NIKHEF 01-018

Aspects of BRST Quantization†

J.W. van Holten*

NIKHEF and Vrije Universiteit

Amsterdam NL

december 20, 2001

Abstract

BRST-methods provide elegant and powerful tools for the construction and analysis of constrained systems, including models of particles, strings and fields. These lectures provide an elementary introduction to the ideas, illustrated with some important physical applications.

† Lectures given at the Summerschool on *Geometry and Topology in Physics* (Rot a/d/ Rot, Germany) sept 2001

* E-mail: v.holten@nikhef.nl

Conventions

In these lecture notes we use the following conventions. Whenever two objects carrying a same index are multiplied (as in $a_i b_i$ or in $u_\mu v^\mu$) the index is a dummy index and is to be summed over its entire range, unless explicitly stated otherwise (summation convention). Symmetrization of objects enclosed is denoted by braces $\{...\}$, anti-symmetrization by square brackets $[...]$; the total weight of such (anti-)symmetrizations is always unity.

In these notes we deal both with classical and quantum hamiltonian systems. To avoid confusion, we use braces $\{\,,\}$ to denote classical Poisson brackets, brackets $[\,,]$ to denote commutators and suffixed brackets $[\,,]_+$ to denote anti-commutators.

The Minkowski metric $\eta_{\mu\nu}$ has signature $(-1, +1, ..., +1)$, the first co-ordinate in a pseudo-cartesian co-ordinate system x^0 being time-like. Arrows above symbols ($\vec{x}$) denote purely spatial vectors (most often 3-dimensional).

Unless stated otherwise, we use natural units in which $c = \hbar = 1$. Therefore we usually do not write these dimensional constants explicitly. However, in a few places where their role as universal constants is not a priori obvious they are included in the equations.

Chapter 1

Symmetries and constraints

The time-evolution of physical systems is described mathematically by differential equations of various degree of complexity, such as Newton's equation in classical mechanics, Maxwell's equations for the electro-magnetic field, or Schrödinger's equation in quantum theory. In most cases these equations have to be supplemented with additional constraints, like initial conditions and/or boundary conditions, which select only one —or sometimes a restricted subset— of the solutions as relevant to the physical system of interest.

Quite often the prefered dynamical equations of a physical system are not formulated directly in terms of observable degrees of freedom, but in terms of more primitive quantities, such as potentials, from which the physical observables are to be constructed in a second separate step of the analysis. As a result, the interpretation of the solutions of the evolution equation is not always straightforward. In some cases certain solutions have to be excluded, as they do not describe physically realizable situations; or it may happen that certain classes of apparently different solutions are physically indistinguishable and describe the same actual history of the system.

The BRST-formalism [1, 2] has been developed specifically to deal with such situations. The roots of this approach to constrained dynamical systems are found in attempts to quantize General Relativity [3, 4] and Yang-Mills theories [5]. Out of these roots has grown an elegant and powerful framework for dealing with quite general classes of constrained systems using ideas borrowed from algebraic geometry.[1]

In these lectures we are going to study some important examples of constrained dynamical systems, and learn how to deal with them so as to be able to extract relevant information about their observable behaviour. In view of the applications to fundamental physics at microscopic scales, the emphasis is on quantum theory. Indeed, this is the domain where the full power and elegance of our methods become most apparent. Nevertheless, many of the ideas and re-

[1]Some reviews can be found in refs. [6]-[14].

sults are applicable in classical dynamics as well, and wherever possible we treat classical and quantum theory in parallel.

1.1 Dynamical systems with constraints

Before delving into the general theory of constrained systems, it is instructive to consider some examples; they provide a background for both the general theory and the applications to follow later.

1. The relativistic particle.
The motion of a relativistic point particle is specified completely by its world line $x^\mu(\tau)$, where x^μ are the position co-ordinates of the particle in some fixed inertial frame, and τ is the proper time, labeling succesive points on the world line. All these concepts must and can be properly defined; in these lectures I trust you to be familiar with them, and my presentation only serves to recall the relevant notions and relations between them.

In the absence of external forces, the motion of a particle w.r.t. an inertial frame satisfies the equation

$$\frac{d^2x^\mu}{d\tau^2} = 0. \tag{1.1}$$

It follows, that the four-velocity $u^\mu = dx^\mu/d\tau$ is constant, and the complete solution of the equations of motion is

$$x^\mu(\tau) = x^\mu(0) + u^\mu \tau. \tag{1.2}$$

A most important observation is, that the four-velocity u^μ is not completely arbitrary, but must satisfy the *physical* requirement

$$u_\mu u^\mu = -c^2, \tag{1.3}$$

where c is a *universal* constant, equal to the velocity of light, for all particles irrespective of their mass, spin, charge or other physical properties. Equivalently, eq.(1.3) states that the proper time is related to the space-time interval traveled by

$$c^2 d\tau^2 = -dx_\mu dx^\mu = c^2 dt^2 - d\vec{x}^{\,2}, \tag{1.4}$$

independent of the physical characteristics of the particle.

The universal condition (1.3) is required not only for free particles, but also in the presence of interactions. When subject to a four-force f^μ the equation of motion (1.1) for a relativistic particle becomes

$$\frac{dp^\mu}{d\tau} = f^\mu, \tag{1.5}$$

where $p^\mu = mu^\mu$ is the four-momentum. Physical forces —e.g., the Lorentz force in the case of the interaction of a charged particle with an electromagnetic field— satisfisy the condition

$$p \cdot f = 0. \tag{1.6}$$

This property together with the equation of motion (1.5) are seen to imply that $p^2 = p_\mu p^\mu$ is a constant along the world line. The constraint (1.3) is then expressed by the statement that

$$p^2 + m^2c^2 = 0, \tag{1.7}$$

with c the same universal constant. Eq. (1.7) defines an invariant hypersurface in momentum space for any particle of given restmass m, which the particle can never leave in the course of its time-evolution.

Returning for simplicity to the case of the free particle, we now show how the equation of motion (1.1) and the constraint (1.3) can both be derived from a single action principle. In addition to the co-ordinates x^μ, the action depends on an auxiliary variable e; it reads

$$S[x^\mu; e] = \frac{m}{2} \int_1^2 \left(\frac{1}{e} \frac{dx_\mu}{d\lambda} \frac{dx^\mu}{d\lambda} - ec^2 \right) d\lambda. \tag{1.8}$$

Here λ is a real parameter taking values in the interval $[\lambda_1, \lambda_2]$, which is mapped by the functions $x^\mu(\lambda)$ into a curve in Minkowski space with fixed end points (x_1^μ, x_2^μ), and $e(\lambda)$ is a nowhere vanishing real function of λ on the same interval.

Before discussing the equations that determine the stationary points of the action, we first observe that by writing it in the equivalent form

$$S[x^\mu; e] = \frac{m}{2} \int_1^2 \left(\frac{dx_\mu}{ed\lambda} \frac{dx^\mu}{ed\lambda} - c^2 \right) ed\lambda, \tag{1.9}$$

it becomes manifest that the action is invariant under a change of parametrization of the real interval $\lambda \to \lambda'(\lambda)$, if the variables (x^μ, e) are transformed simultaneously to (x'^μ, e') according to the rule

$$x'^\mu(\lambda') = x^\mu(\lambda), \qquad e'(\lambda')\, d\lambda' = e(\lambda)\, d\lambda. \tag{1.10}$$

Thus the co-ordinates $x^\mu(\lambda)$ transform as scalar functions on the real line $\mathbf{R}^1$, whilst $e(\lambda)$ transforms as the (single) component of a covariant vector (1-form) in one dimension. For this reason it is often called the *einbein*. For obvious reasons the invariance of the action (1.8) under the transformations (1.10) is called reparametrization invariance.

The condition of stationarity of the action S implies the functional differential equations

$$\frac{\delta S}{\delta x^\mu} = 0, \qquad \frac{\delta S}{\delta e} = 0. \tag{1.11}$$

These equations are equivalent to the ordinary differential equations

$$\frac{1}{e}\frac{d}{d\lambda}\left(\frac{1}{e}\frac{dx^\mu}{d\lambda}\right) = 0, \qquad \left(\frac{1}{e}\frac{dx^\mu}{d\lambda}\right)^2 = -c^2. \tag{1.12}$$

The equations coincide with the equation of motion (1.1) and the constraint (1.3) upon the identification

$$d\tau = e d\lambda, \tag{1.13}$$

a manifestly reparametrization invariant definition of proper time. Recall, that after this identification the constraint (1.3) automatically implies eq.(1.4), hence this definition of proper time coincides with the standard geometrical one.

Exercise 1.1
Use the constraint (1.12) to eliminate e from the action; show that with the choice $e > 0$ (τ increases with increasing λ) it reduces to the Einstein action

$$S_E = -mc \int_1^2 \sqrt{-\frac{dx_\mu}{d\lambda}\frac{dx^\mu}{d\lambda}}\, d\lambda = -mc^2 \int_1^2 d\tau,$$

with $d\tau$ given by eq.(1.4). Deduce that the solutions of the equations of motion are time-like geodesics in Minkowski space. Explain why the choice $e < 0$ can be interpreted as describing anti-particles of the same mass.

2. The electro-magnetic field.
In the absence of charges and currents the evolution of electric and magnetic fields $(\vec{E}, \vec{B})$ is described by the equations

$$\frac{\partial \vec{E}}{\partial t} = \vec{\nabla} \times \vec{B}, \quad \frac{\partial \vec{B}}{\partial t} = -\vec{\nabla} \times \vec{E}. \tag{1.14}$$

Each of the electric and magnetic fields has three components, but only two of them are independent: physical electro-magnetic fields in vacuo are transversely polarized, as expressed by the conditions

$$\vec{\nabla} \cdot \vec{E} = 0, \qquad \vec{\nabla} \cdot \vec{B} = 0. \tag{1.15}$$

The set of four equations (1.14) and (1.15) represent the standard form of Maxwell's equations in empty space.

Repeated use of eqs.(1.14) yields

$$\frac{\partial^2 \vec{E}}{\partial t^2} = -\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) = \Delta \vec{E} - \vec{\nabla}\vec{\nabla} \cdot \vec{E}, \tag{1.16}$$

and an identical equation for $\vec{B}$. However, the transversality conditions (1.15) simplify these equations to the linear wave equations

$$\Box \vec{E} = 0, \qquad \Box \vec{B} = 0, \tag{1.17}$$

with $\Box = \Delta - \partial_t^2$. It follows immediately that free electromagnetic fields satisfy the superposition principle and consist of transverse waves propagating at the speed of light ($c = 1$, in natural units).

Again both the time evolution of the fields and the transversality constraints can be derived from a single action principle, but it is a little bit more subtle than in the case of the particle. For electrodynamics we only introduce auxiliary fields $\vec{A}$ and ϕ to impose the equation of motion and constraint for the electric field; those for the magnetic field then follow automatically. The action is

$$
\begin{aligned}
S_{EM}[\vec{E}, \vec{B}; \vec{A}, \phi] &= \int_1^2 dt\, L_{EM}(\vec{E}, \vec{B}; \vec{A}, \phi), \\
L_{EM} &= \int d^3x \left(-\frac{1}{2} \left(\vec{E}^{\,2} - \vec{B}^{\,2} \right) + \vec{A} \cdot \left(\frac{\partial \vec{E}}{\partial t} - \vec{\nabla} \times \vec{B} \right) - \phi\, \vec{\nabla} \cdot \vec{E} \right).
\end{aligned}
\tag{1.18}
$$

Obviously, stationarity of the action implies

$$
\frac{\delta S}{\delta \vec{A}} = \frac{\partial \vec{E}}{\partial t} - \vec{\nabla} \times \vec{B} = 0, \qquad \frac{\delta S}{\delta \phi} = -\vec{\nabla} \cdot \vec{E} = 0,
\tag{1.19}
$$

reproducing the equation of motion and constraint for the electric field. The other two stationarity conditions are

$$
\frac{\delta S}{\delta \vec{E}} = -\vec{E} - \frac{\partial \vec{A}}{\partial t} + \vec{\nabla} \phi = 0, \qquad \frac{\delta S}{\delta \vec{B}} = \vec{B} - \vec{\nabla} \times \vec{A} = 0,
\tag{1.20}
$$

or equivalently

$$
\vec{E} = -\frac{\partial \vec{A}}{\partial t} + \vec{\nabla} \phi, \qquad \vec{B} = \vec{\nabla} \times \vec{A}.
\tag{1.21}
$$

The second equation (1.21) directly implies the transversality of the magnetic field: $\vec{\nabla} \cdot \vec{B} = 0$. Taking its time derivative one obtains

$$
\frac{\partial \vec{B}}{\partial t} = \vec{\nabla} \times \left(\frac{\partial \vec{A}}{\partial t} - \vec{\nabla} \phi \right) = -\vec{\nabla} \times \vec{E},
\tag{1.22}
$$

where in the middle expression we are free to add the gradient $\vec{\nabla} \phi$, as $\vec{\nabla} \times \vec{\nabla} \phi = 0$ identically.

An important observation is, that the expressions (1.21) for the electric and magnetic fields are invariant under a redefinition of the potentials $\vec{A}$ and ϕ of the form

$$
\vec{A}' = \vec{A} + \vec{\nabla} \Lambda, \qquad \phi' = \phi + \frac{\partial \Lambda}{\partial t},
\tag{1.23}
$$

where $\Lambda(x)$ is an arbitrary scalar function. The transformations (1.23) are the well-known gauge transformations of electrodynamics.

It is easy to verify, that the Lagrangean L_{EM} changes only by a total time derivative under gauge transformations, modulo boundary terms which vanish if the fields vanish sufficiently fast at spatial infinity:

$$L'_{EM} = L_{EM} - \frac{d}{dt} \int d^3x\, \Lambda \vec{\nabla} \cdot \vec{E}. \tag{1.24}$$

As a result the action S_{EM} itself is strictly invariant under gauge transformations, provided $\int d^3x \Lambda \vec{\nabla} \cdot \vec{E}|_{t_1} = \int d^3x \Lambda \vec{\nabla} \cdot \vec{E}|_{t_2}$; however, no physical principle requires such strict invariance of the action. This point we will discuss later in more detail.

We finish this discussion of electro-dynamics by recalling how to write the equations completely in relativistic notation. This is achieved by first collecting the electric and magnetic fields in the anti-symmetric field-strength tensor

$$F_{\mu\nu} = \left(\begin{array}{cccc} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{array} \right), \tag{1.25}$$

and the potentials in a four-vector:

$$A_\mu = (\phi, \vec{A}). \tag{1.26}$$

Eqs.(1.21) then can be written in covariant form as

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{1.27}$$

with the electric field equations (1.19) reading

$$\partial_\mu F^{\mu\nu} = 0. \tag{1.28}$$

The magnetic field equations now follow trivially from (1.27) as

$$\varepsilon^{\mu\nu\kappa\lambda} \partial_\nu F_{\kappa\lambda} = 0. \tag{1.29}$$

Finally, the gauge transformations can be written covariantly as

$$A'_\mu = A_\mu + \partial_\mu \Lambda. \tag{1.30}$$

The invariance of the field strength tensor $F_{\mu\nu}$ under these transformations follows directly from the commutativity of the partial derivatives.

Exercise 1.2
Show that eqs.(1.27)–(1.29) follow from the action

$$S_{cov} = \int d^4x \left(\frac{1}{4} F^{\mu\nu} F_{\mu\nu} - F^{\mu\nu}\, \partial_\mu A_\nu \right).$$

Verify, that this action is equivalent to S_{EM} modulo a total divergence. Check that eliminating $F_{\mu\nu}$ as an independent variable gives the usual standard action

$$S[A_\mu] = -\frac{1}{4} \int d^4x\, F^{\mu\nu}(A) F_{\mu\nu}(A),$$

with $F_{\mu\nu}(A)$ given by the right-hand side of eq.(1.27).

1.2 Symmetries and Noether's theorems

In the preceeding section we have presented two elementary examples of systems whose complete physical behaviour was described conveniently in terms of one or more evolution equations plus one or more constraints. These constraints are needed to select a subset of solutions of the evolution equation as the physically relevant solutions. In both examples we found, that the full set of equations could be derived from an action principle. Also, in both examples the additional (auxiliary) degrees of freedom, necessary to impose the constraints, allowed non-trivial local (space-time dependent) redefinitions of variables leaving the lagrangean invariant, at least up to a total time-derivative.

The examples given can easily be extended to include more complicated but important physical models: the relativistic string, Yang-Mills fields and general relativity are all in this class. However, instead of continuing to produce more examples, at this stage we turn to the general case to derive the relation between local symmetries and constraints, as an extension of Noether's well-known theorem relating (rigid) symmetries and conservation laws.

Before presenting the more general analysis, it must be pointed out that our approach distinguishes in an important way between time- and space-like dimensions; indeed, we have emphasized from the start the distinction between equations of motion (determining the behaviour of a system as a function of time) and constraints, which impose additional requirements. e.g. restricting the spatial behaviour of electro-magnetic fields. This distinction is very natural in the context of hamiltonian dynamics, but potentially at odds with a covariant lagrangean formalism. However, in the examples we have already observed that the not manifestly covariant treatment of electro-dynamics could be translated without too much effort into a covariant one, and that the dynamics of the relativistic particle, including its constraints, was manifestly covariant throughout.

In quantum theory we encounter similar choices in the approach to dynamics, with the operator formalism based on equal-time commutation relations distinguishing space- and time-like behaviour of states and observables, whereas the covariant path-integral formalism allows treatment of space- and time-like dimensions on an equal footing; indeed, upon the analytic continuation of the path-integral to euclidean time the distinction vanishes alltogether. In spite of these differences, the two approaches are equivalent in their physical content.

In the analysis presented here we continue to distinguish between time and space, and between equations of motion and constraints. This is convenient as it allows us to freely employ hamiltonian methods, in particular Poisson brackets in classical dynamics and equal-time commutators in quantum mechanics. Nevertheless, as we hope to make clear, all applications to relativistic models allow a manifestly covariant formulation.

Consider a system described by generalized coordinates $q^i(t)$, where i labels the complete set of physical plus auxiliary degrees of freedom, which may be infinite in number. For the relativistic particle in n-dimensional Minkowski space the $q^i(t)$ represent the n coordinates $x^\mu(\lambda)$ plus the auxiliary variable $e(\lambda)$ (sometimes called the 'einbein'), with λ playing the role of time; for the case of a field theory with N fields $\varphi^a(\vec{x};t)$, $a = 1, ..., N$, the $q^i(t)$ represent the infinite set of field amplitudes $\varphi^a_{\vec{x}}(t)$ at fixed location $\vec{x}$ as function of time t, i.e. the dependence on the spatial co-ordinates $\vec{x}$ is included in the labels i. In such a case summation over i is understood to include integration over space.

Assuming the classical dynamical equations to involve at most second-order time derivatives, the action for our system can now be represented quite generally by an integral

$$S[q^i] = \int_1^2 L(q^i, \dot{q}^i)\, dt, \tag{1.31}$$

where in the case of a field theory L itself is to be represented as an integral of some density over space. An arbitrary variation of the co-ordinates leads to a variation of the action of the form

$$\delta S = \int_1^2 dt\, \delta q^i \left(\frac{\partial L}{\partial q^i} - \frac{d}{dt} \frac{\partial L}{\partial \dot{q}^i} \right) + \left[\delta q^i \frac{\partial L}{\partial \dot{q}^i} \right]_1^2, \tag{1.32}$$

with the boundary terms due to an integration by parts. As usual we define generalized canonical momenta as

$$p_i = \frac{\partial L}{\partial \dot{q}^i}. \tag{1.33}$$

From eq.(1.32) two well-known important consequences follow:
- the action is stationary under variations vanishing at initial and final times: $\delta q^i(t_1) = \delta q^i(t_2) = 0$, if the Euler-Lagrange equations are satisfied:

$$\frac{dp_i}{dt} = \frac{d}{dt} \frac{\partial L}{\partial \dot{q}^i} = \frac{\partial L}{\partial q^i}. \tag{1.34}$$

- for arbitrary variations around the classical paths $q^i_c(t)$ in configuration space: $q^i(t) = q^i_c(t) + \delta q^i(t)$, with $q^i_c(t)$ and its associated momentum $p_{c\,i}(t)$ a solution of the Euler-Lagrange equations, the total variation of the action is

$$\delta S_c = \left[\delta q^i(t) p_{c\,i}(t) \right]_1^2. \tag{1.35}$$

We now define an infinitesimal *symmetry* of the action as a set of continuous transformations $\delta q^i(t)$ (smoothly connected to zero) such that the lagrangean L transforms to first order into a total time derivative:

$$\delta L = \delta q^i \frac{\partial L}{\partial q^i} + \delta \dot{q}^i \frac{\partial L}{\partial \dot{q}^i} = \frac{dB}{dt}, \tag{1.36}$$

where B obviously depends in general on the co-ordinates and the velocities, but also on the variation δq^i. It follows immediately from the definition that

$$\delta S = [B]_1^2\,. \tag{1.37}$$

Observe, that according to our definition a symmetry does *not* require the action to be invariant in a strict sense. Now comparing (1.35) and (1.37) we establish the result that, whenever there exists a set of symmetry transformations δq^i, the physical motions of the system satisfy

$$\left[\delta q^i p_{c\,i} - B_c\right]_1^2 = 0. \tag{1.38}$$

Since the initial and final times (t_1, t_2) on the particular orbit are arbitrary, the result can be stated equivalenty in the form of a conservation law for the quantity inside the brackets.

To formulate it more precisely, let the symmetry variations be parametrized by k linearly independent parameters ϵ^α, $\alpha = 1, ..., k$, possibly depending on time:

$$\delta q^i = R^i[\alpha] = \epsilon^\alpha R_\alpha^{(0)i} + \dot{\epsilon}^\alpha R_\alpha^{(1)i} + ... + \overset{(n)}{\epsilon}{}^\alpha R_\alpha^{(n)\,i} + ..., \tag{1.39}$$

where $\overset{(n)}{\epsilon}{}^\alpha$ denotes the nth time derivative of the parameter. Correspondingly, the lagrangean transforms into the derivative of a function $B[\epsilon]$, with

$$B[\epsilon] = \epsilon^\alpha B_\alpha^{(0)} + \dot{\epsilon}^\alpha B_\alpha^{(1)} + ... + \overset{(n)}{\epsilon}{}^\alpha B_\alpha^{(n)} + ... \tag{1.40}$$

With the help of these expressions we define the 'on shell' quantity[2]

$$\begin{aligned} G[\epsilon] &= p_{c\,i} R_c^i[\epsilon] - B_c[\epsilon] \\ &= \epsilon^\alpha G_\alpha^{(0)} + \dot{\epsilon}^\alpha G_\alpha^{(1)} + ... + \overset{(n)}{\epsilon}{}^\alpha G_\alpha^{(n)} + ..., \end{aligned} \tag{1.41}$$

with component by component $G_\alpha^{(n)} = p_{c\,i} R_{c\,\alpha}^{(n)\,i} - B_{c\,\alpha}^{(n)}$. The conservation law (1.38) can now be stated equivalently as

$$\frac{dG[\epsilon]}{dt} = \epsilon^\alpha \dot{G}_\alpha^{(0)} + \dot{\epsilon}^\alpha \left(G_\alpha^{(0)} + \dot{G}_\alpha^{(1)}\right) + ... + \overset{(n)}{\epsilon}{}^\alpha \left(G_\alpha^{(n-1)} + \dot{G}_\alpha^{(n)}\right) + ... = 0. \tag{1.42}$$

We can now distinguish various situations, of which we consider only the two extreme cases here. First, if the symmetry exists only for $\epsilon =$ constant (a *rigid* symmetry), then all time derivatives of ϵ vanish and $G_\alpha^{(n)} \equiv 0$ for $n \geq 1$, whilst for the lowest component

$$G_\alpha^{(0)} = g_\alpha = \text{constant}, \qquad G[\epsilon] = \epsilon^\alpha g_\alpha, \tag{1.43}$$

[2]An 'on shell' quantity is a quantity defined on a classical trajectory.

as defined on a particular classical trajectory (the value of g_α may be different on different trajectories). Thus, rigid symmetries imply constants of motion; this is Noether's theorem.

Second, if the symmetry exists for arbitrary time-dependent $\epsilon(t)$ (a *local* symmetry), then $\epsilon(t)$ and all its time derivatives at the same instant are independent. As a result

$$\begin{aligned}
\dot{G}^{(0)}_\alpha &= 0, \\
\dot{G}^{(1)}_\alpha &= -G^{(0)}_\alpha, \\
&\dots \\
\dot{G}^{(n)}_\alpha &= -G^{(n-1)}_\alpha, \\
&\dots
\end{aligned} \tag{1.44}$$

Now in general the transformations (1.39) do not depend on arbitrarily high-order derivatives of ϵ, but only on a *finite* number of them: there is some finite N such that $R^{(n)}_\alpha = 0$ for $n \geq N$. Typically, transformations depend at most on the first derivative of ϵ, and $R^{(n)}_\alpha = 0$ for $n \geq 2$. In general, for any finite N all quantities $R^{(n)\,i}$, $B^{(n)}$, $G^{(n)}$ then vanish identically for $n \geq N$. But then $G^{(n)}_\alpha = 0$ for $n = 0, ..., N-1$ as well, as a result of eqs.(1.44). Therefore $G[\epsilon] = 0$ at all times. This is a set of *constraints* relating the coordinates and velocities on a classical trajectory. Moreover, as $dG/dt = 0$, these constraints have the nice property that they are preserved during the time-evolution of the system.

The upshot of this analysis is therefore, that local symmetries imply time-independent constraints. This result is sometimes refered to as Noether's second theorem.

Exercise 1.3
Show that if there is no upper limit on the order of derivatives in the transformation rule (no finite N), one reobtains a conservation law

$$G[\epsilon] = g_\alpha\, \epsilon^\alpha(0) = \text{constant}.$$

Hint: show that $G^{(n)}_\alpha = ((-t)^n/n!)\, g_\alpha$, with g_α a constant, and use the Taylor expansion for $\epsilon(0) = \epsilon(t-t)$ around $\epsilon(t)$.

Group structure of symmetries.
To round off our discussion of symmetries, conservation laws and constraints in the lagrangean formalism, we show that symmetry transformations as defined by eq.(1.36) possess an infinitesimal group structure, i.e. they have a closed commutator algebra (a Lie algebra or some generalization thereof). The proof is simple.

First observe, that performing a second variation of δL gives

$$\begin{aligned}\delta_2\delta_1 L &= \delta_2 q^j \delta_1 q^i \frac{\partial^2 L}{\partial q^j \partial q^i} + \delta_2 \dot{q}^j \delta_1 q^i \frac{\partial^2 L}{\partial \dot{q}^j \partial q^i} + (\delta_2\delta_1 q^i)\frac{\partial L}{\partial q^i} \\ &\quad + \delta_2 \dot{q}^j \delta_1 \dot{q}^i \frac{\partial^2 L}{\partial \dot{q}^j \partial \dot{q}^i} + \delta_2 q^j \delta_1 \dot{q}^i \frac{\partial^2 L}{\partial q^j \partial \dot{q}^i} + (\delta_2\delta_1 \dot{q}^i)\frac{\partial L}{\partial \dot{q}^i} = \frac{d(\delta_2 B_1)}{dt}.\end{aligned} \tag{1.45}$$

By antisymmetrization this immediately gives

$$[\delta_2, \delta_1]\, L = \Big([\delta_1, \delta_2]\, q^i\Big) \frac{\partial L}{\partial q^i} + \Big([\delta_2, \delta_1]\, \dot{q}^i\Big) \frac{\partial L}{\partial \dot{q}^i} = \frac{d}{dt}\left(\delta_2 B_1 - \delta_1 B_2\right). \tag{1.46}$$

By assumption of the completeness of the set of symmetry transformations it follows, that there must exist a symmetry transformation

$$\delta_3\, q^i = [\delta_2, \delta_1]\, q^i, \qquad \delta_3\, \dot{q}^i = [\delta_2, \delta_1]\, \dot{q}^i, \tag{1.47}$$

with the property that the associated $B_3 = \delta_2 B_1 - \delta_1 B_2$. Implementing these conditions gives

$$[\delta_2, \delta_1]\, q^i = R_2^j \frac{\partial R_1^i}{\partial q^j} + \dot{q}^k \frac{\partial R_2^j}{\partial q^k} \frac{\partial R_1^i}{\partial \dot{q}^j} + \ddot{q}^k \frac{\partial R_2^j}{\partial \dot{q}^k} \frac{\partial R_1^i}{\partial \dot{q}^j} - [1 \leftrightarrow 2] = R_3^i, \tag{1.48}$$

where we use a condensed notation $R_a^i \equiv R^i[\epsilon_a]$, $a = 1, 2, 3$. In all standard cases, the symmetry transformations $\delta q^i = R^i$ involve only the coordinates and velocities: $R^i = R^i(q, \dot{q})$. Then R_3 can not contain terms proportional to $\ddot{q}$, and the conditions (1.48) reduce to two separate conditions

$$\begin{aligned}&R_2^j \frac{\partial R_1^i}{\partial q^j} - R_1^j \frac{\partial R_2^i}{\partial q^j} + \dot{q}^k \left(\frac{\partial R_2^j}{\partial q^k} \frac{\partial R_1^i}{\partial \dot{q}^j} - \frac{\partial R_1^j}{\partial q^k} \frac{\partial R_2^i}{\partial \dot{q}^j} \right) = R_3^i, \\ &\frac{\partial R_2^j}{\partial \dot{q}^k} \frac{\partial R_1^i}{\partial \dot{q}^j} - \frac{\partial R_1^j}{\partial \dot{q}^k} \frac{\partial R_2^i}{\partial \dot{q}^j} = 0.\end{aligned} \tag{1.49}$$

Clearly, the parameter ϵ_3 of the transformation on the right-hand side must be an antisymmetric bilinear combination of the other two parameters:

$$\epsilon_3^\alpha = f^\alpha(\epsilon_1, \epsilon_2) = -f^\alpha(\epsilon_2, \epsilon_1). \tag{1.50}$$

1.3 Canonical formalism

The canonical formalism describes dynamics in terms of phase-space coordinates (q^i, p_i) and a Hamiltonian $H(q, p)$, starting form an action

$$S_{can}[q, p] = \int_1^2 \Big(p_i \dot{q}^i - H(q, p) \Big)\, dt. \tag{1.51}$$

Variations of the phase-space coordinates change the action to first order by

$$\delta S_{can} = \int_1^2 dt \left[\delta p_i \left(\dot{q}^i - \frac{\partial H}{\partial p_i} \right) - \delta q^i \left(\dot{p}_i + \frac{\partial H}{\partial q^i} \right) + \frac{d}{dt} \left(p_i \delta q^i \right) \right]. \tag{1.52}$$

The action is stationary under variations vanishing at times (t_1, t_2) if Hamilton's equations of motion are satisfied:

$$\dot{p}_i = \frac{\partial H}{\partial q^i}, \qquad \dot{q}^i = -\frac{\partial H}{\partial p_i}. \tag{1.53}$$

This motivates the introduction of the Poisson brackets

$$\{F, G\} = \frac{\partial F}{\partial q^i} \frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i} \frac{\partial G}{\partial q^i}, \tag{1.54}$$

with allow us to write the time derivative of any phase-space function $G(q,p)$ as

$$\dot{G} = \dot{q}^i \frac{\partial G}{\partial q^i} + \dot{p}_i \frac{\partial G}{\partial p_i} = \{G, H\}. \tag{1.55}$$

It follows immediately, that G is a constant of motion if and only if

$$\{G, H\} = 0, \tag{1.56}$$

everywhere along the trajectory of the physical system in phase space. This is guaranteed to be the case if eq.(1.56) holds everywhere in phase space, but as we discuss below, more subtle situations can arise.

Suppose eq.(1.56) is satisfied; then we can construct variations of (q,p) defined by

$$\delta q^i = \left\{ q^i, G \right\} = \frac{\partial G}{\partial p_i}, \qquad \delta p_i = \{p_i, G\} = -\frac{\partial G}{\partial q^i}, \tag{1.57}$$

which leave the Hamiltonian invariant:

$$\delta H = \delta q^i \frac{\partial H}{\partial q^i} + \delta p_i \frac{\partial H}{\partial p_i} = \frac{\partial G}{\partial p_i} \frac{\partial H}{\partial q^i} - \frac{\partial G}{\partial q^i} \frac{\partial H}{\partial p_i} = \{H, G\} = 0. \tag{1.58}$$

They represent infinitesimal symmetries of the theory provided eq.(1.56), and hence (1.58), is satisfied as an identity, irrespective of whether or not the phase-space coordinates (q,p) satisfy the equations of motion. To see this, consider the variation of the action (1.52) with $(\delta q, \delta p)$ given by (1.57) and $\delta H = 0$ by (1.58):

$$\delta S_{can} = \int_1^2 dt \left[-\frac{\partial G}{\partial q^i} \dot{q}^i - \frac{\partial G}{\partial p_i} \dot{p}_i + \frac{d}{dt} \left(\frac{\partial G}{\partial p_i} p_i \right) \right] = \int_1^2 dt \frac{d}{dt} \left(\frac{\partial G}{\partial p_i} p_i - G \right). \tag{1.59}$$

If we call the quantity inside the parentheses $B(q,p)$, then we have rederived eqs.(1.37) and (1.38); indeed, we then have

$$G = \frac{\partial G}{\partial p_i} p_i - B = \delta q^i p_i - B, \tag{1.60}$$

where we know from eq.(1.55), that G is a constant of motion on classical trajectories (on which Hamilton's equation of motion are satisfied). Observe that, whereas in the lagrangean approach we showed that symmetries imply constants of motion, here we have derived the inverse Noether theorem: constants of motion generate symmetries. An advantage of this derivation over the lagrangean one is, that we have also found explicit expressions for the variations $(\delta q, \delta p)$.

A further advantage is, that the infinitesimal group structure of the tranformations (the commutator algebra) can be checked directly. Indeed, if two symmetry generators G_α and G_β both satisfy (1.56), then the Jacobi identity for Poisson brackets implies

$$\{\{G_\alpha, G_\beta\}, H\} = \{G_\alpha, \{G_\beta, H\}\} - \{G_\beta, \{G_\alpha, H\}\} = 0. \tag{1.61}$$

Hence if the set of generators $\{G_\alpha\}$ is complete, we must have an identity of the form

$$\{G_\alpha, G_\beta\} = P_{\alpha\beta}(G) = -P_{\beta\alpha}(G), \tag{1.62}$$

where the $P_{\alpha\beta}(G)$ are polynomials in the constants of motion G_α:

$$P_{\alpha\beta}(G) = c_{\alpha\beta} + f_{\alpha\beta}^{\;\;\gamma}\, G_\gamma + \frac{1}{2}\, g_{\alpha\beta}^{\;\;\gamma\delta} G_\gamma G_\delta + ... \tag{1.63}$$

The coefficients $c_{\alpha\beta}$, $f_{\alpha\beta}^{\;\;\gamma}$, $g_{\alpha\beta}^{\;\;\gamma\delta}$, ... are constants, having zero Poisson brackets with any phase-space function. As such the first term $c_{\alpha\beta}$ may be called a central charge.

It now follows that the transformation of any phase-space function $F(q,p)$, given by

$$\delta_\alpha F = \{F, G_\alpha\}, \tag{1.64}$$

satisfies the commutation relation

$$\begin{aligned} [\delta_\alpha, \delta_\beta]\, F &= \{\{F, G_\beta\}, G_\alpha\} - \{\{F, G_\alpha\}, G_\beta\} = \{F, \{G_\beta, G_\alpha\}\} \\ &= C_{\beta\alpha}^{\;\;\gamma}(G)\; \delta_\gamma F, \end{aligned} \tag{1.65}$$

where we have introduced the notation

$$C_{\beta\alpha}^{\;\;\gamma}(G) = \frac{\partial P_{\beta\alpha}(G)}{\partial G_\gamma} = f_{\alpha\beta}^{\;\;\gamma} + g_{\alpha\beta}^{\;\;\gamma\delta} G_\delta + ... \tag{1.66}$$

In particular this holds for the coordinates and momenta (q,p) themselves; taking F to be another constraint G_γ, we find from the Jacobi identity for Poisson brackets the consistency condition

$$C_{[\alpha\beta}^{\;\;\delta}\, P_{\gamma]\delta} = f_{[\alpha\beta}^{\;\;\delta}\, c_{\gamma]\delta} + \left(f_{[\alpha\beta}^{\;\;\delta}\, f_{\gamma]\delta}^{\;\;\varepsilon} + g_{[\alpha\beta}^{\;\;\delta\epsilon} c_{\gamma]\delta} \right) G_\varepsilon + ... = 0. \tag{1.67}$$

By the same arguments as in sect. 1.2 (eq.(1.41 and following) it is established, that whenever the theory generated by G_α is a *local* symmetry with time-dependent parameters, the generator G_α turns into a constraint:

$$G_\alpha(q,p) = 0. \tag{1.68}$$

However, compared to the case of rigid symmetries, a subtlety now arises: the constraints $G_\alpha = 0$ define a hypersurface in the phase space to which all physical trajectories of the system are confined. This implies, that it is sufficient for the constraints to commute with the hamiltonian (in the sense of Poisson brackets) on the physical hypersurface (i.e., *on shell*). Off the hypersurface (*off shell*), the bracket of the hamiltonian with the constraints can be anything, as the physical trajectories never enter this part of phase space. Thus the most general allowed algebraic structure defined by the hamiltonian and constraints is

$$\{G_\alpha, G_\beta\} = P_{\alpha\beta}(G), \qquad \{H, G_\alpha\} = Z_\alpha(G), \tag{1.69}$$

where both $P_{\alpha\beta}(G)$ and $Z_\alpha(G)$ are polynomials in the constraints with the property that $P_{\alpha\beta}(0) = Z_\alpha(0) = 0$. This is sufficient to guarantee that in the physical sector of the phase space $\{H, G_\alpha\}|_{G=0} = 0$. Note, that in the case of local symmetries with generators G_α defining constraints, the central charge in the bracket of the constraints must vanish: $c_{\alpha\beta} = 0$. This is a genuine restriction on the existence of local symmetries. A dynamical system with constraints and hamiltonian satisfying eqs.(1.69) is said to be *first class*. Actually, it is quite easy to see that the general first-class algebra of Poisson brackets is more appropriate for systems with local symmetries. Namely, even if the brackets of the constraints and the hamiltonian genuinely vanishes on and off shell, one can always change the hamiltonian of the system by adding a polynomial in the constraints:

$$H' = H + R(G), \qquad R(G) = \rho_0 + \rho_1^\alpha G_\alpha + \frac{1}{2}\,\rho_2^{\alpha\beta} G_\alpha G_\beta + ... \tag{1.70}$$

This leaves the hamiltonian on the physical shell in phase space invariant (up to a constant ρ_0), and therefore the physical trajectories remain the same. Furthermore, even if $\{H, G_\alpha\} = 0$, the new hamiltonian satisfies

$$\{H', G_\alpha\} = \{R(G), G_\alpha\} = Z_\alpha^{(R)}(G) \equiv \rho_1^\beta P_{\beta\alpha}(G) + ..., \tag{1.71}$$

which is of the form (1.69). In addition the equations of motion for the variables (q,p) are changed by a local symmetry transformation only, as

$$(\dot{q^i})' = \left\{q^i, H'\right\} = \left\{q^i, H\right\} + \left\{q^i, G_\alpha\right\} \frac{\partial R}{\partial G_\alpha} = \dot{q}^i + \varepsilon^\alpha \delta_\alpha q^i, \tag{1.72}$$

where ε^α are some —possibly complicated— local functions which may depend on the phase-space coordinates (q,p) themselves. A similar observation holds of course for the momenta p_i. We can actually allow the coefficients $\rho_1^\alpha, \rho_2^{\alpha\beta}, ...$ to be space-time dependent variables themselves, as this does not change the general form of the equations of motion (1.72), whilst variation of the action w.r.t. these new variables will only impose the constraints as equations of motion:

$$\frac{\delta S}{\delta \rho_1^\alpha} = G_\alpha(q, p) = 0, \tag{1.73}$$

in agreement with the dynamics already established.

The same argument shows however, that the part of the hamiltonian depending on the constraints in not unique, and may be changed by terms like $R(G)$. In many cases this allows one to get rid of all or part of $h_\alpha(G)$.

1.4 Quantum dynamics

In quantum dynamics in the canonical operator formalism, one can follow largely the same lines of argument as presented for classical theories in sect. 1.3. Consider a theory of canonical pairs of operators $(\hat{q}, \hat{p})$ with commutation relations

$$\left[\hat{q}^{\,i}, \hat{p}_j\right] = i\delta_j^i, \tag{1.74}$$

and hamiltonian $\hat{H}(\hat{q}, \hat{p})$ such that

$$i\frac{d\hat{q}^{\,i}}{dt} = \left[\hat{q}^{\,i}, \hat{H}\right], \qquad i\frac{d\hat{p}_i}{dt} = \left[\hat{p}_i, \hat{H}\right]. \tag{1.75}$$

The δ-symbol on the right-hand side of (1.74) is to be interpreted in a generalized sense: for continuous parameters (i, j) it represents a Dirac delta-function rather than a Kronecker delta.

In the context of quantum theory, constants of motion become operators $\hat{G}$ which commute with the hamiltonian:

$$\left[\hat{G}, \hat{H}\right] = i\frac{d\hat{G}}{dt} = 0, \tag{1.76}$$

and can therefore be diagonalized on stationary eigenstates. We henceforth assume we have at our disposal a complete set $\{\hat{G}_\alpha\}$ of such constants of motion, in the sense that any operator satisfying (1.76) can be expanded as a polynomial in the operators $\hat{G}_\alpha$.

In analogy to the classical theory, we define infinitesimal symmetry transformations by

$$\delta_\alpha \hat{q}^i = -i\left[\hat{q}^i, \hat{G}_\alpha\right], \qquad \delta_\alpha \hat{p}_i = -i\left[\hat{p}_i, \hat{G}_\alpha\right]. \tag{1.77}$$

By construction they have the property of leaving the hamiltonian invariant:

$$\delta_\alpha \hat{H} = -i\left[\hat{H}, \hat{G}_\alpha\right] = 0. \tag{1.78}$$

Therefore the operators $\hat{G}_\alpha$ are also called symmetry generators. It follows by the Jacobi identity, analogous to eq.(1.61), that the commutator of two such generators commutes again with the hamiltonian, and therefore

$$-i\left[\hat{G}_\alpha, \hat{G}_\beta\right] = P_{\alpha\beta}(\hat{G}) = c_{\alpha\beta} + f_{\alpha\beta}^{\;\;\;\gamma}\,\hat{G}_\gamma + ... \tag{1.79}$$

A calculation along the lines of (1.65) then shows, that for any operator $\hat{F}(\hat{q},\hat{p})$ one has

$$\delta_\alpha \hat{F} = -i\left[\hat{F}, \hat{G}_\alpha\right], \qquad \left[\delta_\alpha, \delta_\beta\right]\hat{F} = i f_{\alpha\beta}^{\;\;\;\gamma}\,\delta_\gamma \hat{F} + ... \tag{1.80}$$

Observe, that compared to the classical theory, in the quantum theory there is an additional potential source for the appearance of central charges in (1.79), to wit the operator ordering on the right-hand side. As a result, even when no central charge is present in the classical theory, such central charges can arise in the quantum theory. This is a source of anomalous behaviour of symmetries in quantum theory.

As in the classical theory, local symmetries impose additional restrictions; if a symmetry generator $\hat{G}[\epsilon]$ involves time-dependent parameters $\epsilon^a(t)$, then its evolution equation (1.76) is modified to:

$$i\frac{d\hat{G}[\epsilon]}{dt} = \left[\hat{G}[\epsilon], \hat{H}\right] + i\,\frac{\partial \hat{G}[\epsilon]}{\partial t}, \tag{1.81}$$

where

$$\frac{\partial \hat{G}[\epsilon]}{\partial t} = \frac{\partial \epsilon^a}{\partial t}\,\frac{\delta \hat{G}[\epsilon]}{\delta \epsilon^a}. \tag{1.82}$$

It follows, that $\hat{G}[\epsilon]$ can generate symmetries of the hamiltonian and be conserved at the same time for arbitrary $\epsilon^a(t)$ only if the functional derivative vanishes:

$$\frac{\delta \hat{G}[\epsilon]}{\delta \epsilon^a(t)} = 0, \tag{1.83}$$

which defines a set of operator constraints, the quantum equivalent of (1.44). The important step in this argument is to realize, that the transformation properties of the evolution operator should be consistent with the Schrödinger equation, which can be true only if both conditions (symmetry and conservation law) hold. To see this, recall that the evolution operator

$$\hat{U}(t,t') = e^{-i(t-t')\hat{H}}, \tag{1.84}$$

is the formal solution of the Schrödinger equation

$$\left(i\frac{\partial}{\partial t} - \hat{H}\right)\hat{U} = 0, \tag{1.85}$$

satisfying the initial condition $\hat{U}(t,t) = \hat{1}$. Now under a symmetry transformation (1.77), (1.80) this equation transforms into

$$\begin{aligned} \delta\left[\left(i\frac{\partial}{\partial t} - \hat{H}\right)\hat{U}\right] &= -i\left[\left(i\frac{\partial}{\partial t} - \hat{H}\right)\hat{U}, \hat{G}[\epsilon]\right] \\ &= -i\left(i\frac{\partial}{\partial t} - \hat{H}\right)\left[\hat{U}, \hat{G}[\epsilon]\right] - i\left[\left(i\frac{\partial}{\partial t} - \hat{H}\right), \hat{G}[\epsilon]\right]\hat{U} \end{aligned} \tag{1.86}$$

For the transformations to respect the Schrödinger equation, the left-hand side of this identity must vanish, hence so must the right-hand side. But the right-hand side vanishes for arbitrary $\epsilon(t)$ if and only if both conditions are met:

$$\left[\hat{H}, \hat{G}[\epsilon]\right] = 0, \quad \text{and} \quad \frac{\partial \hat{G}[\epsilon]}{\partial t} = 0.$$

This is what we set out to prove. Of course, like in the classical hamiltonian formulation, we realize that for generators of local symmetries a more general first-class algebra of commutation relations is allowed, along the lines of eqs.(1.69). Also here, the hamiltonian may then be modified by terms involving only the constraints and, possibly, corresponding lagrange multipliers. The discussion parallels that for the classical case.

1.5 The relativistic particle

In this section and the next we revisit the two examples of constrained systems in sect. 1.1 to illustrate the general principles of symmetries, conservation laws and constraints above. First we consider the relativistic particle.

The starting point of the analysis is the action (1.8):

$$S[x^\mu; e] = \frac{m}{2}\int_1^2 \left(\frac{1}{e}\frac{dx_\mu}{d\lambda}\frac{dx^\mu}{d\lambda} - ec^2\right) d\lambda.$$

Here λ plays the role of system time, and the hamiltonian we construct is the one generating time-evolution in this sense. The canonical momenta are given by

$$p_\mu = \frac{\delta S}{\delta(dx^\mu/d\lambda)} = \frac{m}{e}\frac{dx_\mu}{d\lambda}, \qquad p_e = \frac{\delta S}{\delta(de/d\lambda)} = 0. \tag{1.87}$$

The second equation is a constraint on the extended phase space spanned by the canonical pairs $(x^\mu, p_\mu; e, p_e)$. Next we perform a legendre transformation to obtain the hamiltonian

$$H = \frac{e}{2m}\left(p^2 + m^2c^2\right) + p_e\frac{de}{d\lambda}. \tag{1.88}$$

The last term obviously vanishes upon application of the constraint $p_e = 0$. The canonical (hamiltonian) action now reads

$$S_{can} = \int_1^2 d\lambda \left(p_\mu \frac{dx^\mu}{d\lambda} - \frac{e}{2m} \left(p^2 + m^2 c^2 \right) \right). \tag{1.89}$$

Observe, that the dependence on p_e has dropped out, irrespective of whether we constrain it to vanish or not. The role of the einbein is now clear: it is a lagrange multiplier imposing the dynamical constraint (1.7):

$$p^2 + m^2 c^2 = 0.$$

Note, that in combination with $p_e = 0$, this constraint implies $H = 0$, i.e. the hamiltonian consists *only* of a polynomial in the constraints. This is a general feature of systems with reparametrization invariance, including for example the theory of relativistic strings and general relativity.

In the example of the relativistic particle, we immediately encounter a generic phenomenon: any time we have a constraint on the dynamical variables imposed by a lagrange multiplier (here: e), its associated momentum (here: p_e) is constrained to vanish. It has been shown in a quite general context, that one may always reformulate hamiltonian theories with constraints such that all constraints appear with lagrange multipliers [16]; therefore this pairing of constraints is a generic feature in hamiltonian dynamics. However, as we have already discussed in sect. 1.3, such lagrange multiplier terms do not affect the dynamics, and the multipliers as well as their associated momenta can be eliminated from the physical hamiltonian.

The non-vanishing Poisson brackets of the theory, including the lagrange multipliers, are

$$\{x^\mu, p_\nu\} = \delta^\mu_\nu, \qquad \{e, p_e\} = 1. \tag{1.90}$$

As follows from the hamiltonian treatment, all equations of motion for any quantity $\Phi(x, p; e, p_e)$ can then be obtained from a Poisson bracket with the hamiltonian:

$$\frac{d\Phi}{d\lambda} = \{\Phi, H\}, \tag{1.91}$$

although this equation does not imply any non-trivial information on the dynamics of the lagrange multipliers. Nevertheless, in this formulation of the theory it must be assumed *a priori* that (e, p_e) are allowed to vary; the dynamics can be projected to the hypersurface $p_e = 0$ only after computing Poisson brackets. The alternative is to work with a restricted phase space spanned only by the physical co-ordinates and momenta (x^μ, p_μ). This is achieved by performing a Legendre transformation only with respect to the physical velocities[3]. We first explore the formulation of the theory in the extended phase space.

[3] This is basically a variant of Routh's procedure; see e.g. Goldstein [15], ch. 7.

All possible symmetries of the theory can be determined by solving eq.(1.56):

$$\{G, H\} = 0.$$

Among the solutions we find the generators of the Poincaré group: translations p_μ and Lorentz transformations $M_{\mu\nu} = x_\nu p_\mu - x_\mu p_\nu$. Indeed, the combination of generators

$$G[\epsilon] = \epsilon^\mu p_\mu + \frac{1}{2}\,\epsilon^{\mu\nu} M_{\mu\nu}. \tag{1.92}$$

with constant $(\epsilon^\mu, \epsilon^{\mu\nu})$ produces the expected infinitesimal transformations

$$\delta x^\mu = \{x^\mu, G[\epsilon]\} = \epsilon^\mu + \epsilon^\mu_{\ \ \nu}\, x^\nu, \qquad \delta p_\mu = \{p_\mu, G[\epsilon]\} = \epsilon_\mu^{\ \ \nu}\, p_\nu. \tag{1.93}$$

The commutator algebra of these transformations is well-known to be closed: it is the Lie algebra of the Poincaré group.

Exercise 1.4
Check that the bracket of $G[\epsilon]$ and the hamiltonian H vanishes. Compute the bracket of two Poincaré transformations $G[\epsilon_1]$ and $G[\epsilon_2]$.

For the generation of constraints the local reparametrization invariance of the theory is the one of interest. The infinitesimal form of the transformations (1.10) is obtained by taking $\lambda' = \lambda - \epsilon(\lambda)$, with the result

$$\begin{aligned} \delta x^\mu &= x'^\mu(\lambda) - x^\mu(\lambda) = \epsilon\,\frac{dx^\mu}{d\lambda}, \qquad \delta p_\mu = \epsilon\,\frac{dp_\mu}{d\lambda}, \\ \delta e &= e'(\lambda) - e(\lambda) = \frac{d(e\epsilon)}{d\lambda}. \end{aligned} \tag{1.94}$$

Now recall, that $ed\lambda = d\tau$ is a reparametrization-invariant form. Furthermore, $\epsilon(\lambda)$ is an arbitrary local function of λ. It follows, that without loss of generality we can consider an equivalent set of *covariant* transformations with parameter $\sigma = e\epsilon$:

$$\begin{aligned} \delta_{cov}\, x^\mu &= \frac{\sigma}{e}\,\frac{dx^\mu}{d\lambda}, \qquad \delta_{cov}\, p_\mu = \frac{\sigma}{e}\,\frac{dp_\mu}{d\lambda}, \\ \delta_{cov}\, e &= \frac{d\sigma}{d\lambda}. \end{aligned} \tag{1.95}$$

It is straightforward to check, that under these transformations the canonical lagrangean (the integrand of (1.89)) transforms into a total derivative, and $\delta_{cov}\, S_{can} = [B_{cov}]_1^2$, with

$$B_{cov}[\sigma] = \sigma\left(p_\mu\,\frac{dx^\mu}{ed\lambda} - \frac{1}{2m}(p^2 + m^2c^2)\right). \tag{1.96}$$

Using eq.(1.60), we find that the generator of the local transformations (1.94) is given by

$$G_{cov}[\sigma] = (\delta_{cov} x^\mu) p_\mu + (\delta_{cov} e) p_e - B_{cov} = \frac{\sigma}{2m} \left(p^2 + m^2 c^2 \right) + p_e \frac{d\sigma}{d\lambda}. \qquad (1.97)$$

It is easily verified, that $dG_{cov}/d\lambda = 0$ on physical trajectories for arbitrary $\sigma(\lambda)$ if and only if the two earlier constraints are satisfied at all times:

$$p^2 + m^2 c^2 = 0, \qquad p_e = 0. \qquad (1.98)$$

It is clear that the Poissonbrackets of these constraints among themselves vanish. On the canonical variables, G_{cov} generates the transformations

$$\begin{aligned} \delta_G \, x^\mu &= \{ x^\mu, G_{cov}[\sigma] \} = \frac{\sigma p^\mu}{m}, \quad \delta_G \, p_\mu = \{ p_\mu, G_{cov}[\sigma] \} = 0, \\ \delta_G \, e &= \{ e, G_{cov}[\sigma] \} = \frac{d\sigma}{d\lambda}, \qquad \delta_G \, p_e = \{ p_e, G_{cov}[\sigma] \} = 0. \end{aligned} \qquad (1.99)$$

These transformation rules actually differ from the original ones, eq.(1.95). However, all the differences vanish when applying the equations of motion:

$$\begin{aligned} \delta' x^\mu &= (\delta_{cov} - \delta_G) x^\mu = \frac{\sigma}{m} \left(\frac{m}{e} \frac{dx^\mu}{d\lambda} - p^\mu \right) \approx 0, \\ \delta' p_\mu &= (\delta_{cov} - \delta_G) p_\mu = \frac{\sigma}{e} \frac{dp_\mu}{d\lambda} \approx 0. \end{aligned} \qquad (1.100)$$

The transformations δ' are in fact themselves symmetry transformations of the canonical action, but of a trivial kind: as they vanish on shell, they do not imply any conservation laws or constraints [17]. Therefore the new transformations δ_G are physically equivalent to δ_{cov}.

The upshot of this analysis is, that we can describe the relativistic particle by the hamiltonian (1.88) and the Poisson brackets (1.90), provided we impose on all physical quantities in phase space the constraints (1.98).

A few comments are in order. First, the hamiltonian is by construction the generator of translations in the time coordinate (here: λ); therefore after the general exposure in sects. 1.2 and 1.3 it should not come as a surprise, that when promoting such translations to a local symmetry, the hamiltonian is constrained to vanish.

Secondly, we briefly discuss the other canonical procedure, which takes directly advantage of the the local parametrization invariance (1.10) by using it to fix the einbein; in particular the choice $e = 1$ leads to the identification of λ with proper time: $d\tau = e d\lambda \rightarrow d\tau = d\lambda$. This procedure is called gauge fixing. Now the canonical action becomes simply

$$S_{can}|_{e=1} = \int_1^2 d\tau \left(p \cdot \dot{x} - \frac{1}{2m} \left(p^2 + m^2 c^2 \right) \right). \qquad (1.101)$$

This is a regular action for a hamiltonian system. It is completely Lorentz covariant, only the local reparametrization invariance is lost. As a result, the constraint $p^2 + m^2c^2 = 0$ can no longer be derived from the action; it must now be imposed separately as an external condition. Because we have fixed e, we do not need to introduce its conjugate momentum p_e, and we can work in a restricted physical phase space spanned by the canonical pairs (x^μ, p_μ). Thus, a second consistent way to formulate classical hamiltonian dynamics for the relativistic particle is to use the gauge-fixed hamiltonian and Poisson brackets

$$H_f = \frac{1}{2m}\left(p^2 + m^2c^2\right), \qquad \{x^\mu, p_\nu\} = \delta^\mu_\nu, \tag{1.102}$$

whilst adding the constraint $H_f = 0$ to be satisfied at all (proper) times. Observe, that the remaining constraint implies that one of the momenta p_μ is not independent:

$$p_0^2 = \vec{p}^{\,2} + m^2c^2. \tag{1.103}$$

As this defines a hypersurface in the restricted phase space, the dimensionality of the physical phase space is reduced even further. To deal with this situation, we can again follow two different routes; the first one is to solve the constraint and work in a reduced phase space. The standard procedure for this is to introduce light-cone coordinates $x^\pm = (x^0 \pm x^3)/\sqrt{2}$, with canonically conjugate momenta $p_\pm = (p_0 \pm p_3)/\sqrt{2}$, such that

$$\left\{x^\pm, p_\pm\right\} = 1, \qquad \left\{x^\pm, p_\mp\right\} = 0. \tag{1.104}$$

The constraint (1.103) can then be written

$$2p_+p_- = p_1^2 + p_2^2 + m^2c^2, \tag{1.105}$$

which allows us to eliminate the light-cone co-ordinate x_- and its conjugate momentum $p_- = (p_1^2 + p_2^2 + m^2c^2)/2p_+$. Of course, by this procedure the manifest Lorentz-covariance of the model is lost. Therefore one often prefers an alternative route: to work in the covariant phase space (1.102), and impose the constraint on physical phase space functions only after solving the dynamical equations.

1.6 The electro-magnetic field

The second example to be considered here is the electro-magnetic field. As our starting point we take the action of exercise 1.2, which is the action of eq.(1.18) modified by a total time-derivative, in which the magnetic field has been written in terms of the vector potential as $\vec{B}(A) = \vec{\nabla} \times \vec{A}$:

$$
\begin{aligned}
S_{em}[\phi, \vec{A}, \vec{E}] &= \int_1^2 dt\, L_{em}(\phi, \vec{A}, \vec{E}), \\
L_{em} &= \int d^3x \left(-\frac{1}{2} \left(\vec{E}^2 + [\vec{B}(A)]^2 \right) - \phi\, \vec{\nabla} \cdot \vec{E} - \vec{E} \cdot \frac{\partial \vec{A}}{\partial t} \right)
\end{aligned}
\tag{1.106}
$$

It is clear, that $(\vec{A}, -\vec{E})$ are canonically conjugate; by adding the time derivative we have chosen to let $\vec{A}$ play the role of co-ordinates, whilst the components of $-\vec{E}$ represent the momenta:

$$
\vec{\pi}_A = -\vec{E} = \frac{\delta S_{em}}{\delta(\partial \vec{A}/\partial t)} \tag{1.107}
$$

Also, like the einbein in the case of the relativistic particle, here the scalar potential $\phi = A_0$ plays the role of lagrange multiplier to impose the constraint $\vec{\nabla} \cdot \vec{E} = 0$; therefore its canonical momentum vanishes:

$$
\pi_\phi = \frac{\delta S_{em}}{\delta(\partial \phi/\partial t)} = 0. \tag{1.108}
$$

This is the generic type of constraint for lagrange multipliers, which we encountered also in the case of the relativistic particle. Observe, that the lagrangean (1.106) is already in the canonical form, with the hamiltonian given by

$$
H_{em} = \int d^3x \left(\frac{1}{2} \left(\vec{E}^2 + \vec{B}^2 \right) + \phi\, \vec{\nabla} \cdot \vec{E} + \pi_\phi \frac{\partial \phi}{\partial t} \right). \tag{1.109}
$$

Again, as in the case of the relativistic particle, the last term can be taken to vanish upon imposing the constraint (1.108), but in any case it cancels in the canonical action

$$
\begin{aligned}
S_{em} &= \int_1^2 dt \left(\int d^3x \left[-\vec{E} \cdot \frac{\partial \vec{A}}{\partial t} + \pi_\phi \frac{\partial \phi}{\partial t} \right] - H(\vec{E}, \vec{A}, \pi_\phi, \phi) \right) \\
&= \int_1^2 dt \left(\int d^3x \left[-\vec{E} \cdot \frac{\partial \vec{A}}{\partial t} \right] - H(\vec{E}, \vec{A}, \phi)|_{\pi_\phi = 0} \right)
\end{aligned}
\tag{1.110}
$$

To proceed with the canonical analysis, we have the same choice as in the case of the particle: to keep the full hamiltonian, and include the canonical pair (ϕ, π_ϕ) in an extended phase space; or to use the local gauge invariance to remove ϕ by fixing it at some particular value.

In the first case we have to introduce Poisson brackets

$$
\{A_i(\vec{x}, t), E_j(\vec{y}, t)\} = -\delta_{ij}\, \delta^3(\vec{x} - \vec{y}), \qquad \{\phi(\vec{x}, t), \pi_\phi(\vec{y}, t)\} = \delta^3(\vec{x} - \vec{y}). \tag{1.111}
$$

It is straightforward to check, that the Maxwell equations are reproduced by the brackets with the hamiltonian:

$$\dot{\Phi} = \left\{ \Phi, H \right\}, \tag{1.112}$$

where Φ stands for any of the fields $(\vec{A}, \vec{E}, \phi, \pi_\phi)$ above, although in the sector of the scalar potential the equations are empty of dynamical content.

Among the quantities commuting with the hamiltonian (in the sense of Poisson brackets), the most interesting for our purpose is the generator of the gauge transformations

$$\delta \vec{A} = \vec{\nabla} \Lambda, \qquad \delta \phi = \frac{\partial \Lambda}{\partial t}, \qquad \delta \vec{E} = \delta \vec{B} = 0. \tag{1.113}$$

Its construction proceeds according to eq.(1.60). Actually, the action (1.106) is gauge invariant provided the gauge parameter vanishes sufficiently fast at spatial infinity, as $\delta L_{em} = - \int d^3 x \, \vec{\nabla} \cdot (\vec{E} \partial \Lambda / \partial t)$. Therefore the generator of the gauge transformations is

$$\begin{aligned} G[\Lambda] &= \int d^3 x \left(- \delta \vec{A} \cdot \vec{E} + \delta \phi \, \pi_\phi \right) \\ &= \int d^3 x \left(- \vec{E} \cdot \vec{\nabla} \Lambda + \pi_\phi \frac{\partial \Lambda}{\partial t} \right) = \int d^3 x \left(\Lambda \vec{\nabla} \cdot \vec{E} + \pi_\phi \frac{\partial \Lambda}{\partial t} \right). \end{aligned} \tag{1.114}$$

The gauge transformations (1.113) are reproduced by the Poisson brackets

$$\delta \Phi = \left\{ \Phi, G[\Lambda] \right\}. \tag{1.115}$$

From the result (1.114) it follows, that conservation of $G[\Lambda]$ for arbitrary $\Lambda(\vec{x}, t)$ is due to the constraints

$$\vec{\nabla} \cdot \vec{E} = 0, \qquad \pi_\phi = 0, \tag{1.116}$$

which are necessary and sufficient. These in turn imply that $G[\Lambda] = 0$ itself.

One reason why this treatment might be prefered, is that in a relativistic notation $\phi = A_0$, $\pi_\phi = \pi^0$, the brackets (1.111) take the quasi-covariant form

$$\left\{ A_\mu(\vec{x}, t), \pi^\nu(\vec{y}, t) \right\} = \delta_\mu^\nu \, \delta^3(\vec{x} - \vec{y}), \tag{1.117}$$

and similarly for the generator of the gauge transformations :

$$G[\Lambda] = - \int d^3 x \, \pi^\mu \partial_\mu \Lambda. \tag{1.118}$$

Of course, the three-dimensional δ-function and integral show, that the covariance of these equations is not complete.

The other procedure one can follow, is to use the gauge invariance to set $\phi = \phi_0$, a constant. Without loss of generality this constant can be chosen equal to zero, which just amounts to fixing the zero of the electric potential. In any case, the term $\phi\,\vec{\nabla}\cdot\vec{E}$ vanishes from the action and for the dynamics it suffices to work in the reduced phase space spanned by $(\vec{A},\vec{E})$. In particular, the hamiltonian and Poisson brackets reduce to

$$H_{red} = \int d^3x\, \frac{1}{2}\left(\vec{E}^2 + \vec{B}^2\right), \qquad \{A_i(\vec{x},t), E_j(\vec{y},t)\} = -\delta_{ij}\delta^3(\vec{x}-\vec{y}). \quad (1.119)$$

The constraint $\vec{\nabla}\cdot\vec{E} = 0$ is no longer a consequence of the dynamics, but has to be imposed separately. Of course, its bracket with the hamiltonian still vanishes: $\{H_{red}, \vec{\nabla}\cdot\vec{E}\} = 0$. The constraint actually signifies that one of the components of the canonical momenta (in fact an infinite set: the longitudinal electric field at each point in space) is to vanish; therefore the dimensionality of the physical phase space is again reduced by the constraint. As the constraint is preserved in time (its Poisson bracket with H vanishes), this reduction is consistent. Again, there are two options to proceed: solve the constraint and obtain a phase space spanned by the physical degrees of freedom only, or keep the constraint as a separate condition to be imposed on all solutions of the dynamics. The explicit solution in this case consists of splitting the electric field in transverse and longitudinal parts by projection operators:

$$\vec{E} = \vec{E}_T + \vec{E}_L = \left(1 - \vec{\nabla}\frac{1}{\Delta}\vec{\nabla}\right)\cdot\vec{E} + \vec{\nabla}\frac{1}{\Delta}\vec{\nabla}\cdot\vec{E}, \quad (1.120)$$

and similarly for the vector potential. One can now restrict the phase space to the transverse parts of the fields only; this is equivalent to requiring $\vec{\nabla}\cdot\vec{E} = 0$ and $\vec{\nabla}\cdot\vec{A} = 0$ simultaneously. In practice it is much more convenient to use these constraints as such in computing physical observables, instead of projecting out the longitudinal components explicitly at all intermediate stages. Of course, one then has to check that the final result does not depend on any arbitrary choice of dynamics attributed to the longitudinal fields.

1.7 Yang-Mills theory

Yang-Mills theory is an important extension of Maxwell theory, with a very similar canonical structure. The covariant action is a direct extension of the electromagnetic action in exercise 1.2:

$$S_{YM} = -\frac{1}{4}\int d^4x\, F_{\mu\nu}^a F_a^{\mu\nu}, \quad (1.121)$$

where $F_{\mu\nu}^a$ is the field strength of the Yang-Mills vector potential A_μ^a:

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a - g f_{bc}{}^a A_\mu^b A_\nu^c. \quad (1.122)$$

Here g is the coupling constant, and the coefficients ${f_{bc}}^a$ are the structure constant of a compact Lie algebra g with (anti-hermitean) generators $\{T_a\}$:

$$[T_a, T_b] = {f_{ab}}^c\, T_c. \tag{1.123}$$

The Yang-Mills action (1.121) is invariant under (infinitesimal) local gauge transformations with parameters $\Lambda^a(x)$:

$$\delta A_\mu^a = (D_\mu \Lambda)^a = \partial_\mu \Lambda^a - g{f_{bc}}^a A_\mu^b \Lambda^c, \tag{1.124}$$

under which the field strength $F_{\mu\nu}^a$ transforms as

$$\delta F_{\mu\nu}^a = g{f_{bc}}^a \Lambda^b F_{\mu\nu}^c. \tag{1.125}$$

To obtain a canonical description of the theory, we compute the momenta

$$\pi_a^\mu = \frac{\delta S_{YM}}{\delta \partial_0 A_\mu^a} = -F_a^{0\mu} = \left\{ \begin{array}{cl} -E_a^i, & \mu = i = (1,2,3); \\ 0, & \mu = 0. \end{array} \right. \tag{1.126}$$

Clearly, the last equation is a constraint of the type we have encountered before; indeed, the time component of the vector field, A_0^a, plays the same role of lagrange mutiplier for a Gauss-type constraint as the scalar potential $\phi = A_0$ in electrodynamics, to which the theory reduces in the limit $g \rightarrow 0$. This is brought out most clearly in the hamiltonian formulation of the theory, with action

$$\begin{aligned} S_{YM} &= \int_1^2 dt \left(\int d^3x \left[-\vec{E}_a \cdot \frac{\partial \vec{A}^a}{\partial t} \right] - H_{YM} \right), \\ H_{YM} &= \int d^3x \left(\frac{1}{2} (\vec{E}_a^2 + \vec{B}_a^2) + A_0^a\, (\vec{D} \cdot \vec{E})_a \right). \end{aligned} \tag{1.127}$$

Here we have introduced the notation $\vec{B}^a$ for the magnetic components of the field strength:

$$B_i^a = \frac{1}{2}\, \varepsilon_{ijk} F_{jk}^a. \tag{1.128}$$

In eqs.(1.127) we have left out all terms involving the time-component of the momentum, since they vanish as a result of the constraint $\pi_a^0 = 0$, eq.(1.126). Now A_0^a appearing only linearly, its variation leads to another constraint

$$(\vec{D} \cdot \vec{E})^a = \vec{\nabla} \cdot \vec{E}^a - g{f_{bc}}^a \vec{A}^b \cdot \vec{E}^c = 0. \tag{1.129}$$

As in the other theories we have encountered so far, the constraints come in pairs: one constraint, imposed by a Lagrange multiplier, restricts the physical degrees of freedom; the other constraint is the vanishing of the momentum associated with the Lagrange multiplier.

To obtain the equations of motion, we need to specify the Poisson brackets:

$$\{A_i^a(\vec{x},t), E_{jb}(\vec{y},t)\} = -\delta_{ij}\delta_b^a\,\delta^3(\vec{x}-\vec{y}), \quad \left\{A_0^a,(\vec{x},t), \pi_b^0(\vec{y},t)\right\} = \delta_{ij}\delta_b^a\,\delta^3(\vec{x}-\vec{y}), \tag{1.130}$$

or in quasi-covariant notation

$$\left\{A_\mu^a(\vec{x},t), \pi_b^\nu(\vec{y},t)\right\} = \delta_\mu^\nu\delta_b^a\,\delta^3(\vec{x}-\vec{y}). \tag{1.131}$$

Provided the gauge parameter vanishes sufficiently fast at spatial infinity, the canonical action is gauge invariant:

$$\delta S_{YM} = -\int_1^2 dt \int d^3x\, \vec{\nabla}\cdot\left(\vec{E}_a\frac{\partial\Lambda^a}{\partial t}\right) \simeq 0. \tag{1.132}$$

Therefore it is again straightforward to construct the generator for the local gauge transformations:

$$\begin{aligned} G[\Lambda] &= \int d^3x\,\left(-\delta\vec{A}^a\cdot\vec{E}_a + \delta A_0^a\,\pi_a^0\right) \\ &= \int d^3x\,\pi_a^\mu(D_\mu\Lambda)^a \simeq \int d^3x\,\left(\Lambda_a(\vec{D}\cdot\vec{E})^a + \pi_a^0\,(D_0\Lambda)^a\right). \end{aligned} \tag{1.133}$$

The new aspect of the gauge generators in the case of Yang-Mills theory is, that the constraints satisfy a non-trivial Poisson bracket algebra:

$$\{G[\Lambda_1], G[\Lambda_2]\} = G[\Lambda_3], \tag{1.134}$$

where the parameter on the right-hand side is defined by

$$\Lambda_3 = g f_{bc}{}^a\,\Lambda_1^b\,\Lambda_2^c. \tag{1.135}$$

We can also write the physical part of the constraint algebra in a local form; indeed, let

$$G_a(x) = (\vec{D}\cdot\vec{E})_a(x). \tag{1.136}$$

Then a short calculation leads to the result

$$\{G_a(\vec{x},t), G_b(\vec{y},t)\} = g f_{ab}{}^c\,G_c(\vec{x},t)\,\delta^3(\vec{x}-\vec{y}). \tag{1.137}$$

We observe, that the condition $G[\Lambda] = 0$ is satisfied for arbitrary $\Lambda(x)$ if and only if the two local constraints hold:

$$(\vec{D}\cdot\vec{E})^a = 0, \qquad \pi_a^0 = 0. \tag{1.138}$$

This is sufficient to guarantee that $\{G[\Lambda], H\} = 0$ holds as well. Together with the closure of the algebra of constraints (1.134) this guarantees that the constraints $G[\Lambda] = 0$ are consistent both with the dynamics and among themselves.

Eq.(1.138) is the generalization of the transversality condition (1.116) and removes the same number of momenta (electric field components) from the physical phase space. Unlike the case of electrodynamics however, it is non-linear and can not be solved explicitly. Moreover, the constraint does not determine in closed form the conjugate co-ordinate (the combination of gauge potentials) to be removed from the physical phase space with it. A convenient possibility to impose in classical Yang-Mills theory is the transversality condition $\vec{\nabla} \cdot \vec{A}^a = 0$, which removes the correct number of components of the vector potential and still respects the rigid gauge invariance (with constant parameters Λ^a).

Exercise 1.5
Prove eqs.(1.134) and (1.137).

1.8 The relativistic string

As the last example in this chapter we consider the massless relativistic (bosonic) string, as described by the Polyakov action

$$S_{str} = \int d^2\xi \left(-\frac{1}{2} \sqrt{-g} g^{ab} \, \partial_a X^\mu \partial_b X_\mu \right), \tag{1.139}$$

where $\xi^a = (\xi^0, \xi^1) = (\tau, \sigma)$ are co-ordinates parametrizing the two-dimensional world sheet swept out by the string, g_{ab} is a metric on the world sheet, with g its determinant, and $X^\mu(\xi)$ are the co-ordinates of the string in the D-dimensional embedding space-time (the target space), which for simplicity we take to be flat (Minkowskian). As a generally covariant two-dimensional field theory, the action is manifestly invariant under reparametrizations of the world sheet:

$$X'_\mu(\xi') = X_\mu(\xi), \qquad g'_{ab}(\xi') = g_{cd}(\xi) \, \frac{\partial \xi^c}{\partial \xi'^a} \frac{\partial \xi^d}{\partial \xi'^b}. \tag{1.140}$$

The canonical momenta are

$$\Pi_\mu = \frac{\delta S_{str}}{\delta \partial_0 X^\mu} = -\sqrt{-g} \, \partial^0 X_\mu, \qquad \pi_{ab} = \frac{\delta S_{str}}{\delta \partial_0 g^{ab}} = 0. \tag{1.141}$$

The latter equation brings out, that the inverse metric g^{ab}, or rather the combination $h^{ab} = \sqrt{-g} g^{ab}$, acts as a set of lagrange multipliers, imposing the vanishing of the symmetric energy-momentum tensor:

$$T_{ab} = \frac{2}{\sqrt{-g}} \frac{\delta S_{str}}{\delta g^{ab}} = -\partial_a X^\mu \partial_b X_\mu + \frac{1}{2} \, g_{ab} g^{cd} \partial_c X^\mu \partial_d X_\mu = 0. \tag{1.142}$$

Such a constraint arises because of the local reparametrization invariance of the action. Note however, that the energy-momentum tensor is traceless:

$$T_a^{\;a} = g^{ab} \, T_{ab} = 0. \tag{1.143}$$

and as a result it has only two independent components. The origin of this reduction of the number of constraints is the local Weyl invariance of the action (1.139)

$$g_{ab}(\xi) \to \bar{g}_{ab}(\xi) = e^{\Lambda(\xi)}\, g_{ab}(\xi), \qquad X^\mu(\xi) \to \bar{X}^\mu(\xi) = X^\mu(\xi), \tag{1.144}$$

which leaves h^{ab} invariant: $\bar{h}^{ab} = h^{ab}$. Indeed, h^{ab} itself also has only two independent components, as the negative of its determinant is unity: $-h = -\det h^{ab} = 1$.

The hamiltonian is obtained by Legendre transformation, and taking into account $\pi^{ab} = 0$ it reads

$$\begin{aligned} H &= \frac{1}{2} \int d\sigma \left(\sqrt{-g}\, \left(-g^{00}[\partial_0 X]^2 + g^{11}[\partial_1 X]^2 \right) \,+\, \pi^{ab}\, \partial_0\, g_{ab} \right) \\ &= \int d\sigma\, \left(T^{\,0}_{\;0} + \pi^{ab}\, \partial_0\, g_{ab} \right). \end{aligned} \tag{1.145}$$

The Poisson brackets are

$$\begin{aligned} \{X^\mu(\tau,\sigma), \Pi_\nu(\tau,\sigma')\} &= \delta^\mu_\nu\, \delta(\sigma - \sigma'), \\ \left\{ g_{ab}(\tau,\sigma), \pi^{cd}(\tau,\sigma') \right\} &= \frac{1}{2} \left(\delta^c_a \delta^d_b + \delta^d_a \delta^c_b \right)\, \delta(\sigma - \sigma'). \end{aligned} \tag{1.146}$$

The constraints (1.142) are most conveniently expressed in the hybrid forms (using relations $g = g_{00} g_{11} - g_{01}^2$ and $g_{11} = g g^{00}$):

$$\begin{aligned} g T^{00} &= -T_{11} = \frac{1}{2} \left(\Pi^2 + [\partial_1 X]^2 \right) = 0, \\ \sqrt{-g}\, T^{\,0}_{\;1} &= \Pi \cdot \partial_1 X = 0. \end{aligned} \tag{1.147}$$

These results imply, that the hamiltonian (1.145) actually vanishes, as in the case of the relativistic particle. The reason is also the same: reparametrization invariance, now on a two-dimensional world sheet rather than on a one-dimensional world line.

The infinitesimal form of the transformations (1.140) with $\xi' = \xi - \Lambda(\xi)$ is

$$\begin{aligned} \delta X^\mu(\xi) &= X'^\mu(\xi) - X^\mu(\xi) = \Lambda^a \partial_a X^\mu = \frac{1}{g g^{00}} \left(\sqrt{-g}\, \Lambda^0 \Pi^\mu + \Lambda_1 \partial_\sigma X^\mu \right), \\ \delta g_{ab}(\xi) &= (\partial_a \Lambda^c) g_{cb} + (\partial_b \Lambda^c) g_{ac} + \Lambda^c \partial_c g_{ab} = D_a \Lambda_b + D_b \Lambda_a, \end{aligned} \tag{1.148}$$

where we use the covariant derivative $D_a \Lambda_b = \partial_a \Lambda_b - \Gamma_{ab}^{\;\;c} \Lambda_c$. The generator of these transformations as constructed by our standard procedure now becomes

$$\begin{aligned} G[\Lambda] &= \int d\sigma \left(\Lambda^a \partial_a X \cdot \Pi + \frac{1}{2}\, \Lambda^0 \sqrt{-g}\, g^{ab} \partial_a X \cdot \partial_b X + \pi^{ab} (D_a \Lambda_b + D_b \Lambda_a) \right) \\ &= \int d\sigma\, \left(-\sqrt{-g}\, \Lambda^a T^{\,0}_{\;a} + 2 \pi^{ab} D_a \Lambda_b \right). \end{aligned} \tag{1.149}$$

which has to vanish inorder to represent a canonical symmetry: the constraint $G[\Lambda] = 0$ summarizes all constraints introduced above. The brackets of $G[\Lambda]$ now take the form

$$\{X^\mu, G[\Lambda]\} = \Lambda^a \partial_a X^\mu = \delta X^\mu, \qquad \{g_{ab}, G[\Lambda]\} = D_a \Lambda_b + D_b \Lambda_a = \delta g_{ab}, \quad (1.150)$$

and in particular

$$\Big\{G[\Lambda_1], G[\Lambda_2]\Big\} = G[\Lambda_3], \qquad \Lambda_3^a = \Lambda_{[1}^b \, \partial_b \Lambda_{2]}^a. \qquad (1.151)$$

It takes quite a long and difficult calculation to check this result.

Most practioners of string theory prefer to work in the restricted phase space, in which the metric g_{ab} is not a dynamical variable, and there is no need to introduce its conjugate momentum π^{ab}. Instead, g_{ab} is chosen to have a convenient value by exploiting the reparametrization invariance (1.140) or (1.148):

$$g_{ab} = \rho \, \eta_{ab} = \rho \left(\begin{array}{cc} -1 & 0 \\ 0 & 1 \end{array} \right). \qquad (1.152)$$

Because of the Weyl invariance (1.144) ρ never appears explicitly in any physical quantity, so it does not have to be fixed itself. In particular, the hamiltonian becomes

$$H_{red} = \frac{1}{2} \int d\sigma \left([\partial_0 X]^2 + [\partial_1 X]^2 \right) = \frac{1}{2} \int d\sigma \left(\Pi^2 + [\partial_\sigma X]^2 \right), \qquad (1.153)$$

whilst the constrained gauge generators (1.149) become

$$G_{red}[\Lambda] = \int d\sigma \left(\frac{1}{2} \Lambda^0 \left(\Pi^2 + [\partial_\sigma X]^2 \right) + \Lambda^1 \Pi \cdot \partial_\sigma X \right). \qquad (1.154)$$

Remarkably, these generators still satisfy a closed bracket algebra:

$$\{G_{red}[\Lambda_1], G_{red}[\Lambda_2]\} = G_{red}[\Lambda_3], \qquad (1.155)$$

but the structure constants have changed, as becomes evident from the expression for Λ_3:

$$\begin{aligned} \Lambda_3^0 &= \Lambda_{[1}^1 \, \partial_\sigma \Lambda_{2]}^0 + \Lambda_{[1}^0 \, \partial_\sigma \Lambda_{2]}^1, \\ \Lambda_3^1 &= \Lambda_{[1}^0 \, \partial_\sigma \Lambda_{2]}^0 + \Lambda_{[1}^1 \, \partial_\sigma \Lambda_{2]}^1 \end{aligned} \qquad (1.156)$$

The condition for $G_{red}[\Lambda]$ to generate a symmetry of the hamiltonian H_{red} (and hence to be conserved), is again $G_{red}[\Lambda] = 0$. Observe, that these expressions reduce to those of (1.151) when the Λ^a satisfy

$$\partial_\sigma \Lambda^1 = \partial_\tau \Lambda^0, \qquad \partial_\sigma \Lambda^0 = \partial_\tau \Lambda^1. \qquad (1.157)$$

In terms of the light-cone co-ordinates $u = \tau - \sigma$ or $v = \tau + \sigma$ this can be written:

$$\partial_u(\Lambda^1 + \Lambda^0) = 0, \qquad \partial_v(\Lambda^1 - \Lambda^0) = 0. \tag{1.158}$$

As a result, the algebras are identical for parameters living on only one branch of the (two-dimensional) light-cone:

$$\Lambda^0(u,v) = \Lambda_+(v) - \Lambda_-(u), \qquad \Lambda^1(u,v) = \Lambda_+(v) + \Lambda_-(u), \tag{1.159}$$

with $\Lambda_\pm = (\Lambda_1 \pm \Lambda_0)/2$.

Exercise 1.6
a. Compute the commutator of two infinitesimal transformations (1.148) and show it results in a similar transformation with parameter Λ_3 of eq.(1.151).
b. Prove equations (1.155) and (1.156).

Chapter 2

Canonical BRST construction

Many interesting physical theories incorporate constraints arising from a local gauge symmetry, which forces certain components of the momenta to vanish in the physical phase space. For reparametrization-invariant systems (like the relativistic particle or the relativistic string) these constraints are quadratic in the momenta, whereas in abelian or non-abelian gauge theories of Maxwell-Yang-Mills type they are linear in the momenta (i.e., in the electric components of the field strength).

There are several ways to deal with such constraints. The most obvious one is to solve them and formulate the theory purely in terms of physical degrees of freedom. However, this is possible only in the simplest cases, like the relativistic particle or an unbroken abelian gauge theory (electrodynamics). And even then, there can arise complications such as non-local interactions. Therefore in most cases and for most applications an alternative strategy is more fruitful; this prefered strategy is to keep (some) unphysical degrees of freedom in the theory in such a way that desirable properties of the description, like locality, and rotation or Lorentz-invariance, can be preserved at intermediate stages of calculations. In this chapter we discuss methods for dealing with such a situation, when unphysical degrees of freedom are taken along in the analysis of the dynamics.

The central idea of the BRST construction is to identify the solutions of the constraints with the cohomology classes of a certain nilpotent operator, the BRST operator Ω. To construct this operator we introduce a new class of variables, the ghost variables. For the theories we have discussed in chapter 1, which do not involve fermion fields in essential way (at least from the point of view of constraints), the ghosts are anticommuting variables: odd elements of a Grassmann algebra. However, theories with more general types of gauge symmetries involving fermionic degrees of freedom, like supersymmetry or Siegel's κ-invariance in the theory of superparticles and superstrings, or theories with reducible gauge symmetries, require commuting ghost variables as well. Nevertheless, to bring out the central ideas of the BRST construction as clearly as possible, here we discuss theories with bosonic symmetries only.

2.1 Grassmann variables

The BRST construction involves anticommuting variables, which are odd elements of a Grassmann algebra. The theory of such variables plays an important role in quantum field theory, most prominently in the description of fermion fields as they naturally describe systems satisfying the Pauli exclusion principle. For these reasons we briefly review the basic elements of the theory of anticommuting variables at this point. For more detailed expositions we refer to the references [18, 19].

A Grassmann algebra of rank n is the set of polynomials constructed from elements $\{e, \theta_1, ..., \theta_n\}$ with the properties

$$e^2 = e, \qquad e\theta_i = \theta_i e = \theta_i, \qquad \theta_i\theta_j + \theta_j\theta_i = 0. \tag{2.1}$$

Thus e is the identity element, which will often not be written out explicitly. The elements θ_i are nilpotent: $\theta_i^2 = 0$, whilst for $i \neq j$ the elements θ_i and θ_j anticommute. As a result, a general element of the algebra consists of 2^n terms and takes the form

$$g = \alpha e + \sum_{i=1}^{n} \alpha^i \, \theta_i + \sum_{(i,j)=1}^{n} \frac{1}{2!} \, \alpha^{ij} \, \theta_i\theta_j + ... + \tilde{\alpha} \, \theta_1 ... \theta_n, \tag{2.2}$$

where the coefficients $\alpha^{i_1..i_p}$ are completely antisymmetric in the indices. The elements $\{\theta_i\}$ are called the generators of the algebra. An obvious example of a Grassmann algebra is the algebra of differential forms on an n-dimensional manifold.

On the Grassmann algebra we can define a co-algebra of polynomials in elements $\left\{\bar{\theta}^1, ..., \bar{\theta}^n\right\}$, which together with the unit element e is a Grassmann algebra by itself, but which in addition has the property

$$[\bar{\theta}^i, \theta_j]_+ = \bar{\theta}^i \, \theta_j + \theta_j \, \bar{\theta}^i = \delta_j^i \, e. \tag{2.3}$$

This algebra can be interpreted as the algebra of derivations on the Grassmann algebra spanned by (e, θ_i).

By the property (2.3) the complete set of elements $\left\{e; \theta_i; \bar{\theta}^i\right\}$ is actually turned into a Clifford algebra, which has a (basically unique) representation in terms of Dirac matrices in $2n$-dimensional space. The relation can be established by considering the following complex linear combinations of Grassmann generators:

$$\Gamma_i = \gamma_i = \bar{\theta}^i + \theta_i, \qquad \tilde{\Gamma}_i = \gamma_{i+n} = i\left(\bar{\theta}^i - \theta_i\right), \qquad i = 1, ..., n. \tag{2.4}$$

By construction these elements satisfy the relation

$$[\gamma_a, \gamma_b]_+ = 2 \, \delta_{ab} \, e, \qquad (a, b) = 1, ..., 2n, \tag{2.5}$$

but actually the subsets $\{\Gamma_i\}$ and $\left\{\tilde{\Gamma}_i\right\}$ define two mutually anti-commuting Clifford algebras of rank n:

$$[\Gamma_i, \Gamma_j]_+ = [\tilde{\Gamma}_i, \tilde{\Gamma}_j]_+ = 2\,\delta_{ij}, \qquad [\Gamma_i, \tilde{\Gamma}_j]_+ = 0. \tag{2.6}$$

Of course, the construction can be turned around to construct a Grassmann algebra of rank n and its co-algebra of derivations out of a Clifford algebra of rank $2n$.

In field theory applications we are mostly interested in Grassmann algebras of infinite rank, not only $n \rightarrow \infty$, but particularly also the continuous case

$$[\bar{\theta}(t), \theta(s)]_+ = \delta(t - s), \tag{2.7}$$

where (s, t) are real-valued arguments. Obviously, a Grassmann *variable* ξ is a quantity taking values in a set of linear Grassmann forms $\sum_i \alpha^i \theta_i$ or its continuous generalization $\int_t \alpha(t)\,\theta(t)$. Similarly, one can define derivative operators $\partial/\partial\xi$ as linear operators mapping Grassmann forms of rank p into forms of rank $p-1$, by

$$\frac{\partial}{\partial\xi}\,\xi = 1 - \xi\frac{\partial}{\partial\xi}, \tag{2.8}$$

and its generalization for systems of multi-Grassmann variables. These derivative operators can be constructed as linear forms in $\bar{\theta}^i$ or $\bar{\theta}(t)$.

In addition to differentiation one can also define Grassmann integration. In fact, Grassmann integration is defined as identical with Grassmann differentiation. For a single Grassmann variable, let $f(\xi) = f_0 + \xi f_1$; then one defines

$$\int d\xi\, f(\xi) = f_1. \tag{2.9}$$

This definition satisfies all standard properties of indefinite integrals:

1. linearity:

$$\int d\xi\, \left[\alpha f(\xi) + \beta g(\xi)\right] = \alpha \int d\xi\, f(\xi) + \beta \int d\xi\, g(\xi); \tag{2.10}$$

2. translation invariance:

$$\int d\xi\, f(\xi + \eta) = \int d\xi\, f(\xi); \tag{2.11}$$

3. fundamental theorem of calculus (Gauss-Stokes):

$$\int d\xi \frac{\partial f}{\partial\xi} = 0; \tag{2.12}$$

4. reality: for *real* functions $f(\xi)$ (i.e. $f_{0,1} \in \mathbf{R}$)

$$\int d\xi f(\xi) = f_1 \in \mathbf{R}. \tag{2.13}$$

A particularly useful result is the evaluation of Gaussian Grassmann integrals. First we observed, that

$$\int [d\xi_1 ... d\xi_n] \, \xi_{\alpha_1} ... \xi_{\alpha_n} = \varepsilon_{\alpha_1 ... \alpha_n}. \tag{2.14}$$

From this it follows, that a general Gaussian Grassmann integral is

$$\int [d\xi_1 ... d\xi_n] \exp\left(\frac{1}{2} \, \xi_\alpha A_{\alpha\beta} \, \xi_\beta \right) = \pm \sqrt{|\det A|}. \tag{2.15}$$

This is quite obvious after bringing A into block-diagonal form:

$$A = \left(\begin{array}{cccccc} 0 & \omega_1 & & & & \\ -\omega_1 & 0 & & & & 0 \\ & & 0 & \omega_2 & & \\ & & -\omega_2 & 0 & & \\ & & & & \cdot & \\ & 0 & & & & \cdot \end{array} \right). \tag{2.16}$$

There are then two possibilities:
(i) If the dimensionality of the matrix A is even ($(\alpha, \beta) = 1, ..., 2r$) and none of the characteristic values ω_i vanishes, then every 2×2 block gives a contribution $2\omega_i$ to the exponential:

$$\exp\left(\frac{1}{2} \, \xi_\alpha A_{\alpha\beta} \, \xi_\beta \right) = \exp\left(\sum_{i=1}^{r} \omega_i \, \xi_{2i-1} \xi_{2i} \right) = 1 + ... + \prod_{i=1}^{r} (\omega_i \, \xi_{2i-1} \xi_{2i}). \tag{2.17}$$

The final result is then established by performing the Grassmann integrations, which leaves a non-zero contribution only from the last term, reading

$$\prod_{i=1}^{r} \omega_i = \pm \sqrt{|\det A|}, \tag{2.18}$$

the sign depending on the number of negative characteristic values ω_i.
(ii) If the dimensionality of A is odd, the last block is one-dimensional representing a zero-mode; then the integral vanishes, as does the determinant. Of course, the same is true for even-dimensional A if one of the values ω_i vanishes.

Another useful result is, that one can define a Grassmann-valued delta-function:

$$\delta(\xi - \xi') = -\delta(\xi' - \xi) = \xi - \xi', \tag{2.19}$$

with the properties

$$\int d\xi \, \delta(\xi - \xi') = 1, \qquad \int d\xi \, \delta(\xi - \xi') f(\xi) = f(\xi'). \tag{2.20}$$

The proof follows simply by writing out the integrants and using the fundamental rule of integration (2.9).

2.2 Classical BRST transformations

Consider again a general dynamical system subject to a set of constraints $G_\alpha = 0$, as defined in eqs.(1.41) or (1.60). We take the algebra of constraints to be first-class, as in eq.(1.69):

$$\{G_\alpha, G_\beta\} = P_{\alpha\beta}(G), \qquad \{G_\alpha, H\} = Z_\alpha(G). \tag{2.21}$$

Here $P(G)$ and $Z(G)$ are polynomial expressions in the constraints, such that $P(0) = Z(0) = 0$; in particular this implies that the constant terms vanish: $c_{\alpha\beta} = 0$.

The BRST construction starts with the introduction of canonical pairs of Grassmann degrees of freedom (c^α, b_β), one for each constraint G_α, with Poisson brackets

$$\{c^\alpha, b_\beta\} = \{b_\beta, c^\alpha\} = -i\delta_\beta^\alpha, \tag{2.22}$$

These anti-commuting variables are known as ghosts; the complete Poisson brackets on the extended phase space are given by

$$\{A, B\} = \frac{\partial A}{\partial q^i}\frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q^i} + i(-1)^A \left(\frac{\partial A}{\partial c^\alpha}\frac{\partial B}{\partial b_\alpha} + \frac{\partial A}{\partial b_\alpha}\frac{\partial B}{\partial c^\alpha} \right), \tag{2.23}$$

where $(-1)^A$ denotes the Grassmann parity of A: $+1$, if A is Grassmann-even (commuting), and -1 if A is Grassmann-odd (anti-commuting).

With the help of these ghost degrees of freedom one defines the BRST charge Ω, which has Grassmann parity $(-1)^\Omega = -1$, as

$$\Omega = c^\alpha \left(G_\alpha + M_\alpha \right), \tag{2.24}$$

where M_α is Grassmann-even and of the form

$$\begin{aligned} M_\alpha &= \sum_{n\geq 1} \frac{i^n}{2n!}\, c^{\alpha_1}...c^{\alpha_n} M_{\alpha\alpha_1...\alpha_n}^{\;\;\beta_1...\beta_n} b_{\beta_1}...b_{\beta_n} \\ &= \frac{i}{2}\, c^{\alpha_1} M_{\alpha\alpha_1}^{\;\;\beta_1} b_{\beta_1} - \frac{1}{4}\, c^{\alpha_1} c^{\alpha_2} M_{\alpha\alpha_1\alpha_2}^{\;\;\beta_1\beta_2} b_{\beta_1} b_{\beta_2} + ... \end{aligned} \tag{2.25}$$

The quantities $M_{\alpha\alpha_1...\alpha_p}^{\;\;\beta_1...\beta_p}$ are functions of the classical phase-space variables via the constraints G_α, and are defined such that

$$\{\Omega, \Omega\} = 0. \tag{2.26}$$

As Ω is Grassmann-odd, this is a non-trivial property, from which the BRST charge can be constructed inductively:

$$\begin{aligned} \{\Omega, \Omega\} &= c^\alpha c^\beta \left(P_{\alpha\beta} + M_{\alpha\beta}^{\;\;\gamma} G_\gamma \right) \\ &\quad + i c^\alpha c^\beta c^\gamma \left(\left\{ G_\alpha, M_{\beta\gamma}^{\;\;\delta} \right\} - M_{\alpha\beta}^{\;\;\varepsilon} M_{\gamma\epsilon}^{\;\;\delta} + M_{\alpha\beta\gamma}^{\;\;\delta\varepsilon} G_\varepsilon \right) b_\delta + ... \end{aligned} \tag{2.27}$$

This vanishes if and only if

$$
\begin{aligned}
& M_{\alpha\beta}^{\ \ \gamma} G_\gamma = -P_{\alpha\beta}, \\
& M_{\alpha\beta\gamma}^{\ \ \ \delta\varepsilon} G_\varepsilon = \left\{ M_{[\alpha\beta}^{\ \ \ \delta}, G_{\gamma]} \right\} + M_{[\alpha\beta}^{\ \ \ \varepsilon} M_{\gamma]\varepsilon}^{\ \ \ \delta}, \qquad (2.28) \\
& ...
\end{aligned}
$$

Observe, that the first relation can only be satisfied under the condition $c_{\alpha\beta} = 0$, with the solution

$$
M_{\alpha\beta}^{\ \ \gamma} = f_{\alpha\beta}^{\ \ \gamma} + \frac{1}{2}\, g_{\alpha\beta}^{\ \ \gamma\delta} G_\delta + ... \qquad (2.29)
$$

The same condition guarantees that the second relation can be solved: the bracket on the right-hand side is

$$
\left\{ M_{\alpha\beta}^{\ \ \delta}, G_\gamma \right\} = \frac{\partial M_{\alpha\beta}^{\ \ \delta}}{\partial G_\varepsilon}\, P_{\varepsilon\gamma} = \frac{1}{2}\, g_{\alpha\beta}^{\ \ \delta\varepsilon} f_{\varepsilon\gamma}^{\ \ \sigma} G_\sigma + ... \qquad (2.30)
$$

whilst the Jacobi identity (1.67) implies that

$$
f_{[\alpha\beta}^{\ \ \ \varepsilon} f_{\gamma]\varepsilon}^{\ \ \ \delta} = 0, \qquad (2.31)
$$

and therefore $M_{[\alpha\beta}^{\ \ \ \varepsilon} M_{\gamma]\varepsilon}^{\ \ \ \delta} = \mathcal{O}[G_\sigma]$. This allows to determine $M_{\alpha\beta\gamma}^{\ \ \ \delta\varepsilon}$. Any higher-order terms can be calculated similarly. In practice $P_{\alpha\beta}$ and M_α usually contain only a small number of terms.

Next we observe, that we can extend the classical hamiltonian $H = H_0$ with ghost terms such that

$$
H_c = H_0 + \sum_{n \geq 1} \frac{i^n}{n!}\, c^{\alpha_1} ... c^{\alpha_n}\, h_{\alpha_1 ... \alpha_n}^{(n)\, \beta_1 ... \beta_n}(G)\, b_{\beta_1} ... b_{\beta_n}, \qquad \{\Omega, H_c\} = 0. \qquad (2.32)
$$

Observe, that on the physical hypersurface in the phase space this hamiltonian coincides with the original classical hamiltonian modulo terms which do not affect the time-evolution of the classical phase-space variables (q, p). We illustrate the procedure by constructing the first term:

$$
\begin{aligned}
\{\Omega, H_c\} &= \{c^\alpha G_\alpha, H_0\} + \frac{i}{2} \left\{ c^\alpha G_\alpha, c^\gamma h_\gamma^{(1)\, \beta} b_\beta \right\} + \frac{i}{2} \left\{ c^{\alpha_1} c^{\alpha_2} M_{\alpha_1\alpha_2}^\beta b_\beta, H_0 \right\} + ... \\
&= c^\alpha \left(Z_\alpha - h_\alpha^{(1)\, \beta} G_\beta \right) + ...
\end{aligned}
\qquad (2.33)
$$

Hence the bracket vanishes if the hamiltonian is extended by ghost terms such that

$$
h_\alpha^{(1)\, \beta}(G)\, G_\beta = Z_\alpha(G), \qquad ... \qquad (2.34)
$$

This equation is guaranteed to have a solution by the condition $Z(0) = 0$.

As the BRST charge commutes with the ghost-extended hamiltonian, we can use it to generate ghost-dependent symmetry transformations of the classical phase-space variables: the BRST transformations

$$\begin{aligned} \delta_\Omega\, q^i &= -\left\{ \Omega, q^i \right\} = \frac{\partial \Omega}{\partial p_i} = c^\alpha\, \frac{\partial G_\alpha}{\partial p_i} + \text{ghost extensions}, \\ \delta_\Omega\, p_i &= -\left\{ \Omega, p_i \right\} = -\frac{\partial \Omega}{\partial q_i} = c^\alpha\, \frac{\partial G_\alpha}{\partial q^i} + \text{ghost extensions}. \end{aligned} \tag{2.35}$$

These BRST transformations are just the gauge transformations with the parameters ϵ^α replaced by the ghost variables c^α, plus (possibly) some ghost-dependent extension.

Similarly, one can define BRST transformations of the ghosts:

$$\begin{aligned} \delta_\Omega\, c^\alpha &= -\left\{ \Omega, c^\alpha \right\} = i \frac{\partial \Omega}{\partial b_\alpha} = -\frac{1}{2}\, c^\beta c^\gamma M_{\beta\gamma}^{\;\;\alpha} + ..., \\ \delta_\Omega\, b_\alpha &= -\left\{ \Omega, b_\alpha \right\} = i \frac{\partial \Omega}{\partial c^\alpha} = i G_\alpha - c^\beta M_{\alpha\beta}^{\;\;\gamma}\, b_\gamma + ... \end{aligned} \tag{2.36}$$

An important property of these transformations is their nilpotence:

$$\delta_\Omega^2 = 0. \tag{2.37}$$

This follows most directly from the Jacobi identity for the Poisson brackets of the BRST charge with any phase-space function A:

$$\delta_\Omega^2\, A = \left\{ \Omega, \left\{ \Omega, A \right\} \right\} = -\frac{1}{2}\, \left\{ A, \left\{ \Omega, \Omega \right\} \right\} = 0. \tag{2.38}$$

Thus the BRST variation δ_Ω behaves like an exterior derivative. Next we observe, that gauge invariant physical quantities F have the properties

$$\left\{ F, c^\alpha \right\} = i \frac{\partial F}{\partial b_\alpha} = 0, \qquad \left\{ F, b_\alpha \right\} = i \frac{\partial F}{\partial c^\alpha} = 0, \qquad \left\{ F, G_\alpha \right\} = \delta_\alpha F = 0. \tag{2.39}$$

As a result, such physical quantities must be BRST invariant:

$$\delta_\Omega\, F = -\left\{ \Omega, F \right\} = 0. \tag{2.40}$$

In the terminology of algebraic geometry, such a function F is called BRST closed. Now because of the nilpotence, there are trivial solutions to this condition, of the form

$$F_0 = \delta_\Omega\, F_1 = -\left\{ \Omega, F_1 \right\}. \tag{2.41}$$

These solutions are called BRST exact; they always depend on the ghosts (c^α, b_α), and can not be physically relevant. We conclude, that true physical quantities must be BRST closed, but not BRST exact. Such non-trivial solutions of the BRST condition (2.40) define the BRST cohomology, which is the set

$$\mathcal{H}(\delta_\Omega) = \frac{\mathrm{Ker}(\delta_\Omega)}{\mathrm{Im}(\delta_\Omega)}. \tag{2.42}$$

We will make this more precise later on.

2.3 Examples

As an application of the above construction, we now present the classical BRST charges and transformations for the gauge systems discussed in chapter 1.

1. *Relativistic particle.* We consider the gauge-fixed version of the relativistic particle. Taking $c = 1$, the only constraint is

$$H_0 = \frac{1}{2m}(p^2 + m^2) = 0, \tag{2.43}$$

and hence in this case $P_{\alpha\beta} = 0$. We only introduce one pair of ghost variables, and define

$$\Omega = \frac{c}{2m}(p^2 + m^2). \tag{2.44}$$

It is trivially nilpotent, and the BRST transformations of the phase space variables read

$$\begin{aligned} &\delta_\Omega x^\mu = \{x^\mu, \Omega\} = \frac{cp^\mu}{m}, \quad &&\delta_\Omega p_\mu = \{p_\mu, \Omega\} = 0, \\ &\delta_\Omega c = -\{c, \Omega\} = 0, &&\delta_\Omega b = -\{b, \Omega\} = \frac{i}{2m}(p^2 + m^2) \approx 0. \end{aligned} \tag{2.45}$$

The b-ghost transforms into the constraint, hence it vanishes on the physical hypersurface in the phase space. It is straightforward to verify that $\delta_\Omega^2 = 0$.

2. *Electrodynamics.* In the gauge fixed Maxwell's electrodynamics there is again only a single constraint, and a single pair of ghost fields to be introduced. We define the BRST charge

$$\Omega = \int d^3x\, c\vec{\nabla} \cdot \vec{E}. \tag{2.46}$$

The classical BRST transformations are just ghost-dependend gauge transformations:

$$\begin{aligned} &\delta_\Omega \vec{A} = \left\{\vec{A}, \Omega\right\} = \vec{\nabla} c, \quad &&\delta_\Omega \vec{E} = \left\{\vec{E}, \Omega\right\} = 0, \\ &\delta_\Omega c = -\{c, \Omega\} = 0, &&\delta_\Omega b = -\{b, \Omega\} = i\vec{\nabla} \cdot \vec{E} \approx 0. \end{aligned} \tag{2.47}$$

3. *Yang-Mills theory.* One of the simplest non-trivial systems of constraints is that of Yang-Mills theory, in which the constraints define a local Lie algebra (1.137). The BRST charge becomes

$$\Omega = \int d^3x \left(c^a G_a - \frac{ig}{2}\, c^a c^b f_{ab}{}^c b_c \right), \tag{2.48}$$

with $G_a = (\vec{D} \cdot \vec{E})_a$. It is now non-trivial that the bracket of Ω with itself vanishes; it is true because of the closure of the Lie algebra, and the Jacobi identity for the structure constants.

The classical BRST transformations of the fields become

$$\begin{aligned} &\delta_\Omega \vec{A}^a = \left\{ \vec{A}^a, \Omega \right\} = (\vec{D}c)^a, \qquad \delta_\Omega \vec{E}_a = \left\{ \vec{E}_a, \Omega \right\} = g f_{ab}{}^c c^b \vec{E}_c, \\ &\delta_\Omega c^a = -\left\{ c^a, \Omega \right\} = \frac{g}{2}\, f_{bc}{}^a\, c^b c^c, \quad \delta_\Omega b_a = -\left\{ b_a, \Omega \right\} = i\, G_a + g f_{ab}{}^c\, c^b\, b_c. \end{aligned} \tag{2.49}$$

Again, it can be checked by explicit calculation that $\delta_\Omega^2 = 0$ for all variations (2.49).

Exercise Show that

$$\delta_\Omega\, G_a = g f_{ab}{}^c c^b\, G_c.$$

From this, prove that $\delta_\Omega^2\, b_a = 0$.

4. *Relativistic string.* Finally, we discuss the free relativistic string. We take the reduced constraints (1.154), satisfying the algebra (1.155), (1.156). The the BRST charge takes the form

$$\begin{aligned} \Omega \;=\; & \int d\sigma \left[\frac{1}{2}\, c^0 \left(\Pi^2 + [\partial_\sigma X]^2 \right) + c^1 \Pi \cdot \partial_\sigma X \right. \\ & \left. - i \left(c^1 \partial_\sigma c^0 + c^0 \partial_\sigma c^1 \right) b_0 - i \left(c^0 \partial_\sigma c^0 + c^1 \partial_\sigma c^1 \right) b_1 \right]. \end{aligned} \tag{2.50}$$

The BRST transformations generated by the Poisson brackets of this charge read

$$\begin{aligned} &\delta_\Omega X^\mu = \{ X^\mu, \Omega \} = c^0 \Pi^\mu + c^1 \partial_\sigma X^\mu \approx c^a \partial_a X^\mu, \\ &\delta_\Omega \Pi_\mu = \{ \Pi_\mu, \Omega \} = \partial_\sigma \left(c^0 \partial_\sigma X_\mu + c^1 \Pi_\mu \right) \approx \partial_\sigma \left(\varepsilon^{ab} c_a \partial_b X^\mu \right), \\ &\delta_\Omega c^0 = -\left\{ c^0, \Omega \right\} = c^1 \partial_\sigma c^0 + c^0 \partial_\sigma c^1, \\ &\delta_\Omega c^1 = -\left\{ c^0, \Omega \right\} = c^0 \partial_\sigma c^0 + c^1 \partial_\sigma c^1, \\ &\delta_\Omega b_0 = -\left\{ b_0, \Omega \right\} = \tfrac{i}{2} \left(\Pi^2 + [\partial_\sigma X]^2 \right) + c^1 \partial_\sigma b_0 + c^0 \partial_\sigma b_1 + 2 \partial_\sigma c^1\, b_0 + 2\, \partial_\sigma c^0\, b_1, \\ &\delta_\Omega b_1 = -\left\{ b_1, \Omega \right\} = i\, \Pi \cdot \partial_\sigma X + c^0 \partial_\sigma b_0 + c^1 \partial_\sigma b_1 + 2 \partial_\sigma c^0\, b_0 + 2\, \partial_\sigma c^1\, b_1. \end{aligned} \tag{2.51}$$

A tedious calculation shows, that these transformations are nilpotent indeed: $\delta_\Omega^2 = 0$.

2.4 Quantum BRST cohomology

The construction of a quantum theory for constrained systems poses the following problem: to have a local and/or covariant description of the quantum system, it is advantageous to work in an extended Hilbert space of states, with unphysical components, like gauge and ghost degrees of freedom. Therefore we need first of all a way to characterize physical states within this extended Hilbert space, and secondly a way to construct a unitary evolution operator which does not mix physical and unphysical components. In this section we show, that the BRST construction can solve both these problems [20, 21, 22].

We begin with a quantum system subject to constraints G_α; we impose these constraints on the physical states:

$$G_\alpha |\Psi\rangle = 0, \tag{2.52}$$

implying that physical states are gauge-invariant. In the quantum theory the generators of constraints are operators, which satisfy the commutation relations (1.80):

$$-i\left[G_\alpha, G_\beta\right] = P_{\alpha\beta}(G), \tag{2.53}$$

where we omit the hat on operators for ease of notation.

Next we introduce corresponding ghost field operators (c_α, b_β) with equal-time anti-commutation relations

$$\left[c^\alpha, b_\beta\right]_+ = c^\alpha b_\beta + \beta_\beta c^\alpha = \delta_\beta^\alpha. \tag{2.54}$$

(For simplicity, the time-dependence in the notation has been suppressed). In the ghost-extended Hilbert space we now construct a BRST operator

$$\Omega = c^\alpha \left(G_\alpha + \sum_{n\geq 1} \frac{i^n}{2n!}\, c^{\alpha_1}...c^{\alpha_n} M_{\alpha\alpha_1...\alpha_n}^{\quad\ \ \beta_1...\beta_n} b_{\beta_1}...b_{\beta_n} \right), \tag{2.55}$$

which is required to satisfy the anti-commutation relation

$$\left[\Omega, \Omega\right]_+ = 2\Omega^2 = 0. \tag{2.56}$$

In words, the BRST operator is nilpotent. Working out the square of the BRST operator, we get

$$\begin{aligned} \Omega^2 \;=\; & \frac{i}{2}\, c^\alpha c^\beta \left(-i\left[G_\alpha, G_\beta\right] + M_{\alpha\beta}^{\ \ \gamma} G_\gamma \right) \\ & - \frac{1}{2} c^\alpha c^\beta c^\gamma \left(-i\left[G_\alpha, M_{\beta\gamma}^{\ \ \delta}\right] + M_{\alpha\beta}^{\ \ \varepsilon} M_{\gamma\varepsilon}^{\ \ \delta} + M_{\alpha\beta\gamma}^{\ \ \ \delta\varepsilon} G_\varepsilon \right) b_\delta + ... \end{aligned} \tag{2.57}$$

As a consequence, the coefficients M_α are defined as the solutions of the set of equations

$$
\begin{aligned}
& i\,[G_\alpha, G_\beta] = -P_{\alpha\beta} = M_{\alpha\beta}^{\gamma} G_\gamma, \\
& i\left[G_{[\alpha}, M_{\beta\gamma]}^{\ \ \delta}\right] + M_{[\alpha\beta}^{\ \ \varepsilon} M_{\gamma]\varepsilon}^{\ \ \delta} = M_{\alpha\beta\gamma}^{\ \ \ \delta\varepsilon} G_\varepsilon \\
& ...
\end{aligned}
\tag{2.58}
$$

These are operator versions of the classical equations (2.28). As in the classical case, their solution requires the absence of a central charge: $c_{\alpha\beta} = 0$.

Observe, that the Jacobi identity for the generators G_α implies some restrictions on the higher terms in the expansion of Ω:

$$
\begin{aligned}
0 &= [G_\alpha, [G_\beta, G_\gamma]] + (\text{terms cyclic in } [\alpha\beta\gamma]) = -3i\left[G_{[\alpha}, M_{\beta\gamma]}^{\ \ \delta} G_\delta\right] \\
&= -3\left(i\left[G_{[\alpha}, M_{\beta\gamma]}^{\ \ \delta}\right] + M_{[\alpha\beta}^{\ \ \varepsilon} M_{\alpha]\,\varepsilon}^{\ \ \delta}\right) G_\delta = -\frac{3i}{2}\, M_{\alpha\beta\gamma}^{\ \ \ \delta\varepsilon} M_{\delta\varepsilon}^{\ \ \sigma} G_\sigma.
\end{aligned}
\tag{2.59}
$$

The equality on the first line follows from the first equation (2.58), the last equality from the second one.

To describe the states in the extended Hilbert space, we introduce a ghost-state module, a basis for the ghost states consisting of monomials in the ghost operators c^α:

$$
|[\alpha_1\alpha_2...\alpha_p]\rangle_{gh} = \frac{1}{p!}\, c^{\alpha_1} c^{\alpha_2} ... c^{\alpha_p} |0\rangle_{gh}, \tag{2.60}
$$

with $|0\rangle_{gh}$ the ghost vacuum state annihilated by all b_β. By construction these states are completely anti-symmetric in the indices $[\alpha_1\alpha_2...\alpha_p]$, i.e. the ghosts satisfy Fermi-Dirac statistics, even though they do not carry spin. This confirms their unphysical nature. As a result of this choice of basis, we can decompose an arbitrary state in components with different ghost number (= rank of the ghost polynomial):

$$
|\Psi\rangle = |\Psi^{(0)}\rangle + c^\alpha |\Psi^{(1)}_\alpha\rangle + \frac{1}{2}\, c^\alpha c^\beta |\Psi^{(2)}_{\alpha\beta}\rangle + ... \tag{2.61}
$$

where the states $|\Psi^{(n)}_{\alpha_1...\alpha_n}\rangle$ corresponding to ghost number n are of the form $|\psi^{(n)}_{\alpha_1...\alpha_n}(q)\rangle \times |0\rangle_{gh}$, with $|\psi^{(n)}_{\alpha_1...\alpha_n}(q)\rangle$ states of zero-ghost number, depending only on the degrees of freedom of the constrained (gauge) system; therefore we have

$$
b_\beta |\Psi^{(n)}_{\alpha_1...\alpha_n}\rangle = 0. \tag{2.62}
$$

To do the ghost-counting, it is convenient to introduce the ghost-number operator

$$
N_g = \sum_\alpha c^\alpha b_\alpha, \qquad [N_g, c^\alpha] = c^\alpha, \qquad [N_g, b_\alpha] = -b_\alpha, \tag{2.63}
$$

where as usual the summation over α has to be interpreted in a generalized sense (it includes integration over space when appropriate). It follows, that the BRST operator has ghost number +1:

$$[N_g, \Omega] = \Omega. \tag{2.64}$$

Now consider a BRST-invariant state:

$$\Omega|\Psi\rangle = 0. \tag{2.65}$$

Substitution of the ghost-expansions of Ω and $|\Psi\rangle$ gives

$$\begin{aligned} \Omega|\Psi\rangle &= c^\alpha G_\alpha|\Psi^{(0)}\rangle + \frac{1}{2}\, c^\alpha c^\beta \left(G_\alpha|\Psi^{(1)}_\beta\rangle - G_\beta|\Psi^{(1)}_\alpha\rangle + iM_{\alpha\beta}^{\;\;\gamma}|\Psi^{(1)}_\gamma\rangle \right) \\ &\quad + \frac{1}{2}\, c^\alpha c^\beta c^\gamma \left(G_\alpha|\Psi^{(2)}_{\beta\gamma}\rangle - iM_{\alpha\beta}^{\;\;\delta}|\Psi^{(2)}_{\gamma\delta}\rangle + \frac{1}{2}\, M_{\alpha\beta\gamma}^{\;\;\;\delta\varepsilon}|\Psi^{(2)}_{\delta\varepsilon}\rangle \right) + ... \end{aligned} \tag{2.66}$$

Its vanishing then implies

$$\begin{aligned} & G_\alpha|\Psi^{(0)}\rangle = 0, \\ & G_\alpha|\Psi^{(1)}_\beta\rangle - G_\beta|\Psi^{(1)}_\alpha\rangle + iM_{\alpha\beta}^{\;\;\gamma}|\Psi^{(1)}_\gamma\rangle = 0, \\ & G_{[\alpha}|\Psi^{(2)}_{\beta\gamma]}\rangle - iM_{[\alpha\beta}^{\;\;\;\delta}|\Psi^{(2)}_{\gamma]\delta}\rangle + \frac{1}{2}\, M_{\alpha\beta\gamma}^{\;\;\;\delta\varepsilon}|\Psi^{(2)}_{\delta\varepsilon}\rangle = 0, \\ & ... \end{aligned} \tag{2.67}$$

These conditions admit solutions of the form

$$\begin{aligned} & |\Psi^{(1)}_\alpha\rangle = G_\alpha|\chi^{(0)}\rangle, \\ & |\Psi^{(2)}_{\alpha\beta}\rangle = G_\alpha|\chi^{(1)}_\beta\rangle - G_\beta|\chi^{(1)}_\alpha\rangle + iM_{\alpha\beta}^{\;\;\gamma}|\chi^{(1)}_\gamma\rangle, \\ & ... \end{aligned} \tag{2.68}$$

where the states $|\chi^{(n)}\rangle$ have zero ghost number: $b_\alpha|\chi^{(n)}\rangle = 0$. Substitution of these expressions into eq.(2.61) gives

$$\begin{aligned} |\Psi\rangle &= |\Psi^{(0)}\rangle + c^\alpha G_\alpha|\chi^{(0)}\rangle + c^\alpha c^\beta G_\alpha|\chi^{(1)}_\beta\rangle + \frac{i}{2}\, c^\alpha c^\beta M_{\alpha\beta}^{\;\;\gamma}|\chi^{(1)}_\gamma\rangle \\ &= |\Psi^{(0)}\rangle + \Omega\left(|\chi^{(0)}\rangle + c^\alpha|\chi^{(1)}_\alpha\rangle + ... \right) \\ &= |\Psi^{(0)}\rangle + \Omega\,|\chi\rangle. \end{aligned} \tag{2.69}$$

The second term is trivially BRST invariant because of the nilpotence of the BRST operator: $\Omega^2 = 0$. Assuming that Ω is hermitean, it follows, that $|\Psi\rangle$ is normalized if and only if $|\Psi^{(0)}\rangle$ is:

$$\langle\Psi|\Psi\rangle = \langle\Psi^{(0)}|\Psi^{(0)}\rangle + 2\,\mathrm{Re}\,\langle\chi|\Omega|\Psi^{(0)}\rangle + \langle\chi|\Omega^2|\chi\rangle = \langle\Psi^{(0)}|\Psi^{(0)}\rangle. \tag{2.70}$$

We conclude, that the class of normalizable BRST-invariant states includes the set of states which can be decomposed into a normalizable gauge-invariant state $|\Psi^{(0)}\rangle$ at ghost number zero, plus a trivially invariant zero-norm state $\Omega|\chi\rangle$. These states are members of the BRST cohomology, the classes of states which are BRST invariant (BRST closed) modulo states in the image of Ω (BRST-exact states):

$$\mathcal{H}(\Omega) = \frac{\mathrm{Ker}\,\Omega}{\mathrm{Im}\,\Omega}. \tag{2.71}$$

2.5 BRST-Hodge decomposition of states

We have shown by explicit construction, that physical states can be identified with the BRST-cohomology classes of which the lowest, non-trivial, component has zero ghost-number. However, our analysis does not show to what extent these solutions are unique. In this section we present a general discussion of BRST cohomology to establish conditions for the existence of a direct correspondence between physical states and BRST cohomology classes [23, 24].

We assume that the BRST operator is self-adjoint w.r.t. the physical inner product. An immediate consequence is, that the ghost-extended Hilbert space of states contains zero-norm states. Let

$$|\Lambda\rangle = \Omega|\chi\rangle. \tag{2.72}$$

These states are all orthogonal to each other, including themselves, and thus they have zero-norm indeed:

$$\langle\Lambda'|\Lambda\rangle = \langle\chi'|\Omega^2|\chi\rangle = 0 \quad \Rightarrow \quad \langle\Lambda|\Lambda\rangle = 0. \tag{2.73}$$

Moreover, these states are orthogonal to all normalizable BRST-invariant states:

$$\Omega|\Psi\rangle = 0 \quad \Rightarrow \quad \langle\Lambda|\Psi\rangle = 0. \tag{2.74}$$

Clearly, the BRST-exact states can not be physical. On the other hand, BRST-closed states are defined only modulo BRST-exact states. We prove, that if on the extended Hilbert space $\mathcal{H}_{ext}$ there exists a non-degenerate inner product (*not* the physical inner product), which is also non-degenerate when restricted to the subspace Im Ω of BRST-exact states, then all physical states must be members of the BRST cohomology.

A non-degenerate inner product $(\,,\,)$ on $\mathcal{H}_{ext}$ is an inner product with the property, that

$$(\phi, \chi) = 0, \quad \forall \phi, \quad \Leftrightarrow \quad \chi = 0. \tag{2.75}$$

If the restriction of this inner product to Im Ω is non-degenerate as well, then

$$(\Omega\phi, \Omega\chi) = 0, \quad \forall \phi, \quad \Leftrightarrow \quad \Omega\chi = 0. \tag{2.76}$$

As there are no non-trivial zero-norm states w.r.t. this inner product, the BRST operator can not be self-adjoint; its adjoint, denoted by ${}^*\Omega$ then defines a second nilpotent operator:

$$(\Omega\phi, \chi) = (\phi, {}^*\Omega\chi) \qquad \Rightarrow \qquad (\Omega^2\phi, \chi) = (\phi, {}^*\Omega^2\chi) = 0, \quad \forall \phi. \tag{2.77}$$

The non-degeneracy of the inner product implies that ${}^*\Omega^2 = 0$. The adjoint ${}^*\Omega$ is called the co-BRST operator. Note, that from eq.(2.76) one infers

$$(\phi, {}^*\Omega\,\Omega\chi) = 0, \quad \forall \phi, \quad \Leftrightarrow \quad {}^*\Omega\,\Omega\chi = 0 \quad \Leftrightarrow \quad \Omega\chi = 0. \tag{2.78}$$

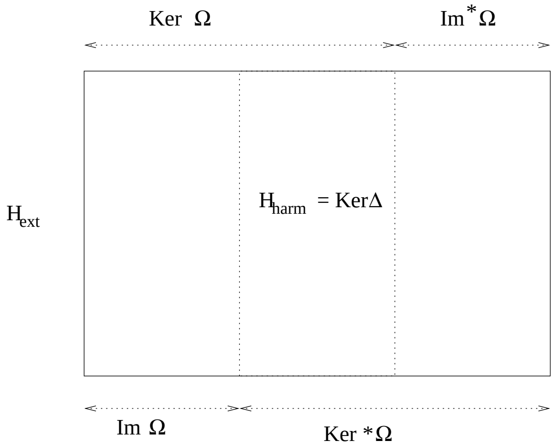

Fig. 1: Decomposition of the extended Hilbert space

It follows immediately, that any BRST-closed vector $\Omega\psi = 0$ is determined uniquely by requiring it to be co-closed as well. Indeed, let ${}^*\Omega\psi = 0$; then

$${}^*\Omega(\psi + \Omega\chi) = 0 \quad \Leftrightarrow \quad {}^*\Omega\,\Omega\chi = 0 \quad \Leftrightarrow \quad \Omega\chi = 0. \tag{2.79}$$

Thus, if we regard the BRST transformations as gauge transformations on states in the extended Hilbert space generated by Ω, then ${}^*\Omega$ represents a gauge-fixing operator determining a single particular state out of the complete BRST orbit. States which are both closed and co-closed are called (BRST) *harmonic*.

Denoting the subspace of harmonic states by $\mathcal{H}_{harm}$, we can now prove the following theorem: the extended Hilbert space $\mathcal{H}_{ext}$ can be decomposed exactly into three subspaces (Fig. 1):

$$\mathcal{H}_{ext} = \mathcal{H}_{harm} + \mathrm{Im}\,\Omega + \mathrm{Im}\,{}^*\Omega. \tag{2.80}$$

Equivalently, any vector in $\mathcal{H}_{ext}$ can be decomposed as

$$\psi = \omega + \Omega\chi + {}^*\Omega\phi, \qquad \text{where} \quad \Omega\omega = {}^*\Omega\omega = 0. \tag{2.81}$$

We sketch the proof. Denote the space of zero modes of the BRST operator (the BRST-closed vectors) by Ker Ω, and the zero modes of the co-BRST operator (co-closed vectors) by Ker ${}^*\Omega$. Then

$$\psi \in \mathrm{Ker}\,\Omega \quad \Leftrightarrow \quad (\Omega\psi, \phi) = 0, \quad \forall\phi, \quad \Leftrightarrow \quad (\psi, {}^*\Omega\phi) = 0, \quad \forall\phi. \tag{2.82}$$

ψ being orthogonal to all vectors in Im ${}^*\Omega$, it follows that

$$\mathrm{Ker}\,\Omega = \left(\mathrm{Im}\,{}^*\Omega\right)^{\perp}, \tag{2.83}$$

the orthoplement of Im ${}^*\Omega$. Similarly we prove

$$\mathrm{Ker}\,{}^*\Omega = \left(\mathrm{Im}\,\Omega\right)^{\perp}. \tag{2.84}$$

Therefore any vector which is not in Im Ω *and* not in Im ${}^*\Omega$ must belong to the orthoplement of both, i.e. to Ker ${}^*\Omega$ and Ker Ω simultaneously; such a vector is therefore harmonic.

Now as the BRST-operator and the co-BRST operator are both nilpotent,

$$\mathrm{Im}\,\Omega \subset \mathrm{Ker}\,\Omega = \left(\mathrm{Im}\,{}^*\Omega\right)^{\perp}, \qquad \mathrm{Im}\,{}^*\Omega \subset \mathrm{Ker}\,{}^*\Omega = \left(\mathrm{Im}\,\Omega\right)^{\perp}. \tag{2.85}$$

Therefore Im Ω and Im ${}^*\Omega$ have no elements in common (recall that the null-vector is not in the space of states). Obviously, they also have no elements in common with their own orthoplements (because of the non-degeneracy of the inner product), and in particular with $\mathcal{H}_{harm}$, which is the set of common states in both orthoplements. This proves the theorem.

We can define a BRST-laplacian Δ_{BRST} as the semi positive definite self-adjoint operator

$$\Delta_{BRST} = (\Omega + {}^*\Omega)^2 = {}^*\Omega\,\Omega + \Omega\,{}^*\Omega, \tag{2.86}$$

which commutes with both Ω and ${}^*\Omega$. Consider its zero-modes ω:

$$\Delta_{BRST}\,\omega = 0 \quad \Leftrightarrow \quad {}^*\Omega\,\Omega\,\omega + \Omega\,{}^*\Omega\,\omega = 0. \tag{2.87}$$

The left-hand side of the last expression is a sum of a vector in Im Ω and one in Im ${}^*\Omega$; as these subspaces are orthogonal w.r.t. the non-degenerate inner product, it follows that

$${}^*\Omega\,\Omega\,\omega = 0 \quad \wedge \quad \Omega\,{}^*\Omega\,\omega = 0, \tag{2.88}$$

separately. This in turn implies $\Omega\omega = 0$ and ${}^*\Omega\omega = 0$, and ω must be a harmonic state:

$$\Delta_{BRST}\,\omega = 0 \quad \Leftrightarrow \quad \omega \in \mathcal{H}_{harm}; \tag{2.89}$$

hence Ker $\Delta_{BRST} = \mathcal{H}_{harm}$. The BRST-Hodge decomposition theorem can therefore be expressed as

$$\mathcal{H}_{ext} = \mathrm{Ker}\, \Delta_{BRST} + \mathrm{Im}\, \Omega + \mathrm{Im}^* \Omega. \tag{2.90}$$

The BRST-laplacian allows us to discuss the representation theory of BRST-transformations. First of all, the BRST-laplacian commutes with the BRST- and co-BRST operators Ω and ${}^*\Omega$:

$$[\Delta_{BRST}, \Omega] = 0, \qquad [\Delta_{BRST}, {}^*\Omega] = 0. \tag{2.91}$$

As a result, BRST-multiplets can be characterized by the eigenvalues of Δ_{BRST}: the action of Ω or ${}^*\Omega$ does not change this eigenvalue. Basically we must then distinguish between zero-modes and non-zero modes of the BRST-laplacian. The zero-modes, the harmonic states, are BRST-singlets:

$$\Omega|\omega\rangle = 0, \qquad {}^*\Omega|\omega\rangle = 0.$$

In contrast, the non-zero modes occur in pairs of BRST- and co-BRST-exact states:

$$\Delta_{BRST}|\phi_\pm\rangle = \lambda^2|\phi_\pm\rangle \quad \Rightarrow \quad \Omega|\phi_+\rangle = \lambda\,|\phi_-\rangle, \quad {}^*\Omega|\phi_-\rangle = \lambda\,|\phi_+\rangle. \tag{2.92}$$

Eq. (2.73) guarantees that $|\phi_\pm\rangle$ have zero (physical) norm; we can however rescale these states such that

$$\langle\phi_-|\phi_+\rangle = \langle\phi_+|\phi_-\rangle = 1. \tag{2.93}$$

It follows, that the linear combinations

$$|\chi_\pm\rangle = \frac{1}{\sqrt{2}}\left(|\phi_+\rangle \pm |\phi_-\rangle\right) \tag{2.94}$$

define a pair of positive- or negative-norm states:

$$\langle\chi_\pm|\chi_\pm\rangle = \pm 1, \qquad \langle\chi_\mp|\chi_\pm\rangle = 0. \tag{2.95}$$

They are eigenstates of the operator $\Omega + {}^*\Omega$ with eigenvalues $(\lambda, -\lambda)$:

$$(\Omega + {}^*\Omega)|\chi_\pm\rangle = \pm\lambda|\chi_\pm\rangle. \tag{2.96}$$

As physical states must have positive norm, all BRST-doublets must be unphysical, and only BRST-singlets (harmonic) states can represent physical states. Conversely, if all harmonic states are to be physical, only the components of the BRST-doublets are allowed to have non-positive norm. Observe, however, that this condition can be violated if the inner product (,) becomes degenerate on the subspace Im Ω; in that case the harmonic gauge does not remove all freedom to make BRST-transformations and zero-norm states can survive in the subspace of harmonic states.

2.6 BRST operator cohomology

The BRST construction replaces a complete set of constraints, imposed by the generators of gauge transformations, by a single condition: BRST invariance. However, the normalizable solutions of the BRST condition (2.65):

$$\Omega|\Psi\rangle = 0, \qquad \langle\Psi|\Psi\rangle = 1,$$

are not unique: from any solution one can construct an infinite set of other solutions

$$|\Psi'\rangle = |\Psi\rangle + \Omega|\chi\rangle, \qquad \langle\Psi'|\Psi'\rangle = 1, \tag{2.97}$$

provided the BRST operator is self-adjoint w.r.t. the physical inner product. Under the conditions discussed in sect. 2.5 the normalizable part of the state vector is unique, hence the transformed state is not physically different from the original one, a nd we actually identify a single physical state with the complete class of solutions (2.97). As observed before, in this respect the quantum theory in the extended Hilbert space behaves much like an abelian gauge theory, with the BRST transformations acting as gauge transformations.

Keeping this in mind, it is clearly necessary that the action of dynamical observables of the theory on physical states is invariant under BRST transformations: an observable $\mathcal{O}$ maps physical states to physical states; therefore if $|\Psi\rangle$ is a physical state, then

$$\Omega\mathcal{O}|\Psi\rangle = [\Omega, \mathcal{O}]\,|\Psi\rangle = 0. \tag{2.98}$$

Again, the solution of this condition for any given observable is not unique: for an observable with ghost number $N_g = 0$, and any operator Φ with ghost number $N_g = -1$,

$$\mathcal{O}' = \mathcal{O} + [\Omega, \Phi]_+ \tag{2.99}$$

also satisfies condition (2.98). The proof follows directly from the Jacobi identity:

$$\Big[\Omega, [\Omega, \Phi]_+\Big] = \Big[\Omega^2, \Phi\Big] = 0. \tag{2.100}$$

This holds in particular for the hamiltonian; indeed, the time-evolution of states in the unphysical sector (the gauge and ghost fields) is not determined a priori, and can be chosen by an appropriate BRST extension of the hamiltonian:

$$H_{ext} = H_{phys} + [\Omega, \Phi]_+\,. \tag{2.101}$$

Here H_{phys} is the hamiltonian of the physical degrees of freedom. The BRST-exact extension $[\Omega, \Phi]_+$ acts only on the unphysical sector, and can be used to define the dynamics of the gauge- and ghost degrees of freedom.

2.7 Lie-algebra cohomology

We illustrate the BRST construction with a simple example: a system of constraints defining an ordinary n-dimensional compact Lie-algebra [25]. The Lie algebra is taken to be a direct sum of semi-simple and abelian $u(1)$ algebras, of the form

$$[G_a, G_b] = i f_{ab}^{\;\;c} G_c, \qquad (a,b,c) = 1, ..., n, \tag{2.102}$$

where the generators G_a are hermitean, and the $f_{ab}^{\;\;c} = -f_{ba}^{\;\;c}$ are real structure constants. We assume the generators normalized such that the Killing metric is unity:

$$-\frac{1}{2}\, f_{ac}^{\;\;d} f_{bd}^{\;\;c} = \delta_{ab}. \tag{2.103}$$

Then $f_{abc} = f_{ab}^{\;\;d}\delta_{dc}$ is completely anti-symmetric. We introduce ghost operators (c^a, b_b) with canonical anti-commutation relations (2.54):

$$[c^a, b_b]_+ = \delta^a_b, \qquad \left[c^a, c^b\right]_+ = [b_a, b_b]_+ = 0.$$

This implies, that in the 'co-ordinate representation', in which the ghosts c^a are represented by Grassmann variables, the b_a can be represented by a Grassmann derivative:

$$b_a = \frac{\partial}{\partial c^a}. \tag{2.104}$$

The nilpotent BRST operator takes the simple form

$$\Omega = c^a G_a - \frac{i}{2}\, c^a c^b f_{ab}^{\;\;c} b_c, \qquad \Omega^2 = 0. \tag{2.105}$$

We define a ghost-extended state space with elements

$$\psi[c] = \sum_{k=0}^{n} \frac{1}{k!}\, c^{a_1} ... c^{a_k}\, \psi^{(k)}_{a_1...a_k}. \tag{2.106}$$

The coefficients $\psi^{(k)}_{a_1..a_k}$ of ghost number k carry completely anti-symmetric product representations of the Lie algebra.

On the state space we introduce an indefinite inner product, with respect to which the ghosts c^a and b_a are self-adjoint; this is realized by the Berezin integral over the ghost variables

$$\langle \phi, \psi \rangle = \int [dc^n ... dc^1]\, \phi^\dagger\, \psi = \frac{1}{n!}\, \varepsilon^{a_1...a_n} \sum_{k=0}^{n} \left(\begin{array}{c} n \\ k \end{array} \right) \phi^{(n-k)\,*}_{a_{n-k}...a_1}\, \psi^{(k)}_{a_{n-k+1}...a_n}. \tag{2.107}$$

In components, the action of the ghosts is given by

$$(c^a \psi)^{(k)}_{a_1...a_k} = \delta^a_{a_1} \psi^{(k-1)}_{a_2 a_3 ... a_k} - \delta^a_{a_2} \psi^{(k-1)}_{a_1 a_3 ... a_k} + ... + (-1)^{k-1} \delta^a_{a_k} \psi^{(k-1)}_{a_1 a_2 ... a_{k-1}}, \tag{2.108}$$

and similarly

$$(b_a \psi)^{(k)}_{a_1 \dots a_k} = \left(\frac{\partial \psi}{\partial c^a} \right)^{(k)}_{a_1 \dots a_k} = \psi^{(k+1)}_{a a_1 \dots a_k}. \tag{2.109}$$

It is now easy to check, that the ghost operators are self-adjoint w.r.t. the inner product (2.107):

$$\langle \phi, c^a \psi \rangle = \langle c^a \phi, \psi \rangle, \qquad \langle \phi, b_a \psi \rangle = \langle b_a \phi, \psi \rangle. \tag{2.110}$$

It follows directly, that the BRST operator (2.105) is self-adjoint as well:

$$\langle \phi, \Omega \psi \rangle = \langle \Omega \phi, \psi \rangle. \tag{2.111}$$

Now we can introduce a second inner product, which is positive definite and therefore manifestly non-degenerate:

$$(\phi, \psi) = \sum_{k=0}^{n} \frac{1}{k!} \left(\phi^{(k)\,*} \right)^{a_1 \dots a_k} \psi^{(k)}_{a_1 \dots a_k}. \tag{2.112}$$

It is related to the first indefinite inner product by Hodge duality: define the Hodge $*$-operator by

$${}^*\psi^{(k)\, a_1 \dots a_k} = \frac{1}{(n-k)!} \, \varepsilon^{a_1 \dots a_k a_{k+1} \dots a_n} \, \psi^{(n-k)}_{a_{k+1} \dots a_n}. \tag{2.113}$$

Furthermore, define the ghost permutation operator $\mathcal{P}$ as the operator which reverses the order of the ghosts in $\psi[c]$; equivalently:

$$(\mathcal{P}\psi)^{(k)}_{a_1 \dots a_k} = \psi^{(k)}_{a_k \dots a_1}. \tag{2.114}$$

Then the two inner products are related by

$$(\phi, \psi) = \langle \mathcal{P}\, {}^*\phi, \psi \rangle. \tag{2.115}$$

An important property of the non-degenerate inner product is, that the ghosts c^a and b_a are adjoint to one another:

$$(\phi, c^a \psi) = (b_a \phi, \psi). \tag{2.116}$$

Then the adjoint of the BRST operator is given by the co-BRST operator

$${}^*\Omega = b_a G^a - \frac{i}{2} \, c^c \, f^{ab}_{\;\;\,c} \, b_a b_b. \tag{2.117}$$

Here raising and lowering indices on the generators and structure constants is done with the help of the Killing metric (δ_{ab} in our normalization). It is easy to check, that ${}^*\Omega^2 = 0$, as expected.

The harmonic states are both BRST- and co-BRST-closed: $\Omega\psi = {}^*\Omega\psi = 0$. They are zero-modes of the BRST-laplacian:

$$\Delta_{BRST} = {}^*\Omega\,\Omega + \Omega{}^*\Omega = \left({}^*\Omega + \Omega\right)^2, \tag{2.118}$$

as follows from the observation that

$$(\psi, \Delta_{BRST}\,\psi) = (\Omega\psi, \Omega\psi) + ({}^*\Omega\psi, {}^*\Omega\psi) = 0 \quad \Leftrightarrow \quad \Omega\psi = {}^*\Omega\psi = 0. \tag{2.119}$$

For the case at hand, these conditions become

$$G_a\psi = 0, \qquad \Sigma_a\psi = 0, \tag{2.120}$$

where Σ_a is defined as

$$\Sigma_a = \Sigma_a^\dagger = -i f_{ab}{}^c c^b b_c. \tag{2.121}$$

From the Jacobi identity it is quite easy to verify that Σ_a defines a representation of the Lie-algebra:

$$[\Sigma_a, \Sigma_b] = i f_{ab}{}^c \Sigma_c, \qquad [G_a, \Sigma_b] = 0. \tag{2.122}$$

The conditions (2.120) are proven as follows. Substitute the explicit expressions for Ω and ${}^*\Omega$ into eq.(2.118) for Δ_{BRST}. After some algebra one then finds

$$\Delta_{BRST} = G^2 + G\cdot\Sigma + \frac{1}{2}\,\Sigma^2 = \frac{1}{2}\,G^2 + \frac{1}{2}\,(G+\Sigma)^2. \tag{2.123}$$

This being a sum of squares, any zero mode must satisfy (2.120). Q.E.D.

Looking for solutions, we observe that in components the second condition reads

$$(\Sigma_a\psi)^{(k)}_{a_1\ldots a_k} = -i f_{a[a_1}{}^b \psi^{(k)}_{a_2\ldots a_k]b} = 0. \tag{2.124}$$

It acts trivially on states of ghost number $k = 0$; hence bona fide solutions are the gauge-invariant states of zero ghost number:

$$\psi = \psi^{(0)}, \qquad G_a\psi^{(0)} = 0. \tag{2.125}$$

However, other solutions with non-zero ghost number exist. A general solution is for example

$$\psi = \frac{1}{3!}\,f_{abc}\,c^a c^b c^c\,\chi, \qquad G_a\chi = 0. \tag{2.126}$$

The 3-ghost state $\psi^{(3)}_{abc} = f_{abc}\chi$ indeed satisfies (2.124) as a result of the Jacobi identity. The states χ are obviously in one-to-one correspondence with the states $\psi^{(0)}$. Hence in general there exist several copies of the space of physical states in the BRST cohomology, at different ghost number. We infer, that in addition to requiring physical states to belong to the BRST cohomology, it is also necessary to fix the ghost number for the definition of physical states to be unique.

Chapter 3

Action formalism

The canonical construction of the BRST cohomology we have described, can be given a basis in the action formulation, either in lagrangean or hamiltonian form. The latter one relates most directly to the canonical bracket formulation. It is then straightforward to switch to a gauge-fixed lagrangean formulation. Once we have the lagrangean formulation, a covariant approach to gauge-fixing and quantization can be developed. In this chapter these constructions are presented, and the relations between various formulations are discussed.

3.1 BRST invariance from Hamilton's principle

We have observed in section 2.6, that the effective hamiltonian in the ghost-extended phase space is defined only modulo BRST-exact terms:

$$H_{eff} = H_c + i \{\Omega, \Psi\} = H_c - i\delta_{\Omega}\Psi, \tag{3.1}$$

where Ψ is a function of the phase space variables with ghost number $N_g(\Psi) = -1$. Moreover, the ghosts (c, b) are canonically conjugate:

$$\{c^{\alpha}, b_{\beta}\} = -i\delta_{\beta}^{\alpha}.$$

Thus we are lead to construct a pseudo-classical action of the form

$$S_{eff} = \int dt \left(p_i \dot{q}^i + i b_{\alpha} \dot{c}^{\alpha} - H_{eff} \right). \tag{3.2}$$

That this is indeed the correct action for our purposes follows from the ghost equations of motion obtained from this action, reading

$$\dot{c}^{\alpha} = -i \frac{\partial H_{eff}}{\partial b_{\alpha}}. \qquad \dot{b}_{\alpha} = -i \frac{\partial H_{eff}}{\partial c^{\alpha}}. \tag{3.3}$$

These equations are in full agreement with the definition of the extended Poisson brackets (2.23):

$$\dot{c}^\alpha = -\left\{H_{eff}, c^\alpha\right\}, \qquad \dot{b}_\alpha = -\left\{H_{eff}, b_\alpha\right\}. \tag{3.4}$$

As H_c is BRST invariant, H_{eff} is BRST-invariant as well: the BRST variations are nilpotent and therefore $\delta_\Omega^2 \Phi = 0$. It is then easy to show, that the action S_{eff} is BRST-symmetric, and the conserved Noether charge is the BRST charge as defined previously:

$$\begin{aligned}
\delta_\Omega S_{eff} &= \int dt \left[\left(\delta_\Omega p_i \dot{q}^i - \delta_\Omega q^i \dot{p}_i + i\delta_\Omega b_\alpha \dot{c}^\alpha + i\delta_\Omega c^\alpha \dot{b}_\alpha - \delta_\Omega H_{eff}\right)\right. \\
&\qquad\qquad \left. + \frac{d}{dt}(p_i \delta_\Omega q^i - i b_\alpha \delta_\Omega c^\alpha)\right] \qquad (3.5) \\
&= \int dt \frac{d}{dt}\left(p_i \delta_\Omega q^i - i b_\alpha \delta_\Omega c^\alpha - \Omega\right).
\end{aligned}$$

To obtain the last equality we have used eqs.(2.35) and (2.36), which can be summarized

$$\delta_\Omega q^i = \frac{\partial \Omega}{\partial p_i}, \qquad \delta_\Omega p_i = -\frac{\partial \Omega}{\partial q^i},$$

$$\delta_\Omega c^\alpha = i\frac{\partial \Omega}{\partial b_\alpha}, \qquad \delta_\Omega b_\alpha = i\frac{\partial \Omega}{\partial c^\alpha}.$$

The action is therefore invariant up to a total time-derivative, and by comparison with eq.(1.59) we conclude, that Ω is the conserved Noether charge.

3.2 Examples

1. *The relativistic particle.* A simple example of the procedure presented above is the relativistic particle [26]. The canonical hamiltonian H_0 is constrained to vanish itself. As a result, the effective hamiltonian is a pure BRST term:

$$H_{eff} = i\left\{\Omega, \Psi\right\}. \tag{3.6}$$

A simple choice for the gauge fermion is $\Psi = b$, which has the correct ghost number $N_g = -1$. With this choice, and the BRST generator Ω of eq.(2.44), the effective hamiltonian is

$$H_{eff} = i\left\{\frac{c}{2m}(p^2 + m^2), b\right\} = \frac{1}{2m}\left(p^2 + m^2\right). \tag{3.7}$$

Then the effective action becomes

$$S_{eff} = \int d\tau \left(p \cdot \dot{x} + ib\dot{c} - \frac{1}{2m}(p^2 + m^2)\right). \tag{3.8}$$

This action is invariant under the BRST transformations (2.45) :

$$\delta_\Omega x^\mu = \{x^\mu, \Omega\} = \frac{cp^\mu}{m}, \quad \delta_\Omega p_\mu = \{p_\mu, \Omega\} = 0,$$

$$\delta_\Omega c = -\{c, \Omega\} = 0, \qquad \delta_\Omega b = -\{b, \Omega\} = \frac{i}{2m}\,(p^2 + m^2),$$

up to a total proper-time derivative:

$$\delta_\Omega S_{eff} = \int d\tau \frac{d}{d\tau}\left[c\left(\frac{p^2 - m^2}{2m}\right)\right]. \tag{3.9}$$

Implementing the Noether construction, the conserved charge resulting from the BRST transformations is

$$\Omega = p \cdot \delta_\Omega x + ib\,\delta_\Omega c - \frac{c}{2m}\,(p^2 - m^2) = \frac{c}{2m}\,(p^2 + m^2). \tag{3.10}$$

Thus we have reobtained the BRST charge from the action (3.8) and the transformations (2.45), confirming that together with the BRST-cohomology principle, they correctly describe the dynamics of the relativistic particle.

From the hamiltonian formulation (3.8) it is straightforward to construct a lagrangean one by using the hamilton equation $p^\mu = m\dot{x}^\mu$ to eliminate the momenta as independent variables; the result is

$$S_{eff} \simeq \int d\tau \left(\frac{m}{2}\,(\dot{x}^2 - 1) + ib\dot{c}\right). \tag{3.11}$$

2. *Maxwell-Yang-Mills theory.* The BRST generator of the Maxwell-Yang-Mills theory in the temporal gauge has been given in (2.48):

$$\Omega = \int d^3x \left(c^a G_a - \frac{ig}{2}\, f_{ab}{}^c c^a c^b b_c \right),$$

with $G_a = (\vec{D}\cdot\vec{E})_a$. The BRST-invariant effective hamiltonian takes the form

$$H_{eff} = \frac{1}{2}\left(\vec{E}_a^2 + \vec{B}_a^2\right) + i\,\{\Omega, \Psi\}. \tag{3.12}$$

A simple choice of the gauge fermion: $\Psi = \lambda^a b_a$, with λ_a some constants, then gives a effective action

$$S_{eff} = \int d^4x \left[-\vec{E}\cdot\frac{\partial\vec{A}}{\partial t} + ib_a\dot{c}^a - \frac{1}{2}\left(\vec{E}_a^2 + \vec{B}_a^2\right) - \lambda^a(\vec{D}\cdot\vec{E})_a + ig\lambda^a f_{ab}{}^c c^b b_c\right]. \tag{3.13}$$

The choice $\lambda^a = 0$ would in effect turn the ghosts into free fields. However, if we eliminate the electric fields $\vec{E}^a$ as independend degrees of freedom by the substitution $E_i^a = F_{i0}^a = \partial_i A_0^a - \partial_0 A_i^a - g f_{bc}{}^a A_i^b A_0^c$, and recalling the classical hamiltonian (1.127), we observe that we might actually interpret λ^a as a constant scalar potential $A_0^a = \lambda^a$, in a BRST-extended relativistic action

$$S_{eff} = \int d^4x \left[-\frac{1}{4} (F_{\mu\nu}^a)^2 + i b_a (D_0 c)^a \right]_{A_0^a = \lambda^a}, \tag{3.14}$$

where $(D_0 c)^a = \partial_0 c^a - g f_{bc}{}^a A_0^b c^c$. The action is invariant under the classical BRST transformations (2.49):

$$\delta_\Omega \vec{A}^a = (\vec{D}c)^a, \qquad \delta_\Omega \vec{E}_a = g f_{ab}{}^c c^b \vec{E}_c,$$

$$\delta_\Omega c^a = \frac{g}{2} f_{bc}{}^a\, c^b c^c, \quad \delta_\Omega b_a = i\, G_a + g f_{ab}{}^c\, c^b\, b_c,$$

with the above BRST generator (2.48) as the conserved Noether charge. All of the above applies to Maxwell electrodynamics as well, except that in an abelian theory there is only a single vector field, and all structure constants vanish: $f_{ab}{}^c = 0$.

3.3 Lagrangean BRST formalism

From the hamiltonian formulation of BRST-invariant dynamical systems it is straightforward to develop an equivalent lagrangean formalism, by eliminating the momenta p_i as independent degrees of freedom. This proceeds as usual by solving Hamilton's equation

$$\dot{q}^i = \frac{\partial H}{\partial p_i},$$

for the momenta in terms of the velocities, and performing the inverse Legendre transformation. We have already seen how this works for the examples of the relativistic particle and the Maxwell-Yang-Mills theory. As the lagrangean is a scalar function under space-time transformations, it is better suited for the development of a manifestly covariant formulation of gauge-fixed BRST-extended dynamics of theories with local symmetries, including Maxwell-Yang-Mills theory and the relativistic particle as well as string theory and general relativity.

The procedure follows quite naturally the steps outlined in the previous sections (3.1 and 3.2):

a. Start from a gauge-invariant lagrangean $L_0(q, \dot{q})$.

b. For each gauge degree of freedom (each gauge parameter), introduce a ghost variable c^a; by definition these ghost variables carry ghost number $N_g[c^a] = +1$. Construct BRST transformations $\delta_\Omega X$ for the extended configuration-space variables $X = (q^i, c^a)$, satisfying the requirement that they leave L_0 invariant (possibly modulo a total derivative), and are nilpotent: $\delta_\Omega^2 X = 0$.

c. Add a trivially BRST-invariant set of terms to the action, of the form $\delta_\Omega \Psi$ for some anti-commuting function Ψ (the gauge fermion).

The last step is to result in an effective lagrangean L_{eff} with net ghost number $N_g[L_{eff}] = 0$. To achieve this, the gauge fermion must have ghost number $N_g[\Psi] = -1$. However, so far we only have introduced dynamical variables with non-negative ghost number: $N_g[q^i, c^a] = (0, +1)$. To solve this problem we introduce anti-commuting anti-ghosts b_a, with ghost number $N_g[b_a] = -1$. The BRST-transforms of these variables must then be commuting objects α_a, with ghost number $N_g[\alpha] = 0$. In order for the BRST-transformations to be nilpotent, we require

$$\delta_\Omega \, b_a = i\alpha_a, \qquad \delta_\Omega \, \alpha_a = 0, \tag{3.15}$$

which indeed trivially satisfy $\delta_\Omega^2 = 0$. The examples of the previous section illustrate this procedure.

1. *Relativistic particle.* The starting point for the description of the relativistic particle was the reparametrization-invariant action (1.8). We identify the integrand as the lagrangean L_0. Next we introduce the Grassmann-odd ghost variable $c(\lambda)$, and define the BRST transformations

$$\delta_\Omega \, x^\mu = c \, \frac{dx^\mu}{d\lambda}, \qquad \delta_\Omega \, e = \frac{d(ce)}{d\lambda}, \qquad \delta_\Omega \, c = c \, \frac{dc}{d\lambda}. \tag{3.16}$$

As $c^2 = 0$, these transformations are nilpotent indeed. In addition, introduce the anti-ghost representation (b, α) with the transformation rules (3.15). We can now construct a gauge fermion. We make the choice

$$\Psi(b, e) = b(e-1) \quad \Rightarrow \quad \delta_\Omega \, \Psi = i\alpha(e-1) - b \, \frac{d(ce)}{d\lambda}. \tag{3.17}$$

As a result, the effective lagrangean (in natural units) becomes

$$L_{eff} = L_0 - i\delta_\Omega \Psi = \frac{m}{2e} \frac{dx_\mu}{d\lambda} \frac{dx^\mu}{d\lambda} - \frac{em}{2} + \alpha(e-1) + ib \, \frac{d(ce)}{d\lambda}. \tag{3.18}$$

Observing that the variable α plays the role of a lagrange multiplier, fixing the einbein to its canonical value $e = 1$ such that $d\lambda = d\tau$, this lagrangean is seen to reproduce the action (3.11):

$$S_{eff} = \int d\tau L_{eff} \simeq \int d\tau \left(\frac{m}{2} \left(\dot{x}^2 - 1 \right) + ib \, \dot{c} \right).$$

2. *Maxwell-Yang-Mills theory.* The covariant classical action of the Maxwell-Yang-Mills theory was presented in eq.(1.121):

$$S_0 = -\frac{1}{4} \int d^4x \, \left(F_{\mu\nu}^a \right)^2.$$

Introducing the ghost fields c^a, we can define nilpotent BRST transformations

$$\delta_\Omega A^a_\mu = (D_\mu c)^a, \qquad \delta_\Omega c^a = \frac{g}{2} f_{bc}{}^a c^b c^c. \tag{3.19}$$

Next we add the anti-ghost BRST multiplets (b_a, α_a), with the transformation rules (3.15). Choose the gauge fermion

$$\Psi(A^a_0, b_a) = b_a(A^a_0 - \lambda^a) \quad \Rightarrow \quad \delta_\Omega \Psi = i\alpha_a(A^a_0 - \lambda^a) - b_a(D_0 c)^a, \tag{3.20}$$

where λ^a are some constants (possibly zero). Adding this to the classical action gives

$$S_{eff} = \int d^4x \left[-\frac{1}{4} (F^a_{\mu\nu})^2 + \alpha_a(A^a_0 - \lambda^a) + ib_a(D_0 c)^a \right]. \tag{3.21}$$

Again, the fields α_a act as lagrange multipliers, fixing the electric potentials to the constant values λ^a. After substitution of these values, the action reduces to the form (3.14).

We have thus demonstrated that the lagrangean and canonical procedures lead to equivalent results; however, we stress that in both cases the procedure involves the choice of a gauge fermion Ψ, restricted by the requirement that it has ghost number $N_g[\Psi] = -1$.

The advantage of the lagrangean formalism is, that it is easier to formulate the theory with different choices of the gauge fermion. In particular, it is possible to make choices of gauge which manifestly respect the Lorentz-invariance of Minkoswki space. This is not an issue for the study of the relativistic particle, but it is an issue in the case of Maxwell-Yang-Mills theory, which we have constructed so far only in the temporal gauge $A^a_0 =$ constant.

We now show how to construct a covariant gauge-fixed and BRST-invariant effective lagrangean for Maxwell-Yang-Mills theory, using the same procedure. In stead of (3.20), we choose the gauge fermion

$$\Psi = b_a \left(\partial \cdot A^a - \frac{\lambda}{2} \alpha^a \right) \quad \Rightarrow \quad \delta_\Omega \Psi = i\alpha_a\, \partial \cdot A^a - \frac{i\lambda}{2} \alpha^2_a - b_a\, \partial \cdot (Dc)^a. \tag{3.22}$$

Here the parameter λ is a arbitrary real number, which can be used to obtain a convenient form of the propagator in perturbation theory. The effective action obtained with this choice of gauge-fixing fermion is, after a partial integration:

$$S_{eff} = \int d^4x \left[-\frac{1}{4} (F^a_{\mu\nu})^2 + \alpha_a\, \partial \cdot A^a - \frac{\lambda}{2} \alpha^2_a - i\partial b_a \cdot (Dc)^a \right]. \tag{3.23}$$

As we have introduced quadratic terms in the bosonic variables α_a, they now behave more like auxiliary fields, rather than lagrange multipliers. Their variational equations lead to the result

$$\alpha^a = \frac{1}{\lambda}\, \partial \cdot A^a. \tag{3.24}$$

Eliminating the auxiliary fields by this equation, the effective action becomes

$$S_{eff} = \int d^4x \left[-\frac{1}{4} (F^a_{\mu\nu})^2 + \frac{1}{2\lambda} (\partial \cdot A^a)^2 - i\partial b_a \cdot (Dc)^a \right]. \tag{3.25}$$

This is the standard form of the Yang-Mills action used in covariant perturbation theory. Observe, that the elimination of the auxiliary field α_a also changes the BRST-transformation of the anti-ghost b_a to:

$$\delta_\Omega\, b^a = \frac{i}{\lambda}\, \partial \cdot A^a \quad \Rightarrow \quad \delta_\Omega^2\, b^a = \frac{i}{\lambda}\, \partial \cdot (Dc)^a \simeq 0. \tag{3.26}$$

The transformation is now nilpotent only after using the ghost field equation.

The BRST-Noether charge can be computed from the action (3.25) by the standard procedure, and leads to the expression

$$\Omega = \int d^3x \left(\pi^\mu_a (D_\mu c)^a - \frac{ig}{2}\, f_{ab}^{\;\;c} c^a c^b \gamma_c \right), \tag{3.27}$$

where π^μ_a is the canonical momentum of the vector potential A^a_μ, and (β^a, γ_a) denote the canonical momenta of the ghost fields (b_a, c^a):

$$\begin{aligned} \pi^i_a &= \frac{\partial \mathcal{L}_{eff}}{\partial \dot{A}^a_i} = -F^{0i}_a = -E^i_a, \quad \pi^0_a = \frac{\partial \mathcal{L}_{eff}}{\partial \dot{A}^a_0} = -\frac{1}{\lambda}\, \partial \cdot A_a, \\ \beta^a &= i \frac{\partial \mathcal{L}_{eff}}{\partial \dot{b}_a} = -(D_0 c)^a, \qquad \gamma_a = i \frac{\partial \mathcal{L}_{eff}}{\partial \dot{c}^a} = \partial_0 b_a. \end{aligned} \tag{3.28}$$

Each ghost field (b_a, c^a) now has its own conjugate momentum, because the ghost terms in the action (3.25) are quadratic in derivatives, rather than linear as before. Note also, that a factor i has been absorbed in the ghost momenta to make them real; this leads to the standard Poisson brackets

$$\{c^a(\vec{x};t), \gamma_b(\vec{y};t)\} = -i\delta^a_b \delta^3(\vec{x} - \vec{y}), \qquad \left\{ b_a(\vec{x};t), \beta^b(\vec{y};t) \right\} = -i\delta^b_a \delta^3(\vec{x} - \vec{y}). \tag{3.29}$$

As our calculation shows, all explicit dependence on (b_a, β^a) has dropped out of the expression (3.27) for the BRST charge.

The parameter λ is still a free parameter, and in actual calculations it is often useful to check partial gauge-independence of physical results, like cross sections, by establishing that they do not depend on this parameter. What needs to be shown more generally is, that physical results do not depend on the choice of gauge fermion. This follows formally from the BRST cohomology being independent of the choice of gauge fermion. Indeed, from the expression (3.27) for Ω we observe that it is of the same form as the one we have used previously in the temporal gauge, even though now π^0_a no longer vanishes identically. In the quantum theory this implies, that the BRST-cohomology classes at ghost number zero correspond to gauge-invariant states, in which

$$\left(\vec{D}\cdot\vec{E}\right)^a = 0, \qquad \partial\cdot A^a = 0. \tag{3.30}$$

The second equation implies, that the time-evolution of the 0-component of the vector potential is fixed completely by the initial conditions and the evolution of the spatial components $\vec{A}^a$. In particular, $A_0^a = \lambda^a = \text{constant}$ is a consistent solution if by a gauge transformation we take the spatial components to satisfy $\vec{\nabla}\cdot\vec{A}^a = 0$.

In actual computations, especially in perturbation theory, the matter is more subtle however: the theory needs to be renormalized, and this implies that the action and BRST-transformation rules have to be adjusted to the introduction of counter terms. To prove the gauge independence of the renormalized theory it must be shown, that the renormalized action still possesses a BRST-invariance, and the cohomology classes at ghost-number zero satisfy the renormalized conditions (3.30). In four-dimensional space-time this can indeed be done for the pure Maxwell-Yang-Mills theory, as there exists a manifestly BRST-invariant regularization scheme (dimensional regularization) in which the theory defined by the action (3.25) is renormalizable by power counting. The result can be extended to gauge theories interacting with scalars and spin-1/2 fermions, except for the case in which the Yang-Mills fields interact with chiral fermions in anomalous representations of the gauge group.

3.4 The master equation

Consider a BRST-invariant action $S_{eff}[\Phi^A] = S_0 + \int dt\,(i\delta_\Omega\Psi)$, where the variables $\Phi^A = (q^i, c^a, b_a, \alpha_a)$ parametrize the extended configuration space of the system, and Ψ is the gauge fermion, which is Grassmann-odd and has ghost number $N_g[\Psi] = -1$. Now by construction

$$\delta_\Omega\Psi = \delta_\Omega\Phi^A\,\frac{\partial\Psi}{\partial\Phi^A}, \tag{3.31}$$

and therefore we can write the effective action also as

$$S_{eff}[\Phi^A] = S_0 + i\int dt\,\left[\delta_\Omega\Phi^A\,\Phi_A^*\right]_{\Phi_A^* = \frac{\partial\Psi}{\partial\Phi^A}}. \tag{3.32}$$

This way of writing considers the action as a functional on a doubled configuration space, parametrized by variables (Φ^A, Φ_A^*), the first set Φ^A being called the *fields*, and the second set Φ_A^* called the *anti-fields*. In the generalized action

$$S^*[\Phi^A, \Phi_A^*] = S_0 + i\int dt\,\delta_\Omega\Phi^A\,\Phi_A^*, \tag{3.33}$$

the anti-fields play the role of sources for the BRST-variations of the fields Φ^A; the effective action S_{eff} is the restriction to the hypersurface $\Sigma[\Psi] : \Phi^*_A = \partial\Psi/\partial\Phi^A$. We observe, that by construction the antifields have Grassmann parity opposite to that of the corresponding fields, and ghost number $N_g[\Phi^*_A] = -(N_g[\Phi^A] + 1)$.

In the doubled configuration space the BRST variations of the fields can be written as

$$i\delta_\Omega \Phi^A = (-1)^A \frac{\delta S^*}{\delta \Phi^*_A}, \tag{3.34}$$

where $(-1)^A$ is the Grassmann parity of the field Φ^A, whilst $-(-1)^A = (-1)^{A+1}$ is the Grassmann parity of the anti-field Φ^*_A. We now define the *anti-bracket* of two functionals $F(\Phi^A, \Phi^*_A)$ and $G(\Phi^A, \Phi^*_A)$ on the large configuration space by

$$(F, G) = (-1)^{F+G+FG} (G, F) = (-1)^{A(F+1)} \left(\frac{\delta F}{\delta \Phi^A} \frac{\delta G}{\delta \Phi^*_A} + (-1)^F \frac{\delta F}{\delta \Phi^*_A} \frac{\delta G}{\delta \Phi^A} \right). \tag{3.35}$$

These brackets are symmetric in F and G if both are Grassmann-even (bosonic), and anti-symmetric in all other cases. Sometimes one introduces the notion of *right derivative*:

$$\frac{F \overleftarrow{\delta}}{\delta \Phi^A} \equiv (-1)^{A(F+1)} \frac{\delta F}{\delta \Phi^A}. \tag{3.36}$$

Then the anti-brackets take the simple form

$$(F, G) = \frac{F \overleftarrow{\delta}}{\delta \Phi^A} \frac{\overrightarrow{\delta} G}{\delta \Phi^*_A} - \frac{F \overleftarrow{\delta}}{\delta \Phi^*_A} \frac{\overrightarrow{\delta} G}{\delta \Phi^A}, \tag{3.37}$$

where the derivatives with a right arrow denote the standard *left* derivatives. In terms of the anti-brackets, the BRST transformations (3.34) can be written in the form

$$i\delta_\Omega \, \Phi^A = (S^*, \Phi^A). \tag{3.38}$$

In analogy, we can define

$$i\delta_\Omega \, \Phi^*_A = (S^*, \Phi^*_A) = (-1)^A \frac{\delta S^*}{\delta \Phi^A}. \tag{3.39}$$

Then the BRST transformation of any functional $Y(\Phi^A, \Phi^*_A)$ is given by

$$i\delta_\Omega Y = (S^*, Y). \tag{3.40}$$

In particular, the BRST-invariance of the action S^* can be expressed as

$$(S^*, S^*) = 0. \tag{3.41}$$

This equation is known as the *master equation*. The formalism presented here was initiated in the work by Zinn-Justin [27] and Batalin and Vilkovisky [28].

Next we observe, that on the physical hypersurface $\Sigma[\Psi]$ the BRST transformations of the antifields are given by the classical field equations; indeed, introducing an anti-commuting parameter μ for infinitesimal BRST transformations

$$i\mu\,\delta_\Omega\Phi^*_A = \frac{\delta S^*}{\delta\Phi^A}\,\mu \quad \stackrel{\Sigma[\Psi]}{\longrightarrow} \quad \frac{\delta S_{eff}}{\delta\Phi^A}\,\mu \simeq 0, \tag{3.42}$$

where the last equality holds only for solutions of the classical field equations. Because of this result, it is customary to redefine the BRST transformations of the antifields such that they vanish:

$$\delta_\Omega\Phi^*_A = 0, \tag{3.43}$$

instead of (3.39). As the BRST transformations are nilpotent, this is consistent with the identification $\Phi^*_A = \partial\Psi/\partial\Phi^A$ in the action; indeed, it now follows that

$$\delta_\Omega\left(\delta_\Omega\Phi^A\,\Phi^*_A\right) = 0, \tag{3.44}$$

which holds before the identification as a result of (3.43), and after the identification because it reduces to $\delta_\Omega^2\Psi = 0$. Note, that the condition for BRST invariance of the action now becomes

$$i\delta_\Omega S^* = \frac{1}{2}\,(S^*, S^*) = 0, \tag{3.45}$$

which still implies the master equation (3.41).

3.5 Path-integral quantization

The construction of BRST-invariant actions $S_{eff} = S^*[\Phi^*_A = \partial\Psi/\partial\Phi^A]$ and the anti-bracket formalism is especially useful in the context of path-integral quantization. The path integral provides a representation of the matrix elements of the evolution operator in the configuration space:

$$\langle q_f, T/2|e^{-iTH}|q_i, -T/2\rangle = \int_{q_i}^{q_f} Dq(t)\, e^{i\int_{-T/2}^{T/2} L(q,\dot{q})dt}. \tag{3.46}$$

In field theory one usually considers the vacuum-to-vacuum amplitude in the presence of sources, which is a generating functional for time-ordered vacuum Green's functions:

$$Z[J] = \int D\Phi\, e^{iS[\Phi]+i\int J\Phi}, \tag{3.47}$$

such that

$$\langle 0|T(\Phi_1...\Phi_k)|0\rangle = \left.\frac{\delta^k Z[J]}{\delta J_1...\delta J_k}\right|_{J=0}. \tag{3.48}$$

The corresponding generating functional $W[J]$ for the connected Green's functions is related to $Z[J]$ by

$$Z[J] = e^{i\,W[J]}. \tag{3.49}$$

For theories with gauge invariances, the evolution operator is constructed from the BRST-invariant hamiltonian; then the action to be used is the in the path integral (3.47) is the BRST invariant action:

$$Z[J] = e^{i\,W[J]} = \int D\Phi^A \;\; e^{i\,S^*[\Phi^A,\Phi^*_A]+i\int J_A\Phi^A}\Big|_{\Phi^*_A=\partial\Psi/\partial\Phi^A}, \tag{3.50}$$

where the sources J_A for the fields are supposed to be BRST invariant themselves. For the complete generating functional to be BRST invariant, it is not sufficient that only the action S^* is BRST invariant, as guaranteed by the master equation (3.41): the functional integration measure must be BRST invariant as well. Under an infinitesimal BRST transformation $\mu\delta_\Omega\Phi^A$ the measure changes by a graded jacobian (superdeterminant) [18, 19]

$$\mathcal{J} = \mathrm{SDet}\left(\delta^A_B + \mu(-1)^B\,\frac{\delta(\delta_\Omega\Phi^A)}{\delta\Phi^B}\right) \approx 1 + \mu\,\mathrm{Tr}\frac{\delta(\delta_\Omega\Phi^A)}{\delta\Phi^B}. \tag{3.51}$$

We now define

$$\frac{\delta(i\delta_\Omega\Phi^A)}{\delta\Phi^A} = (-1)^A\,\frac{\delta^2 S^*}{\delta\Phi^A\delta\Phi^*_A} \equiv \bar{\Delta}S^*. \tag{3.52}$$

The operator $\bar{\Delta}$ defined by

$$\bar{\Delta} = (-1)^A\,\frac{\delta^2}{\delta\Phi^A\delta\Phi^*_A}. \tag{3.53}$$

is a laplacian on the field/anti-field configuration space, with the property $\bar{\Delta}^2 = 0$. The condition of invariance of the measure requires the BRST jacobian (3.51) to be unity:

$$\mathcal{J} = 1 - i\mu\,\bar{\Delta}S^* = 1, \tag{3.54}$$

which reduces to the vanishing of the laplacian of S^*:

$$\bar{\Delta}S^* = 0. \tag{3.55}$$

The two conditions (3.41) and (3.55) imply the BRST invariance of the path integral (3.50). Actually, a somewhat more general situation is possible, in which neither the action nor the functional measure are invariant independently, only the combined functional integral. Let the action generating the BRST transformations be denoted by $W^*[\Phi^A,\Phi^*_A]$:

$$i\delta_\Omega\Phi^A = (W^*,\Phi^A), \qquad i\delta_\Omega\Phi^*_A = 0. \tag{3.56}$$

As a result the graded jacobian for a transformation with parameter μ is

$$\mathrm{SDet}\left(\delta_B^A + \mu(-1)^B\, \frac{\delta(\delta_\Omega \Phi^A)}{\delta \Phi^B}\right) \approx 1 - i\mu\, \bar{\Delta} W^*. \tag{3.57}$$

Then the functional W^* itself needs to satisfy the generalized master equation

$$\frac{1}{2}\,(W^*, W^*) = i\bar{\Delta} W^*, \tag{3.58}$$

for the path-integral to be BRST invariant. This equation can be neatly summarized in the form

$$\bar{\Delta}\, e^{iW^*} = 0. \tag{3.59}$$

Solutions of this equation restricted to the hypersurface $\Phi_A^* = \partial\Psi/\partial\Phi^A$ are acceptable actions for the construction of BRST-invariant path integrals.

A geometrical interpretation of the field/anti-field construction and the master equation has been discussed in refs.[29, 30, 31].

Chapter 4

Applications of BRST methods

In the final chapter of these lecture notes, we turn to some application of BRST-methods other than the perturbative quantization of gauge theories. We deal with two topics; the first is the construction of BRST field theories, presented in the context of the scalar point particle. This is the simplest case [33, 34]; for more complicated ones, like the superparticle [35, 36] or the string [35, 37, 32], we refer to the literature.

The second application concerns the classification of anomalies in gauge theories of the Yang-Mills type. Much progress has been made in this field in recent years [40], of which a summary is presented here.

4.1 BRST Field theory

The examples of the relativistic particle and string show, that in theories with local reparametrization invariance the hamiltonian is one of the generators of gauge symmetries, and as such is constrained to vanish. The same phenomenon also occurs in general relativity, leading to the well-known Wheeler-deWitt equation. In such case the *full* dynamics of the system is actually contained in the BRST cohomology. This opens up the possibility for constructing quantum field theories for particles [32, 33, 34], or strings [32, 35, 37], in a BRST formulation, in which the usual BRST operator becomes the kinetic operator for the fields. This formulation has some formal similarities with the Dirac equation for spin-1/2 fields.

As our starting point we consider the BRST-operator for the relativistic quantum scalar particle, which for free particles, after some rescaling, reads

$$\Omega = c(p^2 + m^2), \qquad \Omega^2 = 0. \tag{4.1}$$

It acts on fields $\Psi(x, c) = \psi_0(x) + c\psi_1(x)$, with the result

$$\Omega\Psi(x, c) = c(p^2 + m^2)\, \psi_0(x). \tag{4.2}$$

As in the case of Lie-algebra cohomology (2.112), we introduce the non-degenerate (positive definite) inner product

$$(\Phi, \Psi) = \int d^d x \, (\phi_0^* \psi_0 + \phi_1^* \psi_1) \,. \tag{4.3}$$

With respect to this inner product the ghosts (b, c) are mutually adjoint:

$$(\Phi, c\Psi) = (b\Phi, \Psi) \quad \leftrightarrow \quad b = c^\dagger. \tag{4.4}$$

Then the BRST operator Ω is not self-adjoint, but rather

$$\Omega^\dagger = b(p^2 + m^2), \qquad \Omega^{\dagger\, 2} = 0. \tag{4.5}$$

Quite generally, we can construct actions for quantum scalar fields coupled to external sources J of the form

$$S_G[J] = \frac{1}{2} \, (\Psi, G\, \Omega \Psi) - (\Psi, J) \,, \tag{4.6}$$

where the operator G is chosen such that

$$G\Omega = (G\Omega)^\dagger = \Omega^\dagger G^\dagger. \tag{4.7}$$

This guarantees that the action is real. From the action we then derive the field equation

$$G\Omega \, \Psi = \Omega^\dagger G^\dagger \, \Psi = J. \tag{4.8}$$

Its consistency requires the co-BRST invariance of the source:

$$\Omega^\dagger J = 0. \tag{4.9}$$

This reflects the invariance of the action and the field equation under BRST transformations

$$\Psi \rightarrow \Psi' = \Psi + \Omega \chi. \tag{4.10}$$

In order to solve the field equation we therefore have to impose a gauge condition, selecting a particular element of the equivalence class of solutions (4.10).

A particularly convenient condition is

$$\Omega G^\dagger \, \Psi = 0. \tag{4.11}$$

In this gauge, the field equation can be rewritten in the form

$$\Delta G^\dagger \, \Psi = \left(\Omega^\dagger \Omega + \Omega \Omega^\dagger \right) G^\dagger \Psi = \Omega \, J. \tag{4.12}$$

Here Δ is the BRST laplacean, which can be inverted using a standard analytic continuation in the complex plane, to give

$$G^\dagger \, \Psi = \frac{1}{\Delta} \, \Omega \, J. \tag{4.13}$$

We interpret the operator $\Delta^{-1}\Omega$ on the right-hand side as the (tree-level) propagator of the field.

We now implement the general scheme (4.6)-(4.13) by choosing the inner product (4.3), and $G = b$. Then

$$G\Omega = bc(p^2 + m^2) = \Omega^\dagger G^\dagger, \tag{4.14}$$

and therefore

$$\frac{1}{2}\,(\Psi, G\,\Omega\Psi) = \frac{1}{2}\int d^dx\,\psi_0^*(p^2 + m^2)\psi_0, \tag{4.15}$$

which is the standard action for a free scalar field[1].

The laplacean for the BRST operators (4.2), (4.5) is

$$\Delta = \Omega\,\Omega^\dagger + \Omega^\dagger\Omega = (p^2 + m^2)^2, \tag{4.16}$$

which is manifestly non-negative, but might give rise to propagators with double poles, or negative residues, indicating the appearance of ghost states. However, in the expression (4.13) for the propagator, one of the poles is canceled by the zero of the BRST operator; in the present context the equation reads

$$c\psi_0 = \frac{1}{(p^2 + m^2)^2}\,c(p^2 + m^2)\,J_0. \tag{4.17}$$

This leads to the desired result

$$\psi_0 = \frac{1}{p^2 + m^2}\,J_0, \tag{4.18}$$

and we recover the standard scalar field theory indeed. It is not very difficult to extend the theory to particles in external gravitational or electromagnetic fields[2], or to spinning particles [33, 38].

However, a different and more difficult problem is the inclusion of self interactions [33]. This question has been addressed mostly in the context of string theory [32, 37]. As it is expected to depend on spin, no unique prescription has been constructed for point particles to date.

4.2 Anomalies and BRST cohomology

In the preceding chapters we have seen how local gauge symmetries are encoded in the BRST-transformations. First, the BRST-transformations of the classical variables correspond to ghost-dependent gauge transformations. Second, the closure of the algebra of the gauge transformations (and the Poisson brackets or

[1] Of course, there is no loss of generality here if we restrict the coefficients ψ_a to be real.

[2] See the discussion in [34], which uses however a less elegant implementation of the action.

commutators of the constraints), as well as the corresponding Jacobi-identities, are part of the condition that the BRST transformations are nilpotent.

It is important to stress, as we observed earlier, that the closure of the classical gauge algebra does not necessarily guarantee the closure of the gauge algebra in the quantum theory, because it may be spoiled by anomalies. Equivalently, in the presence of anomalies there is no nilpotent quantum BRST operator, and no local action satisfying the master equation (3.59). A particular case in point is that of a Yang-Mills field coupled to chiral fermions, as in the electro-weak standard model. In the following we consider chiral gauge theories in some detail.

The action of chiral fermions coupled to an abelian or non-abelian gauge field reads

$$S_F[A] = \int d^4x \, \bar{\psi}_L \not{D} \psi_L. \tag{4.19}$$

Here $D_\mu \psi_L = \partial_\mu \psi_L - g A_\mu^a T_a \psi_L$, with T_a the generators of the gauge group in the representation according to which the spinors ψ_L transform. In the path-integral formulation of quantum field theory the fermions make the following contribution to the effective action for the gauge fields:

$$e^{iW[A]} = \int D\bar{\psi}_L D\psi_L \, e^{iS_F[A]}. \tag{4.20}$$

An infinitesimal local gauge transformation with parameter Λ^a changes the effective action $W[A]$ by

$$\delta(\Lambda) W[A] = \int d^4x \, (D_\mu \Lambda)^a \, \frac{\delta W[A]}{\delta A_\mu^a} = - \int d^4x \, \Lambda^a \left(\partial_\mu \frac{\delta}{\delta A_\mu^a} - g f_{ab}^{\;\;c} A_\mu^b \frac{\delta}{\delta A_\mu^c} \right) W[A], \tag{4.21}$$

assuming boundary terms to vanish. By construction, the fermion action $S_F[A]$ itself is gauge invariant, but this is generally not true for the fermionic functional integration measure. If the measure is not invariant:

$$\begin{aligned} \delta(\Lambda) W[A] &= - \int d^4x \, \Lambda^a \Gamma_a[A] \neq 0, \\ \Gamma_a[A] &= \mathcal{D}_a W[A] \equiv \left(\partial_\mu \frac{\delta}{\delta A_\mu^a} - g f_{ab}^{\;\;c} A_\mu^b \frac{\delta}{\delta A_\mu^c} \right) W[A]. \end{aligned} \tag{4.22}$$

Even though the action $W[A]$ may not be invariant, its variation should still be covariant and satisfy the condition

$$\mathcal{D}_a \Gamma_b[A] - \mathcal{D}_b \Gamma_a[A] = [\mathcal{D}_a, \mathcal{D}_b] \, W[A] = g f_{ab}^{\;\;c} \mathcal{D}_c W[A] = g f_{ab}^{\;\;c} \Gamma_c[A]. \tag{4.23}$$

This consistency condition was first derived by Wess and Zumino [41], and its solutions determine the functional form of the anomalous variation $\Gamma_a[A]$ of the effective action $W[A]$. It can be derived from the BRST cohomology of the gauge theory [39, 44, 40].

To make the connection, observe that the Wess-Zumino consistency condition (4.23) can be rewritten after contraction with ghosts as follows:

$$\begin{aligned} 0 &= \int d^4x\, c^a c^b \left(\mathcal{D}_a\Gamma_b[A] - \mathcal{D}_b\Gamma_a[A] - g f_{ab}^{\;\;c}\Gamma_c[A]\right) \\ &= 2\int d^4x\, c^a c^b \left(\mathcal{D}_a\Gamma_b - \frac{g}{2}\, f_{ab}^{\;\;c}\,\Gamma_c\right) = -2\,\delta_\Omega \int d^4x c^a \Gamma_a, \end{aligned} \qquad (4.24)$$

provided we can ignore boundary terms. The integrand is a 4-form of ghost number +1:

$$I_4^1 = d^4x\, c^a\Gamma_a[A] = \frac{1}{4!}\,\varepsilon_{\mu\nu\kappa\lambda}\, dx^\mu \wedge dx^\nu \wedge dx^\kappa \wedge dx^\lambda\, c^a\Gamma_a[A]. \qquad (4.25)$$

The Wess-Zumino consistency condition (4.24) then implies that non-trivial solutions of this condition must be of the form

$$\delta_\Omega I_4^1 = dI_3^2, \qquad (4.26)$$

where I_3^2 is a 3-form of ghost number +2, vanishing on any boundary of the space-time $\mathcal{M}$.

Now we make a very interesting and useful observation: the BRST construction can be mapped to a standard cohomology problem on a principle fibre bundle with local structure $\mathcal{M}\times G$, where $\mathcal{M}$ is the space-time and G is the gauge group viewed as a manifold [42]. First note, that the gauge field is a function of both the co-ordinates x^μ on the space-time manifold $\mathcal{M}$, and of the parameters Λ^a on the group manifold G. We denote the combined set of these co-ordinates by $\xi = (x, \Lambda)$. To make the dependence on space-time and gauge group explicit, we introduce the Lie-algebra valued 1-form

$$A(x) = dx^\mu A_\mu^a(x) T_a, \qquad (4.27)$$

with T_a a generator of the gauge group, and $A_\mu^a(x)$ the gauge field at the point x in the space-time manifold $\mathcal{M}$. Starting from A, all gauge-equivalent configurations are obtained by local gauge transformations, generated by group elements $a(\xi)$ according to

$$\mathcal{A}(\xi) = -\frac{1}{g}\, a^{-1}(\xi)\, da(\xi) + a^{-1}(\xi)\, A(x)\, a(\xi), \qquad A(x) = \mathcal{A}(x, 0) \qquad (4.28)$$

where d is the ordinary differential operator on the space-time $\mathcal{M}$:

$$da(x, \Lambda) = dx^\mu \frac{\partial a}{\partial x^\mu}(x, \Lambda). \qquad (4.29)$$

Furthermore, the parametrization of the group is chosen such that $a(x, 0) = 1$, the identity element. Then, if $a(\xi)$ is close to the identity:

$$a(\xi) = e^{g\Lambda(x)\cdot T} \approx 1 + g\,\Lambda^a(x) T_a + \mathcal{O}(g^2\Lambda^2), \qquad (4.30)$$

and eq.(4.28) represents the infinitesimally transformed gauge field 1-form (1.124). In the following we interpret $\mathcal{A}(\xi)$ as a particular 1-form living on the fibre bundle with local structure $\mathcal{M} \times G$.

A general one-form $\mathbf{N}$ on the bundle can be decomposed as

$$\mathbf{N}(\xi) = d\xi^i N_i = dx^\mu N_\mu + d\Lambda^a N_a. \tag{4.31}$$

Correspondingly, we introduce the differential operators

$$d = dx^\mu \frac{\partial}{\partial x^\mu}, \qquad s = d\Lambda^a \frac{\partial}{\partial \Lambda^a}, \qquad \mathbf{d} = d + s, \tag{4.32}$$

with the properties

$$d^2 = 0, \qquad s^2 = 0, \qquad \mathbf{d}^2 = ds + sd = 0. \tag{4.33}$$

Next define the left-invariant 1-forms on the group $C(\xi)$ by

$$C = a^{-1}\, sa, \qquad c(x) = C(x, 0). \tag{4.34}$$

By construction, using $sa^{-1} = -a^{-1} sa\, a^{-1}$, these forms satisfy

$$sC = -C^2. \tag{4.35}$$

The action of of the group differential s on the one-form A is

$$s\mathcal{A} = \frac{1}{g}\, DC = \frac{1}{g}\left(dC - g[\mathcal{A}, C]_+ \right). \tag{4.36}$$

Finally, the field strength $\mathcal{F}(\xi)$ for the gauge field $\mathcal{A}$ is defined as the 2-form

$$\mathcal{F} = d\mathcal{A} - g\mathcal{A}^2 = a^{-1} F\, a, \qquad F(x) = \mathcal{F}(x, 0). \tag{4.37}$$

The action of s on $\mathcal{F}$ is given by

$$s\mathcal{F} = [\mathcal{F}, C]. \tag{4.38}$$

Clearly, the above system of equations are in one-to-one correspondence with the BRST transformations of the Yang-Mills fields, described by the Lie-algebra valued one-form $A = dx^\mu A_\mu^a T_a$, and the ghosts described by the Lie-algebra valued grassmann variable $c = c^a T_a$, upon the identification $-gs|_{\Lambda=0} \rightarrow \delta_\Omega$:

$$\begin{aligned}
&-gs\mathcal{A}|_{\Lambda=0} \rightarrow \delta_\Omega A = -dx^\mu \, (D_\mu c)^a T_a = -Dc, \\
&-gsC|_{\Lambda=0} \rightarrow \delta_\Omega c = \frac{g}{2}\, f_{ab}^{\;\;c} c^a c^b\, T_c = \frac{g}{2}\, c^a c^b\, [T_a, T_b] = gc^2. \\
&-gs\mathcal{F}|_{\Lambda=0} \rightarrow \delta_\Omega F = -\frac{g}{2}\, dx^\mu \wedge dx^\nu f_{ab}^{\;\;c} F_{\mu\nu}^a c^b\, T_c = -g[F, c],
\end{aligned} \tag{4.39}$$

provided we take the BRST variational derivative δ_Ω and the ghosts c to anti-commute with the differential operator d:

$$d\delta_\Omega + \delta_\Omega d = 0, \qquad dc + cd = dx^\mu (\partial_\mu c). \tag{4.40}$$

Returning to the Wess-Zumino consistency condition (4.26), we now see that it can be restated as a cohomology problem on the principle fibre bundle on which the 1-form $\mathcal{A}$ lives. This is achieved by mapping the 4-form of ghost number +1 to a particular 5-form on the bundle, which is a local 4-form on $\mathcal{M}$ and a 1-form on G; similarly one maps the 3-form of ghost number +2 to another 5-form which is a local 3-form on $\mathcal{M}$ and a 2-form on G:

$$I_4^1 \rightarrow \omega_4^1, \qquad I_3^2 \rightarrow \omega_3^2, \tag{4.41}$$

where the two 5-forms must be related by

$$-gs\omega_4^1 = d\omega_3^2. \tag{4.42}$$

We now show how to solve this equation as part of a whole chain of equations known as the *descent equations*. The starting point is a set of invariant polynomials known as the Chern characters of order n. They are constructed in terms of the field-strength 2-form:

$$F = dA - gA^2 = \frac{1}{2}\, dx^\mu \wedge dx^\nu F_{\mu\nu}^a\, T_a, \tag{4.43}$$

which satisfies the Bianchi identity

$$DF = dF - g\,[A, F] = 0. \tag{4.44}$$

The two-form F transforms covariantly under gauge transformations (4.28):

$$F \rightarrow a^{-1} F a = \mathcal{F}. \tag{4.45}$$

It follows that the Chern character of order n, defined by

$$Ch_n[A] = \mathrm{Tr}\, F^n = \mathrm{Tr}\, \mathcal{F}^n, \tag{4.46}$$

is an invariant $2n$-form: $Ch_n[A] = Ch_n[\mathcal{A}]$. It is also closed, as a result of the Bianchi identity:

$$d\, Ch_n[A] = n \mathrm{Tr}\, [(DF) F^{n-1}] = 0. \tag{4.47}$$

The solution of this equation is given by the exact $2n$-forms:

$$Ch_n[A] = d\omega_{2n-1}^0[A]. \tag{4.48}$$

Note, that the exact $2n$-form on the right-hand side lies entirely in the local space-time part $\mathcal{M}$ of the bundle, because this is manifestly true for the left-hand side.

Proof of the result (4.48) is to be given; for the time being we take it for granted and continue our argument. First we define a generalized connection on the bundle by

$$\mathbf{A}(\xi) \equiv -\frac{1}{g}\, a^{-1}(\xi)\, \mathbf{d}a(\xi) + a^{-1}(\xi) A(x) a(\xi) = -\frac{1}{g}\, C(\xi) + \mathcal{A}(\xi). \tag{4.49}$$

It follows, that the corresponding field strength on the bundle is

$$\begin{aligned} \mathbf{F} &= \mathbf{dA} - g\mathbf{A}^2 = (d+s)\left(\mathcal{A} - \frac{1}{g}\, C\right) - g\left(\mathcal{A} - \frac{1}{g}\, C\right)^2 \\ &= d\mathcal{A} - g\mathcal{A}^2 = \mathcal{F}. \end{aligned} \tag{4.50}$$

To go from the first to the second line we have used eq.(4.36). This result is sometimes refered to as *the Russian formula* [43]. The result implies, that the components of the generalized field-strength in the directions of the group manifold all vanish.

It is now obvious, that

$$Ch_n[\mathbf{A}] = \mathrm{Tr}\, \mathbf{F}^n = Ch_n[A]; \tag{4.51}$$

moreover $\mathbf{F}$ satisfies the Bianchi identity

$$\mathbf{DF} = \mathbf{dF} - g[\mathbf{A}, \mathbf{F}] = 0. \tag{4.52}$$

Again, this leads us to infer that

$$\mathbf{d}Ch_n[\mathbf{A}] = 0 \quad \Rightarrow \quad Ch_n[\mathbf{A}] = \mathbf{d}\omega^0_{2n-1}[\mathbf{A}] = d\omega^0_{2n-1}[A], \tag{4.53}$$

where the last equality follows from eqs.(4.51) and (4.48). The middle step, which states that the $(2n-1)$-form of which $Ch_n[\mathbf{A}]$ is the total exterior derivative has the same functional form in terms of $\mathbf{A}$, as the one of which it is the exterior space-time derviative has in terms of A, will be justified shortly.

We first conclude the derivation of the chain of descent equations, which follow from the last result by expansion in terms of C:

$$\begin{aligned} d\omega^0_{2n-1}[A] &= (d+s)\, \omega^0_{2n-1}[\mathcal{A} - C/g] \\ &= (d+s)\left(\omega^0_{2n-1}[\mathcal{A}] + \frac{1}{g}\, \omega^1_{2n-2}[\mathcal{A}, C] + ... + \frac{1}{g^{2n-1}}\, \omega^{2n-1}_0[\mathcal{A}, C]\right). \end{aligned} \tag{4.54}$$

Comparing terms of the same degree, we find

$$\begin{aligned}
& d\omega^0_{2n-1}[A] = d\omega^0_{2n-1}[\mathcal{A}], \\
& -gs\omega^0_{2n-1}[\mathcal{A}] = d\omega^1_{2n-2}[\mathcal{A}, C], \\
& -gs\omega^1_{2n-2}[\mathcal{A}, C] = d\omega^2_{2n-3}[\mathcal{A}, C], \\
& ... \\
& -gs\omega^{2n-1}_0[\mathcal{A}, C] = 0.
\end{aligned} \tag{4.55}$$

Obviously, this result carries over to the BRST differentials: with $I^0_n[A] = \omega^0_n[A]$, one obtains

$$\delta_\Omega I^k_m[A, c] = dI^{k+1}_{m-1}[A, c], \qquad m + k = 2n - 1, \;\; k = 0, 1, 2, ..., 2n - 1. \tag{4.56}$$

The first line just states the gauge independence of the Chern character. Taking $n = 3$, we find that the third line is the Wess-Zumino consistency condition (4.42):

$$\delta_\Omega \, I^1_4[A, c] = dI^2_3[A, c].$$

Proofs and solutions

We now show how to derive the result (4.48); this will provide us at the same time with the tools to solve the Wess-Zumino consistency condition. Consider an arbitrary gauge field configuration described by the Lie-algebra valued 1-form A. From this we define a whole family of gauge fields

$$A_t = tA, \qquad t \in [0, 1]. \tag{4.57}$$

It follows, that

$$F_t \equiv F[A_t] = tdA - gt^2A^2 = tF[A] - g(t^2 - t)A^2. \tag{4.58}$$

This field strength 2-form satisfied the appropriate Bianchi identity:

$$D_tF_t = dF_t - g[A_t, F] = 0. \tag{4.59}$$

In addition, one easily derives

$$\frac{dF_t}{dt} = dA - [A_t, A]_+ = D_tA, \tag{4.60}$$

where the anti-commutator of the 1-forms implies a *commutator* of the Lie-algebra elements. Now we can compute the Chern character

$$\begin{aligned}
Ch_n[A] &= \int_0^1 dt \, \frac{d}{dt} \mathrm{Tr} F^n_t = n \int_0^1 dt \, \mathrm{Tr} \left((D_tA) F^{n-1}_t \right) \\
&= nd \int_0^1 dt \, \mathrm{Tr} \left(AF^{n-1}_t \right).
\end{aligned} \tag{4.61}$$

In this derivation we have used both (4.60) and the Bianchi identity (4.59).

It is now straightforward to compute the forms ω_5^0 and ω_4^1. First, taking $n = 3$ in the result (4.61) gives $Ch_3[A] = d\omega_5^0$ with

$$I_5^0[A] = \omega_5^0[A] = 3 \int_0^1 dt \,\mathrm{Tr}\left((D_t A) F_t^2 \right) = \mathrm{Tr}\left(AF^2 + \frac{g}{2}\, A^3 F + \frac{g^2}{10}\, A^5 \right). \quad (4.62)$$

Next, using eqs.(4.39) the BRST differential of this expression gives $\delta_\Omega I_5^0 = dI_4^1$, with

$$I_4^1[A, c] = -\mathrm{Tr}\left(c \left[F^2 + \frac{g}{2}\left(A^2 F + AFA + FA^2 \right) + \frac{g^2}{2}\, A^4 \right] \right). \quad (4.63)$$

This expression determines the anomaly up to a constant of normalization $\mathcal{N}$:

$$\Gamma_a[A] = \mathcal{N}\, \mathrm{Tr}\left(T_a \left[F^2 + \frac{g}{2}\left(A^2 F + AFA + FA^2 \right) + \frac{g^2}{2}\, A^4 \right] \right). \quad (4.64)$$

Of course, the component form depends on the gauge group; for example, for $SU(2) \simeq SO(3)$ it vanishes identically, as is true for any orthogonal group $SO(N)$; in contrast the anomaly does not vanish identically for $SU(N)$, for any $N \geq 3$. In that case it has to be anulled by cancellation between the contributions of chiral fermions in different representations of the gauge group G.

Bibliography

[1] C. Becchi, A. Rouet and R. Stora, Ann. Phys. 98 (1976), 28

[2] I.V. Tyutin, Lebedev preprint FIAN 39 (1975), unpublished

[3] R.P. Feynman, Acta Phys. Pol. 26 (1963), 697

[4] B. DeWitt, Phys. Rev. 162 (1967), 1195

[5] L.D. Faddeev and V.N. Popov, Phys. Lett. B25 (1967), 29

[6] T. Kugo and I. Ojima, Supp. Progr. Theor. Phys. 66 (1979), 1

[7] M. Henneaux, Phys. Rep. 126 (1985), 1

[8] L. Baulieu, Phys. Rep. 129 (1985), 1

[9] B. Kostant and S. Sternberg, Ann. Phys. (NY) 176 (1987), 49

[10] R. Marnelius, in: *Geometrical and Algebraic Aspects of Non-Linear Field Theory*, eds. S. De Filippo, M. Marinaro, G. Marmo and G. Vilasi (North Holland, 1989)

[11] J.W. van Holten, in: *Functional Integration, Geometry and Strings*, eds. Z. Haba and J. Sobczyk (Birkhäuser, 1989), 388

[12] D.M. Gitman and I.V. Tyutin, *Quantization of fields with constraints* (Springer, 1990)

[13] S.K. Blau, Ann. Phys. 205 (1991), 392

[14] M. Henneaux and C. Teitelboim, *Quantization of Gauge Systems* (Princeton Univ. Press, 1992)

[15] H. Goldstein, *Classical Mechanics* (Addison-Wesley, 1950)

[16] L. Fadeev and R. Jackiw, Phys. Rev. Lett. 60 (1988), 1692
R. Jackiw, in: *Constraint Theory and Quantization Methods*,
eds. F. Colomo, L. Lusanna and G. Marmo (World Scientific, 1994), 163

[17] B. de Wit and J.W. van Holten, Phys. Lett. B79 (1978), 389
J.W. van Holten, *On the construction of supergravity theories* (PhD. thesis; Leiden Univ., 1980)

[18] F.A. Berezin, *The method of second quantization* (Academic Press, NY; 1966)

[19] B.S. DeWitt, *Supermanifolds* (Cambridge University Press; 1984)

[20] M. Spiegelglas, Nucl. Phys. B283 (1986), 205

[21] S.A. Frolov and A.A. Slavnov, Phys. Lett. B218 (1989), 461

[22] S. Hwang and R. Marnelius, Nucl. Phys. B320 (1989), 476

[23] A.V. Razumov and G.N. Rybkin, Nucl. Phys. B332 (1990), 209

[24] W. Kalau and J.W. van Holten, Nucl. Phys. B361 (1991), 233

[25] J.W. van Holten, Phys. Rev. Lett. 64 (1990), 2863; Nucl. Phys. B339 (1990), 158

[26] J. Fisch and M. Henneaux, in: *Constraints Theory and Relativistic Dynamics*, eds. G. Longhi and L. Lusanna (World Scientific, Singapore; 1987), 57

[27] J. Zinn-Justin, Springer Lecture Notes in Physics 37 (1975), 1

[28] I.A. Batalin and G.A. Vilkovisky, Phys. Lett. B102 (1981), 27;
Phys. Rev. D28 (1983), 2567

[29] E. Witten, Mod. Phys. Lett. A5 (1990), 487

[30] A. Schwarz, Comm. Math. Phys. 155 (1993), 249

[31] S. Aoyama and S. Vandoren, Mod. Phys. Lett. A8 (1993), 3773

[32] W. Siegel, *Introduction to String Field Theory,* (World Scientific, Singapore, 1988)

[33] H. Hüffel, Phys. Lett. B241 (1990), 369
Int. J. Mod. Phys. A6 (1991), 4985

[34] J.W. van Holten, J. Mod. Phys. A7 (1992), 7119

[35] W. Siegel, Phys. Lett. B151 (1985), 391, 396

[36] S. Aoyama, J. Kowalski-Glikman, L. Lukierski and J.W. van Holten, Phys. Lett. B216 (1989), 133

[37] E. Witten, Nucl. Phys. B268 (1986), 79
Nucl. Phys. B276 (1986), 291

[38] J.W. van Holten, Nucl. Phys. B457 (1995), 375

[39] B. Zumino, in *Relativity, Groups and Topology II* (Les Houches 1983), eds. B.S. DeWitt and R. Stora (North Holland, Amsterdam, 1984)

[40] G. Barnich, F. Brandt and M. Henneaux, Phys. Rep. 338 (2000), 439

[41] J. Wess and B. Zumino, Phys. Lett. B37 (1971), 95

[42] L. Bonora and P. Cotta-Ramusino, Comm. Math. Phys. 87, (1983), 589

[43] R. Stora, in: *New Developments in Quantum Field Theory and Statistical Mechanics*, eds. H. Levy and P. Mitter (Plenum, 1977)

[44] B. Zumino, Y-S. Wu and A. Zee, Nucl. Phys. B239 (1984), 477