Homework Theory 2

Exercises sent to blackboard on Thursday 20 Feb, 17:00
Answers to be submitted individually by Tuesday 10 March 12:00
Please name file as "HW2_Name.pdf"
To: tuning@nikhef.nl
Format: .pdf

1 Bohr's atom model

One of the "problems" that led to the birth of Quantum Mechanics was the fact that electrons do not spiral onto the nucleus.

a) Consider the orbital momentum of the electron, $L = mvr$, and the classic situation of a stable orbit, $\alpha \frac{q_e q_p}{r^2} = \frac{mv^2}{r}$. Write the expression for L in terms of r (eliminating v).

b) Niels Bohr stated in his paper (Phil.Mag **26**, 1, 1913) that *"for a system consisting of a nucleus and an electron rotating round it, ... the angular momentum of the electron round the nucleus is equal to $h/2\pi$"*. What is then the radius of the orbit of the electron? With $E_{kin} = \frac{1}{2}mv^2$ and $E_{pot} = \frac{-\alpha q_e q_p}{r}$, what is the value for the total energy of the orbiting electron?

2 Yukawa's massive force carrier

Yukawa predicted a massive force carrier. Let's find out the predicted mass.

a) The strong force acts only at the scale of the nucleus. The nucleus has a size of $\sim 10^{-15}$m. To what time-scale does this correspond?

b) To what energy scale, i.e. mass scale, does this correspond?
(Hint: use the constants c and/or $\hbar$ to relate the units.)

3 Spinors

We saw that the requirement of a relativistically correct, but linear equation led to the Dirac equation, $(i\gamma^\mu\partial_\mu - m)\psi = 0$, with ψ being a four component spinor.

a) $H\psi = (\vec{\alpha}\cdot\vec{p} + \beta m)\psi$ gives $E^2 = p^2 + m^2$ if the matrices anticommute, $\{\alpha_i, \alpha_j\} = \alpha_i\alpha_j + \alpha_j\alpha_i = 0$. Usually we use the γ matrices, $\gamma = (\beta, \beta\vec{\alpha})$.
Show that indeed $\gamma_1\gamma_2 = \gamma_2\gamma_1$, using the Pauli-Dirac representation,

$$\beta = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}; \quad \vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix}.$$

With $\psi = u(p)e^{-ipx}$ and $p_\mu \to i\partial_\mu$ we get $(\gamma^\mu p_\mu - m)u(p) = 0$. Looking for the eigenvectors, it is easier to go back to the original form; $Hu = (\vec{\alpha}\cdot\vec{p} + \beta m)u = Eu$, which leads to

$$Hu = \begin{pmatrix} m\mathbb{1} & \vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & -m\mathbb{1} \end{pmatrix}\begin{pmatrix} u_A \\ u_B \end{pmatrix} = E\begin{pmatrix} u_A \\ u_B \end{pmatrix}$$

with $\vec{\sigma}$ the Pauli matrices,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \quad \mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

And thus:

$$(\vec{\sigma}\cdot\vec{p})u_B = (E - m)u_A \tag{1}$$
$$(\vec{\sigma}\cdot\vec{p})u_A = (E + m)u_B, \tag{2}$$

where u_A and u_B are two-component objects. Let's inspect this two-fold degeneracy, and find the observable that distinguishes the two components.

b) Consider an electron with the momentum in the z-direction, $\vec{p} = (0, 0, p)$. What do you find for $\vec{\sigma}\cdot\vec{p}$?

c) What is the eigenvalue of $\frac{1}{2}\vec{\sigma}\cdot\hat{p}$ for the eigenfunction

$$\chi = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

with $\hat{p} = \vec{p}/|\vec{p}|$ the vector in the direction of $\vec{p}$ with unit length. What does this value correspond to, you think?

d) Suppose $\hat{p}$ can point in any direction, what is then the meaning of $\frac{1}{2}\vec{\sigma}\cdot\hat{p}$? What are the possible eigenvalues?

e) Let's consider the operator

$$\vec{\Sigma}\cdot\hat{p} \equiv \begin{pmatrix} \vec{\sigma}\cdot\hat{p} & 0 \\ 0 & \vec{\sigma}\cdot\hat{p} \end{pmatrix},$$

What are its eigenvalues for

$$u^{(1)} = \begin{pmatrix} u_A^{(1)} \\ u_B^{(1)} \end{pmatrix} \quad , \quad u^{(2)} = \begin{pmatrix} u_A^{(2)} \\ u_B^{(2)} \end{pmatrix}$$

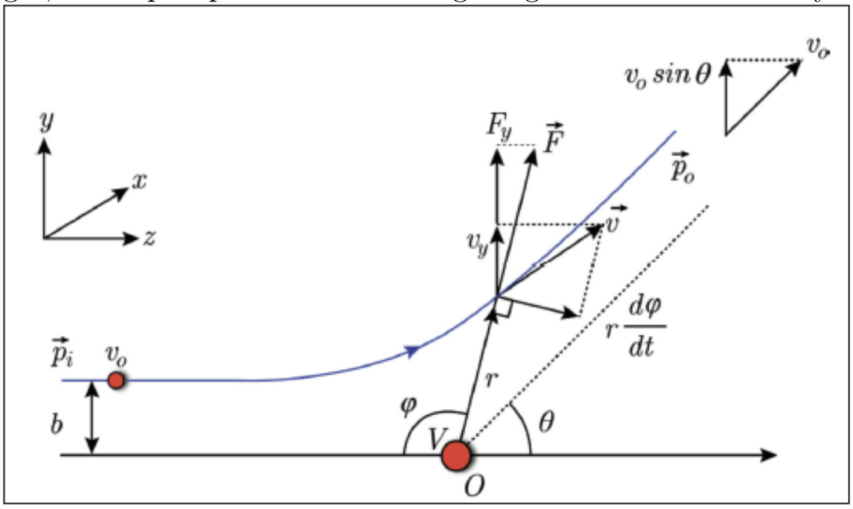
where:

$$u_A^{(1)} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad , \quad u_B^{(1)} = \vec{\sigma}\cdot\vec{p}/(E+m)\begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad u_A^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad , \quad u_B^{(2)} = \vec{\sigma}\cdot\vec{p}/(E+m)\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

(Hint: rotate your frame such that $\vec{p}$ points along the z-axis, such that you only need to worry about p_3.)

4 Rutherford scattering

We calculate the distribution of scattering angles for charged particles on a charged target, like alpha particles scattering off gold nuclei as done by Ernest Rutherford in 1913.



a) The incoming particle arrives with an impact parameter b, and initial velocity v_0. The angular momentum of the initial state is $L = mbv_0$, whereas the angular momentum somewhere after the scatter can be given by $L = mrv_\perp = mr\, d\phi/dr$ Express r in terms of b.

b) The force perpendicular to the direction of the incoming particle is given by $F_y = m\, dv_y/dt$, and $F_y = F\sin\phi = (Z_1 Z_2\alpha/r^2)\sin\phi$.
Give the expression for dv_y/dt, as a function of b (using the result from a).

c) We now multiply both sides with dt, and perform the integral from the start until the end, so the velocity on the left-hand side ranges from $v_y = 0$ to $v_y = v_0 \sin\theta$, and the angle on the right-hand side ranges from $\phi = 0$ to $\phi = \theta$.
Show that

$$\frac{\sin(\theta/2)}{\cos(\theta/2)} = \frac{Z_1 Z_2 \alpha}{m v_0^2} \frac{1}{b}$$

d) For a given surface (ring) of possible incoming particles, $d\sigma = b\, db\, d\phi$, the particle is scattered in a certain solid angle $d\Omega = \sin\theta d\theta d\phi$. Show that the expression for the differential cross section is given by,

$$\frac{d\sigma}{d\Omega} = \frac{b}{\sin\theta} \frac{db}{d\theta} = \left(\frac{Z_1 Z_2 \alpha}{m v_0^2}\right)^2 \frac{1}{4 \sin^4 \frac{\theta}{2}}$$

e) Use the 4-vectors $p_i = (E, 0, 0, m v_0)$ and $p_o = (E, 0, m v_0 \sin\theta, m v_0 \cos\theta)$ for the incoming and outgoing particle, respectively, and express the differential cross section in terms of the 4-momentum transfer $q = p_o - p_i$, instead of θ.

5 Cross section

Let's juggle a bit with cross sections and luminosities.

a) The total cross section for proton-proton scattering at the LHC is about $\sigma_{tot} = 60\,\text{mb}$. To what surface does this cross section correspond? (1 barn $= 10^{-28}\text{m}^2$.) What is the size of an object with similar surface?

b) The cross section for Higgs production at the LHC is approximately $\sigma_{pp\to H+X} = 30\,\text{pb}$. The "luminosity" is the number of particles produced for a given cross-section, and is an important characteristic of the performance of an accelerator. How many Higgs particles are then produced for a total luminosity of $\mathcal{L}_{tot} = 10\ \text{fb}^{-1}$?

c) The "instantaneous" luminosity at the LHC is about $\mathcal{L}_{inst} = 10^{34}\text{s}^{-1}\text{cm}^{-2}$. How many Higgs particles are thus produced per hour?

d) Compare the total proton-proton cross section with the cross section for Higgs production. In what fraction of the proton-proton collisions is a Higgs particle produced?