Coupled pendula

LEP 1.3.25

Related topics

Spiral spring, gravity pendulum, spring constant, torsional vibration, torque, beat, angular velocity, angular acceleration, characteristic frequency.

Principle and task

Two equal gravity pendula with a particular characteristic frequency are coupled by a “soft” spiral spring. The amplitudes of both pendula are recorded as a function of time for various vibrational modes and different coupling factors using a y/t recorder. The coupling factors are determined by different methods.

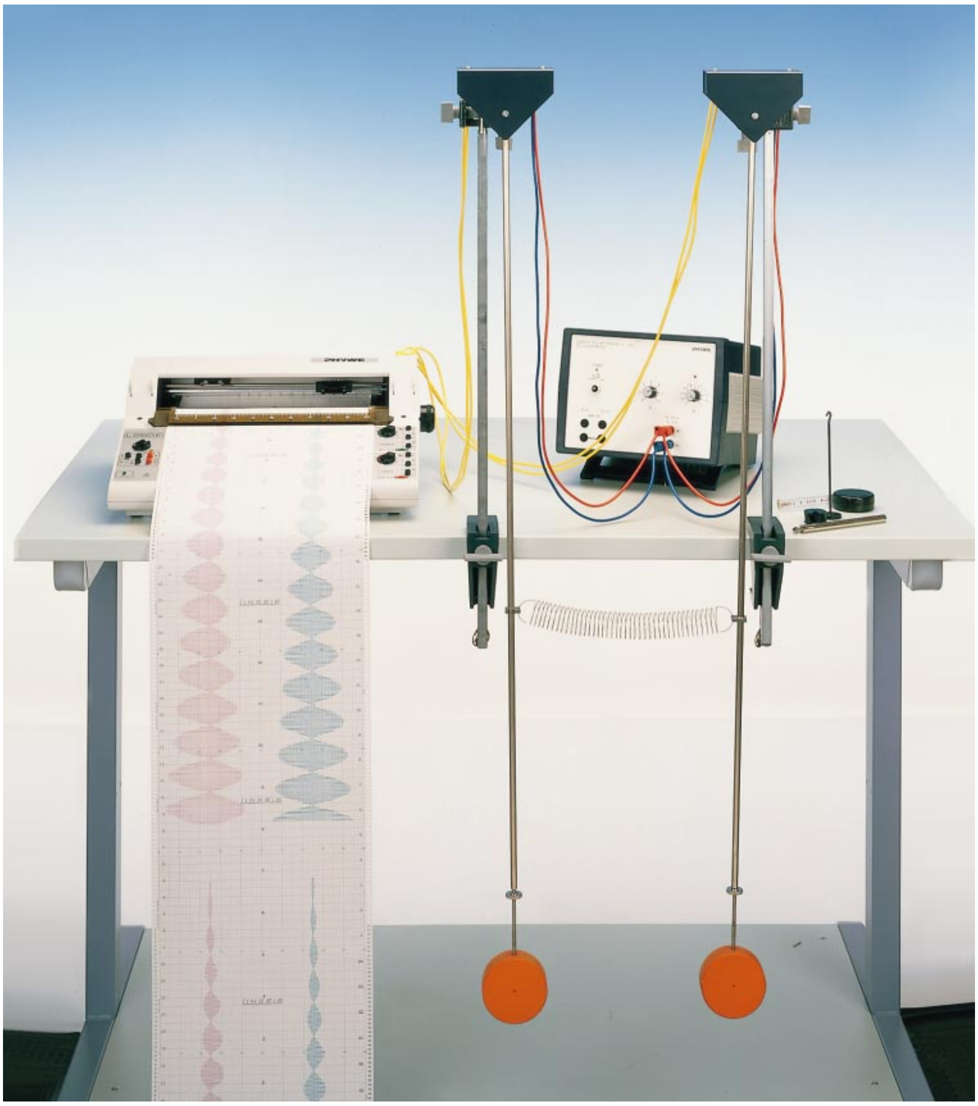
Equipment

Pendulum w. recorder connection	02816.00	2
Helical spring, 3 N/m	02220.00	1
Rod with hook	02051.00	1
Weight holder f. slotted weights	02204.00	1
Slotted weight,10 g, black	02205.01	5
Recorder, tY, 2 channel	11415.95	1
Power supply 0-12V DC/6V,12V AC	13505.93	1
Bench clamp -PASS-	02010.00	2
Support rod -PASS-, square, l 630 mm	02027.55	2
Right angle clamp -PASS-	02040.55	2
Measuring tape, l = 2 m	09936.00	1
Connecting cord, 1000 mm, yellow	07363.02	4
Connecting cord, 500 mm, red	07361.01	2
Connecting cord, 500 mm, blue	07361.04	2

Problems

1. To determine the spring constant of the coupling spring.
2. To determine and to adjust the characteristic frequencies of the uncoupled pendula.
3. To determine the coupling factors for various coupling-lengths using
   a) the apparatus constants
   b) the angular frequencies for “inphase” and “in opposite phase” vibration
   c) the angular frequencies of the beat mode.
4. To check the linear relation between the square of the coupling-lengths and
   a) the particular frequencies of the beat mode
   b) the square of the frequency for “in opposite phase” vibration.
5. To determine the pendulum’s characteristic frequency from the vibrational modes with coupling and to compare this with the characteristic frequency of the uncoupled pendula.

Fig. 1: Experimental set-up for the measurement of the vibrational period of coupled pendula.



Set-up and procedure

Before measurement can begin, the exact value of the spring constant D_F of the coupling spring has to be determined. A supporting rod is fixed to the edge of table by means of a bench clamp. The spring is suspended on the rod from a hook which is attached to the supporting rod via a right angle clamp. Applying Hook's law

$$F = -D_F x$$

the spring constant D_F can be calculated if the extension x of the spring is measured for different slotted weights attached to the spring.

The pendula are then set up without coupling springs as shown in Fig. 1. To record the amplitudes without any time delay one of the pens is pulled out of its holder till the recording of this pen is "on-line" with the recording of the other pen. The "zero"-position of each recording channel is approximately at the center of a paper half. The input sockets of the pendula are now switched in parallel to the DC-output of the power supply unit. The yellow output sockets of the pendula are connected to the recorder. The DC-output voltage of the power supply unit is adjusted to 10 V. For the channels CH 1 and CH 2, a value of 1 V is selected as the measuring range on the recorder. The paper speed should be 1 min/sec. The zero-adjustments have to be reset on the recorder with the help of the potentiometer of the pendula. Finally the pendula are only deflected to such an extent that the individual amplitudes do not exceed the paper width available.

To set the pendula into vibration the pendula rods are touched with the finger-tips on their upper third and simultaneously moved to and fro till the desired amplitudes have been established. In this way transverse vibrations can be avoided. In view of the subsequent experiments with coupled pendula care should be taken already at this stage to ensure that the pendula are oscillating in the same plane.

From the plotted curves the period T_0 is determined several times for each pendulum. The mean values of the periods, $\overline{T}_0$, of both pendula have to be identical within the limits of error. If deviations are observed, the lengths of the pendulum rods have to be adjusted. This is done by detaching the counter nut on the threaded rod of the pendulum weight, adjusting the pendulum length and manually retightening the counter nut.

For the performance of the experiments with coupled pendula, the coupling spring is fixed to the plastic sleeves on the pendulum rods at a point equidistant from the pendulum's fulcrum. Furthermore the "zero"-positions have to be readjusted. It has to be insured that there is no electric conductivity between the pendula.

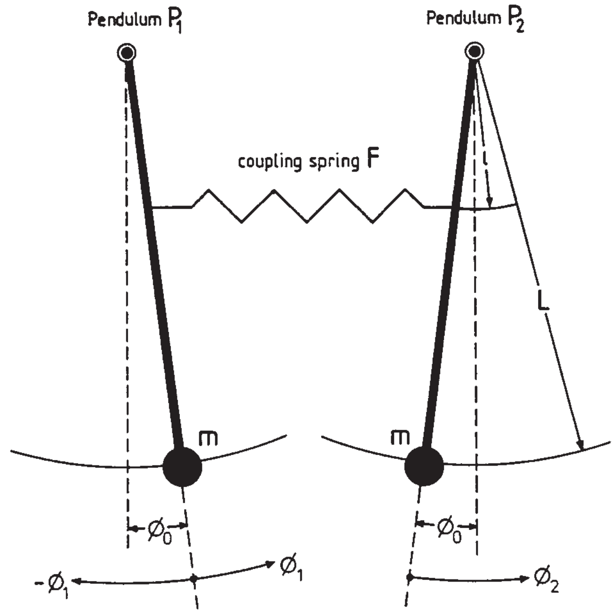
The amplitudes as a function of time are to be recorded for different coupling lenghts lusing the following initial conditions:

A Both pendula are deflected with the same amplitude to the same side and simultaneously released. ("in- phase" vibration)

B. Both pendula are deflected with the same amplitude but in opposite directions and simultaneously released. ("in opposite phase" vibration)

C. One pendulum remains at rest. The second pendulum is deflected and released (beat case). Here satisfactory results can only be achieved if beforehand both pendula have been properly adjusted in such a way that they have in fact the same period $\overline{T}_0$.

Fig. 2: Diagram of coupled pendula at rest.



In all three cases the vibrations have to be recorded for at least three or four minutes. From the plotted curves the mean values for the corresponding vibrational periods can be determined.

Theory and evaluation

If two gravity pendula P_1 and P_2 with the same angular characteristic frequency ω_0 are coupled by a spring, for the position of rest and small angle deviation ~ due to the presence of gravity and spring-tension we have the following torques (Fig. 2):

torque due to gravity:

$$M_{s,0} = m\, g\, L \sin \phi_0 \sim m\, g\, L\phi_0 \qquad (1)$$

torque due to spring-tension:

$$M_{F,0} = -D_F x_0\, l \cos \phi_0 \sim -D_F x_0 l$$

D_F = spring constant
x_0 = extension of the spring
l = coupling length
m = pendulum mass
L = pendulum lengh
g = acceleration due to gravity
ϕ_0 = angle between the vertical and the position of rest

If P_1 is now deflected by ϕ_1 and P_2 by ϕ_2 (see Fig. 2) and subsequently released, we have because of

$$I\ddot{\phi} = M$$

I = moment of inertia of a pendulum around its fulcrum

$$I\ddot{\phi}_1 = M_1 = -\,mgL\phi_1 + D_F l^2(\phi_2 - \phi_1) \quad (2)$$

$$I\ddot{\phi}_2 = M_2 = -\,mgL\phi_2 - D_F l^2(\phi_2 - \phi_1)$$

Introducing the abbreviations

$$\omega_0^2 = \frac{mgL}{I} \text{ and } \Omega^2 = \frac{D_F l^2}{I} \quad (3)$$

we obtain from Eqs. (2)

$$\ddot{\phi}_1 + \omega_0^2\phi_2 = -\,\Omega^2\,(\phi_1 - \phi_2) \quad (4)$$

$$\ddot{\phi}_2 + \omega_0^2\phi_2 = -\,\Omega^2\,(\phi_1 - \phi_2)$$

At $t = 0$ the following three initial conditions are to be realized successively.

A: "inphase" vibration

$$\phi_1 = \phi_2 = \phi_A;\ \phi_1 = \phi_2 = 0$$

B: "in opposite phase" vibration

$$-\,\phi_1 = \phi_2 = \phi_A;\ \phi_1 = \phi_2 = 0 \quad (5)$$

C: beat case

$$\phi_1 = \phi_A;\ \phi_2 = 0;\ \phi_1 = \phi_2 = 0$$

The general solutions of the system of differential equations (4) with the initial conditions (5) are:

A: $$\phi_1(t) = \phi_2(t) = \phi_A \cos \omega_0 t \quad (6a)$$

B: $$\phi_1(t) = \phi_A \cos \sqrt{\omega_0^2 + 2\Omega^2}\, t \quad (6b)$$

$$\phi_2(t) = -\,\phi_A \cos \sqrt{\omega_0^2 + 2\Omega^2}\, t$$

C: $$\phi_1(t) = \phi_A \cos \frac{\sqrt{\omega_0^2 + 2\Omega^2} - \omega_0}{2} \cdot t \quad (6c)$$

$$\cdot \cos \frac{\sqrt{\omega_0^2 + 2\Omega^2} + \omega_0}{2} \cdot t$$

$$\phi_2(t) = -\,\phi_A \sin \frac{\sqrt{\omega_0^2 + 2\Omega^2} - \omega_0}{2} \cdot t$$

$$\cdot \sin \frac{\sqrt{\omega_0^2 + 2\Omega^2} + \omega_0}{2} \cdot t$$

Comment

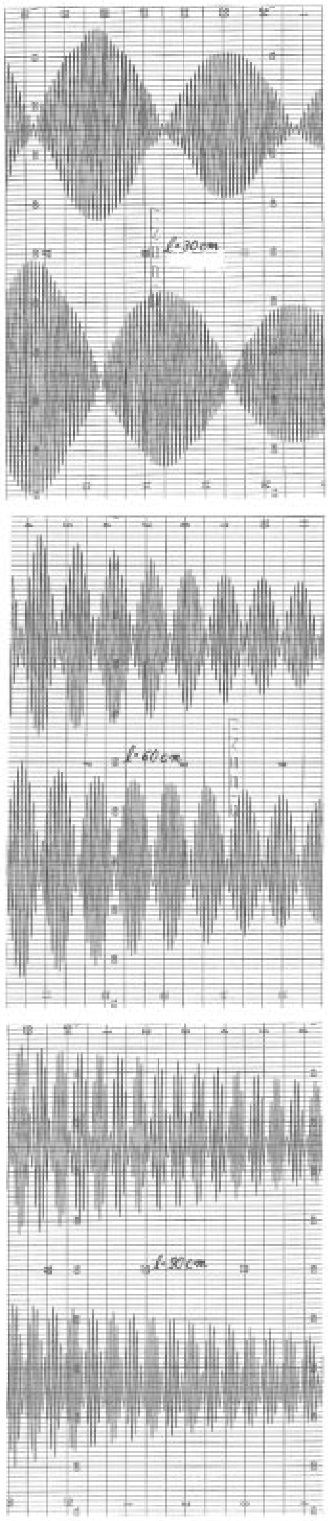
A: "inphase" vibration
Both pendula vibrate inphase with the same amplitude and with the same frequency ω_g. The latter is identical with the angular characteristic frequency w_0 of the uncoupled pendula.

$$\omega_g = \omega_0 \quad (7a)$$

B: "in opposite phase" vibration
Both pendula vibrate with the same amplitude and with the same frequency ω_c but there is a phase-difference of π. In accordance with (3), the angular frequency

$$\omega_c = \sqrt{\omega_0^2 + 2\Omega^2} \quad (7b)$$

depends on the coupling length *l*.



l = 30 cm

l = 60 cm

l = 90 cm

Fig. 3: Amplitude curves of the vibrations of coupled pendula in the beat case for three different coupling lengths *l* as a function of time. Speed of recorder: t = 10 s/Div.

l/cm	$T_g = T_0/s$	T_c/s	$\frac{2\pi}{T_c} = \omega_c/s^{-1}$	T_1/s	$\frac{\pi}{T_1} = \omega_1/s^{-1}$	T_2/s	$\frac{2\pi}{T_2} = \omega_2/s^{-1}$
30	2.024	1.978	3.177	80.75	0.039	2.000	3.142
40	2.019	1.934	3.249	45.74	0.069	1.977	3.178
50	2.028	1.897	3.312	29.79	0.106	1.969	3.191
60	2.027	1.841	3.413	20.76	0.151	1.944	3.232
70	2.028	1.800	3.491	15.59	0.202	1.925	3.264
80	2.023	1.736	3.619	12.29	0.256	1.889	3.326
90	2.021	1.679	3.742	9.85	0.319	1.867	3.365

Tab. 1

l/cm	$K_1 = \frac{D_F l^2}{mgL \pm D_F l^2}$	$K_2 = \frac{\omega_c^2 - \omega_0^2}{\omega_c^2 + \omega_0^2}$	$K_3 = \frac{2\omega_1\omega_2}{\omega_1^2 + \omega_2^2}$	$\frac{(K_1 - K_2)\,100}{K_1}$	$\frac{(K_1 - K_3)\,100}{K_1}$
30	0.0249	0.0245	0.0249	2.5	0
40	0.0435	0.0423	0.0432	2.8	0.7
50	0.0664	0.0636	0.0661	4.2	0.5
60	0.0928	0.0888	0.0934	4.3	0.7
70	0.1223	0.1150	0.1230	6.0	0.6
80	0.1539	0.1415	0.1546	8.1	0.5
90	0.1872	0.1705	0.1878	8.9	0.3

Tab. 2

C: Beat case

For weak coupling, e.g. $\omega_0 \gg \Omega$, the angular frequency of the first factor can be expressed as follows:

$$\omega_1 = \frac{\sqrt{\omega_0^2 + 2\Omega^2} - \omega_0}{2} \simeq \frac{\Omega^2}{2\omega_0} \qquad (8a)$$

For the angular frequency of the second factor we get:

$$\omega_2 = \frac{\sqrt{\omega_0^2 + 2\Omega^2} - \omega_0}{2} \simeq \omega_0 + \frac{\Omega^2}{2\omega_0} \qquad (8b)$$

Subsequently we get:

$$\omega_1 < \omega_2$$

Fig. 3 shows the amplitudes $\phi_1(t)$ and $\phi_3(t)$ of both pendula as a function of time for the beat case and for different coupling lenghts *l*. As coupling factor we define the ratio

$$K = \frac{D_F l^2}{mgL + D_F l^2} \qquad (9)$$

From Eq. (3) and Eq. (9) we get

$$K = \frac{\Omega^2}{\omega_0^2 + \Omega^2} \qquad (10)$$

The coupling factor K of Eq. (10) can be calculated from the frequencies of the individual vibrational modes.

Substituting Eq. (7a) and Eq. (7b) into Eq. (10) results in

$$K = \frac{\omega_c^2 - \omega_g^2}{\omega_c^2 + \omega_g^2} \qquad (11)$$

("in opposite phase" vibration)

Substituting Eq. (8a) and Eq. (8b) into Eq. (10) yields:

$$K = \frac{2\omega_1\omega_2}{\omega_1^2 + \omega_2^2} \qquad (12)$$

(beat case)

To check the influence of coupling length lon the frequencies of the individual vibrational modes, we substitute Eq. (11) and Eq. (12) into Eq. (9). Then we get for the in opposite phase vibration:

$$\omega_c^2 = \frac{2D_F\omega_0^2}{mgL} l^2 + \omega_0^2 \qquad (13)$$

And for the beat case:

$$\omega_1 = \omega_0 \frac{D_F}{2\,mgL} l^2 \qquad (14)$$

as well as

$$\omega_2 = \omega_0 \frac{D_F}{2\,mgL} l^2 + \omega_0 \qquad (15)$$

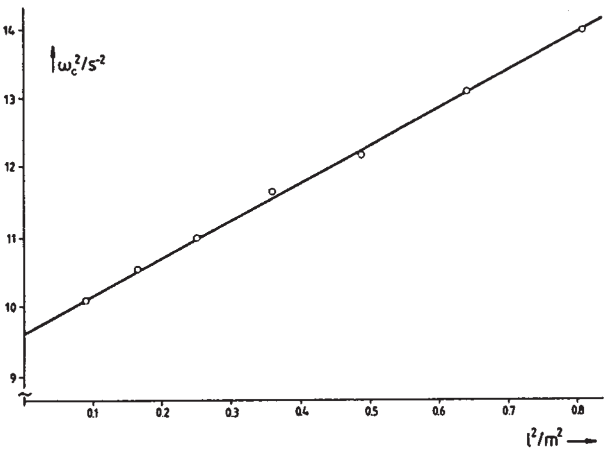
The measurement of the "inphase" vibration of the uncoupled pendula results in the following:

$$\bar{T}_0 = (2.026 \pm 0.003)\text{ s}; \frac{\Delta\bar{T}_0}{\bar{T}_0} \sim \pm 0.15\,\% \qquad (16)$$

or $$\frac{2\pi}{\bar{T}_0} = \bar{\omega}_0 = (3.101 \pm 0.005)\text{ s}^{-1}$$

Tab. 1 shows the mean values of the vibrational periods for different coupling lengths *l* as well as the corresponding angular frequencies.

Fig. 4: Graph of the function $\omega_C^2 = \omega_C^2\,(l^2)$ according to Eq.(13).



From the measured values of the "inphase" vibration (column 2) we get

$$\overline{T}_g \triangleq (2.024 \pm 0.004)\ \text{s};\ \frac{\Delta \overline{T}_0}{\overline{T}_0} \sim \pm 0.2\,\%$$

or $$\frac{2\pi}{\overline{T}_0} = \bar{\omega}_0 = (3.104 \pm 0.006)\ \text{s}^{-1}$$

Column 2 of Tab. 2 contains the coupling factors calculated according to Eq. (9) based on the apparatus constants as a function of the coupling length *l*.

We used:

D_F = 2.9 N/m (measured value)

$L = L_1 = L_2$ = 104 cm

(distance fulcrum center of pendulum weight)

m = 1 kg

(mass of pendulum rod is not included)

g = 9.81 m/s

Columns 3 and 4 contain the coupling factors calculated from the measured angular frequencies for the 'in opposite phase" vibration (Eq. 11) and the beat case (Eq. 12). Columns 5 and 6 show the corresponding percentage deviations.

In Fig. 4 the measured values ω_c^2 of Tab. 1 have been plotted versus l^2. From the regressive line

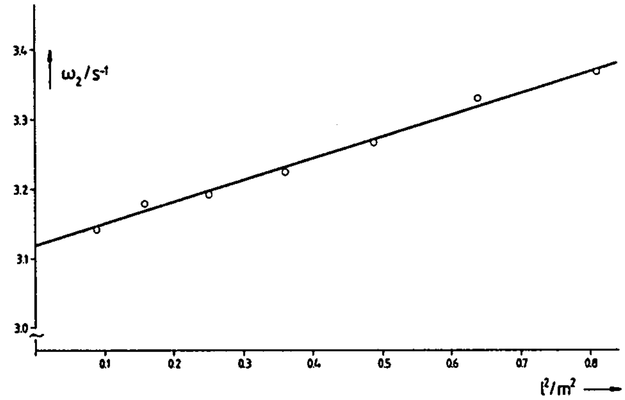
$$y = A + Bx$$

we obtain

$$A = (9.64 \pm 0.58)\ \text{s}^{-2};\ \frac{\Delta A}{A} = \pm 6\,\%$$

$$B = (0.5369 \pm 0.0005)\ \text{s}^{-2}\ \text{m}^{-2};\ \frac{\Delta B}{B} = \pm 0.1\,\%$$

Fig. 5: Graph of the function $\omega_2 = \omega_2\,(l^2)$ according to Eq.(15).



Comparison with Eq. (13) gives

$$\sqrt{A} = \omega_0 = (3.105 \pm 0.093)\ \text{s}^{-1};\ \frac{\Delta\omega_0}{\omega_0} = \pm 3\%$$

In Fig. 5 the measured values ω_0 of Tab. 1 have been plotted versus l^2. The regression line

$$y = A + Bx$$

should confirm Eq. (15). We obtain:

$$A = \omega_0 = (3.12 \pm 0.16)\ \text{s}^{-1};\ \frac{\Delta\omega_0}{\omega_0} = \pm 5\%$$

$$B = (0.309 \pm 0.012)\ \text{s}^{-1}\ \text{m}^{-2};\ \frac{\Delta B}{B} = \pm 4\,\%$$

In Fig. 6 the measured values ω_1 of Tab. 1 are plotted as a function of l^2. The straight line through the origin confirms Eq. (14).

The results obtained for w_{11} using three different vibrational modes for the coupled pendula are in good agreement with the angular characteristic frequency of the uncoupled pendula.

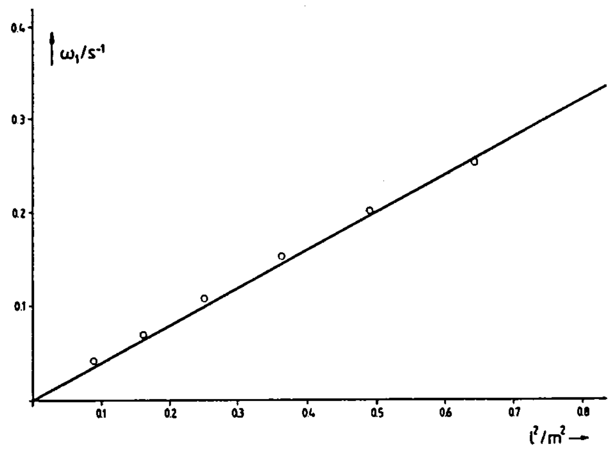


Fig. 6: Graph of the function $\omega_1 = \omega_1\,(l^2)$ according to Eq.(14).