

As under complex conjugation the imaginary part of any expression changes sign, it is obvious that these expressions are anti-symmetric in $[CD]$ and $[AB]$; first, the anti-symmetry in $[CD]$ follows from

$$\mathrm{Im}\,\left[z_C(AB)\, z_D^*(AB)\right] = -\mathrm{Im}\,\left[z_C(AB)\, z_D^*(AB)\right]^* = -\mathrm{Im}\,\left[z_D(AB)\, z_C^*(AB)\right]. \tag{1.76}$$

Similarly the expression bilinear in $\zeta(CD)$ proves the anti-symmetry in $[AB]$. Therefore we can write quite generally

$$\mathrm{Im}\,\left(V^A_{\ C} V^B_{\ D} V^{*D}_{\ A} V^{*C}_{\ B}\right) = \sum_{M,N} \varepsilon^{ABM} \varepsilon_{CDN}\, \mathcal{J}_M^N. \tag{1.77}$$

It only remains to show, that $\mathcal{J}_M^N$ does not depend on the values of (M, N). This follows using the previous observations $\sum_C z_C(AB) = 0$, and $\mathrm{Im}\, |z_C(AB)|^2 = 0$; take any pair of values (AB) and (CD), and let M and N be their respective cyclic complements, i.e. $(ABM) = \mathrm{perm}\ (123)$, and similarly for (CDN). Then

$$z_C(AB) + z_D(AB) + z_N(AB) = 0, \tag{1.78}$$

and as a result

$$\begin{aligned} \varepsilon^{ABM} \varepsilon_{CDN}\, \mathcal{J}_M^N &= \mathrm{Im}\,\left[z_C(AB)\, z_D^*(AB)\right] \;=\; -\mathrm{Im}\,\left[z_N(AB)\, z_D^*(AB)\right] \\ &= -\varepsilon^{ABM} \varepsilon_{NDC}\, \mathcal{J}_M^C \;=\; \varepsilon^{ABM} \varepsilon_{CDN}\, \mathcal{J}_M^C. \end{aligned} \tag{1.79}$$

This proves that $\mathcal{J}_M^N$ is the same for all values of N. Similarly, using the expression bilinear in $\zeta(CD)$ we prove it is independent of the lower index M as

[5] Geometrically this is clear: the area of the triangle is one half times the area of the parallelogram spanned by any two of the three complex numbers (z_1, z_2, z_3).

well. Thus eq.(1.74) follows. Rewritten as in (1.75) it proves the equality of the triangle areas (1.72) for all non-diagonal pairs (AB) and (CD):

$$A^{(AB)}(z_1, z_2, z_3) = -\frac{1}{2}\,\mathcal{J}, \tag{1.80}$$

for any choice of (AB). One can compute the invariant quantity $\mathcal{J}$ in terms of the standard parametrization (1.67); the result is

$$\mathcal{J} = s_1^2 s_2 s_3 c_1 c_2 c_3 \,\sin\delta. \tag{1.81}$$

Thus $\mathcal{J} \neq 0$ is a sufficient condition to have CP-violation ($\delta \neq 0$) in the standard model with three families of quarks and leptons, whilst $\delta = 0$ necessarily implies $\mathcal{J} = 0$. Geometrically, this follows by observing that if there were a basis of fermion fields in which V was real, then the triangles would collapse onto the real line and they would have zero area.

Next we turn to the quark mass matrices, as defined by

$$\mathcal{M}_U^{AB} = v^* f_U^{AB}, \qquad \mathcal{M}_D^{AB} = v f_D^{AB}, \tag{1.82}$$

cf. eq.(1.43). Eq.(1.51) states, that they are diagonalized by the transformations

$$\sum_{M,N} X_{(U)\,M}^{\;A} \mathcal{M}_U^{MN} Y_{(U)\,NB}^{\dagger} = -m_U^A\,\delta_B^A, \qquad \sum_{M,N} X_{(D)\,M}^{\;A} \mathcal{M}_D^{MN} Y_{(D)\,NB}^{\dagger} = -m_D^A\,\delta_B^A. \tag{1.83}$$

Note, that for a hermitean mass matrix $\mathcal{M}$, it follows that $Y = X$, but generically, and in the minimal standard model, this condition is not satisfied. In contrast, by construction the squared mass matrices

$$\mathcal{M}_U^2 = \mathcal{M}_U \mathcal{M}_U^\dagger, \qquad \mathcal{M}_D^2 = \mathcal{M}_D \mathcal{M}_D^\dagger, \tag{1.84}$$

are necessarily hermitean. From eq.(1.83) it follows, that they are diagonalized by the unitary matrices $X_{U,D}$:

$$\begin{aligned} \sum_{M,N} X_{(U)\,M}^{\;A} \mathcal{M}_{U\;\;N}^{2\,M} X_{(U)\,B}^{\dagger\;N} &= (m_U^A)^2 \delta_B^A, \\ \sum_{M,N} X_{(D)\,M}^{\;A} \mathcal{M}_{D\;\;N}^{2\,M} X_{(D)\,B}^{\dagger\;N} &= (m_D^A)^2 \delta_B^A. \end{aligned} \tag{1.85}$$

In general (certainly in the minimal standard model), the unitary transformations X_U and X_D are different, implying they are not diagonal in the same basis of quark-eigenstates. (This is just another way of stating that the mass eigenstates in one of the quark sectors mix the weak eigenstates.) As known from elementary quantum mechanics, matrices which are not diagonal in the same basis do not commute. The degree of non-commutativity is measured by the operator

$$\mathbf{C} = -i \left[\mathcal{M}_U^2, \mathcal{M}_D^2 \right]. \tag{1.86}$$

This operator has interesting properties. In particular, its determinant (the product of its eigenvalues) takes the form

$$\det \mathbf{C} = 2\mathcal{J}\,(m_t^2 - m_c^2)(m_c^2 - m_u^2)(m_u^2 - m_t^2)(m_b^2 - m_s^2)(m_s^2 - m_d^2)(m_d^2 - m_b^2). \tag{1.87}$$

Thus, $\det \mathbf{C} \neq 0$ implies that all quark masses are different, and that CP is violated ($\mathcal{J} \neq 0$). The proof of eq.(1.87) runs as follows. First observe, that $\mathbf{C}$ is hermitean and traceless: $\mathbf{C} = \mathbf{C}^\dagger$, $\mathrm{Tr}\,\mathbf{C} = 0$. Both results follow directly from the defining eq.(1.86). As a result,

$$\det \mathbf{C} = \frac{1}{3}\,\mathrm{Tr}\,\mathbf{C}^3. \tag{1.88}$$

The proof is easily given by diagonalizing $\mathbf{C}$, its eigenvalues λ_A being related by $\lambda_1 + \lambda_2 + \lambda_3 = 0$. Then $\lambda_1^3 + \lambda_2^3 + \lambda_3^3 = 3\lambda_1\lambda_2\lambda_3$. It is left as an exercise to the reader to show this.

Finally, using the definition (1.86) one derives

$$\begin{aligned} \det \mathbf{C} &= i\,\mathrm{Tr}\,[(\mathcal{M}_U^2)^2(\mathcal{M}_D^2)^2\mathcal{M}_U^2\mathcal{M}_D^2 - (\mathcal{M}_D^2)^2(\mathcal{M}_U^2)^2\mathcal{M}_D^2\mathcal{M}_U^2] \\ &= -2\,\mathrm{Im}\,\mathrm{Tr}\,[(\mathcal{M}_U^2)^2(\mathcal{M}_D^2)^2\mathcal{M}_U^2\mathcal{M}_D^2] \end{aligned} \tag{1.89}$$

Next invert eqs.(1.85): $\mathcal{M}^2 = X^\dagger m^2 X$, with m^2 the diagonal mass-squared matrices, and substitute these expressions into eq.(1.89). Observing furhtermore that $X_{(D)}X_{(U)}^\dagger = V$, this leads to the result

$$\begin{aligned} \det \mathbf{C} &= -2\,\mathrm{Im}\,\mathrm{Tr}\,[m_U^4 V^\dagger m_D^4 V m_U^2 V^\dagger m_D^2 V] \\ &= -2 \sum_{A,B,M,N} (m_U^A)^4 (m_D^M)^4 (m_U^B)^2 (m_D^N)^2\,\mathrm{Im}\,[V_M^{*\,A} V_{\;B}^{M} V_N^{*\,B} V_{\;A}^{N}] \\ &= 2\,\mathcal{J} \sum_{\substack{A,\,B,\,M,\,N \\ K,\,L}} (m_U^A)^4 (m_D^M)^4 (m_U^B)^2 (m_D^N)^2\,\varepsilon^{MNK}\varepsilon_{ABL}. \end{aligned} \tag{1.90}$$

Writing out this result in components finally gives expression (1.89); the main step is to note, that

$$\begin{aligned} \sum_{A,B,L} (m_U^A)^4 (m_U^B)^2\,\varepsilon_{ABL} &= m_u^4 m_c^2 - m_c^4 m_u^2 + m_c^4 m_t^2 - m_t^4 m_c^2 + m_t^4 m_u^2 - m_u^4 m_t^2 \\ &= (m_t^2 - m_c^2)(m_c^2 - m_u^2)(m_u^2 - m_t^2), \end{aligned} \tag{1.91}$$

with a similar result for the D-type quarks masses.

1.7 Actions and pseudoclassical fermions

Dirac equations like (1.15) or (1.34) can be interpreted either as classical field equations, with the spinor components ordinary c-number functions on space-time, or as quantum field equations in which the spinors are operators in a state space where they act to change the local fermion density.

The Dirac equation can be derived from an action principle; however, this is possible without modifying the equation only if the spinor components are interpreted as anti-commuting c-numbers, i.e. odd elements of a Grassmann algebra. The issue is, that a real kinetic action for commuting Majorana spinors reduces to a total derivative. Consider two Majorana spinors χ and η; the only real kinetic action for these spinors with a chance to reproduce the kinetic term in the Dirac equation is

$$S_{kin} = \frac{1}{2} \int d^4x \left(\bar{\chi} \partial\!\!\!/ \eta + \bar{\eta} \partial\!\!\!/ \chi \right). \tag{1.92}$$

Indeed,

$$(\bar{\chi} \partial\!\!\!/ \eta)^\dagger = \partial_\mu \eta^\dagger \gamma^{\mu\,\dagger} \gamma_4 \chi = -\partial_\mu \bar{\eta} \gamma^\mu \chi \simeq \bar{\eta} \partial\!\!\!/ \chi, \tag{1.93}$$

where the last equality holds only up to a total derivative; hence the action is real. However, using the Majorana property one obtains for *commuting* spinors

$$\bar{\eta} \partial\!\!\!/ \chi = \bar{\eta} \partial\!\!\!/ C \bar{\chi}^T = \partial_\mu \bar{\chi} C^T \gamma^{\mu\,T} \bar{\eta}^T = \partial_\mu \bar{\chi} \gamma^\mu \eta. \tag{1.94}$$

Substitution of this result into S_{kin} shows that it reduces to a total derivative

$$S_{kin} = \frac{1}{2} \int d^4x \, \partial_\mu \left(\bar{\chi} \gamma^\mu \eta \right). \tag{1.95}$$

Note, that this result applies in particular also if $\chi = \eta$. i.e. if we have only a single Majorana spinor. As Weyl and Dirac spinors can be rewritten in terms of Majorana spinors, the same argument also applies in those cases.

However, for *anti-commuting* spinors, eq.(1.94) is replaced by

$$\bar{\eta} \partial\!\!\!/ \chi = -\partial_\mu \bar{\chi} \gamma^\mu \eta. \tag{1.96}$$

Then the kinetic action becomes

$$S_{kin} = \frac{1}{2} \int d^4x \left(\bar{\chi} \partial\!\!\!/ \eta - \partial_\mu \bar{\chi} \gamma^\mu \eta \right) \equiv \frac{1}{2} \int d^4x \, \bar{\chi} \overleftrightarrow{\partial\!\!\!/} \eta. \tag{1.97}$$

This action can also be used for a single Majorana spinor: $\chi = \eta$. Thus the Dirac action (1.97) applies only to pseudo-classical fermion fields, which can be used in a path-integral formulation of quantum field theory. Commuting classical spinors satisfying a Dirac equation must be interpreted in the context of matrix elements or functional averages. In these notes unless stated otherwise, fermion fields are always taken to be described by anti-commuting spinors.

The action for the fermions in the Standard Model becomes

$$S_{sm,f} \,=\, \frac{1}{2} \int d^4x \left(\bar{\Psi}_{L\,A} \overleftrightarrow{\not{D}} \, \Psi_L^A + \Phi^{AB} \bar{\Psi}_{L\,A} \Psi_{R\,B} + \Phi^{\dagger}_{AB} \bar{\Psi}_R^A \Psi_L^B \right), \tag{1.98}$$

with the scalar field Φ^{AB} in the representation (1.36). Note that by charge conjugation

$$-\bar{\Psi}_{L\,A} \overleftarrow{\not{D}} \, \Psi_L^A \,=\, \bar{\Psi}_R^A \not{D} \Psi_{R\,A}. \tag{1.99}$$

This shows that the kinetic terms in $S_{sm,f}$ are real.

1.8 Anomalies

The classical action for the fermions (1.98) is a functional of the spinor fields $(\Psi_L, \bar{\Psi}_L)$ and the bosonic fields (A_μ, Φ), where A_μ stands collectively for the G_{sm}-gauge fields of the Standard Model, and Φ for the scalars, such as the Higgs doublet H^i and any others which might be needed to describe physical processes at energies close to the Fermi-scale.

It is important that the quantum field theory constructed for these fields respect the local gauge symmetries G_{sm}. The gauge symmetry is needed to decouple unphysical components of the vector fields from physical processses, e.g. negative-norm time-like components, or zero-norm light-like components. The first step to achieve invariance of the quantum field theory is the invariance of the action $S_{sm}[\Psi_L, \bar{\Psi}_L, A_\mu, \Phi]$ under the transformations

$$\begin{aligned}
&\Psi_L' \,=\, U_f[\alpha] \Psi_L, \qquad\qquad \bar{\Psi}_L' \,=\, \bar{\Psi}_L U_f^{-1}[\alpha], \\
&\Phi' \,=\, U_s[\alpha] \Phi, \\
&A_\mu' \,=\, \frac{i}{g}\, U[\alpha] \partial_\mu U^{-1}[\alpha] \,+\, U[\alpha] A_\mu U^{-1}[\alpha].
\end{aligned} \tag{1.100}$$

Here (U_f, U_s) are the representations of G_{sm} on the spinor and scalar fields, respectively. The gauge fields are taken to be Lie-algebra valued: $A_\mu \,=\, A_\mu^A T_A$, with T_A the generators of the algebra, in the appropriate representation for spinor or scalar matter. The transformations of A_μ are then realized by the group elements U in this same representation. Note that the transformation rules are defined to guarantee that

$$(D_\mu \Psi_L)' \,=\, U_f[\alpha]\, D_\mu \Psi_L, \tag{1.101}$$

and similarly for the scalar fields. In this way it is straightforward to achieve gauge invariance of the action $S_{sm,f}$, eq.(1.98).

However, in a quantum field theory invariance of the action is not enough to guarantee gauge-invariance of S-matrix elements. It is the full generating functional of the Green's functions which must respect the gauge symmetries, and in the path-integral formalism this requires gauge invariance of the functional measure in addition to invariance of the action.

To be more precise, consider the fermion determinant

$$Z_f[A_\mu, \Phi] \;=\; \int D\Psi_L D\bar{\Psi}_L \, e^{iS_{sm,f}[\Psi_L, \bar{\Psi}_L, A_\mu, \Phi]}. \tag{1.102}$$

This quantity is a functional of the gauge and scalar fields (A_μ, Φ). It appears in the effective action for the bosonic fields as the factor contributing fermion loops. That is, to compute n-point Green's functions for the vector- and scalar fields one addds to their tree-level action $S_0[A_\mu, \Phi]$ the term

$$\Delta S_{bos}[A_\mu, \Phi] \;=\; -i \ln Z_f[A_\mu, \Phi]. \tag{1.103}$$

The generating functional for the Green's functions of the bosonic fields then becomes

$$Z_{bos}[J^\mu, J, \bar{J}] \;=\; \int DA_\mu \int D\Phi \, e^{iS_0[A_\mu, \Phi] + i\Delta S_{bos}[A_\mu, \Phi] + iJ^\mu A_\mu + i\bar{J}\Phi + i\bar{\Phi}J}. \tag{1.104}$$

As S_0 is itself gauge invariant, the fermion contribution to the effective action must be gauge-invariant independently[6]. Thus we must require

$$Z_f[A'_\mu, \Phi'] \;=\; Z_f[A_\mu, \Phi]. \tag{1.105}$$

Performing the substitution $(A_\mu, \Phi) \rightarrow (A'_\mu, \Phi')$ in the functional integral (1.102), the change in the action can be compensated by performing the corresponding transformations of the fermions, as given by eq.(1.100). However, this then also requires a change of the integration measure

$$D\Psi'_L D\bar{\Psi}'_L \;=\; D\Psi_L D\bar{\Psi}_L \, \mathrm{Det}\, U \, \mathrm{Det}\, U^\dagger. \tag{1.106}$$

As the gauge transformations on the fermion components are unitary, one might expect that using Det $U^\dagger$ = Det U^{-1} = (Det $U)^{-1}$ makes the Jacobian trivially equal to unity. In a finite-dimensional integration space this would indeed be correct. However, as our functional integrals are over infinite-dimensional integration spaces, these manipulations are well-defined only within the context of a consistent regularization procedure. In the case of chiral symmetries often no such consistent procedure respecting all the gauge symmetries exist, and there may be a mismatch between the spectra of U and $U^\dagger$. When such mismatches occur, the Jacobian is not unity and the fermion determinant is not gauge invariant, even though the fermion action is.

[6]We have neglected the contribution of the Faddeev-Popov ghosts; when these are included one requires BRST invariance, rather than gauge invariance; for the matter fields this usually amounts to the same thing.

There exist various procedures for calculating the Jacobian, e.g. using regularized perturbation theory to compute terms in Ward-identities, or computing the Jacobian directly using a gauge-invariant exponential cut-off of type $\exp{(\not{D}/M)^2}$, where the regulator mass M is taken to infinity in the final result. As the anomaly is a sign of non-invariance, usually different procedures give different answers. However, these differences can be absorbed in local counterterms (local polynomial modifications of ΔS_{bos} of order $\hbar$), whilst the essential non-invariant part of the result is universal.

Fig. 3: Triangle diagrams contributing to the chiral anomaly.

In four dimensional space-time, the essential part of ΔS_{bos} potentially violating gauge invariance is in the 3-point function of the vector fields, i.e. the one-loop contribution of the fermions to the trilinear coupling of the Maxwell-Yang-Mills fields. This contribution is represented by the triangle diagrams of fig. 3. For any configuration of three external gauge field there are two such diagrams, differing in the order in which the external gauge fields couple to the fermion loop; equivalently, one can keep the order the same, but reverse the orientation of the fermion lines in the loop. Because of Bose statistics, of the external gauge fields, these contributions add up in the total amplitude. Taking into account the conservation of four-momentum, the external four-momenta of the gauge fields can be chosen as k, q and $-k-q$. Then up to a normalization constant the triangle diagrams are evaluated to give

$$\frac{1}{8}\int d^4p \left[\mathrm{Tr}\left(\frac{(1+\gamma_5)(\not{p}-\not{k})\gamma_\mu T_A (1+\gamma_5)\not{p}\gamma_\nu T_B (1+\gamma_5)(\not{p}+\not{q})\gamma_\lambda T_C}{p^2(p-k)^2(p+q)^2} \right) \right.$$
$$\left. +\ \mathrm{Tr}\left(\frac{(1+\gamma_5)(\not{p}-\not{q})\gamma_\nu T_B (1+\gamma_5)\not{p}\gamma_\mu T_A (1+\gamma_5)(\not{p}+\not{k})\gamma_\lambda T_C}{p^2(p-q)^2(p+k)^2} \right) \right]. \tag{1.107}$$

As it is a dummy integration variable, we can reverse the loop momentum to change its sign: $p \rightarrow -p$, and obtain the equivalent expression

$$\frac{1}{8}\int d^4p \left[\mathrm{Tr}\left(\frac{(1+\gamma_5)(\not{p}-\not{k})\gamma_\mu T_A (1+\gamma_5)\not{p}\gamma_\nu T_B (1+\gamma_5)(\not{p}+\not{q})\gamma_\lambda T_C}{p^2(p-k)^2(p+q)^2} \right) \right.$$
$$\left. -\ \mathrm{Tr}\left(\frac{(1+\gamma_5)(\not{p}+\not{q})\gamma_\nu T_B (1+\gamma_5)\not{p}\gamma_\mu T_A (1+\gamma_5)(\not{p}-\not{k})\gamma_\lambda T_C}{p^2(p+q)^2(p-k)^2} \right) \right]. \tag{1.108}$$

Now insert $\mathcal{C}\mathcal{C}^{-1}$ everywhere between the Dirac matrices in the second term, and use the charge conjugation properties

$$\mathcal{C}\gamma_5\mathcal{C}^{-1} \,=\, \gamma_5^T\,, \qquad \mathcal{C}\gamma_\mu\mathcal{C}^{-1} \,=\, -\gamma_\mu^T.$$

Factorizing the traces over Dirac and Lie-algebra generators, the numerator of the second term then becomes

$$\begin{aligned}
&\mathrm{Tr}\,(T_B T_A T_C)\,\times\,\mathrm{Tr}\,\left((1+\gamma_5)^T(\not{p}+\not{q})^T\gamma_\nu^T(1+\gamma_5)^T\not{p}^T\gamma_\mu^T(1+\gamma_5)^T(\not{p}-\not{k})^T\gamma_\lambda^T\right)\\
&= \mathrm{Tr}\,(T_B T_A T_C)\,\times\,\mathrm{Tr}\,(\gamma_\lambda(\not{p}-\not{k})(1+\gamma_5)\gamma_\mu\not{p}(1+\gamma_5)\gamma_\nu(\not{p}+\not{q})(1+\gamma_5))\\
&= \mathrm{Tr}\,(T_B T_A T_C)\,\times\,\mathrm{Tr}\,((1-\gamma_5)(\not{p}-\not{k})\gamma_\mu(1-\gamma_5)\not{p}\gamma_\nu(1-\gamma_5)(\not{p}+\not{q})\gamma_\lambda)\\
&= 4\,\mathrm{Tr}\,(T_B T_A T_C)\,\times\,\mathrm{Tr}\,((1-\gamma_5)(\not{p}-\not{k})\gamma_\mu\not{p}\gamma_\nu(\not{p}+\not{q})\gamma_\lambda)\,.
\end{aligned} \tag{1.109}$$

Adding now the two contributions in eq.(1.108), the terms which are even in γ_5 have a relative minus sign, and yield

$$\frac{1}{2}\,\mathrm{Tr}\,([T_A,T_B]\,T_C)\int d^4p\,\mathrm{Tr}\left[\frac{(\not{p}-\not{k})\gamma_\mu\not{p}\gamma_\nu(\not{p}+\not{q})\gamma_\lambda}{p^2(p-k)^2(p+q)^2}\right]. \tag{1.110}$$

The Lie-algebra factor in front takes the form

$$\mathrm{Tr}\,([T_A,T_B]\,T_C) \,=\, f_{AB}^{\;\;\;D}\,\mathrm{Tr}\,(T_D T_C) \,=\, f_{ABC}\,N_T, \tag{1.111}$$

where f_{ABC} are the completely covariant structure constants of the gauge algebra, and N_T is a constant depending on the normalization of the generators T_A. This part of the amplitude represents the fermionic 1-loop correction to the tree-level coupling of three gauge bosons. As such it respects the gauge symmetry of the theory.

The terms which are odd in γ_5 have a relative plus sign and add up:

$$\frac{1}{2}\,\mathrm{Tr}\,(\{T_A,T_B\}\,T_C)\,\times\int d^4p\,\mathrm{Tr}\left[\frac{\gamma_5(\not{p}-\not{k})\gamma_\mu\not{p}\gamma_\nu(\not{p}+\not{q})\gamma_\lambda}{p^2(p-k)^2(p+q)^2}\right]. \tag{1.112}$$

Here the braces denote the symmetrized product (anti-commutator):

$$\{T_A,T_B\} = T_A T_B + T_B T_A. \tag{1.113}$$

This part of the amplitude does not respect gauge invariance and produces the main contribution to the chiral anomaly. Therefore a sufficient condition for the anomalies in a gauge theory with chiral fermions to be absent is

$$\sum_r \mathrm{Tr}\,\left(\left\{T_A^{(r)},T_B^{(r)}\right\}T_C^{(r)}\right) \,=\, 0, \tag{1.114}$$

where the sum is to be taken over all representations r of fermions present in the theory, counting possible degeneracies.

In the Standard Model all chiral gauge anomalies cancel, as required. These cancellations take place inside a single family, and therefore the Standard Model allows an arbitrary number of families, as far as anomalies are concerned. To check this result, first consider the potential anomalies in $SU(3)$, to which only the quarks contribute; for a single family:

$$\begin{aligned} \sum_{r=(3,\bar{3})} \operatorname{Tr}\left(\left\{G_a^{(r)}, G_b^{(r)}\right\} G_c^{(r)}\right) &= \sum_{r=(3,\bar{3})} d_{abd}^{(r)} \operatorname{Tr}\left(G_d^{(r)}, G_c^{(r)}\right) \\ &= -i d_{abc}\,(2-2) = 0. \end{aligned} \tag{1.115}$$

Here we gave taken the generators of $SU(3)$ in the triplet and anti-triplet representations as

$$G_a^{(3)} = \frac{i\lambda_a}{2}, \qquad G_a^{(\bar{3})} = -\frac{i\lambda_a}{2}, \tag{1.116}$$

where λ_a are the (hermitean) Gell-Mann matrices. As a consequence

$$d_{abc}^{(3)} = -d_{abc}^{(\bar{3})} = i d_{abc}, \tag{1.117}$$

with d_{abc} the standard totally symmetric d-symbols of $SU(3)$. The cancellation occurs, because there is one $SU(2)$-doublet of triplet quarks, whilst there are two $SU(2)$-singlets of anti-triplet quarks.

Another possible anomaly to which only quarks contribute is the one in $SU(3) \times U_Y(1)$:

$$\begin{aligned} \sum_r \operatorname{Tr}\left(\left\{G_a^{(r)}, G_b^{(r)}\right\} Y^{(r)}\right) &= -\delta_{ab} \sum_r Y^{(r)} \\ &= -\delta_{ab}\left(2 \times \frac{1}{6} - 1 \times \frac{2}{3} + 1 \times \frac{1}{3}\right) = 0. \end{aligned} \tag{1.118}$$

Again the sum is over all triplets and anti-triplets in a single family, containing one $SU(2)$-doublet of triplets with hypercharge $Y = 1/6$, one $SU(2)$-singlet anti-triplet with $Y = -2/3$, and one $SU(2)$-singlet anti-triplet with $Y = 1/3$.

A similar anomaly can occur in $SU(2) \times U_Y(1)$, but it involves both quarks and leptons:

$$\begin{aligned} \sum_r \operatorname{Tr}\left(\left\{T_i^{(r)}, T_j^{(r)}\right\} Y^{(r)}\right) &= -\delta_{ij} \sum_r Y^{(r)} \\ &= -\delta_{ij}\left(3 \times \frac{1}{6} - 1 \times \frac{1}{2}\right) = 0. \end{aligned} \tag{1.119}$$

Here the sum is over all $SU(2)$ doublets, for which $T_i = i\tau_i/2$. Notice, that the contribution of the quarks, $3/6$, is canceled by the $-1/2$ of the leptons.

Without the leptons the electroweak interactions would be anomalous; similarly, the electroweak interactions of the leptons require the existence of a triplet of quark doublets.

Finally, there is a possible anomaly in the pure $U_Y(1)$ sector, involving all chiral fermions in a single family:

$$\begin{aligned}\sum_r \left(Y^{(r)}\right)^3 &= 3\times 2\times\left(\frac{1}{6}\right)^3 + 3\times 1\times\left(-\frac{2}{3}\right)^3 + 3\times 1\times\left(\frac{1}{3}\right)^3 \\ &\quad + 2\times\left(-\frac{1}{2}\right)^3 + 1\times 1\times 1^3 = -\frac{3}{4}+\frac{3}{4} = 0.\end{aligned} \tag{1.120}$$

Here the contributions of the quarks and leptons have been added separately to illustrate the necessity of including all of them in order to achieve complete cancellation.

Chapter 2

Neutrinos

2.1 Neutrinos in the minimal Standard Model

Of the 15 chiral fermions in a single family of quarks and leptons 14 can pair up by Yukawa interactions to massive Dirac fermions: six quarks and one charged lepton. The fifteenth one is the neutrino, and according to the minimal Standard Model (MSM) it remains massless. Thus, in the context of the MSM, if we can count the number of massless neutrinos we can count the number of families.

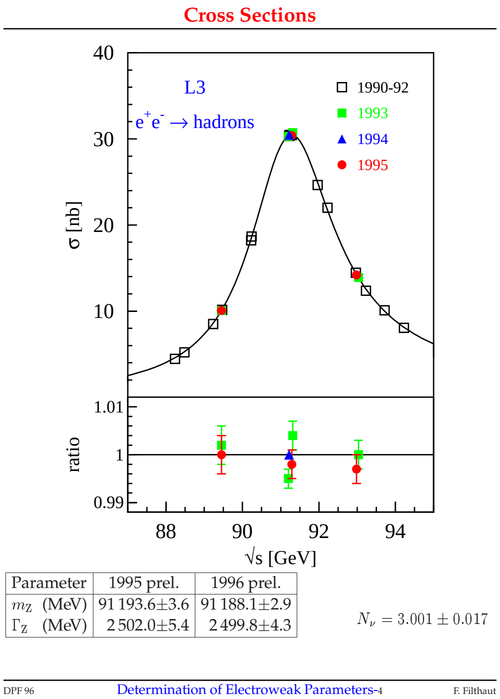

Fig. 3: L3-LEP measurement of the hadronic width of the Z-resonance [2].

There are at least two independent sources of observational evidence that there are three massless, or at least very light, neutrinos. First, there is circumstantial evidence from cosmology. The number of light neutrinos influences the rate of expansion of the universe before and during the period of primordial helium production. A larger number of light neutrinos implies a higher temperature at the time the neutrons freeze out of the cosmic plasma; this in turn implies that a larger fraction of neutrons is left when nucleosynthesis sets in, and more baryonic matter can end up in primordial helium. From the observed fraction of about 25% of baryonic matter captured into primordial helium, cosmologists conclude that the number of light neutrinos is three, or at most four.

A direct determination of the number of different massless/light neutrinos comes from the observed decay of the Z-boson at electron-positron colliders such as LEP. Here the total width of the Z-resonance, as well as the partial widths in different decay channels, can be measured with great precision, and the contribution of unobserved decays to the total width can be determined very accurately. For the decay $Z \rightarrow$ hadrons, values as measured by the L3 experiment are shown in Fig. 3. From these observations it is established that there are 3 types of (nearly) massless left-handed neutrino's into which the Z can decay.

2.2 Neutrino masses I: Majorana neutrinos

The Higgs interactions break the Standard Model gauge group spontaneously to $SU(3) \times U_Q(1)$. As the neutrinos are singlets with respect to this non-broken subgroup, carrying neither color nor electric charge, they can in principle have a mass at low energies from a contribution to their field equations as presented in eq.(1.16):

$$\partial\!\!\!/ \nu_L = -m\, \nu_R, \tag{2.1}$$

with $\nu_R = (\nu_L)^c$. Basically this amounts to considering the neutrino as a Majorana fermion

$$N = \nu_L + (\nu_L)^c = \nu_L + \nu_R, \tag{2.2}$$

satisying an effective Dirac equation

$$(\partial\!\!\!/ + m)\, N = 0. \tag{2.3}$$

The term *effective* here means in the first place that we have neglected the coupling to very massive fields, like the intermediate vector bosons $(W^{\pm}, Z)$ and the physical Higgs field. In the second place it implies, that the mass cannot be a fundamental parameter of the theory, but must result from the spontaneous breaking of G_{sm}, as in the limit of unbroken $SU(2) \times U_Y(1)$ gauge symmetry (in the high-energy limit) the neutrino is not a singlet, but part of a doublet of $SU(2)$ with non-zero hypercharge. In that limit the mass-term in eq.(2.3) must vanish so as not to spoil the unbroken gauge symmetry.

We can also state this differently as follows: the mass m must be the vacuum expectation of a scalar field with quantum numbers such that the transformation properties of the left- and right-hand side of eq.(2.1) agree in the limit of vanishing spontaneous symmetry breaking if we replace the mass by the scalar coupling of which it is the low-energy limit. There is a surprisingly simple way to achieve this. Consider the isospin and hypercharge of the lepton doublet L_L^i and the scalar Higgs doublet H^i in table 1. It is straightforward to see, that the chiral spinors

$$N_L = \varepsilon_{ij} H^i L_L^j, \qquad \bar{N}_L = -\varepsilon^{ij} H_i^* \bar{L}_{Lj}, \tag{2.4}$$

are singlets of $SU(2)$ with vanishing hypercharge: $Y = 0$. Now in the unitary gauge $H^1 = \text{Im } H^2 = 0$, with $\text{Re } H^2 = v + h$, and v is the non-zero vacuum expectation value:

$$H^i = \delta_2^i (v + h) \to \langle H^i \rangle = v \delta_2^i. \tag{2.5}$$

Then this chiral fermion reduces to the standard neutrino component

$$N_L = (v + h) L_L^1 \to v L_L^1 = v \nu_L. \tag{2.6}$$

However, N_L being a singlet of the full gauge group G_{sm}, we can write down a Dirac equation with an invariant mass-term of Majorana type for it:

$$\partial\!\!\!/ N_L = -m N_R \quad \Leftrightarrow \quad (\partial\!\!\!/ + m) N = 0, \tag{2.7}$$

with $N_R = (N_L)^c$, and $N = N_L + N_R$. In the context of quantum field theory, with the fermion fields represented by anti-commuting spinors, we can derive the above equation from a lagrangean

$$\mathcal{L}_N^{(0)} = \frac{1}{2|v|^2} \left(\bar{N}_L \overleftrightarrow{\partial\!\!\!/} N_L + m \bar{N}_L N_R + m^* \bar{N}_R N_L \right). \tag{2.8}$$

Notice that it is possible to add a further term

$$\Delta \mathcal{L}_N = \frac{1}{2|v|^2} \varepsilon^{ik} \varepsilon_{jl} \bar{L}_{Li} \gamma^\mu L_L^j \left(H_k^* \overleftrightarrow{D}_\mu H^l \right). \tag{2.9}$$

As the Higgs field is constant in the vacuum, and its v.e.v. respects the electromagnetic $U_Q(1)$, this does not change the low-energy limit of eq.(2.8). Then in the unitary gauge the combined terms give

$$\mathcal{L}_N^{(0)} + \Delta \mathcal{L}_N = \frac{1}{2|v|^2} \left(-\varepsilon^{ik} \varepsilon_{jl} H_k^* H^l \bar{L}_{Li} D\!\!\!\!/ L_L^j + m \varepsilon^{ik} \varepsilon^{jl} H_k^* H_l^* \bar{L}_{Li} L_{Rj} + \text{h.c.} \right). \tag{2.10}$$

To this we can still add the standard kinetic terms for the lepton doublet; we then get a third gauge-invariant lagrangean for a massive Majorana neutrino:

$$\mathcal{L}_N^{(1)} = \frac{1}{4} \left(\delta_j^i - |v|^{-2} \varepsilon^{ik} \varepsilon_{jl} H_k^* H^l \right) \bar{L}_{Li} D\!\!\!\!/ L_L^j + \frac{m}{4|v|^2} \varepsilon^{ik} \varepsilon^{jl} H_k^* H_l^* \bar{L}_{Li} L_{Rj} + \text{h.c.} \tag{2.11}$$

Clearly, a large set of consistent gauge-invariant lagrangeans can be written down, all of which have the same low-energy limit describing a single massive Majorana neutrino. It is equally clear, that all of these lagrangeans are of non-renormalizable type and require an explicit cut-off to be introduced.

Alternatively, one could write a renormalizable theory with a new complex triplet scalar field ${\Phi^i}_j = (\Phi_a \tau_a/2)^i{}_j$, with τ_a the Pauli matrices. Then one can construct the Yukawa term

$$\Delta\mathcal{L}_{tr} \,=\, f_{tr} {\Phi^i}_k \, \varepsilon^{kj} \bar{L}_{L\,i} L_{R\,j} \,+\, \text{h.c.}, \tag{2.12}$$

which is invariant under G_{sm} provided one assigns the triplet of scalars a hypercharge $Y = -1$. The component ${\Phi^1}_2 = (\Phi_1 - i\Phi_2)/2$ is then electrically neutral:

$$[Q, \Phi_1 - i\Phi_2] \,=\, 0. \tag{2.13}$$

If this scalar component gets a vacuum expectation value, a mass for the neutral leptons is generated, in a way respecting the conservation of electric charge. However, apart from a rather strong experimental bound on the existence of a triplet Higgs, especially under the condition that the photon is to remain massless, this requires the introduction of a second scale of symmetry breaking into the theory.

In all cases, massive Majorana neutrinos are a sign of a new scale in physics where new interactions must become relevant, and where the Standard Model is no longer applicable in its minimal form.

2.3 Neutrino masses II: Dirac neutrinos

An alternative to the construction of models with massive Majorana neutrinos is to introduce independent right-handed neutrinos which are singlets with respect to the full Standard Model gauge group G_{sm}. In this scenario eq.(2.1) applies, but without the charge-conjugation relation between ν_L and ν_R. Therefore a second equation is needed to complete the model:

$$\not{\partial} \nu_R \,=\, -m\nu_L. \tag{2.14}$$

This equation is obtained by requiring that ν_L and ν_R are the chiral projections of a Dirac spinor ν:

$$\nu_L \,=\, \frac{1}{\sqrt{2}} \left(1 + \gamma_5\right) \nu, \qquad \nu_R \,=\, \frac{1}{\sqrt{2}} \left(1 - \gamma_5\right) \nu, \tag{2.15}$$

which satisfies the equation

$$\left(\not{\partial} + m\right) \nu \,=\, 0. \tag{2.16}$$

Taking chiral projections this equation reproduces both eq.(2.1) and eq.(2.14). Eq.(2.16) can be derived from an action principle with

$$\mathcal{L} = \bar{\nu}(\partial\!\!\!/ + m)\nu = \frac{1}{2}\,\bar{\nu}_L \partial\!\!\!/ \nu_L + \frac{m}{2}\,\bar{\nu}_L \nu_R + \text{h.c.} \tag{2.17}$$

As the right-handed neutrino does not couple to any of the gauge fields, it can only interact via a Yukawa coupling with the Higgs field and the lepton doublet. This is completely analogous to the coupling of the right-handed U-quark to the quark doublet. The mass of Dirac neutrinos can then be induced by the Higgs mechanism via this Yukawa interaction. Thus Dirac neutrinos can be fitted entirely into the frame work of the minimal Standard Model, and moreover the Standard Model extended with right-handed neutrinos is still renormalizable and requires no new scale of physics to act as a cut-off.

Extending the MSM with right-handed neutrinos implies that a family of fermions now form a 16-dimensional representation of the gauge group G_{sm}:

$$\Psi_L = \left(Q_L^{\alpha i}, U_{L\,\alpha}, D_{L\,\alpha}, L_L^i, N_L, E_L \right), \tag{2.18}$$

with the representation content

$$\left\{ (3,2)_{1/6}, (\bar{3},1)_{-2/3}, (\bar{3},1)_{1/3}, (1,2)_{-1/2}, (1,1)_0, (1,1)_1 \right\}. \tag{2.19}$$

Here N_L is the charge conjugate spinor of the right-handed neutrino field: $N_L = (\nu_R)^c$. With N_f generations these fields again satisfy the Dirac equation (1.34), but now with a matrix of scalar fields of the form

$$\Phi^{AB} = \begin{bmatrix} 0 & f_U^{AB}\varepsilon^{ij}H_j^*\delta_\beta^\alpha & f_D^{AB}H^i\delta_\beta^\alpha & 0 & 0 & 0 \\ f_U^{BA}\varepsilon^{ji}H_i^*\delta_\alpha^\beta & 0 & 0 & 0 & 0 & 0 \\ f_D^{BA}H^j\delta_\alpha^\beta & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & f_N^{AB}\varepsilon^{ij}H_j^* & f_L^{AB}H^i \\ 0 & 0 & 0 & f_N^{BA}\varepsilon^{ji}H_i^* & 0 & 0 \\ 0 & 0 & 0 & f_L^{BA}H^j & 0 & 0 \end{bmatrix}. \tag{2.20}$$

These changes only affect the lepton sector; the field equations for the quarks are as in (1.37). The lepton equations then take the form

$$\begin{aligned} D\!\!\!\!/\, L_L^{A\,i} &= f_N^{AB}\,\varepsilon^{ij}\,H_j^*\,N_{R\,B} + f_L^{AB}\,H^i E_{R\,B}, \\ D\!\!\!\!/\, N_L^A &= f_N^{BA}\,\varepsilon^{ij}\,H_j^*\,L_{R\,Bi}, \\ D\!\!\!\!/\, E_L^A &= f_L^{BA}\,H^i\,L_{R\,Bi}. \end{aligned} \tag{2.21}$$

The vacuum expectation (1.41) of the Higgs field then induces the masses, as anticipated above. The complete analogy with the quark sector also implies, that after diagonalization of the Yukawa matrices vf_L^{AB} and vf_N^{AB} all mixing between the lepton families has disappeared from the gauge interactions, except in the couplings to the charged weak vector bosons. In that sector there appears a new leptonic CKM matrix, which can be parametrized completely like, but independend of, the quark mixing matrix.

Once right-handed neutrinos are incorporated into the model, Dirac masses induced via the Higgs mechanism are not the only possible new contribution to the lepton mass matrix. Even within the context of renormalizable models one can introduce a Majorana mass term for the right-handed neutrinos, as this is a gauge singlet by itself. We here consider the effects for a single family. We take the most general case, in which both the standard right-handed neutrinos as well as the left-handed ones have a Majorana mass. The corresponding quadratic part of the lagrangean is

$$\Delta\mathcal{L}_\nu = \frac{1}{2}\,\bar{\nu}_L^{(d)}\partial\!\!\!/\nu_L^{(d)} + \frac{1}{2}\,\bar{\nu}_L^{(s)}\partial\!\!\!/\nu_L^{(s)} + \frac{1}{4}\left(\bar{\nu}_R^{(d)}, \bar{\nu}_R^{(s)}\right)\left(\begin{array}{cc} \mu_d & M \\ M & \mu_s \end{array}\right)\left(\begin{array}{c} \nu_L^{(d)} \\ \nu_L^{(s)} \end{array}\right) + \text{h.c.} \tag{2.22}$$

The notation is as follows. The left-handed neutrino which is part of the doublet is called $\nu_L^{(d)}$, whereas the singlet neutrino is denoted by $\nu_L^{(s)}$; $\nu_R^{(d)}$ and $\nu_R^{(s)}$ are their charge conjugates. The matrix elements (μ_d, μ_s) are the corresponding Majorana mass parameters, whilst the off-diagonal elements M represent the Dirac mass terms.

For the analysis and construction of mass eigenstates it is convenient to rewrite expression (2.22) in terms of Majorana spinors. First we define $\nu^{(d)}$ and $\nu^{(s)}$ as the self-conjugate combinations

$$\nu_d = \frac{1}{\sqrt{2}}\left(\nu_L^{(d)} + \nu_R^{(d)}\right), \qquad \nu_s = \frac{1}{\sqrt{2}}\left(\nu_L^{(s)} + \nu_R^{(s)}\right). \tag{2.23}$$

Furthermore, we write the complex matrix elements in eq.(2.22) in terms of the (positive) real moduli (m_r, m) and phases (α_r, β):

$$\begin{aligned} \mu_r &= m_r e^{i\alpha_r}, \qquad\qquad r = (d,s), \\ M &= m e^{i\beta}. \end{aligned} \tag{2.24}$$

Then the free neutrino lagrangean takes the form

$$\Delta\mathcal{L}_\nu = \frac{1}{2}\,\bar{\nu}_d\partial\!\!\!/\nu_d + \frac{1}{2}\,\bar{\nu}_s\partial\!\!\!/\nu_s + \frac{1}{2}\left(\bar{\nu}_d, \bar{\nu}_s\right)\left(\begin{array}{cc} m_d e^{i\alpha_d\gamma_5} & m e^{i\beta\gamma_5} \\ m e^{i\beta\gamma_5} & m_s e^{i\alpha_s\gamma_5} \end{array}\right)\left(\begin{array}{c} \nu_d \\ \nu_s \end{array}\right). \tag{2.25}$$

The most general transformation of the Majorana spinor pair (ν_d, ν_s) leaving the kinetic terms diagonal and properly normalized is of the form

$$\left(\begin{array}{c} \nu_d \\ \nu_s \end{array}\right) = \left(\begin{array}{cc} \cos\theta e^{-i\sigma\gamma_5} & -\sin\theta e^{-i\varepsilon\gamma_5} \\ \sin\theta e^{-i\rho\gamma_5} & \cos\theta e^{-i\eta\gamma_5} \end{array}\right)\left(\begin{array}{c} \nu_d' \\ \nu_s' \end{array}\right), \tag{2.26}$$

provided the phases are related by $\rho - \eta = \sigma - \varepsilon \bmod 2\pi n$. If we now use such a transformation to attempt diagonalization of the mass matrix the result is

$$\frac{1}{2}\,(\bar{\nu}'_d, \bar{\nu}'_s) \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix} \begin{pmatrix} \nu'_d \\ \nu'_s \end{pmatrix}, \tag{2.27}$$

with the elements given by

$$\begin{aligned} M_{11} &= m_d \cos^2\theta e^{i(\alpha_d - 2\sigma)\gamma_5} + m_s \sin^2\theta e^{i(\alpha_s - 2\rho)\gamma_5} - 2m \sin\theta\cos\theta e^{i(\beta-\sigma-\rho)\gamma_5}, \\ M_{22} &= m_s \cos^2\theta e^{i(\alpha_s - 2\eta)\gamma_5} + m_d \sin^2\theta e^{i(\alpha_d - 2\varepsilon)\gamma_5} - 2m \sin\theta\cos\theta e^{i(\beta-\varepsilon-\eta)\gamma_5}, \\ M_{12} &= M_{21} = \sin\theta\cos\theta \left(m_s e^{i(\alpha_s-\eta-\rho)\gamma_5} - m_d e^{i(\alpha_d-\varepsilon-\sigma)\gamma_5}\right) \\ &\quad + m\left(\cos^2\theta e^{i(\beta-\eta-\sigma)\gamma_5} - \sin^2\theta e^{i(\beta-\varepsilon-\rho)\gamma_5}\right). \end{aligned} \tag{2.28}$$

The requirements for the off-diagonal terms to vanish: $M_{12} = M_{21} = 0$, imply two constraints:

$$\begin{aligned} \tan 2\theta &= \frac{2m\cos(\beta-\eta-\sigma)}{m_d\cos(\alpha_d-\varepsilon-\sigma) - m_s\cos(\alpha_s-\eta-\rho)} \\ &= \frac{2m\sin(\beta-\eta-\sigma)}{m_d\sin(\alpha_d-\varepsilon-\sigma) - m_s\sin(\alpha_s-\eta-\rho)}. \end{aligned} \tag{2.29}$$

It follows, that firstly

$$\frac{m_d}{m_s} = \frac{\sin(\alpha_s-\beta+\sigma-\rho)}{\sin(\alpha_d-\beta-\sigma+\rho)}, \tag{2.30}$$

or equivalently:

$$\tan(\sigma-\rho) = \frac{m_d\sin(\alpha_d-\beta) - m_s\sin(\alpha_s-\beta)}{m_d\cos(\alpha_d-\beta) + m_s\cos(\alpha_s-\beta)}; \tag{2.31}$$

and secondly:

$$\tan 2\theta = \frac{2m\cos(\beta-(\sigma-\rho)-(\eta+\rho))}{m_d\cos(\alpha_d-2(\sigma-\rho)-(\eta+\rho)) - m_s\cos(\alpha_s-(\eta+\rho))}. \tag{2.32}$$

Here we have used the above relation $\sigma - \rho = \varepsilon - \eta$ to eliminate ε. It follows we can still fix two independ parameters, e.g. σ and η, so as to make the blocks M_{11} and M_{22} real and diagonal. The precise conditions read

$$\begin{aligned} m_d\cos^2\theta\sin(\alpha_d-2\sigma) + m_s\sin^2\theta\sin(\alpha_s-2\rho) &= 2m\sin\theta\cos\theta\cos(\beta-\sigma-\rho), \\ m_s\cos^2\theta\sin(\alpha_s-2\eta) + m_d\sin^2\theta\sin(\alpha_d-2\varepsilon) &= 2m\sin\theta\cos\theta\cos(\beta-\varepsilon-\eta). \end{aligned} \tag{2.33}$$

As an example, consider the case in which

$$\beta = \frac{1}{2}(\alpha_d + \alpha_s). \tag{2.34}$$

Then we can take

$$\sigma = \varepsilon = \frac{1}{2}\alpha_d, \qquad \rho = \eta = \frac{1}{2}\alpha_s. \tag{2.35}$$

It follows that

$$\operatorname{cotg} 2\theta = \frac{m_d - m_s}{2m}. \tag{2.36}$$

Furthermore, the diagonal elements of the mass matrix, $M_+ = M_{11}$ and $M_- = M_{22}$, are real with values

$$M_\pm = \frac{1}{2}\left(m_d + m_s \pm \sqrt{4m^2 + (m_d - m_s)^2}\right). \tag{2.37}$$

The corresponding eigenstates are

$$\begin{aligned} \nu_+ &= \nu'_d = \cos\theta\, e^{i\alpha_d\gamma_5/2}\,\nu_d + \sin\theta\, e^{i\alpha_s\gamma_5/2}\,\nu_s, \\ \nu_- &= \nu'_s = \cos\theta\, e^{i\alpha_s\gamma_5/2}\,\nu_s - \sin\theta\, e^{i\alpha_d\gamma_5/2}\,\nu_d. \end{aligned} \tag{2.38}$$

As we have observed before, a Majorana mass parameter for the left-handed neutrino in an $SU(2)\times U_Y(1)$ invariant theory is suppressed, as it is not renormalizable in a theory with only Higgs doublets. On the other hand, there is no constraint on the mass of the singlet neutrino, which can be large in principle. Therefore a situation of interest is the one with the hierarchy $m_s \gg m \gg m_d$. In this case the mixing is naturally small, and the mass eigenvalues have a large hierarchy:

$$M_+ \approx m_s, \qquad M_- \approx \frac{m_d}{2} - \frac{m^2}{m_s}. \tag{2.39}$$

The natural value of the mass of the lightest neutrino is suppressed with respect to the Dirac mass parameter m by a factor $m/m_s \ll 1$. This can explain in a natural way why the masses of the neutrinos in the Standard Model (which are part of $SU(2)$ doublets) are so much smaller than those of the corresponding charged leptons, for which only Dirac masses are allowed and no such suppression is possible. This mechanism for generating a naturally small mass for the neutrino of the Standard Model by allowing a very large mass for a singlet neutrino is sometimes called the 'see-saw' mechanism.

Chapter 3

Gauge bosons and broken symmetries

3.1 Higgs fields

In the minimal standard model all fermion masses are generated by the vacuum expectation value of a single scalar field multiplet, the Higgs doublet. This has been made explicit in eq.(1.43), representing the v.e.v. of the Yukawa interactions (1.36). Neutrino masses of Dirac type can be obtained using the same scalar doublet from the Yukawa interactions (2.20). Even the Majorana mass terms for the left-handed neutrino, being part of the lepton doublet, can be obtained from the standard Higgs doublet, at the price of using non-renormalizable interactions as in the lagrangean (2.11). Only the Majorana mass terms for right-handed neutrinos, being strict gauge singlets, are free parameters, for which the Brout-Englert-Higgs mechanism of mass generation is not required.

However, as mentioned at the end of section (2.2), a renormalizable Majorana mass term for the left-handed neutrino requires a real triplet of scalar fields, rather than a complex doublet, as in eq.(2.12). Such a triplet is not attractive for phenomenological model building, unless the associated scalar particles are very heavy, as it tends to predict values of the ρ-parameter, defined by[1]

$$\rho = \frac{M_W^2}{M_Z^2 \cos^2 \theta_w}, \tag{3.1}$$

which are in conflict with the measured value $\rho = 1.0107 \pm 0.0006$ [3]. The reason for this conflict is simple. The covariant derivative of a real triplet of scalar fields $\Phi = i\boldsymbol{\phi} \cdot \boldsymbol{\tau}/2$ is

$$D_\mu \boldsymbol{\phi} = \partial_\mu \boldsymbol{\phi} - g_2 \mathbf{W}_\mu \times \boldsymbol{\phi}. \tag{3.2}$$

[1] Cf. eq.(1.62).

Therefore the kinetic action of the scalar field contributes a term

$$-\frac{g_2^2}{2}\left[\mathbf{W}_\mu^2\,\boldsymbol{\phi}^2\;-\;(\mathbf{W}\cdot\boldsymbol{\phi})^2\right], \tag{3.3}$$

to the quadratic self-interaction of the $SU(2)$ gauge bosons. It does not contribute to the $U_Y(1)$ mass terms, because for a real triplet the hypercharge $Y = 0$. For the same reason the electric charge of the scalars is

$$Q = T_3 = (+1, 0, -1), \tag{3.4}$$

with the zero eigenvalue corresponding to the component ϕ^3, whilst the charged components are $\phi^\pm = (\phi^1 \mp i\phi^2)/\sqrt{2}$. For these scalar fields to develop a vacuum expectation value respecting electric charge conservation, i.e., the symmetry $U_Q(1)$, the v.e.v. must be in the third isospin component:

$$\langle\phi^a\rangle\;=\;v_3\delta_3^a. \tag{3.5}$$

This then contributes to the gauge boson masses a term

$$\Delta\mathcal{L}_m[W]\;=\;-\frac{g_2^2v_3^2}{2}\left((W_\mu^1)^2+(W_\mu^2)^2\right)\;=\;-\,g_2^2v_3^2\,W^+\cdot W^-, \tag{3.6}$$

with $W^\pm = (W^1 \pm iW^2)/\sqrt{2}$ the charged W-fields. As a result, the real scalar triplet contributes to the charged vector boson mass M_W, but not to the neutral Z-boson mass M_Z. For this reason a triplet scalar changes the mass ratio M_W/M_Z, which reflects itself in a change of the ρ-parameter from its canonical value $\rho = 1$ in the MSM at tree level.

Whilst triplets must be absent or negligeable, complex Higgs doublets do not create similar problems, and additional Higgs doublets are allowed in extensions of the minimal Standard Model. A quick look at equation (1.36) or its extension (2.20), recalling that the only essential aspect of the field content of Φ^{AB} is its symmetrical character, teaches us that without changing the particle spectrum we could have used three or four independent Higgs doublets per family. If we call them $(H^A_{(U)}, H^A_{(D)}, H^A_{(L)})$ respectively, we can write

$$\Phi^{AB}\;=\;\begin{bmatrix} 0 & f_U^{AB}\,H^{A\,i}_{(U)}\,\delta^\alpha_\beta & f_D^{AB}\,H^{A\,i}_{(D)}\,\delta^\alpha_\beta & 0 & 0 \\ f_U^{BA}\,H^{B\,j}_{(U)}\,\delta^\beta_\alpha & 0 & 0 & 0 & 0 \\ f_D^{BA}\,H^{B\,j}_{(D)}\,\delta^\beta_\alpha & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & f_L^{AB}\,H^{A\,i}_{(L)} \\ 0 & 0 & 0 & f_L^{BA}\,H^{B\,j}_{(L)} & 0 \end{bmatrix}. \tag{3.7}$$

The corresponding Dirac equations become:

$$\begin{aligned}
\not{D} Q_L^{A\alpha i} &= f_U^{AB}\, H_{(U)}^{A\,i}\, U_{R\,B}^{\alpha} + f_D^{AB} H_{(D)}^{A\,i}\, D_{R\,B}^{\alpha},\\
\not{D} U_{L\,\alpha}^{A} &= f_U^{BA}\, H_{(U)}^{B\,i}\, Q_{R\,B\alpha i},\\
\not{D} D_{L\,\alpha}^{A} &= f_D^{BA}\, H_{(D)}^{B\,i}\, Q_{R\,B\alpha i},\\
\not{D} L_L^{A\,i} &= f_L^{AB}\, H_{(L)}^{A\,i}\, E_{R\,B},\\
\not{D} E_L^{A} &= f_L^{BA}\, H_{(L)}^{B\,i} L_{R\,Bi}.
\end{aligned} \tag{3.8}$$

For singlet neutrinos a straightforward extension can be given. We observe, that the scalar doublets $(H_{(D)}^A, H_{(L)}^A)$ are just copies of the original Higgs doublet with hypercharge $Y = 1/2$. However, the doublets $H_{(U)}^{A\,i}$, and also any $H_{(N)}^{A\,i}$, take the place of the conjugate doublet $\varepsilon^{ij} H_j^*$, and hence must have opposite hypercharge $Y = -1/2$. Vacuum expectation values of all electrically neutral scalar components —the lower $T_3 = -1/2$ components of the D- and L-doublets, the upper $T_3 = +1/2$ components of the U- and N-doublets— now generate masses for the various charge eigenstates of the quarks and leptons, as before.

Consider now the effect of the proliferation of Higgs doublets on the vector boson masses. The covariant derivatives of the scalar doublets read

$$D_\mu H_{(F)} = \left(\partial_\mu - \frac{ig_2}{2}\, \mathbf{W}_\mu \cdot \boldsymbol{\tau} - \frac{ig_1}{2}\, (-1)^F B_\mu \right) H_{(F)}, \tag{3.9}$$

with

$$(-1)^F = \left\{ \begin{array}{ll} +1, & \text{for } F = (D, L); \\ -1, & \text{for } F = (U, N). \end{array} \right. \tag{3.10}$$

The quadratic gauge-boson terms giving rise to the vector boson masses become

$$- \sum_{F,A} \left(\frac{g_2^2}{4}\, \mathbf{W}_\mu^2 + \frac{g_1^2}{4}\, B_\mu^2 \right) \left| H_{(F)}^A \right|^2 + \frac{g_1 g_2}{2} \sum_{F,A} (-1)^F\, B^\mu \mathbf{W}_\mu \cdot \left(H_{(F)\,A}^{\dagger} \boldsymbol{\tau} H_{(F)}^A \right). \tag{3.11}$$

Now either the upper or the lower component of $H_{(F)}^A$ gets a non-zero v.e.v., depending on F; the other component has no v.e.v.:

$$\langle H_{(F)}^A \rangle = \left(\begin{array}{c} 0 \\ v_{(F)}^A \end{array} \right), \qquad F = (D, L), \tag{3.12}$$

and

$$\langle H_{(F)}^A \rangle = \left(\begin{array}{c} v_{(F)}^A \\ 0 \end{array} \right), \qquad F = (U, N). \tag{3.13}$$

Therefore only the diagonal terms with τ_3 can contribute to the last term in eq.(3.11):

$$\langle \left(H^{\dagger}_{(F)\,A} \tau_a H^A_{(F)} \right) \rangle \,=\, -\,(-1)^F\, \left| v^A_{(F)} \right|^2\, \delta^3_a. \tag{3.14}$$

As a result, all factors $(-1)^F$ cancel, and the result is

$$\begin{aligned} \Delta\mathcal{L}_m[W] \;&=\; -\sum_{A,F} \left| v^A_{(F)} \right|^2 \left(\frac{g_2^2}{4}\, \mathbf{W}^2_\mu \,+\, \frac{g_1^2}{4}\, B^2_\mu \,-\, \frac{g_1 g_2}{2}\, B^\mu W^3_\mu. \right) \\ &=\; -\sum_{A,F} \left| v^A_{(F)} \right|^2 \left(\frac{g_2^2}{2}\, W^+ \cdot W^- \,+\, \frac{g_2^2}{4\cos^2\theta_w}\, Z^2_\mu \right), \end{aligned} \tag{3.15}$$

where we have defined as usual

$$Z_\mu \,=\, \cos\theta_w\, W^3_\mu \,-\, \sin\theta_w\, B_\mu. \tag{3.16}$$

Observe, that as far as the vector boson mass-terms are concerned it is the same as coupling to a single Higgs doublet with v.e.v.

$$|v|^2 \,=\, \sum_{A,F} \left| v^A_{(F)} \right|^2. \tag{3.17}$$

Thus the vector boson masses are

$$M^2_W \,=\, \frac{g_2^2}{2} \sum_{A,F} \left| v^A_{(F)} \right|^2, \qquad M^2_Z \,=\, \frac{g_2^2}{2\cos^2\theta_w} \sum_{A,F} \left| v^A_{(F)} \right|^2. \tag{3.18}$$

In particular, at the tree level we obtain

$$\rho \,=\, \frac{M^2_W}{M^2_Z \cos^2\theta_w} \,=\, 1. \tag{3.19}$$

For the vector bosons it makes no difference in principle how many scalar doublets there are, as long as the total amount of $SU(2)\times U_Y(1)$ symmetry breaking, as given by $\sum |v|^2$ is the same. Even so, there are differences in the fermion sector. Actually, the Yukawa couplings in the fermion sector can be used to *distinguish* the various scalar doublets. Clearly, the individual scalar vacuum expectation values $v^A_{(F)}$ are each smaller than the equivalent single one $|v|$:

$$\frac{|v^A_{(F)}|}{|v|} \,=\, \frac{|v^A_{(F)}|}{\sqrt{\sum_{A,F} |v^A_{(F)}|^2}} \,\leq\, 1. \tag{3.20}$$

Now the fermion masses are the eigenvalues of $v^A_{(F)} f^{AB}_{(F)}$, and given what they are, if the v.e.v.'s are smaller, the Yukawa coupling parameters $f^{AB}_{(F)}$ must be larger. Thus implies, that the scalar-fermion coupling is stronger when there are more

Higgs doublets. In fact, if there would be too many scalar doublets, the Yukawa couplings might actually become of order unity, and the perturbation appproach to the analysis of the Standard Model would break down. This is particularly true for the heaviest quark doublet, as the mass of the t-quark indicates that its Yukawa coupling must be close to unity even if there is only a single Higgs doublet.

Another effect of having more than one Higgs doublet is, that there will be massive charged scalars, as only two charged and one neutral Higgs are absorbed by the massive vector-boson states according to the Brout-Englert-Higgs mechanism. Therefore, although there is no objection in principle against the existence if more than one Higgs doublet, phenomenology favors at best a small number of them.

3.2 Abelian vs. non-abelian gauge theories

There are several important differences between abelian gauge theories like QED, in which local $U_Q(1)$ symmetry is realized, and non-abelian gauge theories like the $SU(3) \times SU(2)$ sector of the Standard Model. These differences result from the non-linear nature of the non-abelian gauge theories and their local symmetries.

In the first place, in non-abelian gauge theories the gauge fields carry (non-abelian) charges themselves. Therefore, even in the absence of any charged fermionic or scalar matter, a non-abelian gauge field theory is a non-trivial interacting quantum field theory. In contrast, abelian gauge fields like the electromagnetic field are neutral and have no direct self-interactions.

A second result from the non-linear nature of non-abelian gauge field theories is that their self-coupling as well as the coupling to fermionic and scalar matter is fixed in terms of a single coupling parameter. This happens, because the unitary representations of a non-abelian compact Lie algebra form a discrete set, with a unique normalization of the generators in terms of the Cartan-Killing metric. The transformations of the Lie-algebra valued gauge fields take the form

$$A'_\mu = \frac{1}{g} \left(\partial_\mu U \right) U^{-1} + U A_\mu U^{-1}, \tag{3.21}$$

as dictated by the covariant transformation rule of the field strength:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - g \left[A_\mu, A_\nu \right] \quad \Rightarrow \quad F'_{\mu\nu} = U F_{\mu\nu} U^{-1}. \tag{3.22}$$

The inhomogeneous term with $1/g$ in the transformation rule (3.21) can now only cancel in the transformation of covariant derivatives if these are constructed with the same coupling parameter g; in detail:

$$D_\mu \phi = \left(\partial_\mu - g A_\mu \right) \phi \quad \Rightarrow \quad \left(D_\mu \phi \right)' = U D_\mu \phi, \tag{3.23}$$

if and only if the g in the covariant derivative is the same as the one in the transformation (3.23). This argument is valid for any unitary representation of the gauge group. Therefore all couplings of the gauge fields, whether self-coupling or coupling to charged matter, are fixed directly by the choice of representation.

However, for abelian gauge fields there is a continuum of irreducible unitary representations of $U(1)$ from which one can choose, all one-dimensional. These representations are characterized by the value of the weight y, which defines the generator of the $U(1)$ transformations:

$$\phi' = U_y[\alpha]\,\phi, \qquad U_y[\alpha] = e^{i\alpha y}. \tag{3.24}$$

Thus even though the coupling parameter g may be universal (it may in fact alway be chosen equal to unity by a rescaling gauge parameter α) the covariant derivative

$$D_\mu\phi = (\partial_\mu - igyA_\mu)\,\phi \equiv \left(\partial_\mu - i\frac{q}{\hbar}A_\mu\right)\phi, \tag{3.25}$$

allows for a continuum of charge values $q = gy\hbar$.

It is therefore rather surprising to observe that in nature the values of the hypercharges (or equivalently, electric charges) do not seem to form a continuum, or a random set, but a regular discrete set in terms of a fundamental unit, as evident from table 1 in sect.(1.4):

$$Y = \frac{n}{6}, \qquad n = (1, -4, 2, -3, 6), \tag{3.26}$$

identical in each family; in addition it is required that $n = \pm 3$ for any Higgs doublet, and $n = 0$ for a (real) Higgs triplet or a singlet neutrino. For the electric charges the result is even a bit more puzzling, as the smallest charges (those of the quarks) do not appear free in nature, and all observed charges of subatomic particles are integer multiples of the electron charge. This clearly indicates that there is something to be explained here.

3.3 Short-distance behaviour of gauge theories

The self-interactions of non-abelian gauge theories have important consequences for both long and the short-distance behaviour of these theories. Even in the absence of any fermionic or bosonic matter, the self-interactions of non-abelian gauge bosons produce vacuum fluctuations which modify the properties of the vacuum state, evident for example in the case of QCD in its chromo-dielectric properties. In particular, whereas in electrodynamics vacuum fluctuations involving charged particles tend to screen electric charges at large distances by vacuum polarization, in the case of non-abelian chromodynamics the vacuum polarization of the color field produces anti-screening effects creating strong interactions at large distances, whilst they become increasingly weaker on smaller scales. These strong interactions at large distance are at the origin of the phenomenon of color confinement, the concept introduced to explain the failure to oberve free color charges, such as free gluons, in nature. The fact that such a scale dependence arises in the pure quantum Yang-Mills theory is surprising, as the classical theory is scale invariant depending only a single dimensionless coupling constant g.

In the perturbative approach to quantum field theories these effects manifest themselves in the behaviour of running coupling parameters. Consider the one-loop correction to the gauge coupling. In a multiplicative renormalization scheme, the renormalized coupling parameter g_R is related to the bare one g_0 by

$$g_R^2 = Z^2 g_0^2, \tag{3.27}$$

where Z is obtained from a non-trivial combination of the vertex and propagator renomalization factors. It can be computed in dimensional regularization, with the result that in the one-loop approximation

$$g_R^2 = g_0^2 + a g_0^4 \left(\frac{\mu^2}{-q^2} \right)^{\frac{\epsilon}{2}} \left(\frac{2}{\epsilon} + \text{terms finite in } \lim \epsilon \rightarrow 0 \right) + \mathcal{O}(g_0^6). \tag{3.28}$$

Here $-q^2$ is the external four-momentum at which the interaction is probed. Furthermore, $\epsilon = 4 - d$, and μ is a scale introduced to regularize the theory in $d - \epsilon$ dimensions whilst keeping the coupling parameter dimensionless. The constant a depends on the specific theory.

Now as μ is a regulator mass-scale with no physical meaning, we can remove it from the theory by fixing a normalization condition: one must take from experiment the value of g_R^2 at a given external momentum $-q^2$; this then implicitly fixes a relation between the bare coupling g_0^2 and μ for any non-zero value of ϵ. This information can now be used to determine unambiguously the value of the coupling when probed at another value of the external momentum, say $-q^2 = -Q^2$. Using the result

$$\left(\frac{\mu^2}{-Q^2} \right)^{\frac{\epsilon}{2}} = \left(\frac{\mu^2}{-q^2} \right)^{\frac{\epsilon}{2}} \left(\frac{q^2}{Q^2} \right)^{\frac{\epsilon}{2}} = \left(\frac{\mu^2}{-q^2} \right)^{\frac{\epsilon}{2}} \left(1 + \frac{\epsilon}{2} \ln \frac{q^2}{Q^2} + ... \right), \tag{3.29}$$

one finds that in the limit $\epsilon \rightarrow 0$, up to this order of expansion in g_0:

$$\begin{aligned} g_R^2(-Q^2) &= g_R^2(-q^2) + a g_R^4(-q^2) \ln \frac{q^2}{Q^2} \\ &= g_R^2(-q^2) \left(1 - a g_R^2(-q^2) \ln \frac{Q^2}{q^2} \right) \\ &= \frac{g_R^2(-q^2)}{1 + a g_R^2(-q^2) \ln \dfrac{Q^2}{q^2}}. \end{aligned} \tag{3.30}$$

Basically what has happened is, that whereas originally one had a relation between a renormalized physical coupling parameter g_R and a bare coupling g_0, imagined as something like 'the coupling when all interactions are switched of', we now instead have an expression for the value of the physical coupling as a function of the scale at which it is probed. That is, we have exchanged an unobservable dimensionless parameter g_0 for a physical scale $-q^2$.

For a pure non-abelian Yang-Mills theory one finds that

$$a = \frac{11}{48\pi^2} C_2^{(ad)}, \tag{3.31}$$

where $C_2^{(ad)}$ is the eigenvalue of the quadratic Casimir operator in the adjoint representation. Because this value of a is positive, it follows form eq.(3.30) that for $(Q^2/q^2) > 1$ the effective value of the coupling $g_R^2(-Q^2)$ is *smaller* than $g_R^2(-q^2)$. Of course, for an abelian gauge theory the absence of self-interactions implies $a = 0$, and there is no scale-dependence of the coupling parameter in the absence of charged matter fields.

Clearly, the introduction of charged fermions and/or scalars into the theory affects the value of a, because they contribute to the renormalization by their vacuum polarization effects. Their contribution to a turns out to be negative; more precisely, for Weyl fermions in representations F and s calars in representations S of the gauge group, one obtains

$$a = \frac{1}{48\pi^2} \left(11C_2^{(ad)} - 2\sum_F C_2(F) - \sum_S C_2(S) \right). \tag{3.32}$$

The constants $C_2(R)$, $R = (F, S)$, are defined again in terms of the quadratic Casimir operators in the corresponding representations:

$$\text{Tr}\,(T_A(R)T_B(R)) = C_2(R)\,\delta_{AB}. \tag{3.33}$$

For instance, in the case of QCD with 8 gluons in the adjoint representation of $SU(3)$, and N_f quark triplets, one has

$$a = \frac{33 - 2N_f}{48\pi^2}. \tag{3.34}$$

With $N_f = 6$, the number observed and in agreement with the width of the Z-boson, this remains positive. Hence QCD is asymptotically free: its coupling decreases at large energy scales to arbitrarily small values. This is to be contrasted with QED, where only the charged fermions contribute to a to produce a negative value for a. It follows, that the fine-structure parameter $\alpha(-q^2)$ grows with energy, and QED becomes a strongly interacting theory at some very high energy scale.

3.4 Charge quantization

There are several different mechanisms which can explain the quantization of $U(1)$ charges, as observed in nature. However, in many models it turns out that various mechanisms are at work at the same time, suggesting different mechanisms to be related.

The most straightforward solution of the quantization problem is by embedding the $U(1)$ transformations in a larger compact and semi-simple non-abelian Lie group G. As the couplings in such a theory are fixed by the representation content, the charges characterizing specific $U(1)$ subgroups are fixed as well.

Consider for example the embedding of $U(1)$ in $SU(2) \simeq SO(3)$. Suppose that an $SU(2)$ symmetry with generators $\{T_i\}_{i=1}^3$ is broken spontaneously to $U(1)$, the generator of which we label as T_3. Even though the physical states no longer have to form degenerate mutiplets of $SU(2)$, they are still eigenstates of T_3, with eigenvalues quantized as even or odd half-integers:

$$t_3 \,=\, 0,\, \pm\frac{1}{2},\, \pm 1,\, \pm\frac{3}{2},\, ... \tag{3.35}$$

In sect.(3.1) we discussed the interaction between $SU(2)$ gauge bosons and a real triplet of scalar fields. In this case hypercharge with its associated gauge field plays no role, and we ignore it for time being. In a world without hypercharge, electric charge equals the third isospin component: $Q = T_3$. The components of the scalar triplet have eigenvalues $T_3 = (+1, 0, -1)$. Now let $SU(2)$ be broken to $U_Q(1)$ by a v.e.v. for ϕ_3, as in eq.(3.5). The other two components of the scalar triplet can be taken to vanish by applying a local $SU(2)$ gauge transformation to the unitary gauge. Thus

$$\boldsymbol{\phi} \,=\, \left(\begin{array}{c} 0 \\ 0 \\ v_3 + \varphi \end{array} \right), \tag{3.36}$$

where v_3 is the v.e.v. and φ a real neutral massive scalar, the Higgs-Kibble boson. At the same time, the charged vector bosons $W^{\pm}$ become massive, with

$$M_W^2 \,=\, g_2^2 v_3^2, \tag{3.37}$$

whilst the neutral vector boson W^3 remains massless. In this simple hyperchargeless world W^3 would have to be the electromagnetic $U_Q(1)$ gauge field, and its quanta the photons; there would be no massive neutral vector boson Z. In summary, the spectrum of physical particles in this model, in the absence of fermions, consists of two massive charged vector bosons $W^{\pm}$ with $Q = \pm 1$, a neutral massless photon W^3, and a neutral massive scalar φ. These particles do not form representations of $SU(2)$, but electric charge is conserved and quantized.

The observations remain true if one adds matter fields in representations of $SU(2)$. Suppose, as an illustration of principle, that one adds a triplet $\boldsymbol{\lambda}$ and a singlet χ of Majorana fermions. These representations are real, and one can construct both a gauge invariant Yukawa coupling and gauge-invariant Majorana masses:

$$\begin{aligned}\Delta\mathcal{L}_{Yuk} &= -\frac{\mu}{2}\,\bar{\chi}\chi - \frac{m}{2}\,\bar{\boldsymbol{\lambda}}\cdot\boldsymbol{\lambda} - f\boldsymbol{\phi}\cdot\bar{\chi}\,\boldsymbol{\lambda} \\ &\to -\frac{\mu}{2}\,\bar{\chi}\chi - \frac{m}{2}\,\bar{\boldsymbol{\lambda}}\cdot\boldsymbol{\lambda} - f\,(v_3+\varphi)\,\bar{\chi}\,\lambda_3 \\ &= -m\,\bar{\psi}\psi - m_+\,\bar{\omega}_+\omega_+ - m_-\,\bar{\omega}_-\omega_- - \ldots\end{aligned} \tag{3.38}$$

Here the unitary gauge has been applied, reducing the Yukawa interactions to those for the neutral Majorana fermions

$$\omega^+ = \frac{1}{\sqrt{2}}\,(\sin\theta\lambda_3 + \cos\theta\chi)\,, \qquad \omega^- = \frac{1}{\sqrt{2}}\,(\cos\theta\lambda_3 - \sin\theta\chi)\,. \tag{3.39}$$

with mixing angle

$$\tan 2\theta = \frac{2fv_3}{\mu - m}, \tag{3.40}$$

and with masses

$$m_\pm = \frac{1}{2}\left(\mu + m \pm \sqrt{(\mu-m)^2 + 4f^2v_3^2}\right). \tag{3.41}$$

In contrast, the charged Dirac fermion with $Q = +1$:

$$\psi = \frac{1}{\sqrt{2}}\,(\lambda_1 - i\lambda_2)\,, \tag{3.42}$$

keeps the same mass m. Observe, that $SU(2)$ is manifestly broken by the mixing of the singlet and triplet representations and the splitting of the masses (m, m_+, m_-). However, as the $U_Q(1)$ generated by T_3 is not broken, only states with the same charge $Q = T_3$ can mix, and charge is both conserved and quantized. Note also, that in spite of the spontaneous symmetry breaking the average of the fermion masses is unchanged, at least at the tree level:

$$\overline{m} = \frac{1}{4}\,(m_+ + m_- + 2m) = \frac{1}{4}\,(\mu + 3m)\,. \tag{3.43}$$

In the context of the Standard Model this mechanism to explain charge quantization can be realized if the hypercharge $U_Y(1)$ symmetry is part of a large broken non-abelian symmetry group. We return to this scenario in the next chapter.

3.5 Monopoles

We have seen that $U(1)$ charges can be quantized if the charge operator, the generator of $U(1)$, is part of a non-abelian Lie algebra, requiring it to be a hermitean operator with a discrete spectrum of eigenvalues.

A completely independent argument for the quantization of electric charge can be derived if one allows for the existence of magnetic monopoles [4]. Suppose localized magnetic charges exists, such that for any spherical volume including such a charge one has

$$\int d^3\mathbf{r}\, \boldsymbol{\nabla} \cdot \mathbf{B} \,=\, 4\pi\kappa. \tag{3.44}$$

Note that $\varepsilon_0 c\kappa = \kappa/(\mu_0 c)$ has the dimension of electric charge, i.e. can be measured in coulombs. Asymptotically the magnetic field must look like

$$\mathbf{B} \,=\, \frac{\kappa \mathbf{r}}{r^3}, \tag{3.45}$$

which can be obtained from a vector potential

$$A^{\pm} \,=\, A^{\pm}_{\mu} dx^{\mu} \,=\, -\kappa \left(\cos\theta \pm 1 \right)\, d\varphi, \tag{3.46}$$

with A^+ defined in the domain $0 < \theta \leq \pi$ and A^- in the domain $0 \leq \theta < \pi$. In the overlap region the two vector potentials are related by a gauge transformation:

$$A^+ \,-\, A^- \,=\, \frac{i}{g}\, e^{-i2\kappa g} d e^{i2\kappa g}. \tag{3.47}$$

Hence they describe the same magnetic field. We consider the non-relativistic motion of a particle with charge $q = gy\hbar$ in the background of a heavy monopole, so we can take the monopole to be stationary [5]. The Hamiltonian of the particle is

$$H \,=\, \frac{1}{2m} \left(\mathbf{p} - q\mathbf{A} \right)^2. \tag{3.48}$$

The conserved total angular momentum of the charge-monopole system is then given by the expression

$$\mathbf{J} \,=\, \mathbf{r} \times \left(\mathbf{p} - q\mathbf{A} \right) \,-\, q\kappa \frac{\mathbf{r}}{r}. \tag{3.49}$$

These operators satisfy the standard commutation relations:

$$[H, \mathbf{J}] \,=\, 0, \qquad\qquad [J_i, J_j] \,=\, i\hbar \varepsilon_{ijk} J_k. \tag{3.50}$$

The unitary representations of this $so(3)$ algebra are the ones with highest weight $j = 2n$, n an integer, such that

$$\mathbf{J}^2\, |j, j_z\rangle \,=\, j(j+1)\hbar^2\, |j, j_z\rangle, \qquad J_z\, |j, j_z\rangle \,=\, j_z \hbar\, |j, j_z\rangle, \tag{3.51}$$

where j_z takes integer or half-integer values $j_z = j, j-1, ..., -j$. Now we observe, that

$$J_z \,=\, x p_y \,-\, y p_x \,-\, q \left(x A_y - y A_x \right) \,-\, q\kappa \frac{z}{r} \,=\, -i\hbar \frac{\partial}{\partial \varphi} \pm q\kappa, \tag{3.52}$$

with the $\pm$-sign corresponding to the domains of $A^{\pm}$. Now in each of these domains φ takes all values on the circle $S^1 = \{0 \leq \varphi < 2\pi\}$. Single-valuedness of the wave functions then requires the eigenstates of J_z to be eigenstates of $-i\hbar\,\partial/\partial\varphi$ with eigenvalue $m\hbar$, $m \in \mathbf{Z}$. It follows, that

$$q\kappa \,=\, (j_z - m)\,\hbar \quad \Rightarrow \quad 2q\kappa \,=\, l\hbar, \tag{3.53}$$

with l an integer. Equivalently,

$$y \,=\, \frac{q}{g\hbar} \,=\, \frac{l}{2g\kappa}. \tag{3.54}$$

Therefore the values of the $U(1)$ charges must be quantized in units $1/(2g\kappa)$.

Many gauge theories with spontaneously broken symmetries actually admit magnetic monopole solutions [6, 7], i.e. static finite energy solutions with non-zero magnetic charge. In fact, the $SO(3)$ gauge model with a real triplet of Higgs scalars provides an example. In an $SO(3)$ invariant potential the symmetry is broken if the lowest energy state is obtained for

$$\boldsymbol{\phi}^2 \,=\, v^2 \,\neq\, 0. \tag{3.55}$$

In the vacuum, $\boldsymbol{\phi}$ is constant, and we may take it to be aligned in the ϕ_3 direction in the internal charge space, as in eq.(3.5). However, one can imagine other finite-energy solutions in which ϕ asymptocially becomes constant in magnitude, but not in orientation. For instance, one can consider field congifurations of the 'hedgehog' type:

$$\phi^a \,=\, \delta^a_i\, \frac{r^i}{r}\, F(r), \tag{3.56}$$

which one may think of as pointing radially outward. It connects smoothly to a solution in empty space at large distances satisfying the condition (3.55) if

$$\lim_{r\rightarrow\infty} F(r) \,=\, v. \tag{3.57}$$

Such a field cannot be deformed continuously to the vacuum solution, because of its topological distinct character: it has a non-zero winding number, whilst the vacuum solution has vanishing winding number.

An invariant electro-magnetic field strength can be defined as the gauge field strength in the unbroken direction modulo a scalar-field contribution, by

$$\mathcal{F}_{\mu\nu} \,=\, \frac{\boldsymbol{\phi}}{\sqrt{\boldsymbol{\phi}^2}} \cdot \left(\mathbf{F}_{\mu\nu} + \frac{1}{g\boldsymbol{\phi}^2}\, D_\mu\boldsymbol{\phi} \times D_\nu\boldsymbol{\phi} \right). \tag{3.58}$$

It certainly reduces to the electro-magnetic field strength $F^3_{\mu\nu}$ in the vacuum. It turns out, that there exist solutions for the $SO(3)$ Yang-Mills equations with the scalar field of the form (3.56) such that $\mathcal{F}_{\mu\nu}$ has the magnetic monopole form (3.45). This can be seen from the field equation for the dual field strength:

$$\frac{1}{2}\,\varepsilon^{\mu\nu\kappa\lambda}\,\partial_\nu \mathcal{F}_{\kappa\lambda} \;=\; -\frac{4\pi}{g}\,s^\mu, \tag{3.59}$$

with the magentic current given by

$$s^\mu \;=\; \frac{1}{4\pi(\boldsymbol{\phi}^2)^{\frac{3}{2}}}\,\varepsilon^{\mu\nu\kappa\lambda}\,D_\nu\boldsymbol{\phi}\cdot\left(D_\kappa\boldsymbol{\phi}\times D_\lambda\boldsymbol{\phi}\right). \tag{3.60}$$

Taking the time component of eq.(3.59) we get

$$\boldsymbol{\nabla}\cdot\mathbf{B} \;=\; \frac{4\pi}{g}\,s^0, \tag{3.61}$$

and the magnetic charge is

$$\kappa \;=\; \frac{1}{g}\int d^3\mathbf{r}\,s^0. \tag{3.62}$$

As the charged particles in the theory —like the $W^\pm$ and the fermions $\lambda^\pm$ we have discussed— have charge $q = \pm g\hbar$, the magnetic charges of the monopole solutions must satisfy

$$\kappa \;=\; \frac{l}{2g} \quad\Leftrightarrow\quad 2\int d^3\mathbf{r}\,s^0 \;=\; l. \tag{3.63}$$

This condition is satisfied because the integral represents the winding number, which takes integer values.

We finally observe, that stable monopoles do not appear in the Standard Model. First of all, there is good evidence that there is no real Higgs triplet, because of the value of the ρ-parameter. Moreover, independend $U(1)$ gauge transformations can generally connect sectors of the theory with different winding number, thereby destablizing any such solutions.

Chapter 4

Grand Unification

4.1 Gauge unification

The Standard Model with symmetry breaking mediated by one or several $SU(2)$ doublets of scalars is in excellent agreement with experimental data. However, to explain charge quantization the $U_Y(1)$ hypercharge symmetry should be part of a larger non-Abelian gauge group and/or the Standard Model must be extended to include magnetic monopoles.

We have also observed, that in gauge theories like the Standard Model coupling parameters are effectively scale dependend. As the gauge group of the Standard Model is not simple, it has three coupling parameters, one for each factor group. Moreover, as each of these gauge groups $SU(3)$, $SU(2)$ and $U_Y(1)$ affects a different set of chiral fermions and/or scalars, the three couplings (g_1, g_2, g_3) each depend on the scale in a different way. Remarkably, if one follows the evolution of these couplings, assuming now new particles to start participating at some higher scale $-Q^2 \gg (\Lambda_{Fermi} \sim 100 \text{ GeV})^2$, they come quite close to a common value at energies somewhere in the range between 10^{14} and 10^{16} GeV, provided the hypercharge coupling parameter g_1 is scaled by a factor $\sqrt{5/3}$. Below we explain the theoretical motivation for this factor. Here we just observe, that a factor of this magnitude is required by the *data* to make the couplings converge at a single scale.

Both observations suggest, that at energies above the Fermi scale Λ_{Fermi} the Standard Model group $G_{sm} = SU(3) \times SU(2) \times U_Y(1)$ may be the unbroken subgroup of a larger semi-simple gauge group G_{GUT} which governs physics at the high scale $\Lambda_{GUT} \sim 10^{15}$ GeV mentioned above. In such a scenario the known electro-weak and color interactions are unified in a gauge theory with a single coupling constant. Models of the fundamental interactions incorporating this unification are called Grand Unified Theories (hence the label *GUT*).

Of course, there are a number of phenomenological constraints on models of grand unification. The gauge group G_{GUT} must be large enough to include

G_{sm}, and semi-simple to explain charge quantization (and also to allow for stable monopole solutions). At the same time, the representations must be small enough not to include any new observable light particles, i.e. new particles participating in the electro-weak intereactions with masses in the range of the Fermi-scale, which contradict experimental data.

Most models meeting these criteria are based on one of three simple Lie groups which can break spontaneously to G_{sm} whilst accomodating the known gauge bosons, quarks and leptons, and Higgs-doublets, as light excitations at the level of Λ_{GUT}. These groups are $SU(5)$, $SO(10)$ and E_6. These groups form a hierarchy in the sense that they allow a chain of breaking to subgroups as shown in fig. 4.

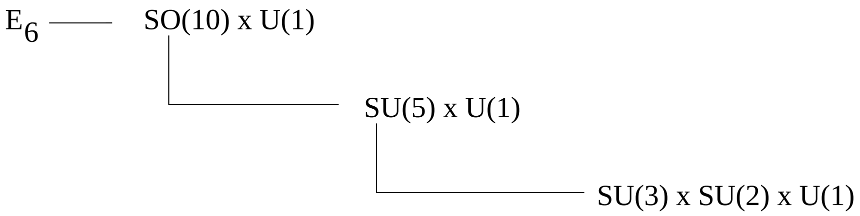

Fig. 4: Chains of breaking of GUT groups.

This scheme is not the only way to break the larger gauge groups, but still it gives good insight into the structure of representations needed to achieve the gauge unification. Therefore fitting the known particles into representations of $SU(5)$ is a useful first step towards unification with larger gauge groups as well.

As the chiral fermion families each have the same set of quantum numbers under G_{sm}, this will also be the case under the GUT group G_{GUT}. Thus GUTs explain the structure of a single family in terms of representations of the unified gauge group G_{GUT}; however, they do not explain the number of families: in the standard grand unification scenarios there is no quantum number distinguishing the families, other than the mass. In particular there is no gauge interaction coupling to a family quantum number.

4.2 SU(5)

Unification of the gauge interactions in a single spontaneously broken $SU(5)$ gauge theory requires the matter fields (chiral fermions, Higgs scalars) to fit into representations of SU(5). These are labeled as follows. The fundamental representation $\mathbf{5}$ and its conjugate $\bar{\mathbf{5}}$, are denoted with an upper, resp. lower index $I, J, .. = 1, ..., 5$. For example, Ψ_I denotes a $\bar{\mathbf{5}}$. All other representations can then be labeled as tensors of rank (p, q), with p the number of fundamental representation indices, and q the number of conjugate indices. Thus the anti-symmetric tensor of rank $(2, 0)$, denoted by $\Psi^{IJ} = -\Psi^{JI}$, describes a complex 10-dimensional representation. Indeed, such a tensor has $\frac{1}{2} \times 5 \times 4 = 10$ independend complex components.

The elements of the Lie algebra $su(5)$ are represented in the fundamental representation by the traceless anti-hermitean 5×5 matrices, of which there are 24 independend ones. Thus the infinitesimal $SU(5)$ transformations on a field Ψ^I or its conjugate Ψ^*_I are

$$\delta\Psi^I \,=\, \Lambda^I{}_J\,\Psi^J, \qquad \delta\Psi^*_I \,=\, -\Psi^*_J\,\Lambda^J{}_I, \tag{4.1}$$

with the traceless and anti-hermitean matrix $\Lambda^I{}_J$ of the form

$$\Lambda \,=\, \left(\begin{array}{cc} \Lambda_3 & X \\ -X^\dagger & \Lambda_2 \end{array} \right). \tag{4.2}$$

Here Λ_3 and Λ_2 represent elements of the Lie-algebra $s[u(3) \times u(2)]$, i.e. infinitesimal elements of the fundamental representations of the groups $U(3)$ and $U(2)$ such that

$$\mathrm{Tr}\,\Lambda \,=\, \mathrm{Tr}_3\Lambda_3 \,+\, \mathrm{Tr}_2\Lambda_2 \,=\, 0. \tag{4.3}$$

Here $\mathrm{Tr}_{(2,3)}$ represent the 2-, respectively 3-dimensional subtraces over de blocks to which $\Lambda_{(2,3)}$ belong. The complex 3×2 blocks X and $X^\dagger$ represent the generators of the symmetries which are broken when $SU(5) \rightarrow S[U(3) \times U(2)]$.

We define the generators of $SU(5)$ in the fundamental representation to have the normalization

$$\mathrm{Tr}_{fund}\,(T_A T_B) \,=\, -\,\frac{1}{2}\,\delta_{AB}. \tag{4.4}$$

Then the infinitesimal transformations in the $S[U(3) \times U(2)]$ subgroup can be written as

$$\left(\begin{array}{cc} \Lambda_3 & 0 \\ 0 & \Lambda_2 \end{array} \right) \,=\, \alpha^a T_a \,+\, \alpha^i T_i + \alpha^0 T_0, \tag{4.5}$$

where we have made a decomposition into generators T_a of $SU(3)$, T_i of $SU(2)$, and T_0 of the (traceless) $U(1)$. Using the conventions of chapter 1, in particular as defined in section 1.8, we choose a basis for these generators as follows:

$$\begin{array}{l} T_a \,=\, \left(\begin{array}{cc} G_a & 0 \\ 0 & 0 \end{array} \right) \,=\, \left(\begin{array}{cc} \frac{i}{2}\,\lambda_a & 0 \\ 0 & 0 \end{array} \right), \\ \\ T_i \,=\, \left(\begin{array}{cc} 0 & 0 \\ 0 & \frac{i}{2}\,\tau_i \end{array} \right), \quad T_0 \,=\, \left(\begin{array}{cc} \frac{-i}{\sqrt{15}}\,1_3 & 0 \\ 0 & \frac{3i}{2\sqrt{15}}\,1_2 \end{array} \right), \end{array} \tag{4.6}$$

with $(1_3, 1_2)$ the unit matrices in the respective blocks. It is straightforward to check, that this basis is anti-hermitean: $T^\dagger_A = -T_A$, traceless: $\mathrm{Tr}\,(T_A) = 0$, and normalized according to (4.4).

Now its is clear, that $SU(3)$-triplet and anti-triplet representations, as well as $SU(2)$-doublets are included in the representations of $SU(5)$, like the **5** and the $\bar{\mathbf{5}}$. However, the weak hypercharges do not (and can not) correspond to the eigenvalues of T_0, as these are not rational. Fortunately, this can be mended by a simple rescaling. That is, to get agreement with the hypercharge assignments in the Standard Model we must change the $U_Y(1)$ weight by a factor $\sqrt{5/3}$:

$$Y = -i\sqrt{\frac{5}{3}}\,T_0 = \left(\begin{array}{cc} -\frac{1}{3} & 0 \\ & \\ 0 & \frac{1}{2} \end{array} \right). \tag{4.7}$$

As a consequence, one must rescale the gauge coupling by the inverse factor:

$$g_5 = \sqrt{\frac{5}{3}}\,g_1 \quad \Rightarrow \quad g_5 T_0 = i g_1 Y. \tag{4.8}$$

This provides a natural explanation for the factor $\sqrt{5/3}$ observed to be required by a unification of the three gauge couplings is in the Standard Model.

With the factor $\sqrt{3/5}$ in eq.(4.7) a minimal Standard Model family of chiral fermions can be fitted into two representations of $SU(5)$: a **10** and a $\bar{\mathbf{5}}$. As we have observed, a **10** of left-handed fermions is described by an anti-symmetric (2,0) tensor Ψ_L^{IJ} of chiral spinors. If we split the labels (I,J) into subgroup labels $I = (\alpha, i)$, $J = (\beta, j)$ of $SU(3)$ and $SU(2)$, respectively, we can define the sub-tensors

$$\Psi^{IJ} = \left(\begin{array}{cc} \psi^{\alpha\beta} & \psi^{\alpha j} \\ -\psi^{i\beta} & \psi^{ij} \end{array} \right). \tag{4.9}$$

Now the anti-symmetric $SU(3)$-tensor $\psi^{\alpha\beta} = -\psi^{\beta\alpha}$ represents an anti-triplet $\bar{\mathbf{3}}$, the equivalence being defined via the duality relation

$$\psi_\alpha = \frac{1}{2}\,\varepsilon_{\alpha\beta\gamma}\,\psi^{\beta\gamma}. \tag{4.10}$$

Thus its $SU(3) \times SU(2)$ content is $(\bar{3}, 1)$. Similarly, the antisymmetric $SU(2)$-tensor $\psi^{ij} = -\psi^{ji}$ is equivalent to a singlet:

$$\psi = \frac{1}{2}\,\varepsilon_{ij}\,\psi^{ij}, \tag{4.11}$$

which is labeled as $(1,1)$ under $SU(3) \times SU(2)$. Finally the off-diagonal components $\psi^{\alpha j}$ represent a genuine $(3,2)$ representation of $SU(3) \times SU(2)$. Turning to the $U(1)$ eigenvalues, we observe that

$$Y\,\psi^{\alpha\beta} = -\frac{2}{3}\,\psi^{\alpha\beta}, \qquad Y\,\psi^{\alpha j} = \frac{1}{6}\,\psi^{\alpha j}, \qquad Y\,\psi^{ij} = \psi^{ij}. \tag{4.12}$$

It follows, that the complete $SU(3) \times SU(2) \times U_Y(1)$ content of the **10** of $SU(5)$ is

$$\mathbf{10} = \left((\bar{3}, 1)_{-2/3}, (3,2)_{1/6}, (1,1)_1 \right). \tag{4.13}$$

By comparison with table 1, we identify these components with the chiral fermions U_L, Q_L and E_L, respectively:

$$\Psi_L^{IJ} = \begin{pmatrix} \varepsilon^{\alpha\beta\gamma} U_{L\,\gamma} & Q_L^{\alpha j} \\ & \\ -Q_L^{\beta i} & \varepsilon^{ij} E_L \end{pmatrix}. \tag{4.14}$$

The remaining chiral fermions D_L and L_L can now be fitted into a $\bar{\mathbf{5}}$ of $SU(5)$, which we denote by Ψ_I. This is easy to check; writing

$$\Psi_I = (\psi_\alpha, \psi_j), \tag{4.15}$$

it follows directly, that the $SU(3) \times SU(2)$ content of the component fields is $(\bar{3}, 1)$ for ψ_α, and $(1, 2)$ for ψ_j, the latter being equivalent as a representation to $\varepsilon_{jk}\psi^k$. Moreover, from (4.7) the hypercharges are seen to be 1/3 and -1/2 respectively. Therefore the full $SU(3) \times SU(2) \times U_Y(1)$ content of the $\bar{\mathbf{5}}$ of $SU(5)$ is

$$\bar{\mathbf{5}} = \Big((\bar{3}, 1)_{1/3}, (1, 2)_{-1/2} \Big). \tag{4.16}$$

This confirms the assignment

$$\Psi_{L\,I} = \Big(D_{L\,\alpha}, \varepsilon_{ij}\, L_L^j \Big). \tag{4.17}$$

Thus the the chiral fermion families fit nicely into representations of $SU(5)$. We also observe, that additional singlet neutrinos can be fitted trivially into the theory as singlets of $SU(5)$: $\mathbf{1} = (1, 1)_0$.

4.3 Symmetry breaking in SU(5)

Grand unified models have attractive features. Turning around the previous argument about the unification of running gauge couplings, and using that in the $SU(5)$ model at the GUT-scale one has the relation

$$g_1 = \sqrt{\frac{3}{5}}\, g_5, \qquad g_2 = g_5 \tag{4.18}$$

one can *calculate* the weak mixing angle:

$$\sin^2 \theta_w = \frac{g_1^2}{g_1^2 + g_2^2} = \frac{3}{8} = 0.375. \tag{4.19}$$

This is relatively far from the measured value (as referred to the $\overline{\mathrm{MS}}$ scheme):

$$\sin^2 \theta_w = 0.2311. \tag{4.20}$$

However, the measured value refers to the ratio of couplings measured at the scale M_Z^2, whereas eq.(4.19) refers to the unification scale. The running of the couplings can take care of the difference, depending on the assumptions one makes for Λ_{GUT} and the number and type of particles participating in the standard-model interactions with masses below the GUT-scale. In this sense gauge unification can explain the value of the weak mixing at the Fermi scale.

In contrast, the amount of symmetry breaking, as expressed by the the v.e.v. of the Higgs doublet in the Standard Model, is not fixed by the unification: in the minimal $SU(5)$ model it remains a free parameter. Thus the minimal $SU(5)$ model has two describe symmetry breaking in (at least) two independend steps: the breaking of $SU(5) \rightarrow SU(3) \times SU(2) \times U_Y(1)$ at the scale Λ_{GUT}, and the breaking of $SU(2) \times U_Y(1) \rightarrow U_Q(1)$ at the Fermi scale Λ_{Fermi}. The difference between these scales of symmetry breaking is huge:

$$\frac{\Lambda_{Fermi}^2}{\Lambda_{GUT}^2} \sim 10^{-27\pm1}. \tag{4.21}$$

This huge hierarchy implies, that to a first approximation we can neglect the low-energy symmetry breaking and the masses of the standard-model fermion families at the GUT scale. Therefore the first step in the breaking of $SU(5)$ must preserve all of the $SU(3) \times SU(2) \times U_Y(1)$ symmetry. As a result we have two constraints on any scalar multiplet breaking $SU(5)$: it must break $G_{GUT} \rightarrow G_{sm}$, and it is not allowed to couple to the $\mathbf{10}$ and $\bar{\mathbf{5}}$ representations of chiral fermions in such a way as to generate masses for these particles at the GUT scale.

The simplest way to achieve this is by using a real scalar multiplet Φ^A in the adjoint representation ($\mathbf{24}$). Equivalently, we may represent it as a hermitean 5×5 matrix

$$\Phi^I{}_J = \Phi^A \left(-iT_A\right)^I{}_J, \tag{4.22}$$

the T_A being the generators of $SU(5)$ in the fundamental representation. The only way vacuum expectation value of this scalar field which does not break G_{sm} is proportional to T_0:

$$\langle \Phi \rangle = -iv_5\, T_0 = v_5 \sqrt{\frac{3}{5}}\, Y. \tag{4.23}$$

This is rather obvious, as T_0 is proportional to the hypercharge Y, which commutes with all elements of $SU(3) \times SU(2) \times U_Y(1)$ by construction. At the same time, this generator does not commute with any of the $SU(5)$ generators in the off-diagonal sector:

$$[Y, X] = -\frac{5}{6}\, X. \tag{4.24}$$

From this it follows that the complex vector bosons in the broken sector of $SU(5)$ have the non-singlet G_{sm} quantum numbers of the representation $(3,2)_{-5/6}$. As a result all of the gauge bosons in the coset $SU(5)/(SU(3) \times SU(2) \times U(1))$ become massive at the GUT scale. This achieves the first part of the symmetry breaking.

The second requirement is automatically fulfilled, as the adjoint can not produce an $SU(5)$-invariant coupling between chiral fermions in the $\bar{\mathbf{5}}$ and/or $\mathbf{10}$. A brief glance at eq.(1.34) should be sufficient to convince oneself of this fact.

Applying this equation to the chiral fermion representations of $SU(5)$ actually teaches us, that covariant Yukawa couplings for a $\mathbf{10}$, $\bar{\mathbf{5}}$ and $\mathbf{1}$ of fermions, denoted by $(\Psi_L^{IJ}, \Psi_{L\,I}, \Psi_L)$ respectively, must be of the form

$$\begin{aligned}
\not{D}\Psi_L^{IJ} &= \tfrac{1}{2}\, f_1\, \Phi^{IJKL}\, \Psi_{R\,KL} \,+\, f_2\, \Phi^{IJ}{}_K\, \Psi_R^K \,+\, f_3\, \Phi^{[IJ]}\, \Psi_L, \\
\not{D}\Psi_{L\,I} &= \tfrac{1}{2}\, f_2\, \Phi^{JK}{}_I\, \Psi_{R\,JK} \,+\, f_4\, \Phi_{(IJ)}\, \Psi_R^J \,+\, f_5\, \Phi_I\, \Psi_R, \\
\not{D}\Psi_L &= f_3\, \Phi^{[IJ]}\, \Psi_{R\,IJ} \,+\, f_5\, \Phi_I \Psi_R^I \,+\, f_6\, \Phi\, \Psi_R.
\end{aligned} \tag{4.25}$$

Here $\Phi_{(IJ)}$ and $\Phi^{[IJ]}$ are distinct symmetric and anti-symmetric tensors of rank (0,2) and (2,0), respectively. Interestingly, we can achieve a set of non-trivial Yukawa couplings of this form starting from a single scalar Φ^I, by taking

$$\Phi^{IJKL} \,=\, \varepsilon^{IJKLM}\, \Phi^\dagger_M, \qquad \Phi^{IJ}{}_K \,=\, \delta^I_K \Phi^J - \delta^J_K \Phi^I, \tag{4.26}$$

and $\Phi_{(IJ)} = \Phi^{[IJ]} = 0$, or what effectively amounts to the same: $f_3 = f_4 = 0$. This is the simplest way to reproduce the minimal Standard Model, as Φ^I can be decomposed under G_{sm} as

$$\Phi^I \,=\, \Big(H^\alpha, H^i\Big) \,=\, \Big((3,1)_{-1/3}, (1,2)_{1/2}\Big)\,. \tag{4.27}$$

Therefore it contains the standard Higgs doublet, cf. table 1, and all masses are generated in precisely the same way as in the minimal Standard Model if the $\mathbf{5}$ gets a v.e.v.:

$$\langle \Phi^I \rangle \,=\, v\, \delta^I_5 \,=\, (0, 0, 0, 0, v)\,. \tag{4.28}$$

Of course, the self couplings of the Higgs fields must be such that the color-charged Higgs H^α becomes very heavy, as it has not been seen in deep-inelastic parton processes in the nucleon.

A singlet scalar Φ, which couples only to the singlet neutrino Ψ_L by the Yukawa coupling f_6, is also allowed as it provides the Majorana mass μ_s of equation (2.22):

$$f_6\, \langle \Phi \rangle \,=\, \mu_s. \tag{4.29}$$

A Dirac mass for the neutrinos is generated by the Yukawa coupling f_5.

Clearly, the standard mass matrices of the minimal Standard Model are generated by the Yukawa couplings f_1 and f_2. It is instructive to consider these in some more detail. We take the v.e.v. (4.28) of the Higgs scalar, with the field content of Ψ_L^{IJ} and $\Psi_{L\,I}$ as in eqs.(4.14) and (4.17). We observe that by hermitean conjugation we have

$$\begin{aligned}
\Psi_{RIJ} &= -\begin{pmatrix} \varepsilon_{\alpha\beta\gamma} U_R^\gamma & Q_{R\alpha j} \\ -Q_{R\beta i} & \varepsilon_{ij} E_R \end{pmatrix}, \\
\Psi_R^I &= \begin{pmatrix} D_R^\alpha, & -\varepsilon^{ij} L_{Rj} \end{pmatrix}.
\end{aligned} \tag{4.30}$$

Substitution of these expressions into eqs.(4.25) then gives

$$\begin{aligned}
\not{D} U_{L\alpha} &= -f_1\, v^*\, Q_{R\alpha 1}, & \not{D} Q_L^{\alpha 1} &= -f_1\, v^*\, U_R^\alpha, \\
\not{D} D_{L\alpha} &= -f_2\, v\, Q_{R\alpha 2}, & \not{D} Q_L^{\alpha 2} &= -f_2\, v\, D_R^\alpha, \\
\not{D} E_L &= -f_2\, v\, L_{R2}, & \not{D} L_L^2 &= -f_2\, v\, E_R.
\end{aligned} \tag{4.31}$$

For a singlet neutrino $\Psi_L = N_L$ one finds in addition:

$$\not{D} L_L^1 = f_5\, v^* N_R, \qquad \not{D} N_L = f_5\, v^* L_{R1} + \mu_s\, N_R. \tag{4.32}$$

Generalizing this to the case of N_f families, we obtain a mass matrix

$$M^{AB} = \begin{bmatrix}
0 & -f_1^{AB}\, v^*\, \delta_\beta^\alpha & -f_2^{AB}\, v\, \delta_\beta^\alpha & 0 & 0 & 0 \\
-f_1^{BA}\, v^*\, \delta_\alpha^\beta & 0 & 0 & 0 & 0 & 0 \\
-f_2^{BA}\, v\, \delta_\alpha^\beta & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & f_5^{AB}\, v^* & -f_2^{AB}\, v \\
0 & 0 & 0 & f_5^{BA}\, v^* & \mu_s^{AB} & 0 \\
0 & 0 & 0 & -f_2^{BA}\, v & 0 & 0
\end{bmatrix}. \tag{4.33}$$

Comparison with eqs.(1.43) and (2.20) then leads to an identification of Yukawa couplings $f_U^{AB} = -f_1^{AB}$ and $f_N^{AB} = f_5^{AB}$, whilst $f_L^{AB} = f_D^{AB} = -f_2^{AB}$, at least at the scale of $SU(5)$ unification. Hence the $SU(5)$ unification predicts a relation between the masses of the D-type quarks and the charged leptons; in the absence of mixing of the D-quarks:

$$m_d = m_e, \qquad m_s = m_\mu, \qquad m_b = m_\tau. \tag{4.34}$$

These relations have to be modified because of:
- CKM mixing of the D-quarks, and possibly of the charged leptons, if neutrinos

have a non-zero mass;
- renormalization effects, running masses to be extrapolated from Λ_{GUT} to well below Λ_{Fermi};
- QCD effects: no free quark masses can be measured, so the masses have to be estimated from the masses of bound states like the ρ meson, the J/ψ or the Y.
Although such corrections are hard to compute accurately, it is generally agreed that the predictions (4.34) are not well met by the data. This is one respect in which $SU(5)$ is not very satisfactory. One can think of several remidies to resolve this discrepancy, such as
- other scalars, i.e. switching on the Yukawa couplings (f_3, f_4); in general this does not work very well;
- allow for more particles at intermediate scales between Λ_{Fermi} and Λ_{GUT} contributing to renormalization effects; however, this also effects the running of the gauge couplings;
- take other (larger) unification groups G_{GUT}, such as $SO(10)$ or E_6.
Some of these options will be discussed further below.

4.4 Proton decay

In grand unified theories quarks and leptons are part of the same gauge multiplets; therefore the gauge interactions can mediate exchange of quantum numbers between them, turning quarks into leptons and vice-versa. As a result baryon number is not conserved in grand unified theories, and particles like protons and neutrons are not stable, at least as a matter of principle. On the other hand, many attempts to detect the decay of nucleons by oberving the sudden production of charged particles like positrons or muons have been undertaken in the past 20 years, so far without result. The present observational limit on the life time of the proton is [3]:

$$\tau_{proton} > 1.6 \times 10^{25}\, \text{yrs}, \tag{4.35}$$

independend of the mode of decay, whilst specific modes of decay have limits orders of magnitude larger, e.g.

$$\tau(p \to \pi^0 e^+) > 5.5 \times 10^{32}\, \text{yrs}. \tag{4.36}$$

Thus baryon decay is definitely highly suppressed and provides another kind of experimental tests of unification models, in addition to e.g. fermion mass spectra.

The first steps in producing an estimate of the proton life-time are relatively straightforward. As the proton decay is mediated by the vector bosons carrying non-adjoint $SU(3) \times SU(2) \times U_Y(1)$ charges, the effective low-energy interaction can be decribed as a point-like four-fermion interaction of the Fermi-type, like in β-decay, except that the Fermi-constant, which is proportional to $1/M_W^2$, is replaced by a similar decay constant

$$\frac{G_X}{\sqrt{2}} = \frac{g_5^2}{8M_X^2}. \tag{4.37}$$

Here g_5 is the unified gauge coupling constant, and M_X a mass of the order of 10^{15} GeV or higher. For example, one of the contibutions to the four-fermion interaction

$$\Delta\mathcal{L}_{4f} \sim \left(\overline{\Psi}_{L\,IK}\gamma^\mu\Psi_L^{JK}\right)\left(\overline{\Psi}_L^I\gamma_\mu\Psi_{L\,J}\right), \tag{4.38}$$

is the one obtained by heavy gauge boson exchange as depicted in fig. 5.

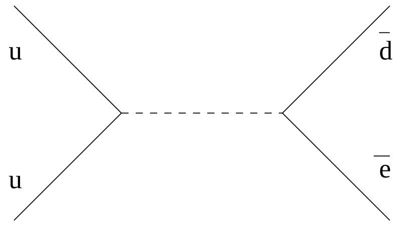

Fig. 5: Vector boson exchange mediating proton decay in $SU(5)$

From this observation it follows, that at a semi-quantitative level the decay probability of the proton will for dimensional reasons be of the form

$$\Gamma_p = \frac{1}{\tau_p} = C\,\frac{g_5^4 m_p^5}{M_X^4}, \tag{4.39}$$

with m_p taken as the proton mass and C a dimensionless parameter. It is clear that this will give very large decay times for the proton for a large range of values of C. For $C = \mathcal{O}(1)$ and $g_5 = \mathcal{O}(10^{-1})$ one naively finds $\tau_p = \mathcal{O}(10^{35}\,\text{yrs})$.

Nevertheless, more precise estimates within the minimal $SU(5)$ model —which involve highly non-trivial computations— produce considerably shorter life-times, corresponding to $C = \mathcal{O}(10^{-6})$ for the decay $p^+ \to \pi^0 e^+$. This puts the life time of the proton in minimal $SU(5)$ well below the experimental limit.

Furthermore, precision measurements of the running couplings of G_{sm} indicate [9], that in the minimal $SU(5)$ model —with no new physics between the Standard Model and Λ_{GUT}— there is no exact matching of the gauge couplings at a single scale. Together with the unrealistic mass-relations for the quarks and leptons, this is generally considered to rule out the minimal $SU(5)$ scenario. However, some caution is necessary as additional heavy particles participating in the strong and electro-weak interactions mediated by the $SU(3) \times SU(2) \times U_Y(1)$ gauge bosons below the GUT scale can easily change some of the parameters going into the calculation, like the precise value of M_X. This holds for example for the color-triplet partners of the Higgs doublet in $SU(5)$, in particular if more than one Higgs multiplet is involved in the spontaneous break-down of G_{sm}. It holds even more strongly for supersymmetric extensions of $SU(5)$, for which the gauge couplings are known to converge within the experimental limits.

4.5 Beyond $SU(5)$

Extending the spectrum of fermions and scalars participating in the gauge interactions below Λ_{GUT} is one remedy to solve the known discrepancies between the $SU(5)$ model and the experimental data. An additional possibility is to introduce more vector bosons by enlarging the GUT symmetry group G_{GUT}. For such a scenario there are some interesting theoretical motivations.

Consider again the process illustrated in fig. 5. A remarkable aspect of this process is, that it violates both baryon number and lepton number by one unit (quarks having baryon number 1/3, antiquarks having baryon number -1/3), but that the difference of baryon and lepton number is conserved:

$$\Delta B \,-\, \Delta L \,=\, 0. \tag{4.40}$$

This turns out to be a general feature of the GUT-models based on $SU(5)$, either as the fundamental gauge group, or as an intermediate step in the breaking of a larger group like $SO(10)$ or E_6, like in fig. 4. In fact the simplest explanation of $B-L$ conservation is to observe that in $SO(10)$ unification, the first step in the spontaneous symmetry breaking:

$$SO(10) \,\rightarrow\, SU(5) \times U(1), \tag{4.41}$$

the $U(1)$ is precisely the generator of $B-L$ transformations:

$$q \,\rightarrow\, e^{i\alpha/3}\, q, \qquad l \,\rightarrow\, e^{-i\alpha}\, l, \tag{4.42}$$

with q any quark and l any lepton. This explains why it is an exact symmetry compatible with the $SU(5)$ gauge interactions.

The $SO(10)$ breaking scheme (4.41) also helps to understand why the $SU(5)$ model with the minimal matter content of chiral fermions in families consisting of $\mathbf{10}+\bar{\mathbf{5}}$ representations is anomaly free. The basic observation is, that in $SO(10)$ a family of chiral fermions fills out one fundamental representation, the chiral spinor representation $\mathbf{16}$, which decomposes under $SU(5)$ as

$$\mathbf{16}_{SO(10)} \,\rightarrow\, [\mathbf{10}+\bar{\mathbf{5}}+\mathbf{1}]_{SU(5)}\,. \tag{4.43}$$

Now the generators of $SO(10)$ in the standard 32-dimensional spinor representation are

$$\Sigma_{MN} \,=\, \frac{1}{4}\,[\Gamma_M, \Gamma_N]\,, \tag{4.44}$$

with $\{\Gamma_M\}_{M=1}^{10}$ the Dirac matrices in 10 dimensions. The chiral spinor $\mathbf{16}$ is obtained by projection of the Dirac represention using

$$\frac{1}{2}\,(1+\Gamma_{11})\,, \tag{4.45}$$

and their $SO(10)$ transformations are generated by

$$\Sigma^{+}_{MN} = \frac{1}{2} (1 + \Gamma_{11}) \Sigma_{MN}. \tag{4.46}$$

All eigenvalues of these 10-dimensional rotation matrices are ± 1, with equal numbers of positive and negative ones; hence Tr $(\Sigma^{+}_{MN}) = 0$. It follows immediately, that

$$\mathrm{Tr} \left(\Sigma^{+}_{MN} \left\{ \Sigma^{+}_{KL}, \Sigma^{+}_{PQ} \right\} \right) = 0. \tag{4.47}$$

Thus the representation **16** of 4-dimensional Weyl spinors in $SO(10)$ is anomaly free, and so is its conjugate $\overline{\mathbf{16}}$. As all representations of $SO(10)$ are constructed from these 16-dimensional spinor representations, it follows that actually all representations of $SO(10)$ are anomaly free in a 4-dimensional gauge theory.

From the decomposition (4.43) of the representation **16** under $SU(5)$ we observe, that the singlet neutrino is automatically present in $SO(10)$, and no longer optional as in $SU(5)$. Of course, being a singlet in $SU(5)$ it does not participate in the $SU(5)$ gauge interactions, and does not contribute to $SU(5)$ anomalies. Thus in $SU(5)$ the gauge anomalies of chiral spinors cancel between the **10** and the $\bar{\mathbf{5}}$.

Similar properties as for $SO(10$ also hold in the case of E_6: all representations of E_6 are anomaly free; the fundamental representation **27** is complex, and decomposes under $SO(10)$ into

$$\mathbf{27}_{E_6} \rightarrow [\mathbf{16} + \mathbf{10} + \mathbf{1}]_{SO(10)}. \tag{4.48}$$

Therefore E_6 is a suitable gauge theory for a grand unified model, able to incorporate one family of quarks and leptons in a single multiplet, containing additional fermions as well: at least one more singlet neutrino, plus a **10** of $SO(10)$. As more intermediate stages of symmetry breaking are involved, the scale at which the $SO(10)$ or E_6 gauge symmetry is realized is also higher than the unification scale of $SU(5)$. This helps to solve some problems, e.g. the proton life time becomes larger, within the experimentally allowed range. On the other hand not only the number of fermions increases, due to larger multiplets, but also the number of scalars necessary to break the gauge group G_{GUT} in several steps down to G_{sm}. This makes the models much more complex and more difficult to analyse.

Appendix A

Conventions and notations

Vectors, tensors and spinors
In this section I collect my conventions and notations. Contravariant Lorentz vectors are written as $v^\mu = (v^0, \mathbf{v})$, with $\mathbf{v}$ the spatial part of the vector. Indices are raised, lowered and contracted with the Minkowski metric

$$\eta_{\mu\nu} \,=\, \eta^{\mu\nu} \,=\, \mathrm{diag}(-1, +1, +1, +1). \tag{A.1}$$

The 4-momentum of a free particle of mass m then satisfies

$$p_\mu p^\mu \,+\, m^2c^2 \,=\, -\,\frac{E^2}{c^2} \,+\, \mathbf{p}^2 \,+\, m^2c^2 \,=\, 0. \tag{A.2}$$

Usually I employ natural units: $\hbar = c = 1$.

A Lorentz transformation is represented on a 4-vector by a matrix $\Lambda^\mu{}_\nu$ with the property

$$\eta_{\mu\nu}\Lambda^\mu{}_\kappa\Lambda^\mu{}_\lambda \,=\, \eta_{\kappa\lambda}. \tag{A.3}$$

An infinitesimal Lorentz transformation then is of the form

$$\Lambda^\mu{}_\nu \,=\, \delta^\mu_\nu \,+\, \omega^\mu{}_\nu, \qquad \text{with} \quad \omega_{\mu\nu} \,+\, \omega_{\nu\mu} \,=\, 0. \tag{A.4}$$

We can write an infinitesimal Lorentz transformation on a 4-vector as

$$\delta v^\mu \,=\, v'^{\,\mu} \,-\, v^\mu \,=\, \frac{1}{2}\,\omega^{\alpha\beta}\,(\Sigma_{\alpha\beta})^\mu{}_\nu\, v^\nu. \tag{A.5}$$

where the generators $\Sigma_{\alpha\beta}$ are 4×4-matrices with components

$$(\Sigma_{\alpha\beta})^\mu{}_\nu \,=\, \delta^\mu_\alpha\,\eta_{\beta\nu} \,-\, \delta^\mu_\beta\,\eta_{\alpha\nu}. \tag{A.6}$$

They satisfy the commutation relations

$$[\Sigma_{\alpha\beta}, \Sigma_{\gamma\delta}] \,=\, \eta_{\beta\gamma}\,\Sigma_{\alpha\delta} \,-\, \eta_{\alpha\gamma}\,\Sigma_{\beta\delta} \,-\, \eta_{\beta\delta}\,\Sigma_{\alpha\gamma} \,+\, \eta_{\alpha\delta}\,\Sigma_{\beta\gamma}. \tag{A.7}$$

In this equation we have suppressed the indices refering to the matrix components. This defines the structure constants of the Lorentz algebra.

Lorentz tensors are objects $T^{\mu\nu\cdots}$ transforming in a direct product representation of the vector representation defined by (A.5), (A.6). In addition, there also exists a spinor representation. This representation is constructed as follows. We start from the Clifford algebra in N dimensions with generating elements (e, Γ_μ), where $\mu = 1, ..., N$, such that

$$
\begin{aligned}
e^2 &= e, \qquad e\,\Gamma_\mu = \Gamma_\mu\, e = \Gamma_\mu, \\
\{\Gamma_\mu, \Gamma_\nu\} &= \Gamma_\mu \Gamma_\nu + \Gamma_\nu \Gamma_\mu = 2\,\eta_{\mu\nu}\, e.
\end{aligned}
\tag{A.8}
$$

In the following e, being the unit element, will usually not be written explicitly. Because of the last property, all products of unequal elements Γ_μ are completely antisymmetric:

$$
\Gamma^{(2)}_{\mu\nu} \equiv \Gamma_\mu \Gamma_\nu = -\,\Gamma^{(2)}_{\nu\mu}, \qquad \mu \neq \nu,
\tag{A.9}
$$

and similarly for higher products $\Gamma^{(k)}_{\mu_1 ... \mu_k} = \Gamma_{\mu_1} ... \Gamma_{\mu_k}$, for $\mu_1 \neq ... \neq \mu_k$. If in a product two identical generating elements appear, they can be permuted to multiply each other directly, so as to produce a unit element e. If this is then multiplied by a monomial of Γ_μ's, with all remaining indices different, it is again completely antisymmetric in these indices; if not, the procedure can be repeated. Therefore an arbitrary element R of the Clifford algebra is a finite polynomial of at most 2^N terms for $\mu = 1, ..., N$:

$$
R[a] = a_{(0)} e + a^{\mu}_{(1)} \Gamma_\mu + \frac{1}{2!}\, a^{\mu_1 \mu_2}_{(2)} \Gamma^{(2)}_{\mu_1 \mu_2} + ... + a_* \Gamma^*.
\tag{A.10}
$$

Here the star denotes the dual of the last (2^Nth) product element:

$$
\Gamma^* = \frac{1}{N!}\, \varepsilon^{\mu_1 ... \mu_N}\, \Gamma^{(N)}_{\mu_1 ... \mu_N},
\tag{A.11}
$$

with $\varepsilon^{\mu_1 ... \mu_N}$ the N-dimensional permutationsymbol.

In four space-time dimensions the Clifford algebra has only one irreducible representation (up to a similarity transformation), defined by the 4×4 Dirac matrices $\{\gamma_\mu\}^3_{\mu=0}$. A convenient basis in many applications is

$$
\boldsymbol{\gamma} \simeq \begin{pmatrix} 0 & -i\boldsymbol{\sigma} \\ i\boldsymbol{\sigma} & 0 \end{pmatrix}, \qquad \gamma^0 \simeq \begin{pmatrix} i1_2 & 0 \\ 0 & -i1_2 \end{pmatrix}.
\tag{A.12}
$$

Here $\simeq$ represents equality up to a similarity transformation, $\boldsymbol{\sigma}$ are the 2×2 Pauli-matrices, and 1_2 is the 2×2 unit matrix. These conventions are such that the space-like generators of the Clifford algebra are represented by the hermitean matrices: $\boldsymbol{\gamma}^\dagger = \boldsymbol{\gamma}$, the time-like generator being anti-hermitean: $(\gamma^0)^\dagger = -\gamma^0 = \gamma_0$. These relations can be summarized in the single property

$$\left(\gamma^0\gamma^\mu\right)^\dagger = \gamma^0\gamma^\mu, \quad \text{or} \quad \gamma^{\mu\,\dagger} = \gamma^0\gamma^\mu\gamma^0. \tag{A.13}$$

Another useful relation is

$$\gamma_\mu = \gamma^{\mu\,\dagger} = \gamma^0\gamma^\mu\gamma^0. \tag{A.14}$$

In some applications it is convenient to work with hermitean matrices; these are obtained by introducing

$$\gamma_4 = i\gamma_0 \simeq \left(\begin{array}{cc} 1_2 & 0 \\ 0 & -1_2 \end{array} \right). \tag{A.15}$$

The Dirac matrices ($\boldsymbol{\gamma}$,γ_4) define a basis for the euclidean (Wick-rotated) Clifford algebra in 4 dimensions. The elements ψ of the 4-dimensional vector space on which these matrices act as linear operators are called *spinors*; in this terminology, the space itself is called *spinor space*.

The unit element of the Clifford algebra is represented in spinor space by the 4×4 unit matrix 1_4; the dual of the last element is represented by

$$\gamma_5 = \frac{i}{4!}\varepsilon_{\kappa\lambda\mu\nu}\gamma^\kappa\gamma^\lambda\gamma^\mu\gamma^\nu = \gamma_1\gamma_2\gamma_3\gamma_4. \tag{A.16}$$

The permutation symbol is here defined implicitly to satisfy the convention

$$\varepsilon_{0123} = -\varepsilon^{0123} = 1. \tag{A.17}$$

By construction $\gamma_5 = \gamma_5^\dagger$. As $\gamma_5^2 = 1_4$, its eigenvalues are ± 1. Moreover, γ_5 is traceless; therefore it has 2 positive and two negative eigenvalues. This can also be seen explicitly from the representation

$$\gamma_5 \simeq \left(\begin{array}{cc} 0 & -1_2 \\ -1_2 & 0 \end{array} \right). \tag{A.18}$$

Observe, that the definition of the Clifford algebra implies

$$\gamma_5\gamma_\mu = -\gamma_\mu\gamma_5. \tag{A.19}$$

Of the remaining 10 elements of the basis of the Clifford algebra, 4 correspond to the anti-symmetric 3-index elements $\Gamma^{(3)}_{\lambda\mu\nu}$, dual to the pseudo-vector like elements $\Gamma_\mu\Gamma^*$:

$$i\gamma_5\gamma_\mu = \frac{1}{3!}\,\varepsilon_{\mu\nu\kappa\lambda}\,\gamma^\nu\gamma^\kappa\gamma^\lambda. \tag{A.20}$$

The other 6 elements $\Gamma^{(2)}_{\mu\nu}$ are represented by the anti-symmetric 2-index set of anti-hermitean matrices

$$\sigma_{\mu\nu} = \frac{1}{4}\left[\gamma_\mu, \gamma_\nu\right]. \tag{A.21}$$

A straightforward calculation shows, that these matrices satisfy the commutation relations of the Lorentz algebra (A.7)

$$[\sigma_{\alpha\beta}, \sigma_{\gamma\delta}] = \eta_{\alpha\delta}\,\sigma_{\beta\gamma} - \eta_{\beta\delta}\,\sigma_{\alpha\gamma} - \eta_{\alpha\gamma}\,\sigma_{\beta\delta} + \eta_{\beta\gamma}\,\sigma_{\alpha\delta}. \tag{A.22}$$

Therefore the they generate an irreducible representation of the Lorentz transformations on spinors; in infinitesimal form:

$$\delta\psi = \psi' - \psi = \frac{1}{2}\,\omega^{\alpha\beta}\,\sigma_{\alpha\beta}\,\psi. \tag{A.23}$$

The σ-matrices satisfy the duality relations

$$\frac{1}{2}\,\varepsilon_{\mu\nu\kappa\lambda}\sigma^{\kappa\lambda} = i\gamma_5\,\sigma_{\mu\nu}, \qquad \frac{1}{2}\,\varepsilon^{\mu\nu\kappa\lambda}\sigma_{\kappa\lambda} = i\gamma_5\,\sigma^{\mu\nu}. \tag{A.24}$$

Charge conjugation

From a complex scalar field φ one can construct a 4-current

$$j_\mu = -i\varphi^*\,\overleftrightarrow{\partial}_\mu\,\varphi. \tag{A.25}$$

If the scalar field satisfies an equation

$$\Box\varphi = \frac{\partial U}{\partial\varphi^*}, \tag{A.26}$$

with $U[\varphi^*, \varphi]$ a real potential function, and with the complex conjugate equation for φ^*, then the 4-divergence of the current becomes

$$\partial_\mu j^\mu = -i\left(\varphi^*\frac{\partial U}{\partial\varphi^*} - \varphi\frac{\partial U}{\partial\varphi}\right). \tag{A.27}$$

Clearly, if $\varphi\,\partial U/\partial\varphi$ is real:

$$\varphi\frac{\partial U}{\partial\varphi} = \varphi^*\frac{\partial U}{\partial\varphi^*}, \tag{A.28}$$

then the 4-divergence of the current vanishes:

$$\partial_\mu j^\mu = 0. \tag{A.29}$$

From a divergence-free current we can construct a conserved charge:

$$Q = \int d^3x\, j^0(t, \mathbf{x}). \tag{A.30}$$

Here the integral is over a complete 3-dimensional hypersurface of equal time. Provided the boundary terms vanish, it is straightforward to check that

$$\frac{dQ}{dt} = \int d^3x\, \partial_0 j^0 = \int d^3x\, \boldsymbol{\nabla} \cdot \mathbf{j} = 0. \tag{A.31}$$

By construction the 4-current j_μ as defined in eq.(A.25), and hence the charge Q, are real. However, note that the substitution $\varphi \rightarrow \varphi^*$ and vice-versa, makes the current and the charge change sign:

$$\varphi \leftrightarrow \varphi^* \qquad \Rightarrow \qquad Q \rightarrow -Q. \tag{A.32}$$

Therefore the anti-linear transformation defined by interchange of the scalar field and its complex conjugate is called charge conjugation, and we write

$$\varphi^c \equiv \varphi^*. \tag{A.33}$$

Note however, that charge conjugation only transforms the fields, not the other complex factors in the theory. In particular, the imaginary unit i in front of the current is unaffected by charge conjugation:

$$j_\mu^c = \left(-i\varphi^* \overleftrightarrow{\partial_\mu} \varphi \right)^c = -i\varphi \overleftrightarrow{\partial_\mu} \varphi^* = i\varphi^* \overleftrightarrow{\partial_\mu} \varphi = -j_\mu. \tag{A.34}$$

Finally, observe that the current j_μ and the charge Q vanish identically if the fields are real: $\varphi^c = \varphi \rightarrow j_\mu = Q = 0$.

Next we consider the case of fermions. From the hermitean conjugation property (A.13) we infer, that from a 4-component spinor ψ and its hermitean conjugate $\psi^\dagger$ we van construct a real 4-current

$$j_f^\mu = -\psi^\dagger \gamma^0 \gamma^\mu \psi. \tag{A.35}$$

Its is customary to introduce the notation $\bar{\psi} = \psi^\dagger \gamma_4 = -i\psi^\dagger \gamma^0$; then

$$j_f^\mu = -i\,\bar{\psi}\gamma^\mu \psi. \tag{A.36}$$

One can check that under the Lorentz transformations (A.23) the current transforms as a true 4-vector. Also observe that as γ_4 is hermitean, with the definition of $\bar{\psi}$ the quantity

$$S_f = \bar{\psi}\psi \tag{A.37}$$

is a real Lorentz scalar.

Now assume that the spinor satisfies a field equation of the form

$$\gamma^\mu \partial_\mu \psi = U(x)\psi, \tag{A.38}$$

for some scalar function U. From the conjugation property (A.13) it follows that

$$\partial_\mu \left(\bar{\psi}\,\gamma^\mu\right) \;=\; -\,\bar{\psi}\,U^*(x). \tag{A.39}$$

Then the 4-divergence of the fermion current is

$$\partial_\mu\, j_f^\mu \;=\; -i\,\left(U - U^*\right)\,\bar{\psi}\psi. \tag{A.40}$$

It follows, that the current is 4-divergence free if $U(x)$ is real:

$$U \;=\; U^* \qquad \Rightarrow \qquad \partial_\mu\, j_f^\mu \;=\; 0. \tag{A.41}$$

As in the case of the scalar current we can then define the real conserved charge

$$Q_f \;=\; \int d^3x\, j^0(t,\mathbf{x}) \;=\; \int d^3x\,\psi^\dagger\psi. \tag{A.42}$$

As for the complex scalar field, we can define an anti-linear operation $\psi \to \psi^c$ such that the current j_f^μ and the charge Q_f change sign. This charge conjugation transformation is defined by

$$\psi \;\rightarrow\; \psi^c \;\equiv\; \mathcal{C}\bar{\psi}^T \;\Leftrightarrow\; \bar{\psi}^c \;=\; \psi^T\mathcal{C}^T. \tag{A.43}$$

where T denotes transposition in spinor space, and $\mathcal{C}$ is the charge conjugation matrix defined by the property

$$\mathcal{C}^{-1}\gamma^\mu\mathcal{C} \;=\; -\,\gamma^{\mu\,T}, \qquad \mathcal{C}^T \;=\; -\mathcal{C}^{-1}. \tag{A.44}$$

First, we show that such a charge conjugation operation indeed changes the sign of the current:

$$\begin{aligned}\left(j_f^\mu\right)^c \;&=\; -i\,\bar{\psi}^c\gamma^\mu\psi^c \;=\; -i\psi^T\mathcal{C}^T\gamma^\mu\mathcal{C}\bar{\psi}^T \\ &=\; i\psi^T\mathcal{C}^{-1}\gamma^\mu\mathcal{C}\bar{\psi}^T \;=\; -i\psi^T\gamma^{\mu\,T}\bar{\psi}^T \\ &=\; i\,\left(\bar{\psi}\gamma^\mu\psi\right)^T \;=\; i\bar{\psi}\gamma^\mu\psi \;=\; -\,j_f^\mu.\end{aligned} \tag{A.45}$$

A crucial step here is from the second to the third line, where we take the components of the spinors to be odd elements of a Grassmann algebra, i.e. they are anti-commuting c-numbers. It then remains to show that a charge conjugation matrix with the properties (A.44) exists. It is sufficient to prove this in a specific basis, which we take to be the one defined in eq.(A.12). In this basis we have

$$\mathcal{C} \;=\; \mathcal{C}^\dagger \;=\; \mathcal{C}^{-1} \;=\; -\mathcal{C}^T \;=\; \begin{pmatrix} 0 & \sigma_2 \\ \sigma_2 & 0 \end{pmatrix}. \tag{A.46}$$

Like for scalar fields, a spinor which is self charge-conjugate has a vanishing current and charge:

$$\psi^c = \psi \qquad \Rightarrow \qquad j^\mu_f = Q_f = 0. \tag{A.47}$$

Such chargeless spinors are known as Majorana spinors.

Spin and helicity

The behaviour of free massive particles under rotations is characterized by their spin. The spin can always be determined unambiguously by a measurement in the rest frame of the particle, in which according to quantum mechanics one can measure the *total spin* σ, given by the eigenvalue of the Casimir operator

$$\Sigma^2 = \sigma(\sigma + 1)\hbar^2, \tag{A.48}$$

and one of the components, conventionally taken to be the z-component:

$$\Sigma_z = \sigma_z \hbar. \tag{A.49}$$

The representation theory of the rotation group shows, that the quantum number σ takes half-integer values: $0, \frac{1}{2}, 1, ...$, whilst $\sigma_z = -\sigma, -\sigma + 1, ..., +\sigma$. The spin degeneracy of any free massive particle state is therefore $2\sigma + 1$. In the following we again take $\hbar = c = 1$.

For massless particles on the other hand, a transformation to the restframe does not exist, and the rotation group leaving the equations of motion invariant is reduced to rotations in the plane perpendicular to the direction of propagation. As these plane rotations are abelian, all irreducible unitary representations are one-dimensional; this implies that states are characterized by a single quantum number, which is the angular momentum in the direction of propagation. This quantity is usually called the helicity.

We now illustrate these concepts for the case of fermions. A massive free fermion state of momentum p_μ satisfies the Dirac equation

$$(-i\gamma \cdot p + m)\ \psi(p) = 0. \tag{A.50}$$

Multiplying the equation by $(i\gamma \cdot p + m)$ gives

$$\left(p_\mu p^\mu + m^2\right)\ \psi(p) = 0. \tag{A.51}$$

This shows that physical states live on the mass shell. In the rest frame of the particle $\mathbf{p} = 0$ and $E = m$, hence

$$m\ (-i\gamma_0 + 1)\ \psi(\mathbf{p} = 0) = 0. \tag{A.52}$$

The combination of gamma matrices inside the parentheses actually defines a projection operator on the lower components of the spinor:

$$-i\gamma_0 + 1 \,=\, 2 \left(\begin{array}{cc} 0 & 0 \\ 0 & 1 \end{array} \right). \tag{A.53}$$

It follows that the spinor in the rest frame has two free components:

$$\psi(\mathbf{p} = 0) \,=\, \left(\begin{array}{c} \xi \\ 0 \end{array} \right), \tag{A.54}$$

with ξ a (two-component) Pauli spinor.

The Lorentz transformations on the components of the spinor ψ are generated by the spinorial matrices $\sigma_{\mu\nu}$. The rotation group is the spatial part of the Lorentz group; therefore we can identify the generators of real 3-dimensional rotations with

$$\Sigma_i \,=\, -\frac{i}{2}\,\varepsilon_{ijk}\sigma_{jk}, \qquad [\Sigma_i, \Sigma_j] \,=\, i\varepsilon_{ijk}\Sigma_k. \tag{A.55}$$

In the standard representation we then find

$$\boldsymbol{\Sigma}^2 \,=\, \frac{1}{4} \left(\begin{array}{cc} \boldsymbol{\sigma}^2 & 0 \\ 0 & \boldsymbol{\sigma}^2 \end{array} \right) \,=\, \frac{3}{4}\, 1_4, \qquad \Sigma_3 \,=\, \frac{1}{2} \left(\begin{array}{cc} \sigma_3 & 0 \\ 0 & \sigma_3 \end{array} \right). \tag{A.56}$$

It follows directly that the states

$$\xi_+ \,=\, \left(\begin{array}{c} 1 \\ 0 \end{array} \right), \qquad \xi_- \,=\, \left(\begin{array}{c} 0 \\ 1 \end{array} \right), \tag{A.57}$$

are eigenstates of the total spin $\boldsymbol{\Sigma}^2$ with eigenvalue $\sigma(\sigma+1)$, $\sigma = 1/2$, and of the third spin component Σ_3 with eigenvalue $\pm 1/2$, respectively.

Next we consider massless fermions. We take the direction of propagation to be the z-direction; then the four momentum is of the form

$$p^\mu \,=\, (p, 0, 0, p), \qquad p_\mu p^\mu \,=\, 0. \tag{A.58}$$

Now the Dirac equation implies

$$p\,\left(\gamma_0 + \gamma_3\right)\,\psi(p) \,=\, 0. \tag{A.59}$$

In the standard basis the solutions of this equation are

$$\psi(p) \,=\, \left(\begin{array}{c} \xi \\ \eta \end{array} \right), \qquad \eta \,=\, \sigma_3 \xi. \tag{A.60}$$

Equivalently, we can write

$$\psi(p) = \left(\begin{array}{cc} 1 & -\sigma_3 \\ \sigma_3 & -1 \end{array} \right) \left(\begin{array}{c} \xi \\ 0 \end{array} \right). \tag{A.61}$$

Again, the physical states can be characterized completely in terms of a two-component spinor ξ. Next we construct the Pauli-Lubanski vector

$$J_\mu = -\frac{i}{2}\, \varepsilon_{\mu\nu\kappa\lambda}\, p^\nu \sigma^{\kappa\lambda}. \tag{A.62}$$

Comparison with (A.55) shows, that the time component of this vector is

$$J_0 = \mathbf{p} \cdot \mathbf{\Sigma}, \tag{A.63}$$

representing the projection of the spin on the 3-momentum $\mathbf{p}$. For the massless case with 4-momentum (A.58) in particular one obtains

$$J_0 = -J_3 = \frac{p}{2} \left(\begin{array}{cc} \sigma_3 & 0 \\ 0 & \sigma_3 \end{array} \right). \tag{A.64}$$

Its action on the massless physical spinors (A.61) is

$$J_0 \psi(p) = p \left(\begin{array}{cc} 1 & -\sigma_3 \\ \sigma_3 & -1 \end{array} \right) \left(\begin{array}{c} \lambda\xi \\ 0 \end{array} \right), \tag{A.65}$$

with

$$\lambda = \frac{\mathbf{p} \cdot \boldsymbol{\sigma}}{2p} \simeq \frac{1}{2}\, \sigma_3. \tag{A.66}$$

We conclude that massless fermion states defined by two-component spinors $\xi_\pm$ as in (A.57), are eigenstates of the helicity operator λ, with eigenvalues $\pm 1/2$.

A covariant way to write this result is by introducing the time-reversed momentum 4-vector

$$\bar{p}^\mu = (-p, 0, 0, p), \qquad \bar{p}_\mu \bar{p}^\mu = 0, \qquad \bar{p} \cdot p = 2p. \tag{A.67}$$

We can then define the helicity operator

$$\Lambda(p) = -\frac{\bar{p} \cdot J}{\bar{p} \cdot p} \rightarrow \Lambda(p) = \frac{J_0}{p} = \Sigma_3, \tag{A.68}$$

where the last expressions hold for the particular choice of 4-momentum p^μ above. Hence Λ is the four-component generalization of the two-component helicity operator λ. Note that for massless particles there is no physically realizable Lorentz transformation which can change a positive helicity state into a negative helicity state; this would correspond to a reversal of the direction of the momentum, interchanging the past and future light cone. Therefore the helicity of massless particles, as illustrated here for the fermions, is invariant under Lorentz transformations.

Chirality
The hermitean Dirac matrix γ_5 commutes with all the generators of the Lorentz transformations:

$$[\sigma_{\mu\nu}, \gamma_5] \, = \, 0. \tag{A.69}$$

Therefore fermion states of specific spin or helicity can be simultaneous eigenstates of γ_5. We have already seen that its eigenvalues are ± 1. In fact, because γ_5 is a square root of unity, we can decompose any spinor ψ automatically in eigenspinors of γ_5 by using projection operators $(1 \pm \gamma_5)/2$:

$$\psi \, = \, \frac{1}{2} \, (1 + \gamma_5) \, \psi \, + \, \frac{1}{2} \, (1 - \gamma_5) \, \psi \, \equiv \, \psi_+ \, + \, \psi_-. \tag{A.70}$$

Here the components $\psi_\pm$ are the eigenstates

$$\gamma_5 \, \psi_\pm \, = \, \pm \, \psi_\pm. \tag{A.71}$$

For reasons to become clear, the eigenvalue of γ_5 is called the *chirality* of a spinor; basically it can be positive or negative (± 1).

There are two properties of chiral spinors which somewhat limit their usefulness. First, eigenspinors of γ_5 are necessarily complex, or more precisely: they cannot be self-conjugate under charge conjugation. Consider for example a positive chirality spinor $\psi_+ = \frac{1}{2} \, (1 + \gamma_5) \, \psi$; its charge-conjugate is

$$\begin{aligned} (\psi_+)^c \; &= \; \frac{1}{2} \mathcal{C} \, \left(\psi^\dagger (1 + \gamma_5)^\dagger \gamma_4 \right)^T \, = \, \frac{1}{2} \mathcal{C} \, \left(\bar{\psi} (1 - \gamma_5) \right)^T \\ &= \; \frac{1}{2} \mathcal{C} \, (1 - \gamma_5)^T \, \mathcal{C}^{-1} \mathcal{C} \, \bar{\psi}^T \, = \, \frac{1}{2} \, (1 - \gamma_5) \, \psi^c \, = \, (\psi^c)_-. \end{aligned} \tag{A.72}$$

Thus, the charge conjugate of a positive-chirality spinor is a negative-chirality charge-conjugate spinor: charge conjugation switches the chirality from positive to negative or vice-versa. This is often expressed by the statement that Majorana spinors can not be chiral, nor can chiral spinors be Majorana.

The second limitation on the use of chiral spinors is, that physical spinors (i.e., those satisfying the Dirac equation) can only be chiral if they are massless. This is easy to see, as γ_5 does not commute with the Dirac operator $\gamma \cdot p$, and they do not have any common eigenstates, with the possible exception of zero-modes. As the eigenvalues of γ_5 are only ± 1, this means only zero-modes of the Dirac operator have can be states of specific chirality. Indeed

$$\gamma_5 \, \gamma \cdot p \, \psi_\pm \, = \, - \, \gamma \cdot p \, \gamma_5 \, \psi_\pm \, = \, \mp \, \gamma \cdot p \, \psi_\pm, \tag{A.73}$$

showing that the Dirac operator also switches the chirality of as spinor, turning positive-chirality spinors into negative-chirality ones. Equivalently, applying the chiral projection operators to the Dirac equation gives

$$i\,\gamma\cdot p\,\psi_{\pm}\,=\,m\,\psi_{\mp}. \tag{A.74}$$

Therefore a chiral spinor can only satisfy the Dirac equation if its mass vanishes:

$$\gamma\cdot p\,\psi_{\pm}\,=\,0. \tag{A.75}$$

For such physical chiral spinors, describing massless fermions, the chirality is actually directly related to the helicity. Indeed, consider the action of γ_5 on the massless spinor states (A.60). It is easy to establish that

$$\gamma_5\psi(p)\,=\,\left(\begin{array}{c}-\sigma_3\xi\\ -\xi\end{array}\right)\,=\,-2\,\Sigma_3\,\psi(p)\,=\,-2\,\Lambda(p)\,\psi(p). \tag{A.76}$$

Therefore for massless spinors the eigenstates of γ_5 can be identified with the helicity eigenstates, with the positive eigenvalue of γ_5 corresponding to left-handed states, and the negative eigenvalue to right-handed states. This is the reason for the terminology of right- and left-handed, or chiral, spinors. We observe, that the physical chiral spinors $\psi_{\pm}$ are defined by the two-component spinors $\xi_{\mp}$, respectively. Note, that in the literature one often encounters the labels (L,R) for positive/negative helicity spinors: $\psi_L\,=\,\psi_+$ and $\psi_R\,=\,\psi_-$. However, there is some confusion on the meaning of the conjugate spinors: some authors use $\bar{\psi}_L$ to denote the conjugate of the left-handed spinor, others use it to the denote the left-handed projection of the conjugate spinor. We use the first convention:

$$\bar{\psi}_{L,R}\,=\,(\psi_{L,R})^{\dagger}\gamma_4 \quad\Rightarrow\quad \left\{\begin{array}{l}\bar{\psi}_L\gamma_5\,=\,-\,\bar{\psi}_L,\\ \bar{\psi}_R\gamma_5\,=\,\bar{\psi}_R.\end{array}\right. \tag{A.77}$$

Bibliography

[1] C. Jarlskog (ed.), *CP-Violation* (World Scientific; Singapore, 1989)

[2] F. Filthaut, *Precise Determination of Electroweak Parameters with the L3 Detector at LEP*, Ann. ¡eeting of the Division of Particles and Fields of the APS, Minneapolis (1996)

[3] C. Caso et al., *Particle data tables and reviews*; Eur. Phys. J. C3 (1998), 1; URL: http://pdg.lbl.gov/

[4] P.A.M. Dirac, Proc. R. Soc. A133 (1931), 60

[5] T.T. Wu and C.N. Yang, Nucl. Phys. B107 (1976), 365

[6] G. 't Hooft, Nucl. Phys. B79 (1974), 276

[7] A.M. Polyakov, JETP Lett. 20 (1974), 194

[8] H. Georgi and S.L. Glashow, Phys. Rev. Lett. 32 (1974), 438

[9] U. Amaldi, W. de Boer and H. Furstenau, Phys. Lett. B260 (1991), 447