Homework Lecture 6 (Theory 4, Gauge symmetry)

Exercises sent to brightspace on Wednesday 18 March, 12:00
Answers to be submitted individually by Tuesday 21 April 12:00
Please name file as “HW4_Name.pdf”
To: tuning@nikhef.nl
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1 Dirac equation from the Lagrangian

The Dirac equation was found by Paul Dirac by constructing the equation ofxs motion that is both relativistically correct (like the Klein-Gordon equation), and linear in d/dt (like the Schrödinger equation) to avoid negative-energy solutions,

$$(i\gamma^\mu \partial_\mu - m)\psi(x) = 0.$$

Hamilton's principle of stationary (or “least”) action says that the “path” taken by the system between times t_1 and t_2, is the one for which the change in action is minimal. (The action S is obtained from the time-integral of the Lagrangian, i.e. by integrating the diffence of kinetic and potential energy of the system over time.) This requirement is equivalent to the Euler-Lagrange equation

$$\frac{\partial \mathcal{L}}{\partial \psi(x)} = \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi(x))}.$$

Show that the Euler-Lagrange equation of the Lagrangian

$$\mathcal{L} = i\bar{\psi}\gamma_\mu \partial^\mu \psi - m\bar{\psi}\psi$$

leads to the Dirac equation, and its adjoint, $(i\gamma^\mu \partial_\mu + m)\bar{\psi}(x) = 0$. Note that you need to consider ψ and $\bar{\psi}$ as independent fields.

2 Massless gauge bosons

a) The Lagrangian that describes the fermions in QED is

$$\mathcal{L}_{\text{QED,fermion}} = \bar{\psi}(i\gamma^\mu D_\mu - m)\psi.$$

Show that the Lagrangian is invariant under the local gauge transformation

$$\begin{aligned} \psi &\rightarrow \psi' = e^{ie\alpha(x)}\psi, \\ \text{with} \quad A^\mu &\rightarrow A'^\mu = A^\mu - \partial^\mu \alpha(x). \end{aligned} \tag{1}$$

b) Adding the term that describes the free photons (which "by the way" lead to the Maxwell equations $\partial_\mu F^{\mu\nu} = j^\nu$), gives

$$\mathcal{L}_{\text{QED}} = \bar{\psi}(i\gamma^\mu D_\mu - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

If the photon would have a mass, the corresponding mass term would be $\mathcal{L}_{\gamma\,\text{mass}} = \frac{1}{2}m^2 A^\mu A_\mu$. Local gauge invariance implies that the Lagrangian remains unchanged under the transformation

$$A^\mu \rightarrow A'^\mu = A^\mu - \partial^\mu \alpha(x)$$

Show that the mass term of the photon violates local gauge invariance.

3 Self-interacting gauge bosons

Instead of the "simple" phase factor in QED, see Eq. 1, we will now consider a rotation in isospin space

$$\psi \quad \rightarrow \quad \psi' = e^{\frac{i}{2}\vec{\tau}\cdot\vec{\alpha}(x)}\psi, \tag{2}$$

with ψ a two-component object in isospin space.

a) In order to keep the Lagrangian invariant under this gauge transformation, the covariant derivative $D_\mu = \mathbb{1}\partial_\mu + igB_\mu$ is introduced, with B_μ a (2×2) matrix. It can be expressed in terms of the three gauge fields $\vec{b}_\mu(x) = \Big(b_{\mu,1}(x), b_{\mu,2}(x), b_{\mu,3}(x)\Big)$. Write B_μ as a $(2{\times}2)$ matrix, using the Pauli matrices $\vec{\tau}$, starting from $B_\mu = \frac{1}{2}\vec{\tau}\cdot\vec{b}_\mu(x)$.

b) Again, we wish the Lagrangian to stay invariant under the gauge transformation. Let's investigate again what happens with the Lagrangian under the gauge transformation

$$\psi \quad \rightarrow \quad \psi' = e^{\frac{i}{2}\vec{\tau}\cdot\vec{\alpha}(x)}\psi, . \tag{3}$$

We wish that again the derivative behaves like:

$$D_\mu\psi \rightarrow D'_\mu\psi' = e^{\frac{i}{2}\vec{\tau}\cdot\vec{\alpha}(x)}(D_\mu\psi),$$

(i.e. that you can "pull the exponent through" the derivative), such that $\mathcal{L}' = \mathcal{L}$. We will find what then the transformation of the B_μ field should be.
Write out $D'_\mu\psi'$ (using $D_\mu = \mathbb{1}\partial_\mu + igB_\mu$ and $\psi' = e^{\frac{i}{2}\vec{\tau}\cdot\vec{\alpha}(x)}\psi \equiv G\psi$) in terms of B'_μ and G.

c) If you compare your answer to the desired result

$$D'_\mu \psi' = G(D_\mu \psi),$$

show that you then find the following gauge field transformation for B_μ:

$$B'_\mu = G(B_\mu)G^{-1} + \frac{i}{g}(\partial_\mu G)G^{-1}$$

d) (EXTRA) If the gauge transformation is "very small", we can use the approximation (Taylor expansion) $e^{ix} \approx 1 + ix$,

$$e^{\frac{i}{2}\vec{\tau}\cdot\vec{\alpha}(x)} = G \approx \mathbb{1} + \frac{i}{2}\vec{\tau}\cdot\vec{\alpha}(x),$$

to demonstrate that the three $\vec{b}_\mu$ fields transform as

$$\vec{b}'_\mu = \vec{b}_\mu - \vec{\alpha} \times \vec{b}_\mu - \frac{1}{g}\partial_\mu \vec{\alpha}.$$

In other words, the transformation of each of the three $\vec{b}_\mu$ fields, involve the other $\vec{b}_\mu$ fields.
What is the consequence of this for the phenomenology (behaviour) of the gauge fields?