Homework Theory 1

Exercises sent to Brightspace on Tueday 11 February 2020, 17:00
Answers to be submitted individually by Tuesday 18 February 2020 11:00am
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1 Lorentz transformation

a) The Galilean transformation of the space coordinate, from coordinate system S to system S', with relative velocity v, is given by $x' = x - vt$. What is the Galilean transformation of the time coordinate, between two inertial observers?

b) The Galilean transformation of the space coordinate x, from system S' to S, is given by $x = x' - vt$. Let's find the corresponding transformation if we assume that the speed of light is equal in systems S and S', ie. $x' = ct'$ and $x = ct$. We modify the Galilean transformation rules, by $x' = \gamma(x - vt)$ and find the expression for γ :

$$x' = \gamma(x - vt) \quad \overset{x=ct}{=} \quad \gamma(ct - vt) \tag{1}$$

$$x = \gamma(x' + vt') \quad \overset{x'=ct'}{=} \quad \gamma(ct' + vt') \tag{2}$$

This leads to:

$$\frac{x'}{\gamma} = \frac{ct'}{\gamma} = \frac{\gamma(ct - vt)}{\gamma} = (ct - vt) \tag{3}$$

Eliminate t in the above expression, and give the expression for γ.

c) Rewrite the Lorentztransformation,

$$x' = \gamma(x - vt) \tag{4}$$

$$t' = \gamma(t - \frac{v}{c}x), \tag{5}$$

expressing the velocity as a fraction of the speed of light, $\beta = v/c$, and the time-coordinate as $x^0 \equiv ct$.

d) The time-coordinate, and three space coordinates can be expressed as 4-vectors $x^\mu = (ct, x, y, z)$. Show that the quantity $I = \Sigma_{\mu=0,3}\Sigma_{\nu=0,3}g_{\mu\nu}x^\mu x^\nu = x_\mu x^\mu$ is invariant, ie. that $I = I'$. (Apply a boost in the direction of x^1.)

e) Suppose you want to build a muon collider, and you want to keep your muons about 30 minutes in your accelerator before they decay. What boost (ie. value for γ) is then needed for the muons? (The lifetime of muons is 2.2 μs.) To what beam energy does this correspond? (The mass of the muon is 106 MeV/c^2.)

2 Relativistic momentum

Given 4-vector calculus, we know that $p_\mu p^\mu = E^2/c^2 - \vec{p}^2 = m_0^2 c^2$.

a) Show that you get in trouble when you use $E = mc^2$ and $\vec{p} = m\vec{v}$.

b) Show that $E = \gamma m_0 c^2$ and $\vec{p} = \gamma m_0 \vec{v}$ obey $E^2/c^2 - \vec{p}^2 = m_0^2 c^2$.

Notice that energy and momentum are "treated" in the same way; both get an extra factor γ. Such an "identical treatment" is known as covariance, and implies that the Lorentz transformations and 4-vector description yield a consistent picture.

It is tempting to write the substitution $m = \gamma m_0$, to yield the original formulas $E = mc^2$ and $\vec{p} = m\vec{v}$. This is sometimes referred to as "relativistic mass". However, Albert Einstein himself wrote on 19 June 1948 in a letter to Lincoln Barnet (quote from L.B. Okun (1989), p. 42): "It is not good to introduce the concept of the mass $m = \gamma m_0$ of a moving body for which no clear definition can be given. It is better to introduce no other mass concept than the rest mass m_0. Instead of introducing m it is better to mention the expression for the momentum and energy of a body in motion."

So, from now on every m we use, refers to the rest mass m_0. And we will use natural units, $c = 1$. Hence, $E = \gamma m$!

3 Center-of-mass energy

a) Not only the space and time can be expressed as a 4-vector, but also energy and momentum can be expressed as 4-vectors, $p^\mu = (E/c, p_x, p_y, p_z)$. Because $p_\mu p^\mu$ is invariant, this means that the rest-mass m_0 of a particle does not change under Lorentz transformations. Show that $p_\mu p^\mu = m_0^2 c^2$.

b) Let's consider two colliding particles a and b, with 4-momenta p_a^μ and p_b^μ. We will use natural units, with $c = 1$ and $\hbar = 1$, so $p_a^\mu = (E_a, \vec{p}_a)$. We take the masses of the two colliding particles equal, $m_a = m_b = m$, and we sit in the center-of-mass frame of the system, $\vec{p}_a = -\vec{p}_b$. What are the four components of the sum of the two 4-vectors, $p_{tot}^\mu = (p_a^\mu + p_b^\mu)$?

c) The 'invariant mass' of the combined system, is often called the 'center-of-mass energy' of the collision. If the energy of both particles a and b is 4 TeV, what is then the center-of-mass energy, $\sqrt{s} \equiv \sqrt{p_{tot}^\mu p_{\mu,tot}}$?

d) Let's consider a fixed-target collision of two protons. One proton has an energy of 4 TeV, and 4-vector p_a^μ, whereas the other proton is at rest, with 4-vector p_b^μ. What are the four components of the sum of the two 4-vectors, $p_{tot}^\mu = (p_a^\mu + p_b^\mu)$? Give the expression for the center-of-mass energy of this system.

e) People were afraid that the earth would be destroyed at the start of the LHC, planning for collisions with beams of 7 TeV each. The earth has been bombarded for billions of years with cosmic rays. What is the center-of-mass energy of the highest energetic cosmic rays (10^{21} eV) hitting the atmosphere? Was the fear justified?

f) What is the energy of a cosmic ray hitting the atmosphere, that corresponds to the center-of-mass energy of collisions of two lead-ions ^{208}Pb with energies of 2.24 TeV per nucleon?

g) Consider relatively low-energy proton-proton collisions, with opposite and equal momenta (ie. the center-of-mass system is at rest). In the process $p+p \rightarrow p+p+p+\bar{p}$ an extra proton-antiproton pair is created. What is the minimum energy of the protons to create two extra (anti)protons?