PHYWE	**Fourier optics – 2f Arrangement**	**LEP 2.6.11**

Related topics

Fourier transform, lenses, Fraunhofer diffraction, index of refraction, Huygens' principle.

Principle and task

The electric field distribution of light in a specific plane (object plane) is Fourier transformed into the 2f configuration.

Equipment

Optical base plate w. rubber ft.	08700.00	1
Laser, He-Ne 0.2/1.0 mW, 220 VAC*	08180.93	1
Adjusting support 35×35 mm	08711.00	2
Surface mirror 30×30 mm	08711.01	2
Magnetic foot f. opt. base plt.	08710.00	7
Holder f. diaphr./beam splitter	08719.00	1
Lens, mounted, f +150 mm	08022.01	1
Lens, mounted, f +100 mm	08021.01	1
Lensholder f. optical base plate	08723.00	2
Screen, white, 150×150 mm	09826.00	1
Diffraction grating, 50 lines/mm	08543.00	1
Screen, with diffracting elements	08577.02	1
Qbjective 25× N.A. 0.45	62470.00	1
Sliding device, horizontal	08713.00	1
xy shifting device	08714.00	2
Adapter ring device	08714.01	1
Pin hole 30 um	08743.00	1
Rule, plastic, 200 mm	09937.01	1

***Alternative**

He/Ne Laser, 5 mW with holder	08701.00	1
Power supply f. laser head 5 mW	08702.93	1

Objective

Investigation of the Fourier transform by a convex lens for different diffraction objects in a 2f set-up.

Set-up and procedure

— In the following, the pairs of numbers in brackets refer to the coordinates on the optical base plate in accordance with Fig. 1b. These coordinates are intended to help with coarse adjustment.

— Perform the experimental set-up according to fig. 1a or 1b. The recommended set-up height (beam path height) is 130 mm.

— The **E25x** beam expansion system (magnetic foot at [1,6]) and the lens $\mathbf{L_0}$ [1,3] are not to be used for the first beam adjustment.

— When adjusting the beam path with the adjustable mirrors $\mathbf{M_1}$ [1,8] and $\mathbf{M_2}$ [1,1], the beam is set along the 1. x and 1. y coordinates of the base plate.

— Now place the **E25x** [1,6] beam expansion system without its objective and pinhole, but equipped instead only with the adjustment diaphragm, in the beam path. Orient it such that the beam passes through the circular stops without obstruction. Now replace these diaphragms with the objective and the pinhole diaphragm. Move the pinhole diaphragm toward the focus of the objective. In the process, first ensure that a maximum of diffuse light strikes

Fig. 1a: Experimental set-up for fundamental principle of Fourier optics.

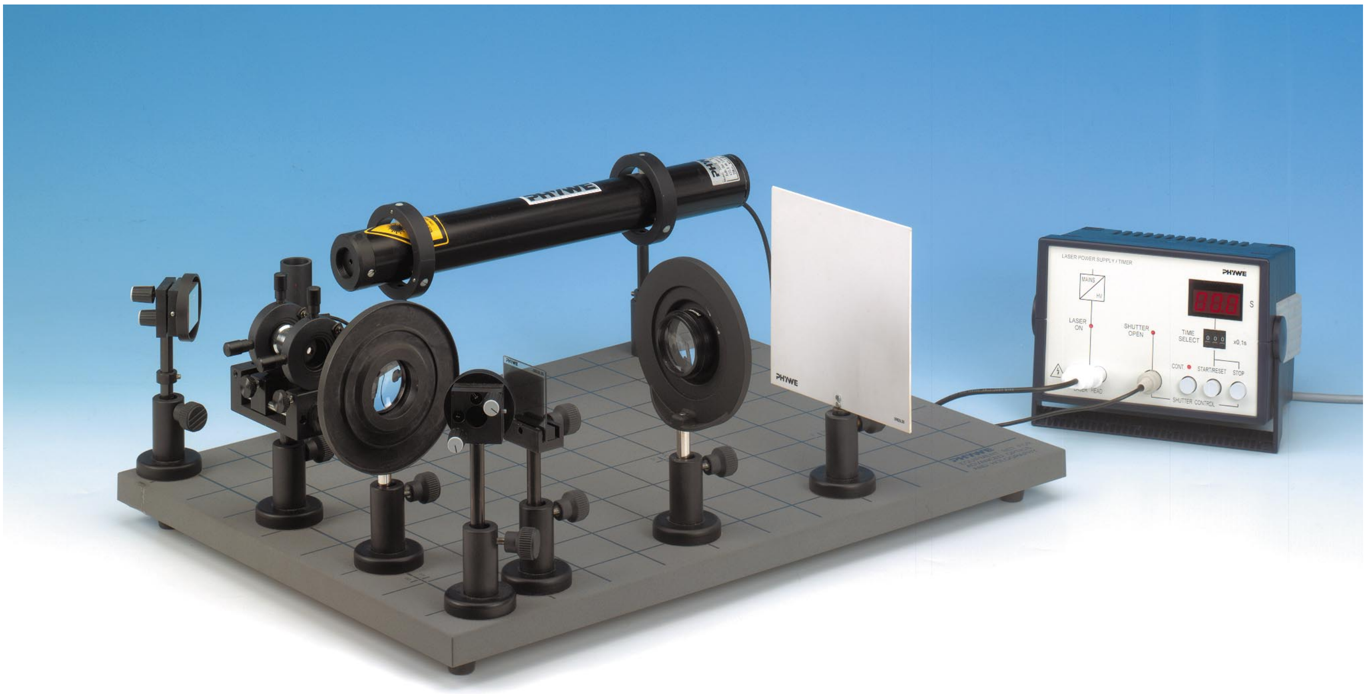
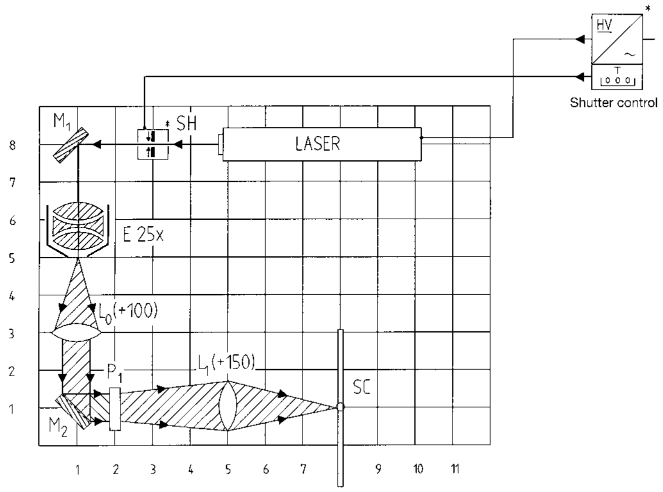


Fig.1b: Experimental set-up for the fundamental principles of Fourier optic (2f set-up).
*only required for the 5 mW laser!



the pinhole diaphragm and later the expanded beam. Successively adjust the lateral positions of the objective and the pinhole diaphragm while approaching the focus in order to ultimately provide an expanded beam without diffraction phenomena. The $\mathbf{L_0}$ [1,3] (f = +100 mm) is now positioned at a distance exactly equal to the focal length behind the pinhole diaphragm such that parallel light now emerges from the lens. No divergence of the light spot should occur with increasing separation. (testing for parallelism via the light spot's diameter with a ruler at various distances behind the lens $\mathbf{L_0}$ in a range of approximately 1 m).

— Now set-up the additional optical components.

Set-up and procedure: (in accordance with Fig.1a and 1b)

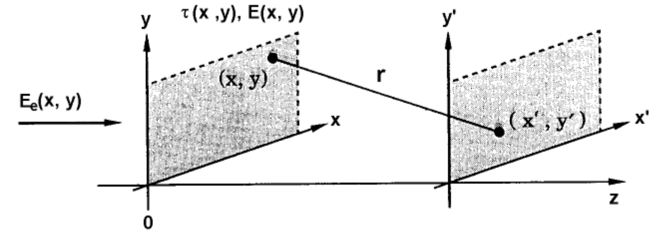
Place a plate holder $\mathbf{P_1}$ [2,1] in the object plane. Position the lens $\mathbf{L_1}$ [5,1] at the focus (f = 150 mm) and the screen **Sc** [8,1] at the same distance behind the lens.

— (a) As a first partial experiment observe the plane wave itself (the light spot), i.e. no diffracting structures are placed in the object plane. According to the theory, a point should appear in the Fourier plane **Sc** behind the lens. This is also the focus; this fact can be checked by changing the screen's distance from the lens.)

— (b) Now clamp the diaphragm with diffraction objects into the plate holder $\mathbf{P_1}$ in the object plane. While doing so, adjust its height and lateral position in such a manner that the light spot strikes the slit which has a slit width of 0.2 mm. The Fourier transform of the slit can be seen on the screen as the typical diffraction pattern of a slit (compare with the theory).

— (c) The diffraction grating (50 lines/mm) now serves as a diffracting structure; clamp it in the plate holder $\mathbf{P_1}$. Conclusions about the slit separation can be made from the separation of the diffraction maxima in the Fourier plane Sc behind the lens $\mathbf{L_1}$ (see theory).

Fig.2: A plane wave E_e(x,y) is diffracted in the plane with τ (x,y) for z=0.



Theory and evaluation

The Fourier transform plays a major role in the natural sciences. In the majority of cases, one deals with Fourier transforms in a time range, which supplies us with the spectral composition of a time signal. This concept can be extended in two aspects:

1. In our case a spatial signal and not a temporal signal is transformed.
2. A two-dimensional transform is performed.
   From this, the following is obtained:

$$\tilde{E}(\nu_x, \nu_y) \cdot \tilde{F}\,[E(x,y)]\,(\nu_x, \nu_y) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} E(x,y)e^{-2\pi i\,(\nu_x \cdot x + \nu_y \cdot y)} d_x d_y \quad (1)$$

where ν_x and ν_y are spatial frequencies.

Scalar diffraction theory

In Fig. 2 we observe a plane wave which is diffracted in one plane. For this wave in the xy plane directly behind the plane z = 0 with the following transmission distribution τ (x,y):

$$E(x,y) = \tau(x,y)\, E_e(x,y)$$

where $E_e(x,y)$: electric field distribution of the incident wave.
The further expansion can be described by the assumption that a spherical wave emanates from each point $(x,y,0)$ behind the diffracting structure (Huygens' principle). This leads to Kirchhoff's diffraction integral:

$$E(x',y',z) = \frac{1}{i\lambda}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} E(x,y)\,\frac{e^{ikr}}{r}\,\cos(\mathbf{n},\mathbf{r})\,d_x d_y \quad (2)$$

with λ = spherical wave length
n = normal vector of the (x, y) plane
k = wave number = $\frac{2\pi}{\lambda}$

Equation (2) corresponds to a accumulation of spherical waves, where the factor $1/(i\lambda)$ is a phase and amplitude factor and cos $(\mathbf{n},\mathbf{r})$ a directional factor which results from the Maxwell field equations.
The Fresnel approximation (observations in a remote radiation field) considers only rays which occupy a small angle to the optical axis (z axis), i.e. $|x|,|y| \ll z$ and $|x'|,|y'| \ll z$. In this case, the directional factor can be neglected and the $1/r$ dependence becomes: $1/r = 1/z$. In the exponential function, this

Fig. 3: Relationships between spatial frequencies and the diffraction angle.

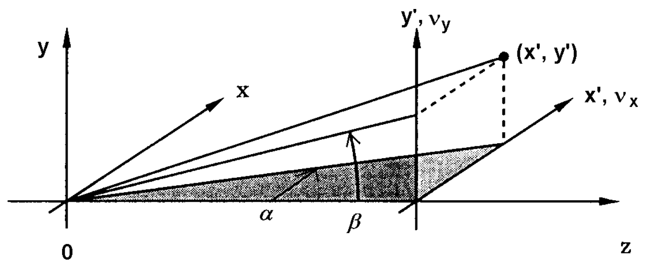
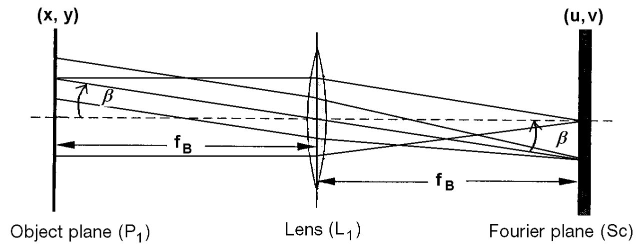


Fig. 4: Experimental set-up with supplement for direct measurement of the initial velocity of the ball.



cannot be performed as easily since even small changes in *r* result in large phase changes. To achieve this, the roots in

$$r = \sqrt{(x'-x)^2 + (y'-y)^2 + z^2} = z\sqrt{1 + \frac{(x'-x)^2}{z^2} + \frac{(y'-y)^2}{z^2}}$$

are expanded into a series and one obtains:

$$r = z + \frac{(x'-x)^2}{2z} + \frac{(y'-y)^2}{2z}$$

This results in the Fresnel approximation of the diffraction integral

$$E(x',y',z) = \frac{e^{ikz}}{i\lambda} \cdot \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} E(x,y) \cdot e^{i\frac{k}{2z}((x'-x)^2 + (y'-y)^2)} d_x d_y \tag{3}$$

For long distances from the diffracting plane with concurrent finite expansion of the diffracting structure, one obtains the *Fraunhofer approximation:*

$$E(x',y',z) = C(x',y',z) \cdot \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} E(x,y) \cdot e^{-2\pi i\left(\frac{x'}{\lambda z}x + \frac{y'}{\lambda z}y\right)} d_x d_y \tag{4}$$

$$\text{with } C(x',y',z) = \frac{e^{ikz}}{i\lambda z} \cdot e^{i\frac{\pi}{\lambda z}(x'^2+y'^2)}$$

with the spatial frequencies as new coordinates:

$$\nu_x = \frac{x'}{\lambda z}\ ;\ \nu_y = \frac{y'}{\lambda z} \tag{5}$$

Consequently, the field distribution in the plane of observation (x',y',z) is shown by the following:

$$E'(x',y',z) = C(\lambda z\nu_x, \lambda z\nu_y, z)\,\tilde{F}\,[E(x,y)](\nu_x,\nu_y) = \tilde{E}(\nu_x,\nu_y) \tag{6}$$

The electric field distribution in the plane (x',y') for z = const is thus established by a Fourier transform of the field strength distribution in the diffracting plane after multiplication with a quadratic phase factor exp $((i\pi/\lambda\ z)\ (x^2+y^2))$.
The spatial frequencies are proportional to the corresponding diffraction angles (see Fig. 3), where:

$$\nu_x = \frac{x'}{\lambda z} = \frac{\tan\alpha}{\lambda} \approx \frac{\alpha}{\lambda}$$

$$\nu_y = \frac{y'}{\lambda z} = \frac{\tan\beta}{\lambda} \approx \frac{\beta}{\lambda}$$

Through the making of a photographic recording or through observation of the diffraction image with one eye, the intensity formation disappears due to the phase information of the light in the plane (x',y',z). As a consequence, only the intensity distribution (this corresponds to the power spectrum) can be observed. As a result the phase factor *C* (Equation 6) drops out of the operation. Therefore, the following results:

$$I(\nu_x, \nu_y) = \frac{1}{\lambda^2 z^2}\ |\ \tilde{F}\ [E(x,y)]\ (\nu_x, \nu_y)\ |^2 \tag{7}$$

Fourier transform by a lens

A biconvex lens exactly performs a two-dimensional Fourier transform from the front to the rear focal plane if the diffracting structure (entry field strength distribution) lies in the front focal plane (see Fig. 4). In this process, the coordinates *v* and *u* correspond to the angles β and α with the following correlations:

$$\nu_x = \frac{x'}{\lambda z} = \frac{\alpha}{\lambda} = \frac{u}{\lambda f_B} \tag{8}$$

$$\nu_y = \frac{y'}{\lambda z} = \frac{\beta}{\lambda} = \frac{v}{\lambda f_B}$$

This means that the lens projects the image of the remote radiation field in the rear focal plane:

$$\tilde{E}(u,v) = A\,(u,v,f_B) \cdot \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} E(x,y) \cdot e^{-2\pi i\left(\frac{u}{\lambda f_B}x + \frac{v}{\lambda f_B}\cdot y\right)} d_x d_y \tag{9}$$

The phase factor *A* becomes independent of *u* and *v*, if the entry field distribution is positioned exactly in the front focal plane. Thus, the *complex amplitude* spectrum results:

$$\tilde{E}(u,v) \sim \tilde{F}\ [E(x,y)]\ (u,v)$$

Again the power spectrum is recorded or observed:

$$I(u,v) = |\tilde{E}(u,v)|^2 \sim |\ \tilde{F}\ [E(x,y)]|^2 \tag{10}$$

It, too, is independent of the phase factor *A* and thus becomes independent of the position of the diffraction structure in the front focal plane.
Additionally, equation 8 shows that the larger the focal length of the lens is, the more extensive the diffraction image in the (u,v) plane is.

Fig. 5: Spectra of a plane wave.
(a) for the direction of light propagation parallel to the optical axis.
(b) for slanted incidence of the plane wave with reference to the optical axis.

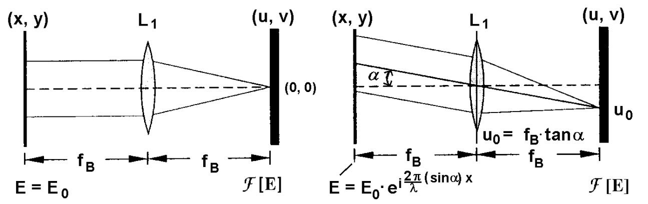
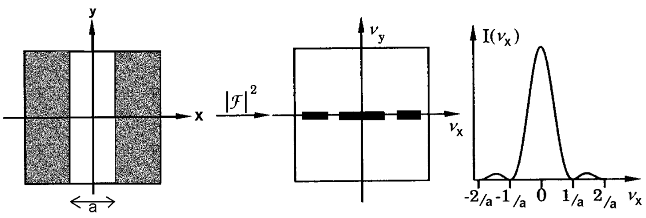


Fig. 6: Infinitely long slit with the width a and its Fourier spectrum.



Examples of Fourier spectra

(a) Plane wave:
A plane wave which propagates itself in the direction of the optical axis (z axis) (Fig. 5) is distinguished in the object plane – (x,y) plane – by a constant amplitude. Thus, the following results for the Fourier transform:

$$E(x,y) = E_0 \qquad (11)$$

$$\text{and } \tilde{F}\,[E(x,y)] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} E_0 e^{-2\pi i\,(\nu_x x + \nu_y y)} d_x d_y$$

$$= E_0 \cdot \delta(\nu_x)\,\delta(\nu_y)$$

This is a point on the focal plane at $(\nu_x,\nu_y) = (0,0)$, which shifts at slanted incidence by an angle α to the optical axis on the rear focal plane (see Fig. 5) with $\nu_x = \sin\alpha/\lambda$.

(b) Infinitely long slit with finite width
If the diffracting structure is an infinite slit which is transilluminated by a plane wave, this slit is mathematically described by a rectangular function rect perpendicular to the slit direction and having the same width a:

$$E(x,y) = \text{rect}\left(\frac{x}{a}\right) = E_0 \begin{cases} 1 & \text{for } |x| < a/2 \\ 0 & \text{else} \end{cases}$$

In the rear focal plane the following spectrum then results:

$$\tilde{F}\,[E(x,y)] = E_0 \int_{-\infty}^{+\infty}\int_{-a/2}^{+a/2} e^{-2\pi i\,(\nu_x x + \nu_y \cdot y)} d_x d_y \qquad (12)$$

$$= E_0 \cdot \delta(\nu_y)\,\frac{\sin(\pi \cdot \nu_x a)}{\pi \nu_x}\,(\nu_y)$$

$$= E_0 \cdot a\delta(\nu_y)\,\text{sinc}\,(a\,\nu_x)$$

with the definition of the slit function sinc:

$$\text{sinc}\,(x) = \frac{\sin(\pi \cdot x)}{\pi \cdot x}$$

For infinitely long extension of the slit, one obtains on extension in the slit direction in the spectrum. This changes for a finite length of the slit.
The zero points of the Sinc function are located at $\ldots -2/a, -1/a, 1/a, 2/a, \ldots$(see Fig.6).

(c) Grid:
A grid is a composite diffracting structure. It consists of a periodic sequence (to be represented by a so-called *comb function comb*) of individual identical slit functions sinc.

The grid consists of M slits having a width a and a slit separation d ($>a$) in the x direction. As a result, the field strength distribution can be in the front focal plane can be represented as follows:

$$E(x,y) = E_0 \sum_{m=1}^{M} \text{rect}\left(\frac{x}{a} - \frac{m \cdot d}{a}\right) = E_0 \left[\sum_{m=1}^{M} \delta(x - m \cdot d)\right] * \text{rect}\,\frac{x}{a}$$

where the Fourier transform of a convolution product $(E_1 * E_2)$ is given by:

$$\tilde{F}\,[(E_1 * E_2)\,(x,y)]\,(\nu_x,\nu_y) = \tilde{F}\,[E_1(x,y)]\,(\nu_x,\nu_y) \cdot \tilde{F}\,[E_2(x,y)]\,(\nu_x,\nu_y)$$

Using the calculation rules for Fourier transforms, the following spectrum results in the rear focal plane of the lens:

$$\tilde{F}\,(E) = E_0 \cdot \delta(\nu_y) \cdot \frac{\sin(\pi a \nu_x)}{\pi \nu_x} \sum_{m=1}^{M} e^{-2\pi i\,m \cdot d \nu_x} \qquad (13)$$

$$= E_0\,\delta(\nu_y)\,a \cdot \text{sinc}\,(a\nu_x) \cdot e^{-\pi i d \nu_x (M+1)}\,\frac{\sin(\pi M d \nu_x)}{\sin(\pi \cdot d\,\nu_x)}$$

Due to the intensity formation the phase factor is cancelled:

$$I\,(\nu_x,\nu_y) = |E_0|^2 \delta(\nu_y) a^2 \,\text{sinc}^2 (a\nu_x)\,\frac{\sin^2(\pi \cdot M \cdot d\,\nu_x)}{\sin^2(\pi\,d\,\nu_x)} \qquad (14)$$

In Fig. 7, a grid with its corresponding spectrum (and the corresponding intensity distributions) is presented.

One sees on the spectrum that the envelope curve is formed by the spectrum of the individual slit which has a width a. The finer structure is produced by the periodicity, which is determined by the grid constant Md.

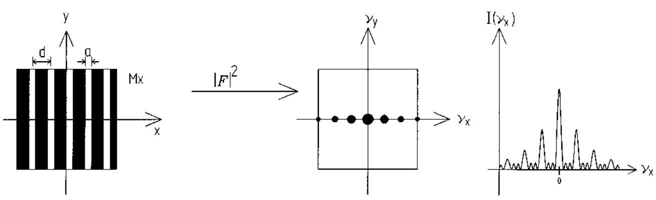


Fig. 7: Grating consisting of M slits and its Fourier spectrum.