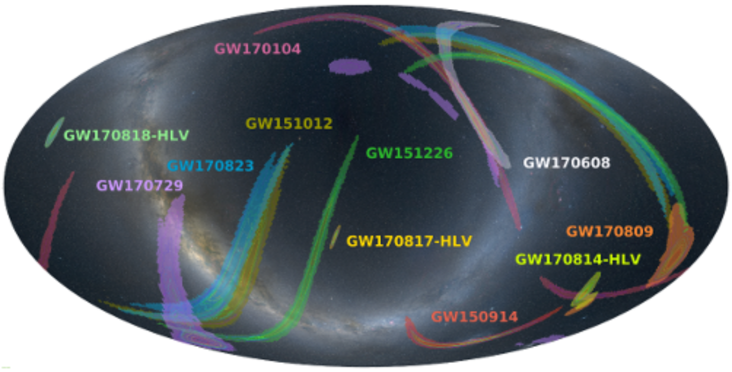
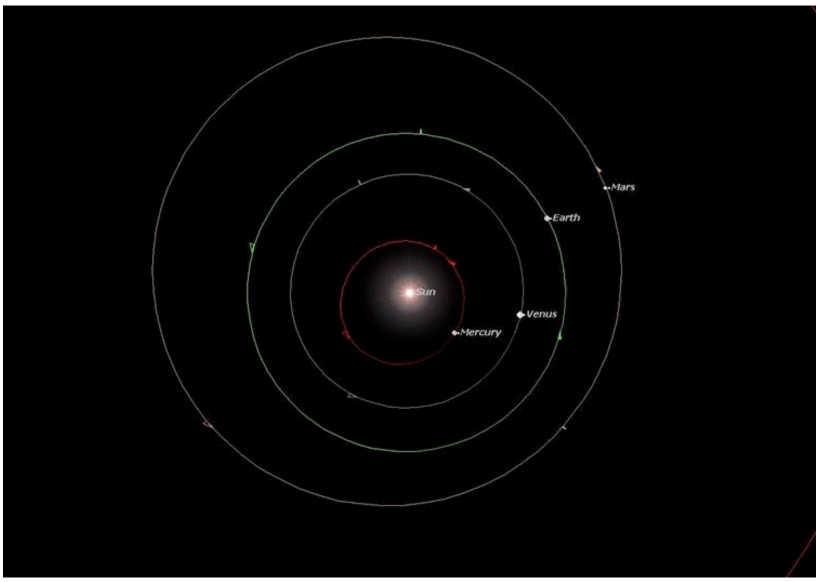
Gravitational waves



Jan W. van Holten

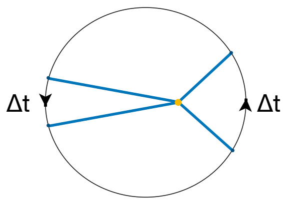
Bonn, Febr. 2023

Kepler's laws of planetary orbits



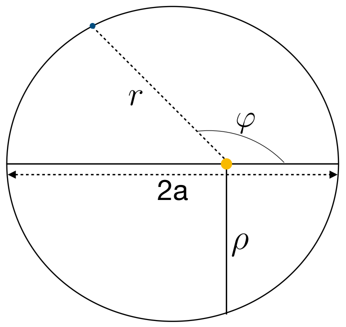
inner solar system

- planets move on elliptic orbits with the sun in a focal point
- the radius connecting the planet and the sun sweeps out equal areas in equal times



- the square of the period of the orbit is proportional to the cube of the major axis

Newton's theory of gravity



$$r = \frac{\rho}{1 - e\cos\varphi}$$

- masses interact by a *central force*
  → conservation of angular momentum
- inverse square law
  → period-radius relation

$$\mathbf{F} = \frac{GmM}{r^2}\hat{\mathbf{r}} \quad \longrightarrow \quad \frac{a^3}{T^2} = \frac{GM}{4\pi^2}$$

	a^3/T^2
Mercury	3.362
Venus	3.362
Earth	3.362
Mars	3.362
	(10^{18} m^3 s^{-2})

actually motion around common center of mass

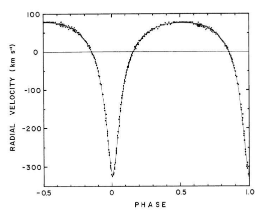
$$\longrightarrow \quad M = m_1 + m_2 \qquad m = \frac{m_1 m_2}{m_1 + m_2}$$

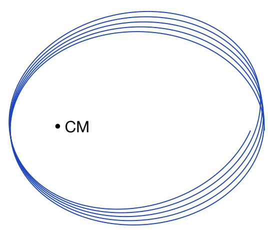
Instantaneous action at a distance

Binary pulsar 1913 +16:
violations of Kepler's laws and evidence for GR

Binary pulsar data:

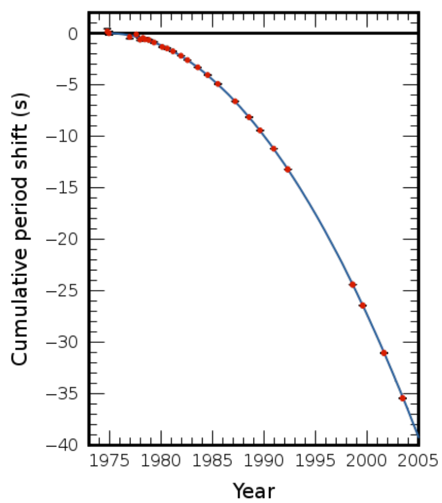
masses	$m_1 = 1.441\,M_\odot, \quad m_2 = 1.387\,M_\odot$
period	$7.75\,\mathrm{hr}$
semi-major axis	$a = 1\,950\,000\,\mathrm{km}$
eccentricity	$e = 0.617$
distance	$6.4\,\mathrm{kpc}$



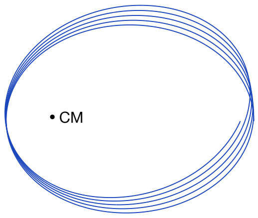


quasi-elliptic orbit

- precesses $4.2°/\mathrm{yr}$ → corrections to Newton's law
- shrinks $\Delta a = 3.5\,\mathrm{m/yr}$ → emission of gravitational waves $7.35 \times 10^{24}\ \mathrm{W}$



Binary black holes



$M_{BH}^2 \simeq 50\, m_1 m_2$

GW energy flux of binaries ~ product of the masses

→ binary black holes with masses $M_{BH} = 10\, M_{\odot}$
same orbit would emit ~ 50 times more energy:
3.7×10^{26} W ~ solar luminosity in e.m. radiation

- frequencies of these massive compact binaries are very low:

$f_{binary} \sim 0.7 \times 10^{-4}$ Hz $\qquad f_{light} \sim 0.5 \times 10^{6}$ Hz

$\lambda_{binary} \sim 4 \times 10^{9}$ km $\qquad \lambda_{light} \sim 600$ nm

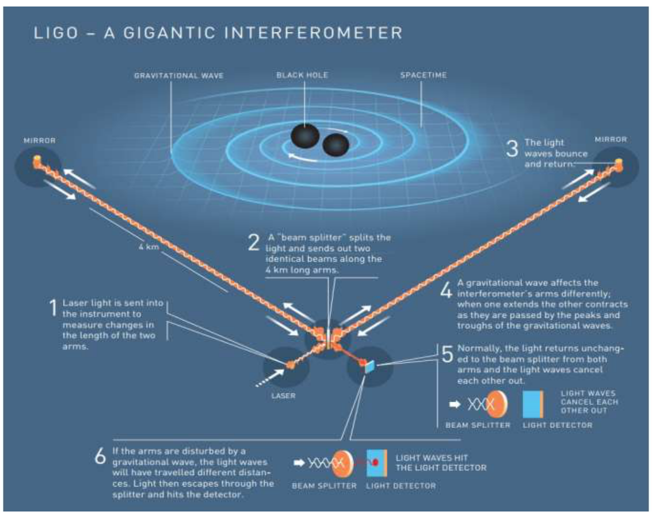
- Gravitational forces are extremely weak:

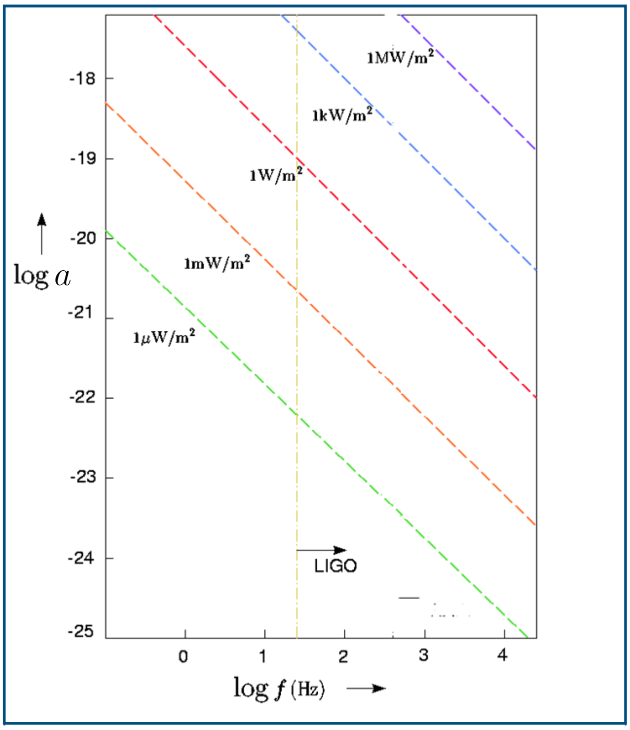
in hydrogen atom $\dfrac{F_{newton}}{F_{coulomb}} \simeq 0.45 \times 10^{-39}$

gravitational radiation very difficult to detect!

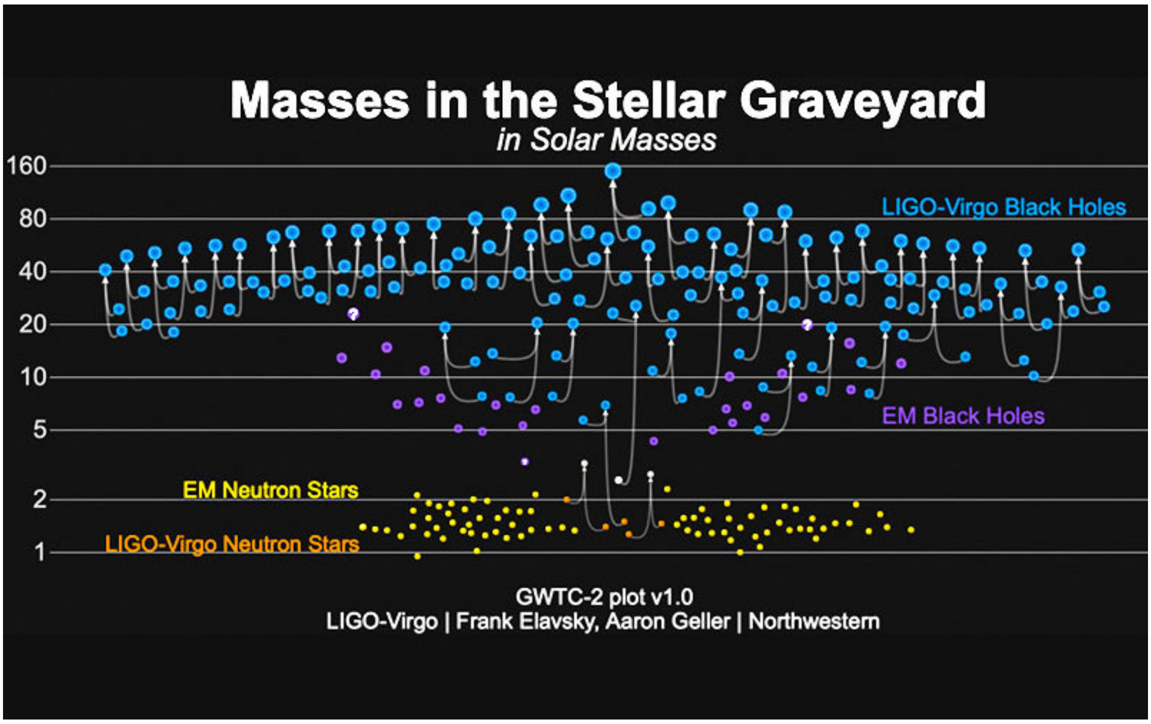
Direct detection of gravitational waves

Amplitudes expressed by metric deformations (strain) $a = \dfrac{\Delta l}{l}$





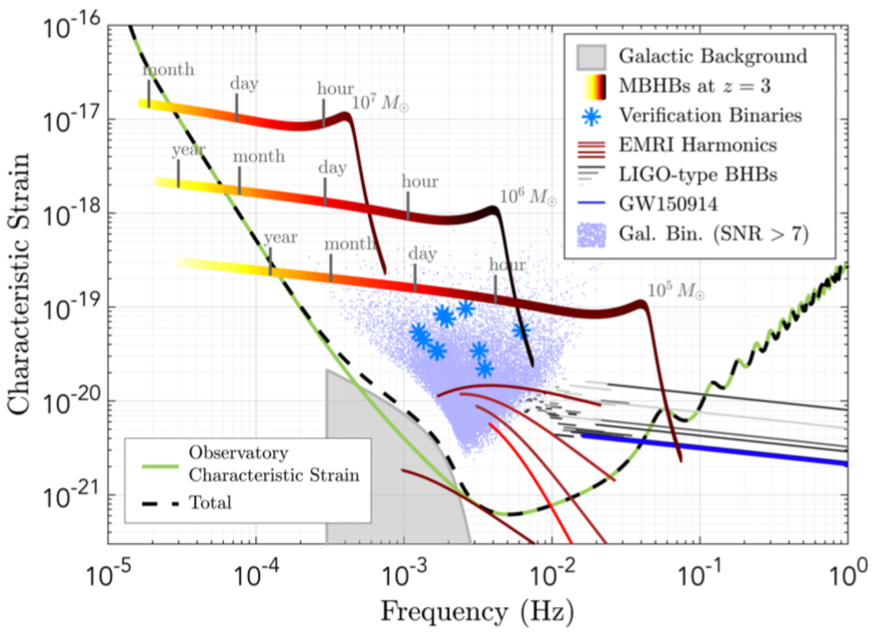
Selection of observed mergers of black-hole and neutron-star binaries (LIGO-Virgo-KAGRA)

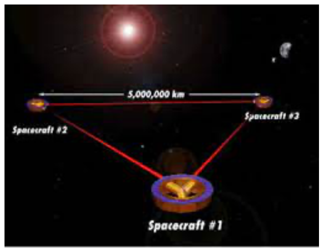


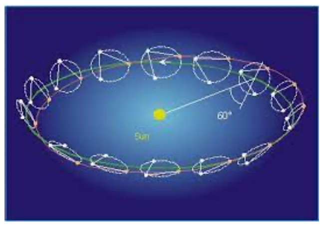
mergers last up to minutes and reach frequencies up to ~ 300 Hz.

Observing the inspiral phase of compact binaries:

space mission (LISA)

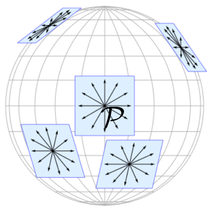




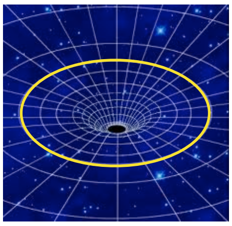


General Relativity

gravitational field ⟷ space-time geometry



local flat geometry + gravitational forces ⟷ global curvature



Static Schwarzschild geometry:

non-euclidean relation between circumference and radius of circular orbits ⟶ corrections to Newton's law

Gravitational fields are dynamical ⟷ geometry can *fluctuate*

fluctuations can propagate as *gravitational waves*

Tools for dynamical space-time: differential geometry

metric: line element $ds^2 = g_{\mu\nu}(x)\, dx^\mu dx^\nu$ (signature $(-,+,+,+)$)

connection: geodesics $\ddot{x}^\mu + \Gamma_{\lambda\nu}{}^{\mu}\, \dot{x}^\lambda \dot{x}^\nu = 0$

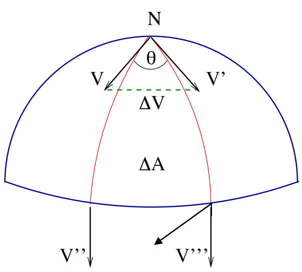
with $\Gamma_{\lambda\nu}{}^{\mu} = \frac{1}{2} g^{\mu\kappa} \left(\partial_\lambda g_{\kappa\nu} + \partial_\nu g_{\kappa\lambda} - \partial_\kappa g_{\lambda\nu} \right)$

⟷ covariant derivative $\nabla_\nu A_\mu = \partial_\nu A_\mu - \Gamma_{\nu\mu}{}^{\lambda} A_\lambda$ (parallel displacement)

in particular $\nabla_\lambda g_{\mu\nu} = 0$

curvature: Riemann tensor

non-commuting covariant derivatives $[\nabla_\mu, \nabla_\lambda] A_\nu = -R_{\mu\lambda\nu}{}^{\kappa} A_\kappa$



with $R_{\mu\lambda\nu}{}^{\kappa} = \partial_\mu \Gamma_{\lambda\nu}{}^{\kappa} - \partial_\lambda \Gamma_{\mu\nu}{}^{\kappa} - \Gamma_{\mu\nu}{}^{\sigma} \Gamma_{\lambda\sigma}{}^{\kappa} + \Gamma_{\lambda\nu}{}^{\sigma} \Gamma_{\mu\sigma}{}^{\kappa}$

$$= (\partial_\mu \Gamma_\lambda - \partial_\lambda \Gamma_\mu - [\Gamma_\mu, \Gamma_\lambda])_\nu{}^{\kappa}$$

Einstein equations

Ricci tensor and scalar $\quad R_{\mu\nu} = R_{\nu\mu} = R_{\mu\lambda\nu}{}^{\lambda}, \qquad R = R_{\mu}{}^{\mu}$

Bianchi identity $\quad \nabla_\sigma R_{\mu\lambda\nu}{}^{\kappa} + \nabla_\mu R_{\lambda\sigma\nu}{}^{\kappa} + \nabla_\lambda R_{\sigma\mu\nu}{}^{\kappa} = 0$

$$\longrightarrow \quad \nabla^\mu R_{\mu\nu} = \frac{1}{2} \nabla_\nu R$$

Einstein tensor $\quad G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \quad \longrightarrow \quad \nabla^\mu G_{\mu\nu} = 0$

$$\boxed{G_{\mu\nu} = -\kappa^2 T_{\mu\nu}}$$

energy-momentum tensor

$$\kappa^2 = \frac{\hbar}{m_{planck}^2 c^3} = \frac{8\pi G}{c^4} \simeq 2.1 \times 10^{-43} \ \text{kg}^{-1} \ \text{m}^{-1} \ \text{s}^2$$

local energy-momentum conservation: $\quad \nabla^\mu T_{\mu\nu} = 0$

Weak gravity: linear approximation of GR

space-time geometry: small deviations from Minkowski space

$$g_{\mu\nu} = \eta_{\mu\nu} + 2\kappa h_{\mu\nu}$$

Minkowski metric $\eta = \text{diag}(-1, +1, +1, +1)$ — metric fluctuation (e.g., grav. wave)

using $\eta^{\mu\nu}$ and $\eta_{\mu\nu}$ to raise and lower indices

$\longleftrightarrow$ treat $h_{\mu\nu}$ as a symmetric tensor field in Minkowski space

in this approximation:

$$\Gamma_{\mu\nu}{}^{\lambda} = \kappa \left(\partial_\mu h_\nu{}^\lambda + \partial_\nu h_\mu{}^\lambda - \partial^\lambda h_{\mu\nu} \right) + \mathcal{O}(\kappa^2)$$

$$R_{\mu\nu\kappa}{}^{\lambda} = \kappa \left(\partial_\mu \partial_\kappa h_\nu{}^\lambda - \partial_\nu \partial_\kappa h_\mu{}^\lambda + \partial^\lambda \partial_\nu h_{\mu\kappa} - \partial^\lambda \partial_\mu h_{\nu\kappa} \right) + \mathcal{O}(\kappa^2)$$

Einstein equations:

$$\boxed{\frac{1}{\kappa}\left(R_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} R \right) = \Box h_{\mu\nu} - \partial_\mu \partial^\lambda h_{\lambda\nu} - \partial_\nu \partial^\lambda h_{\lambda\mu} + \partial_\mu \partial_\nu h_\lambda^\lambda - \eta_{\mu\nu} \left(\Box h_\lambda^\lambda - \partial^\kappa \partial^\lambda h_{\kappa\lambda} \right) = -\kappa T_{\mu\nu}}$$

In linear approximation:

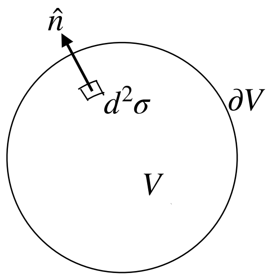
Ricci tensor: $R_{\mu\nu} = \kappa \left(\Box h_{\mu\nu} - \partial_\mu \partial^\lambda h_{\lambda\nu} - \partial_\nu \partial^\lambda h_{\lambda\mu} + \partial_\mu \partial^\nu h_\lambda^\lambda\right)$

Ricci scalar: $R = R_\mu^{\ \mu} = 2\kappa \left(\Box h_\mu^{\ \mu} - \partial^\mu \partial^\nu h_{\mu\nu}\right)$

Bianchi identity: $\longrightarrow \quad \partial^\mu \left(R_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} R\right) = 0 \quad \longleftrightarrow \quad \partial^\mu T_{\mu\nu} = 0$

Energy-momentum conservation: $P_\mu = \int_V d^3x\, T_{0\mu}$

$$\longrightarrow \quad \frac{dP_\mu}{dt} = \int_V d^3x\, \partial_0 T_{0\mu} = \int_V d^3x\, \partial_k T_{k\mu} = \oint_{\partial V} d^2\sigma\, T_{n\mu}$$



$T_{n\mu} = \hat{n}_k T_{k\mu}$

forced by Einstein equations

Energy-momentum conservation of matter related to gauge symmetry of Einstein equations

local gauge transformations $\quad h_{\mu\nu} \to h'_{\mu\nu} = h_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu$

for arbitrary vector field $\xi_\mu(x)$ leave Riemann tensor invariant: $R'^{\ \ \lambda}_{\mu\kappa\nu} = R_{\mu\kappa\nu}^{\ \ \ \lambda}$

Gauge fixing $\longrightarrow$ *De Donder gauge:* $\quad \partial_\nu h^\nu_{\ \mu} = \frac{1}{2} \partial_\mu h^\nu_{\ \nu}$

reduces Einstein equations to inhomogeneous wave equation:

$$\Box \left(h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h^\lambda_{\ \lambda}\right) = -\kappa T_{\mu\nu} \quad \longleftrightarrow \quad \Box h_{\mu\nu} = -\kappa \left(T_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} T^\lambda_{\ \lambda}\right)$$

Field redefinition:

$$\underline{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h^\lambda_{\ \lambda} \quad \longleftrightarrow \quad h_{\mu\nu} = \underline{h}_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} \underline{h}^\lambda_{\ \lambda}.$$

simplification: $\boxed{\Box \underline{h}_{\mu\nu} = -\kappa T_{\mu\nu}}$

Action

Einstein: $$S[g] = \frac{1}{16\pi G}\int d^4x\sqrt{-g}\,R \simeq \frac{1}{\kappa^2}\int d^4x\, g^{\mu\nu}\left(\Gamma_{\mu\lambda}{}^{\kappa}\Gamma_{\nu\kappa}{}^{\lambda} - \Gamma_{\mu\nu}{}^{\lambda}\Gamma_{\lambda\kappa}{}^{\kappa}\right)$$

[Proof: use $$\frac{1}{2}\int\sqrt{-g}\; g^{\mu\nu}\partial_\lambda\Gamma^\lambda_{\mu\nu} = \int\sqrt{-g}\; g^{\mu\nu}\left(\Gamma^\kappa_{\mu\lambda}\Gamma^\lambda_{\kappa\nu} - \frac{1}{2}\Gamma^\lambda_{\mu\nu}\Gamma^\kappa_{\lambda\kappa}\right),$$
$$\int\sqrt{-g}\; g^{\mu\nu}\partial_\mu\Gamma^\lambda_{\nu\lambda} = \int\sqrt{-g}\; g^{\mu\nu}\Gamma^\lambda_{\mu\nu}\Gamma^\kappa_{\lambda\kappa}.$$]

weak-gravity limit:

$$S[h] = \int d^4x\left[-\frac{1}{2}\partial^\lambda h^{\mu\nu}\partial_\lambda h_{\mu\nu} + \partial^\mu h^{\nu\lambda}\partial_\nu h_{\mu\lambda} - \partial^\mu h_{\mu\nu}\partial^\nu h_\lambda{}^\lambda + \frac{1}{2}\partial^\lambda h_\mu{}^\mu\partial_\lambda h_\nu{}^\nu\right]$$

This action is invariant (up to boundary terms) under the gauge transformations

$$h_{\mu\nu} \to h'_{\mu\nu} = h_{\mu\nu} + \partial_\mu\xi_\nu + \partial_\nu\xi_\mu$$

interaction term: $$S_{int} = \kappa\int d^4x\, h_{\mu\nu}T^{\mu\nu}$$ invariant if $\partial_\mu T^{\mu\nu} = 0$

Rewriting action and field equation in terms of $\underline{h}_{\mu\nu}$

$$S[\underline{h}] = \int d^4x\left[-\frac{1}{2}\partial^\lambda\underline{h}^{\mu\nu}\partial_\lambda\underline{h}_{\mu\nu} + \partial^\mu\underline{h}^{\nu\lambda}\partial_\nu\underline{h}_{\mu\lambda} + \frac{1}{4}\partial^\lambda\underline{h}_\mu{}^\mu\partial_\lambda\underline{h}_\nu{}^\nu + \kappa\underline{h}_{\mu\nu}\left(T^{\mu\nu} - \frac{1}{2}\eta^{\mu\nu}T_\lambda{}^\lambda\right)\right]$$

after some rewriting $\longrightarrow$ $$\Box\underline{h}_{\mu\nu} - \partial_\mu\partial^\lambda\underline{h}_{\lambda\nu} - \partial_\nu\partial^\lambda\underline{h}_{\lambda\mu} + \eta_{\mu\nu}\partial^\kappa\partial^\lambda\underline{h}_{\kappa\lambda} = -\kappa T_{\mu\nu}$$

invariant under $$\underline{h}_{\mu\nu} \to \underline{h}'_{\mu\nu} = \underline{h}_{\mu\nu} + \partial_\mu\xi_\nu + \partial_\nu\xi_\mu - \eta_{\mu\nu}\partial^\lambda\xi_\lambda$$

Free fields: plane-wave decomposition

$$\underline{h}_{\mu\nu} = \int\frac{d^4k}{(2\pi)^2}\,\varepsilon_{\mu\nu}(k)e^{ik\cdot x} \qquad \varepsilon^*_{\mu\nu}(k) = \varepsilon_{\mu\nu}(-k)$$

$\longrightarrow$ $$k^2\varepsilon_{\mu\nu} - k_\mu k^\lambda\varepsilon_{\lambda\nu} - k_\nu k^\lambda\varepsilon_{\lambda\mu} + \eta_{\mu\nu}k^\kappa k^\lambda\varepsilon_{\kappa\lambda} = 0$$

invariant under $$\varepsilon'_{\mu\nu} = \varepsilon_{\mu\nu} + k_\mu\alpha_\nu + k_\nu\alpha_\mu - \eta_{\mu\nu}k^\lambda\alpha_\lambda.$$

with $$\xi_\mu = i\int\frac{d^4k}{(2\pi)^2}\,\alpha_\mu(k)e^{-ik\cdot x}$$

gauge fixing

take $k^\mu \varepsilon_{\mu\nu} = 0$ $\longrightarrow$ $k^2 \varepsilon_{\mu\nu} = 0$

$$\varepsilon_{\mu\nu}(k) = e_{\mu\nu}(\mathbf{k},\omega)\,\delta(k^2)$$

with $\omega = \sqrt{\mathbf{k}^2}$

$$k^\mu e_{\mu 0} = \omega e_{00} + k_i e_{i0} = 0 \qquad k^\mu e_{\mu j} = \omega e_{0j} + k_i e_{ij} = 0$$

residual transformations: use $\alpha_\mu(k) = a_\mu(\mathbf{k},\omega_\mathbf{k})\,\delta(k^2)$

$$e'_{00} = e_{00} - \omega a_0 + \mathbf{k}\cdot\mathbf{a}, \qquad e'_{0i} = e_{0i} - \omega a_i + k_i a_0$$

$$e'_{ij} = e_{ij} + k_i a_j + k_j a_i - \delta_{ij}\left(\omega a_0 + \mathbf{k}\cdot\mathbf{a}\right)$$

can arrange $\longrightarrow$

$$e'_{00} = e'_{0i} = e'_{ii} = 0$$

$$k_j e'_{ji} = 0$$

(*TT* - gauge)

reality: $e^*_{\mu\nu}(\mathbf{k},\omega) = e_{\mu\nu}(-\mathbf{k},-\omega)$

$$\longrightarrow \quad \underline{h}_{\mu\nu} = \int \frac{d^3k}{8\pi^2\omega}\left(e_{\mu\nu}(\mathbf{k})e^{i(\mathbf{k}\cdot\mathbf{x}-\omega t)} + e^*_{\mu\nu}(\mathbf{k})e^{-i(\mathbf{k}\cdot\mathbf{x}-\omega t)}\right)$$

in canonical quantum gravity $\longrightarrow$ annihilation/creation operators

in *TT* - gauge:

$$\underline{h}_{00} = \underline{h}_{0i} = \underline{h}_{ii} = 0$$

$$\partial_j \underline{h}_{ji} = 0$$

in this gauge $\underline{h}^\lambda{}_\lambda = 0$ $\longrightarrow$ $\underline{h}_{\mu\nu} = h_{\mu\nu}$

(recall: this holds for *free* fields = external lines in QG)

Results for monochromatic waves

- the free physical plane-wave amplitudes can be restricted to the sets $e_{\mu\nu}(\mathbf{k},\omega)$ subject to the following conditions

$$e_{00} = e_{0i} = e_{ii} = 0, \qquad k_j e_{ji} = 0$$

- without loss of generality we can take the z-direction in the direction of $\mathbf{k}$:

$$\mathbf{k} = (0,0,\omega) \longrightarrow e_{i3} = 0$$

- the resulting physical amplitude takes the form

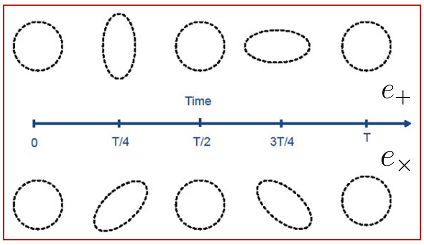
$$e'_{\mu\nu}(\mathbf{k},\omega) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & e_+(\omega) & e_\times(\omega) & 0 \\ 0 & e_\times(\omega) & -e_+(\omega) & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

- there are only 2 independent physical free-wave modes, transverse to the wave-propagation:

$$g_{\mu\nu} = \eta_{\mu\nu} + 2\kappa \underline{h}_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 + 2\kappa e_+ \cos(\mathbf{k}\cdot\mathbf{x} - \omega t) & 2\kappa e_\times \cos(\mathbf{k}\cdot\mathbf{x} - \omega t) & 0 \\ 0 & 2\kappa e_\times \cos(\mathbf{k}\cdot\mathbf{x} - \omega t) & 1 - 2\kappa e_+ \cos(\mathbf{k}\cdot\mathbf{x} - \omega t) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Detection principle

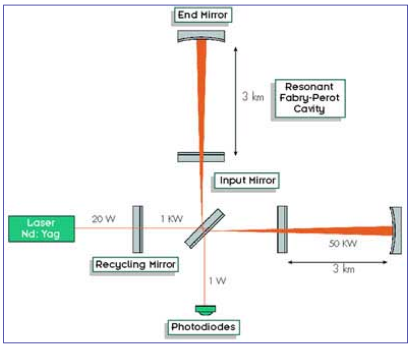
Distances between points on a ring in the x-y-plane change by passage of a monochromatic gravitational wave in the z-direction as follows:



$$\begin{aligned} ds^2 &= g_{\mu\nu} dx^\mu dx^\nu \\ &= -dt^2 + dx^2 + dy^2 + dz^2 + 2\kappa e_+ \cos\omega(z-t)\,(dx^2 - dy^2) + 4\kappa e_\times \cos\omega(z-t)\,dx dy \end{aligned}$$

(fix $z = 0$)

This provides a method to detect gravitational waves by comparing distances in 2 perpendicular directions using interferometry:



+-mode: one arm gets longer, the other shorter
→ difference in travelling time of the laser beams: phase difference creates change in the output of the interferometer

(disadvantage: does not see diagonal x-mode
↓
need more than one detector with different orientations)

Virgo detector (Pisa, It.)
arm-length: 3 km

curvature dynamics
in the linear approximation

$$R_{\mu\kappa\nu\lambda} = \kappa\left(\partial_\mu\partial_\kappa h_{\nu\lambda} - \partial_\mu\partial_\lambda h_{\nu\kappa} - \partial_\nu\partial_\kappa h_{\mu\lambda} + \partial_\nu\partial_\lambda h_{\mu\kappa}\right)$$

is invariant under $h_{\mu\nu} \to h'_{\mu\nu} = h_{\mu\nu} + \partial_\mu\xi_\nu + \partial_\nu\xi_\mu$

Bianchi identity: $\partial_\sigma R_{\mu\kappa\nu\lambda} + \partial_\mu R_{\kappa\sigma\nu\lambda} + \partial_\kappa R_{\sigma\mu\nu\lambda} = 0$

$$\rightarrow \left[\begin{array}{l} \partial^\mu R_{\mu\kappa\nu\lambda} = \partial_\nu R_{\lambda\kappa} - \partial_\lambda R_{\nu\kappa} \\ \Box R_{\mu\kappa\nu\lambda} = \partial_\mu\partial_\kappa R_{\nu\lambda} - \partial_\mu\partial_\lambda R_{\nu\kappa} - \partial_\nu\partial_\kappa R_{\mu\lambda} + \partial_\nu\partial_\lambda R_{\mu\kappa} \end{array}\right.$$

Einstein equations: $\Box R_{\mu\kappa\nu\lambda} = -8\pi G\left(\partial_\mu\partial_\kappa \underline{T}_{\nu\lambda} - \partial_\mu\partial_\lambda \underline{T}_{\nu\kappa} - \partial_\nu\partial_\kappa \underline{T}_{\mu\lambda} + \partial_\nu\partial_\lambda \underline{T}_{\mu\nu}\right)$

$$\underline{T}_{\mu\nu} = T_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}T_\lambda{}^\lambda$$

(Follows directly from definition using De Donder gauge)

space-time split

spatial components: $R_{klmn} = \varepsilon_{kli}\varepsilon_{mnj}P_{ij}$

$\longleftrightarrow$ $P_{ij} = P_{ji} = \frac{1}{4}\varepsilon_{ikl}\varepsilon_{jmn}R_{klmn} = \kappa\,\varepsilon_{ikl}\varepsilon_{jmn}\,\partial_k\partial_m h_{ln}$

Bianchi identity: $\partial_i P_{ij} = \varepsilon_{ikl}\left(\partial_i R_{klmn}\right)\varepsilon_{jmn} = 0$

free fields: $P_{jj} = \frac{1}{2}R = 0$

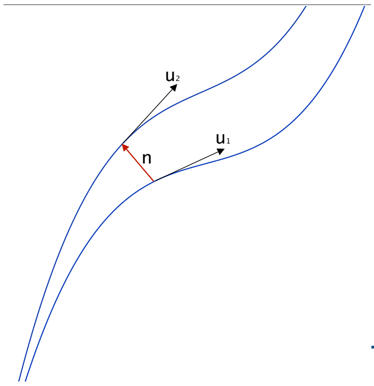
↓

time-components: $R_{0jkl} = \left(\frac{\partial_0\partial_i}{\Delta}P_{mn}\right)\varepsilon_{mij}\varepsilon_{nkl}$

$$R_{0i0j} = -P_{ij} = (\partial_0^2 h_{ij})^{TT}$$

- In empty space all components of the Riemann (= Weyl) tensor are expressed in terms of 2 independent d.o.f. contained in transverse and traceless gauge invariant 3-tensor P_{ij}
- These d.o.f. are expressed by the TT-part of h_{ij}

detection principle: geodesic deviation



continuous set of geodesics $x^\mu(\tau;\lambda)$

proper velocity $u^\mu = \frac{dx^\mu}{d\tau}$

geodesic deviation $n^\mu = \frac{dx^\mu}{d\lambda}$

$$\longrightarrow \quad \frac{Dn^\mu}{D\tau} = u^\nu \nabla_\nu n^\mu = n^\lambda \nabla_\lambda u^\mu = \frac{Du^\mu}{D\lambda}$$

$$\frac{D^2 n^\mu}{D\tau^2} = u^\lambda \nabla_\lambda (u^\nu \nabla_\nu n^\mu) = R_{\lambda\nu\kappa}{}^\mu u^\lambda u^\kappa n^\nu$$

in rest frame of free falling test mass: $u^\mu = (1,0,0,0)$ and $\Gamma_{\lambda\nu}{}^\mu = 0$

$$\longrightarrow \quad \frac{d^2 n^0}{d\tau^2} = 0, \quad \frac{d^2 n^i}{d\tau^2} = P_{ij} n^j = -(\ddot{h}_{ij})^{TT} n^j$$

Energy-momentum and angular momentum of gravitational radiation

For *free* gravitational waves in the TT-gauge, in a volume *V*, the following quantities are conserved modulo boundary terms:

$$E_V = \int_V d^3x\, \mathcal{E}(x,t) \qquad \mathcal{E} = \frac{1}{2}(\partial_t h_{ij})^2 + \frac{1}{2}(\partial_k h_{ij})^2 \quad \text{energy density}$$

$$P_{V\,i} = \int_V d^3x\, \Pi_i(x,t) \qquad \Pi_i = -\,\partial_i h_{mn}\, \partial_t h_{mn} \quad \text{momentum density}$$

$$L_{V\,i} = \int_V d^3x\, \Lambda_i(x,t) \qquad \Lambda_i = \varepsilon_{ijk}\left[2h_{jm}\partial_t h_{km} - x_j \partial_k h_{mn} \partial_t h_{mn}\right] \quad \text{angular-momentum density}$$

Each integrand satisfies an equation of continuity:

$$\partial_t \mathcal{E} = -\partial_i \Pi_i$$

$$\partial_t \Pi_i = -\partial_k \mathcal{S}_{ki}$$

$$\partial_t \Lambda_i = -\partial_k \mathcal{J}_{ki}$$

where Π_i as above, and

$$\mathcal{S}_{ki} = \partial_k h_{mn} \partial_i h_{mn} + \frac{1}{2}\left[(\partial_t h_{mn})^2 - (\partial_j h_{mn})^2\right]$$

$$\mathcal{J}_{ki} = \varepsilon_{ijl}\left[h_{ln} \overleftrightarrow{\partial_k} h_{jn} + x_j \partial_l h_{mn} \partial_k h_{mn} + \frac{1}{2}\delta_{kl} x_j ((\partial_t h_{mn})^2 - (\partial_p h_{mn})^2)\right]$$

$$\longrightarrow \quad \left(\frac{dE_V}{dt}, \frac{dP_{V\,i}}{dt}, \frac{dL_{V\,i}}{dt}\right) = -\oint_{\partial V} d^2\sigma\, (\Pi_n, \mathcal{S}_{ni}, \mathcal{J}_{ni}) = 0$$

modulo flow of gravitational-wave energy/momentum/angular momentum across the boundary of *V*

Energy flux

Taking the volume to be a large sphere of radius *r* : $V = S_r$

the surface element becomes a spherical surface element: $d^2\sigma = r^2 \sin\theta d\theta d\varphi = r^2 d\Omega$

then we can write for the outward radial energy flux:

$$\frac{dE}{r^2 d\Omega dt} = \Pi_n = \partial_r h_{ij} \partial_t h_{ij}$$

-Taking monochromatic plane waves in the ***z***-direction through an area element $dA = dxdy$ in the plane $z = 0$:

$$\frac{dE}{dAdt} = \Pi_z = \partial_z h_{ij} \partial_t h_{ij} \underset{(z=0)}{=} -\frac{2\omega^2}{c}\left(e_+^2 + e_\times^2\right)\sin^2 \omega t$$

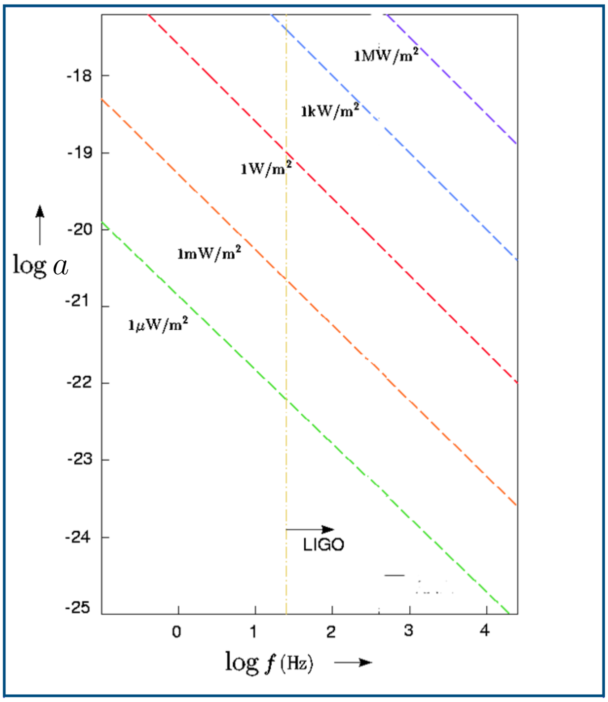
(temporarily restoring c)

recall:

$$\underline{h}_{11} = -\underline{h}_{22} = e_+ \cos\omega(z - ct)$$
$$\underline{h}_{12} = \underline{h}_{21} = e_\times \cos\omega(z - ct)$$

Energy densities in monochromatic plane waves

Averaging the flux of plane waves over an integral number of cycles: $\omega T = 2\pi n$



flux $\Phi = \dfrac{\pi c^3 f^2}{8G} a^2$

$$\frac{\pi c^3}{8G} = 1.6\times 10^{35}\,\mathrm{kg/s} = 1.4\times 10^{52}\,\mathrm{W}/c^2$$

$$\overline{\frac{dE}{dAdt}} = \frac{1}{T}\int_0^T dt\, \left.\frac{dE}{dAdt}\right|_{z=0} = -\frac{\omega^2}{c}\left(e_+^2 + e_\times^2\right)$$

amplitude of metric variations: $a_{+,\times} = 2\kappa e_{+,\times}$

frequency: $\omega = 2\pi f$

→ combine in expression for flux

$$\overline{\frac{dE}{dAdt}} = \frac{\pi c^3 f^2}{8G}\left(a_+^2 + a_\times^2\right)$$

full amplitude: $a = \sqrt{a_+^2 + a_\times^2}$

Even small amplitudes correspond to large fluxes: extreme energy densities create tiny deformations of space:

space is `stiffest substance' known

Other fluxes

Outward momentum flux: $$\frac{dP_i}{r^2 d\Omega dt} = \mathcal{S}_{ni}$$

Outward angular momentum flux: $$\frac{dL_i}{r^2 d\Omega dt} = \mathcal{J}_{ni}$$

1. For radial flow out of spherical volume S_r involving fields $h_{ij}(t-r)$:

integrated momentum flux vanishes: $$\frac{dP_i}{dt} = -\oint_{\partial S_r} d^2\sigma\, \mathcal{S}_{ni} = 0$$

as momentum density $\mathcal{S}_{ni}$ on the boundary surface in direction of propagation, i.e. radially outward:

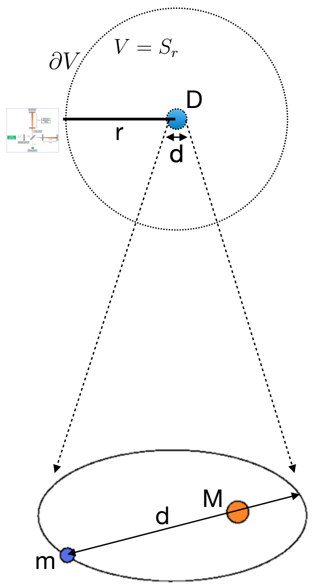
$\mathcal{S}_{ni} \propto \hat{r}_i$ ⟶ integrating over a full sphere contributions from opposite points cancel

2. This argument does *not* hold for angular momentum:

$$\frac{dL_i}{dt} = -\oint_{\partial S_r} d^2\sigma\, \mathcal{J}_{ni} \neq 0$$

as $\mathcal{J}_{ni}$ directed *orthogonal* to direction of propagation: *tangent* to surface

Sources of gravitational waves



binary star system

here: consider an isolated source of maximal size d observed from a distance r with $r \gg d$

e.g., a binary system of compact objects like white dwarfs, neutron stars or black holes

note: for PSR 1913+16 $\frac{d}{r} \sim 10^{-8}$

Assume the observer is at rest w.r.t. to the CM of the source; if not: signals are Doppler shifted.

We have to solve an inhomogeneous wave equation if type

$$\Box\,\phi(\mathbf{x},t) = \rho(\mathbf{x},t)$$

Retarded (causal) solution:

$$\phi(\mathbf{x},t) = -\frac{1}{4\pi}\int_{V=S_r} d^3x'\,\frac{\rho(\mathbf{x}',t-|\mathbf{x}'-\mathbf{x}|)}{|\mathbf{x}'-\mathbf{x}|}$$

position of source element

position of observer

distance between source element and observer

solving the gravitational-wave equation

$$\Box \underline{h}_{\mu\nu} = -\kappa T_{\mu\nu} \longrightarrow \underline{h}_{\mu\nu}(\mathbf{x},t) = \frac{\kappa}{4\pi}\int_{S_r} d^3x' \, \frac{T_{\mu\nu}(\mathbf{x}', t-|\mathbf{x}'-\mathbf{x}|)}{|\mathbf{x}'-\mathbf{x}|}$$

to evaluate, note:

- $\underline{h}_{\mu\nu}(\mathbf{x},t)$ observed in far region where $T_{\mu\nu}(\mathbf{x},t) = 0$
- in that region $\Box \underline{h}_{\mu\nu} = 0$

and the *TT* - gauge applies: $\underline{h}_{\mu\nu}(\mathbf{x},t) = h_{\mu\nu}(\mathbf{x},t)$

- only spherical waves falling off as $1/r$ survive:

$$h_{ij}(\mathbf{x},t) \sim \int dk \, e_{ij}(k)\frac{e^{ik(r-t)}}{r} \quad \text{and} \quad h_{00} = h_{0i} = 0$$

with $h_{jj} = 0$ and $\hat{r}_i h_{ij} = 0$

$$h_{jj} = 0 \;\updownarrow\; e_{jj} = 0 \qquad \hat{r}_i h_{ij} = 0 \;\updownarrow\; k_i e_{ij} = 0$$

General form of amplitude

$$h_{\mu\nu} = \frac{\kappa}{4\pi r}\int_{S_r} d^3x' \, T_{\mu\nu}(\mathbf{x}', t-r)$$

energy-momentum conservation:

$$\partial_0 h_{0\mu} = \frac{\kappa}{4\pi r}\int_{S_r} d^3x' \, \partial_0 T_{0\mu}(\mathbf{x}', t-r)$$

$$= \frac{\kappa}{4\pi r}\int_{S_r} d^3x' \, \partial_i' T_{i\mu}(\mathbf{x}', t-r) = 0$$

TT - gauge:

$$h_{ij} = \frac{\kappa}{4\pi r}(\delta_{ik} - \hat{r}_i\hat{r}_k)(\delta_{jl} - \hat{r}_j\hat{r}_l)\left(I_{kl} + \frac{1}{2}\delta_{kl}\hat{r}\cdot I\cdot\hat{r}\right)$$

with $h_{kk} = 0 \longleftrightarrow I_{kk} = 0$

and $I_{ij}(t-r) = \int_{S_r} d^3x' \left(T_{ij} - \frac{1}{3}\delta_{ij}T_{kk}\right)_{t-r}$

Quadrupole approximation

use: $\partial_0^2 T_{00} = \partial_0 \partial_i T_{i0} = \partial_i \partial_j T_{ij}$

$$\longrightarrow \quad \frac{1}{2}\partial_0^2 \int d^3x\, x_i x_j T_{00} = \frac{1}{2}\int d^3x\, x_i x_j\, \partial_k \partial_l T_{kl} = \int d^3x\, T_{ij}$$

for non-relativistic sources $T_{00}(\mathbf{x},t) = \rho(\mathbf{x},t)$ (mass density)

$$\longrightarrow \quad I_{ij} = \frac{1}{2}\frac{\partial^2 Q_{ij}}{\partial t^2}$$

$$Q_{ij}(t-r) = \int_{S_r} d^3x' \left(x'_i x'_j - \frac{1}{3}\delta_{ij}\mathbf{x}'^2 \right) \rho(\mathbf{x}', t-r)$$

(mass quadrupole)

final result:

$$h_{ij}(\mathbf{x},t) = \frac{\kappa}{4\pi r}(\delta_{ik} - \hat{r}_i \hat{r}_k)(\delta_{jl} - \hat{r}_j \hat{r}_l)\frac{\partial^2}{\partial t^2}\left(Q_{kl} + \frac{1}{2}\delta_{kl}\hat{r}\cdot Q \cdot \hat{r} \right)_{t-r}$$

Differential fluxes of energy, momentum and angular momentum

$$\frac{dE}{d\Omega dt} = -\frac{G}{8\pi c^5}\left[\mathrm{Tr}\,\dddot{Q}^2 - 2\hat{\mathsf{r}}\cdot\dddot{Q}^2\cdot\hat{\mathsf{r}} + \frac{1}{2}(\hat{\mathsf{r}}\cdot\dddot{Q}\cdot\hat{\mathsf{r}})^2 \right]$$

$$\frac{dP_i}{d\Omega dt} = \frac{G}{8\pi c^6}\hat{\mathsf{r}}_k \left[\mathrm{Tr}\,\dddot{Q}^2 - 2\hat{\mathsf{r}}\cdot\dddot{Q}^2\cdot\hat{\mathsf{r}} + \frac{1}{2}(\hat{\mathsf{r}}\cdot\dddot{Q}\cdot\hat{\mathsf{r}})^2 \right] = -\frac{1}{c}\frac{dE}{d\Omega dt}$$

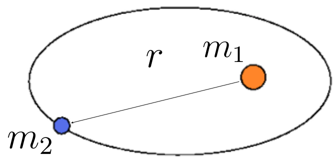
$$\frac{dL_i}{d\Omega dt} = -\frac{G}{4\pi c^5}\varepsilon_{kij}\Bigg[\left(\ddot{Q}\cdot\dddot{Q}\right)_{ij} - \left(\ddot{Q}\cdot\hat{\mathsf{r}}\right)_i \left(\dddot{Q}\cdot\hat{\mathsf{r}}\right)_j + \hat{\mathsf{r}}_i \left(\ddot{Q}\cdot\dddot{Q}\cdot\hat{\mathsf{r}} - \frac{1}{2}\ddot{Q}\cdot\hat{\mathsf{r}}\,\hat{r}\cdot\dddot{Q}\cdot\hat{\mathsf{r}} \right)_j \Bigg]$$

Integrated fluxes

$$\frac{dE}{dt} = -\frac{G}{5c^5}\mathrm{Tr}\,\dddot{Q}^2 \qquad \frac{dP_k}{dt} = 0$$

$$\frac{dL_i}{dt} = -\frac{2G}{5c^5}\varepsilon_{kij}\left(\ddot{Q}\cdot\dddot{Q}\right)_{ij}$$

Compact binaries



binary systems

$$M = m_1 + m_2 \quad \mu = \frac{m_1 m_2}{m_1 + m_2}$$

$$r = |\mathbf{r}_2 - \mathbf{r}_1|$$

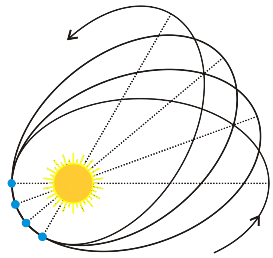
3 stages:

formation	(n-body interactions)
inspiral	(quasi-newtonian)
merger	(relativistic)

quasi-newtonian: Kepler + precession

$$r(\varphi) = \frac{\rho}{1 - e \cos n\varphi}$$

eccentricity — precession rate

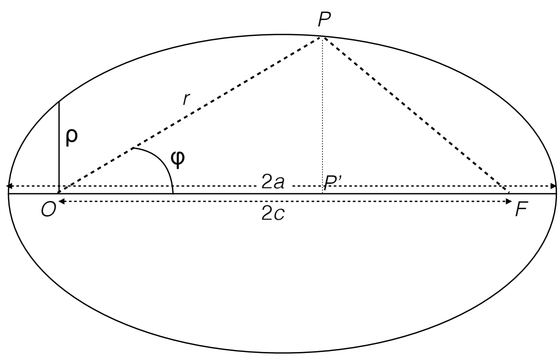


find central force with such orbits

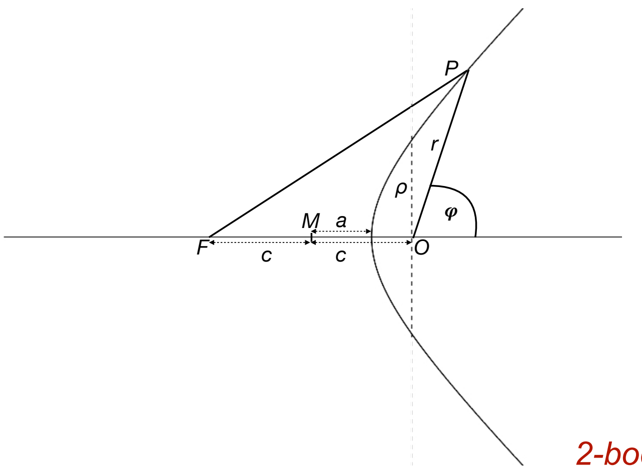
$$\mathbf{F}(r) = F(r)\hat{\mathbf{r}} \longrightarrow r^2\dot{\varphi} = \ell$$

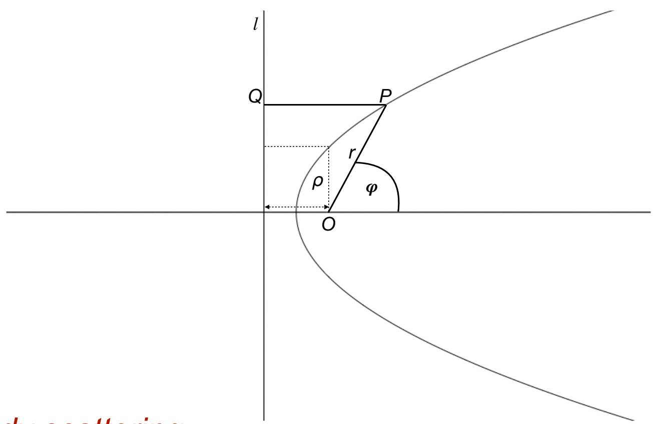
quasi-newtonian plane orbits

$$r = \frac{\rho}{1 - e \cos n\varphi}$$



bound system





2-body scattering

$$F(r) = -\frac{GM\mu}{r^2} - \frac{\beta\mu}{r^3}$$

$$\dot{\varphi} = \frac{\ell}{r^2}$$ ← angular momentum/unit of mass

$$\dot{r} = r'\dot{\varphi} = -\frac{en\ell}{\rho}\sin n\varphi$$ radial velocity

$$\frac{1}{n^2} = 1 + \frac{\beta}{GM\rho} = \frac{\ell^2}{GM\rho}$$ precession

$$\longrightarrow \quad n\ell = \sqrt{GM\rho}$$

Relativistic precession (Schwarzschild) $\quad \frac{1}{n^2} \simeq 1 + \frac{6GM}{c^2\rho}$

but precession may also arise because of many-body forces.

Quadrupole approximation

$$Q_{ij}(t) = m_1\left(r_{1i}r_{1j} - \frac{1}{3}\,\delta_{ij}\,\mathbf{r}_1^2\right) + m_2\left(r_{2i}r_{2j} - \frac{1}{3}\,\delta_{ij}\,\mathbf{r}_2^2\right)$$

$$\xrightarrow[CM]{} \quad \mu\left(r_ir_j - \frac{1}{3}\,\delta_{ij}r^2\right)$$

T T - gauge:

$$2\kappa h_{ij}(\vec{\mathsf{r}},t) = \frac{2G}{c^4\mathsf{r}}\left[\ddot{Q}_{ij} - \hat{\mathsf{r}}_i(\ddot{Q}\cdot\hat{\mathsf{r}})_j - \hat{\mathsf{r}}_j(\ddot{Q}\cdot\hat{\mathsf{r}})_i + \frac{1}{2}\,(\delta_{ij} + \hat{\mathsf{r}}_i\hat{\mathsf{r}}_j)\hat{\mathsf{r}}\cdot Q\cdot\hat{\mathsf{r}}\right]_{t_{ret}}$$

Note: $r = |\mathbf{r}_2 - \mathbf{r}_1| = \dfrac{\rho}{1 - e\cos n\varphi}$ = separation between masses

$\vec{\mathsf{r}} = \mathsf{r}\,\hat{\mathsf{r}}$ with $\mathsf{r} = |\mathbf{x}|$ = distance to observer

$\hat{\mathsf{r}} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ = direction of observer

kinematical relations

$$r(\varphi) = \frac{\rho}{1 - e\cos n\varphi}$$

$$\dot{\varphi} = \frac{\ell}{r^2} \qquad \dot{r} = r'(\varphi)\dot{\varphi} = -\frac{n\ell}{\rho}\sqrt{e^2 - \left(1 - \frac{\rho}{r}\right)^2}$$

$$\ddot{\varphi} = -\frac{2\ell}{r^3}\,\dot{r} \qquad \ddot{r} = -\frac{n\ell^2}{\rho r^2}\left(1 - \frac{\rho}{r}\right)$$

$$\longrightarrow \quad \ddot{\mathbf{Q}} = \frac{\mu n\ell^2}{\rho^2}\left[n\left(e^2 - 1 + \frac{\rho}{r}\right)\mathbf{E} + n\left((e^2 - 1 + \frac{\rho}{r} - \frac{2\rho^2}{r^2}\right)\mathbf{M} - \frac{2\rho}{r}\sqrt{e^2 - 1 + \frac{2\rho}{r} - \frac{\rho^2}{r^2}}\,\mathbf{N}\right]$$

where

$$\mathbf{E} = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{1}{3} & 0 \\ 0 & 0 & -\frac{2}{3} \end{bmatrix} \qquad \mathbf{M} = \begin{bmatrix} \cos 2\varphi & \sin 2\varphi & 0 \\ \sin 2\varphi & -\cos 2\varphi & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad \mathbf{N} = \begin{bmatrix} -\sin 2\varphi & \cos 2\varphi & 0 \\ \cos 2\varphi & \sin 2\varphi & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Example I: circular orbits

$$\mathbf{r} = R\,(\cos\omega t, \sin\omega t, 0) \quad \text{with} \quad \omega^2 = \frac{GM}{R^3}$$

$$2\kappa h_{ij}^{(x)} = \frac{2G^2 M\mu}{c^4 \mathsf{r} R}\begin{pmatrix} 0 & 0 & 0 \\ 0 & \cos 2\omega t & 0 \\ 0 & 0 & -\cos 2\omega t \end{pmatrix}_{t_{ret}}$$

+-mode only

$$2\kappa h_{ij}^{(y)} = \frac{2G^2 M\mu}{c^4 \mathsf{r} R}\begin{pmatrix} -\cos 2\omega t & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \cos 2\omega t \end{pmatrix}_{t_{ret}}$$

+-mode only

180° out of phase

$$2\kappa h_{ij}^{(z)} = -\frac{4G^2 M\mu}{c^4 \mathsf{r} R}\begin{pmatrix} \cos 2\omega t & \sin 2\omega t & 0 \\ \sin 2\omega t & -\cos 2\omega t & 0 \\ 0 & 0 & 0 \end{pmatrix}_{t_{ret}}$$

+- and x-mode
90° out of phase

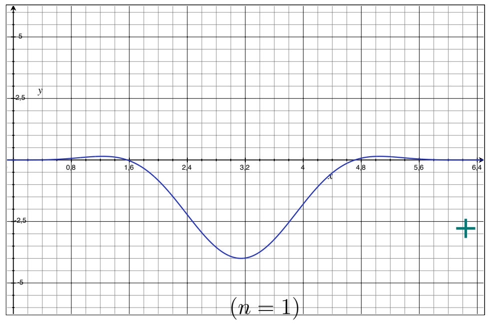
Binary neutron stars of Hulse-Taylor type : $\frac{G^2 M\mu}{c^4 R\,\mathsf{r}} = \frac{2\times 10^{-19}}{\mathsf{r}[\mathrm{lyr}]}$

The frequency of gravitational waves = 2 x orbital frequency

Example II: parabolic orbits

$$2\kappa h_{ij}^{(x)} = -\frac{2G^2 M\mu}{c^4 \mathsf{r}\rho}\,(1-\cos\varphi)^2\cos\varphi \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

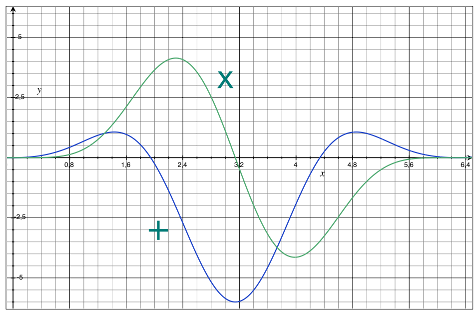
+-mode only



$$2\kappa h_{ij}^{(z)} = \frac{GM\mu}{c^4 \mathsf{r}\rho}\,(1-\cos\varphi)$$

$$\times \begin{pmatrix} 2\cos\varphi - \cos 2\varphi & 2\sin\varphi - \sin 2\varphi & 0 \\ 2\sin\varphi - \sin 2\varphi & -2\cos\varphi + \cos 2\varphi & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

+- and x-mode
90° out of phase



Parabolic orbit representing the motion of the effective 1-body reduced mass in the CM frame of two masses scattering in the x-y-plane: Graphs: gravitational-wave amplitudes seen along the axis of the parabola (x-axis) and from above (along the z-axis) as a function of the evolution parameter φ, measuring the progression of the effective mass in its orbit.

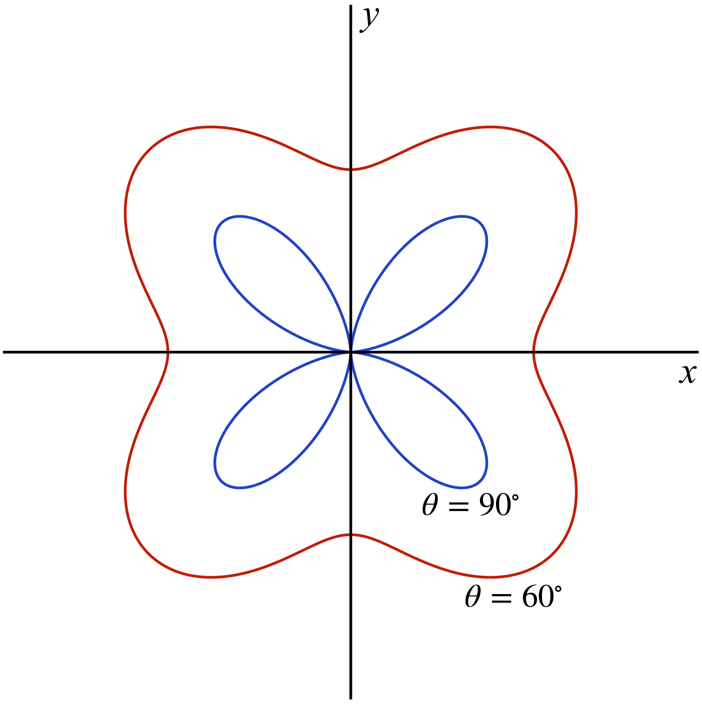
Energy loss

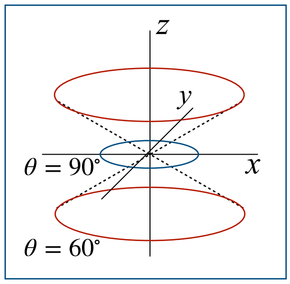
Differential flux
$$\frac{dE}{d\Omega dt} = -\frac{G}{8\pi c^5}\left[\mathrm{Tr}\,\dddot{Q}^2 - 2\hat{\mathsf{r}}\cdot\dddot{Q}^2\cdot\hat{\mathsf{r}} + \frac{1}{2}(\hat{\mathsf{r}}\cdot\dddot{Q}\cdot\hat{\mathsf{r}})^2\right]$$

$$\begin{aligned}\frac{dE}{d\Omega dt} = -\frac{G\mu^2\ell^6}{8\pi c^5 r^8}\Big[&2A^2 + 2B^2 \\ &- 2\sin^2\theta\left(A^2 + B^2 + A^2\cos 2(\phi-\varphi) + AB\sin 2(\phi-\varphi)\right) \\ &+ \frac{1}{2}\sin^4\theta\left(A^2 + B^2 + 2A^2\cos 2(\phi-\varphi) + 2AB\sin 2(\phi-\varphi)\right. \\ &\left. + \left(A^2 - B^2\right)\cos^2 2(\phi-\varphi) + 2AB\sin 2(\phi-\varphi)\cos 2(\phi-\varphi)\right)\Big]\end{aligned}$$

$$A = \frac{n^3 r^2}{\rho^2}\sqrt{(e^2-1) + \frac{2\rho}{r} - \frac{\rho^2}{r^2}} \qquad B = -\frac{4n^2 r}{\rho} + 4(n^2-1)$$

normalised angular distribution of intensity of radiation in periastron and apastron for PSR 1913+16





view perpendicular to orbital plane
absolute intensity periastron:apastron: 220:1

Total flux $$\frac{dE}{dt} = -\frac{G}{5c^5}\,\mathrm{Tr}\,\dddot{Q}^2$$

$$\rightarrow \quad \frac{dE}{dt} = -\frac{1}{30n^6}\left(\frac{2GM}{c^2\rho}\right)^4 \frac{\mu^2 c^3}{M\rho}\left[n^6\left(e^2-1\right)\frac{\rho^4}{r^4} + 2n^6\frac{\rho^5}{r^5} - n^4\left(n^2-12\right)\frac{\rho^6}{r^6} - 24n^2\left(n^2-1\right)\frac{\rho^7}{r^7} + 12(n^2-1)^2\frac{\rho^8}{r^8}\right]$$

$$\xrightarrow{n=1} \quad \frac{dE}{dt} = -\frac{1}{30}\left(\frac{2GM}{c^2\rho}\right)^4 \frac{\mu^2 c^3}{M\rho}\left[\left(e^2-1\right)\frac{\rho^4}{r^4} + 2\frac{\rho^5}{r^5} + 11\frac{\rho^6}{r^6}\right]$$

(Peters-Mathews, 1963)

closed orbits ($e < 1$): energy loss per period

$$\Delta E = -\frac{4\pi\sqrt{2}}{5n^6}\left(\frac{2GM}{c^2\rho}\right)^{7/2}\frac{\mu^2c^2}{M}\left[1 + \frac{e^2}{24}\left(n^6 + 12n^4 - 120n^2 + 180\right) + \frac{e^4}{96}\left(n^6 + 216n^4 - 720n^2 + 540\right) + \frac{5e^6}{16}\left(n^2-1\right)^2\right]$$

[PSR 1913 + 16 (Hulse-Taylor) 3.16×10^{46} **J**]

average power: ~ 10^{25} W

circular orbits:

$$\frac{dE}{dt} = -\frac{32G^4M^3\mu^2}{5c^5R^5} \longrightarrow \Delta E = -\frac{4\pi\sqrt{2}}{5}\left(\frac{2GM}{c^2R}\right)^{7/2}\frac{\mu^2c^2}{M}$$

orbital energy: $E = -\dfrac{GM\mu}{2R}$

$$\longrightarrow \frac{\Delta E}{E} = \frac{\pi\sqrt{2}}{5}\frac{\mu}{M}\left(\frac{2GM}{c^2R}\right)^{5/2}$$

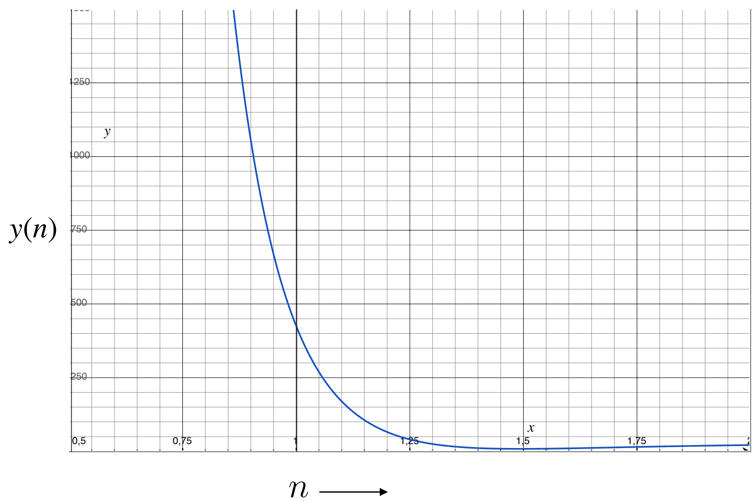
rate of inspiral: $\dfrac{dR}{dt} = \dfrac{dE/dt}{dE/dR} = -\dfrac{64}{5}\dfrac{G^3M^2\mu}{c^5R^3}$

$$\longrightarrow \boxed{R(t) = 4\left[\frac{G^3M^2\mu}{5c^5}(t_0 - t)\right]^{1/4}}$$

parabolic orbits:

$$\Delta E = -\frac{\pi\sqrt{2}}{120}\left(\frac{2GM}{c^2\rho}\right)^{7/2}\frac{\mu^2c^2}{M}\underbrace{\left(5 + \frac{294}{n^2} - \frac{1260}{n^4} + \frac{1386}{n^6}\right)}_{y(n)}$$

for $n = 1$: ~ 4.5 times the energy/turn of equivalent circular orbit



but: any radiative energy loss results in capture!

Loss of angular momentum

Differential flux $$\frac{dL_i}{d\Omega dt} = -\frac{G}{4\pi c^5}\varepsilon_{kij}\left[\left(\ddot{Q}\cdot\dddot{Q}\right)_{ij} - \left(\ddot{Q}\cdot\hat{\mathrm{r}}\right)_i\left(\dddot{Q}\cdot\hat{\mathrm{r}}\right)_j + \hat{\mathrm{r}}_i\left(\ddot{Q}\cdot\dddot{Q}\cdot\hat{\mathrm{r}} - \frac{1}{2}\ddot{Q}\cdot\hat{\mathrm{r}}\,\hat{r}\cdot\dddot{Q}\cdot\hat{\mathrm{r}}\right)_j\right]$$

Total flux

$$\longrightarrow \quad \frac{dL_z}{dt} = -\frac{4G}{5c^5}\left(\ddot{Q}\cdot\dddot{Q}\right)_{xy}$$

$$= -\frac{8G\mu^2\ell^5}{5c^5r^6}\left[n^4(1-e^2)\frac{r^3}{\rho^3} - 2n^2(n^2-1)(1-e^2)\frac{r^2}{\rho^2} + n^2(n^2+2)\frac{r}{\rho} - 4(n^2-1)\right]$$

$$\frac{dL_x}{dt} = \frac{dL_y}{dt} = 0$$

closed orbits (*e* < *1*): angular momentum loss per period

$$\Delta L_z = -\frac{8\pi}{5n^5}\left(\frac{2GM}{c^2\rho}\right)^3\frac{\mu^2\rho c}{M}\left[1 + \frac{e^2}{8}\left(3n^4 - 20n^2 + 24\right) + \frac{e^4}{8}\left(2n^2-3\right)\left(n^2-1\right)\right]$$

circular orbits:

$$\frac{dL_z}{dt} = -\frac{32\,G^{7/2}M^{5/2}\mu^2}{5c^5R^{7/2}} \quad\longrightarrow\quad \Delta L_z = -\frac{8\pi}{5}\left(\frac{2GM}{c^2R}\right)^3\frac{\mu^2Rc}{M}$$

orbital angular momentum: $L_z = \mu\sqrt{GMR}$

$$\longrightarrow \quad \frac{\Delta L_z}{L_z} = -\frac{4\pi\sqrt{2}}{5}\frac{\mu}{M}\left(\frac{2GM}{c^2R}\right)^{5/2}$$

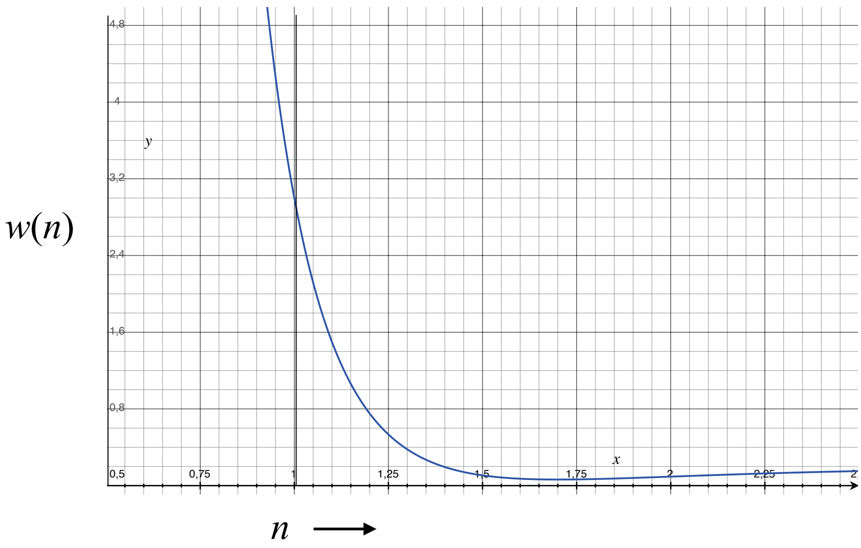
rate of inspiral: $$\frac{dR}{dt} = \frac{dL_z/dt}{dL_z/dR} = -\frac{64}{5}\frac{G^3M^2\mu}{c^5R^3}$$

$$\longrightarrow \quad \boxed{R(t) = 4\left[\frac{G^3M^2\mu}{5c^5}(t_0 - t)\right]^{1/4}}$$

parabolic orbits:

$$\Delta L_z = -\pi \left(\frac{2GM}{c^2 \rho}\right)^3 \frac{\mu^2 \rho c}{M} \underbrace{\left[\frac{7}{n^5} - \frac{5}{n^3} + \frac{1}{n}\right]}_{w(n)}$$

for $n = 1$: ~ 2 times the angular momentum/turn of equivalent circular orbit



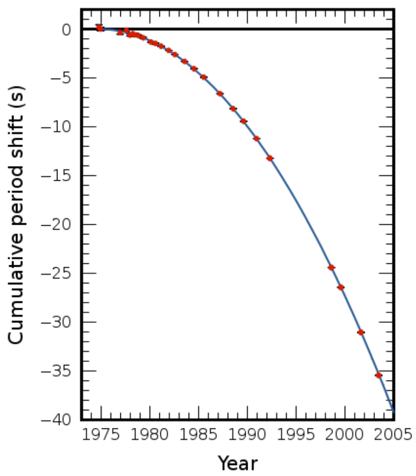
bound orbits: $a = \dfrac{\rho}{1 - e^2}$

$$\longrightarrow \quad \frac{da}{dt} = \frac{1}{1 - e^2}\frac{d\rho}{dt} + \frac{2\rho e}{(1 - e^2)^2}\frac{de}{dt}$$

Kepler: $\omega^2 = \dfrac{4\pi^2}{T^2} = \dfrac{GM}{a^3}$

$$\longrightarrow \quad \frac{dT}{dt} = 3\pi \sqrt{\frac{a}{GM}} \frac{da}{dt}$$

$$\frac{\Delta T}{T} = -\frac{192\pi}{5}\frac{\mu}{M}\left(\frac{2\pi GM}{c^3 T}\right)^{5/3} \left(1 - e^2\right)^{-7/2} \left[\frac{1}{n^6} + \frac{e^2}{24}\left(1 + \frac{12}{n^2} - \frac{120}{n^4} + \frac{180}{n^6}\right) + \frac{e^4}{96}\left(1 + \frac{216}{n^2} - \frac{720}{n^4} + \frac{540}{n^6}\right) + \frac{5e^6}{16n^6}\left(n^2 - 1\right)^2\right]$$



for PSR1913+16:

$$\frac{\Delta T}{T} = 2.40 \times 10^{-12} \quad \longrightarrow \quad \Delta T = 0.67 \times 10^{-7}\ \mathrm{s/orbit}$$

$$\frac{\Delta a}{a} = 1.60 \times 10^{-12} \quad \longrightarrow \quad \Delta a = 3.1\ \mathrm{mm/orbit}$$

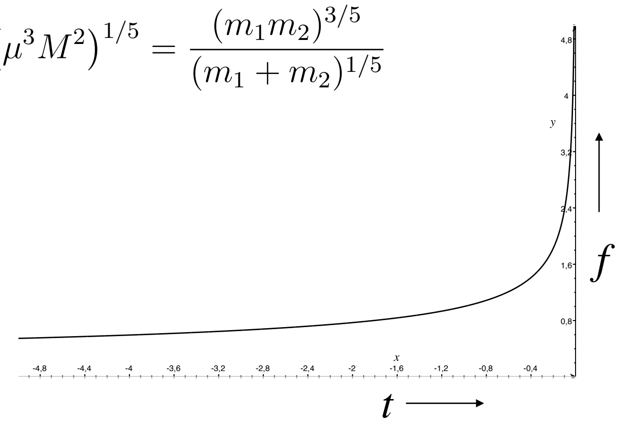
Circular orbits

$$\rho = R, \;\; e = 0, \;\; n = 1 \quad \longrightarrow \quad R(t) = R_S \left[\frac{32\mu}{5M} \frac{c(t_0 - t)}{R_S} \right]^{1/4}$$

$\longleftrightarrow$ time till coalescence: $$c(t_0 - t) = \frac{5}{32} \frac{\mu}{M} \frac{R^4}{R_S^3} \qquad (R \gg R_S)$$

orbital frequency: $$f(t) = \frac{c}{16\pi} \left(\frac{G\mathcal{M}}{c^2} \right)^{-5/8} \left(\frac{5}{c(t_0 - t)} \right)^{3/8}$$

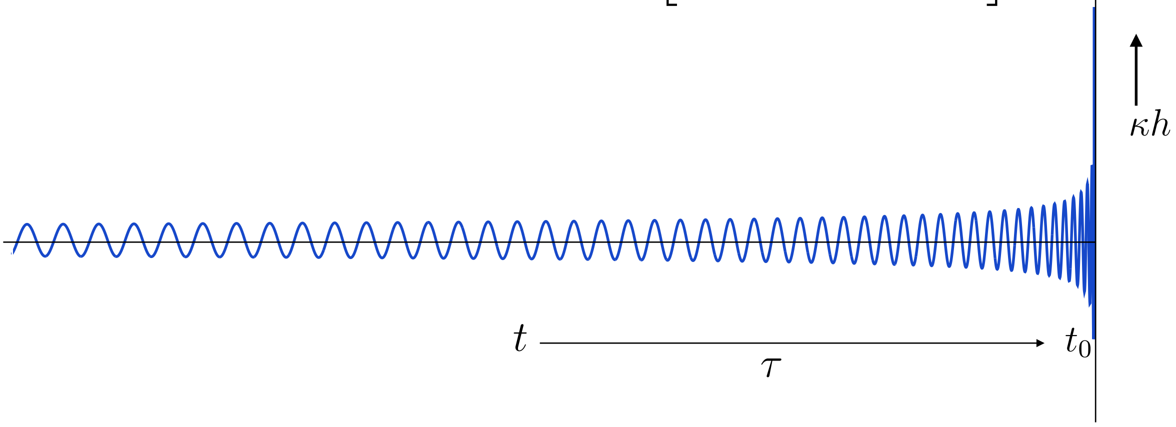
chirp mass $$\mathcal{M} \equiv \left(\mu^3 M^2 \right)^{1/5} = \frac{(m_1 m_2)^{3/5}}{(m_1 + m_2)^{1/5}}$$



wave form

$$\tau = (t_0 - t)_{ret} = t_0 - t \quad \longrightarrow$$

$$\begin{aligned} 2\kappa h_+(\tau) &= \frac{2G^2 M \mu}{c^4 R(\tau) \mathrm{r}} \cos \left[2\pi \int_0^\tau d\tau' f(\tau') \right] \\ &= \frac{1}{2\mathrm{r}} \left(\frac{G\mathcal{M}}{c^2} \right) \left(\frac{5G\mathcal{M}}{c^3 \tau} \right)^{1/4} \cos \left[2 \left(\frac{5G\mathcal{M}}{c^3 \tau} \right)^{-5/8} + \Phi_0 \right] \end{aligned}$$



A sample of binary merger signals